

GENERALIZED SZEGÖ THEOREMS AND ASYMPTOTICS OF CUMULANTS BY GRAPHICAL METHODS

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ABSTRACT. We obtain some general asymptotics results about a class of deterministic sums called “sums with dependent indices,” which generalize a classical theorem of Szegő. The above type of sums is encountered when establishing convergence to the Gaussian distribution of sums of Wick products by the method of cumulants. Our asymptotic results reduce in this situation the proof of the central limit theorem to the study of the connectivity of a family of associated graphs.

INTRODUCTION

The origins of this work are in a series of papers: Breuer and Major [BM], Giraitis and Surgailis [G1] and especially Fox and Taqqu [F2], in which the authors established convergence to the normal of certain sums by studying the asymptotic behavior of their cumulants. In Fox and Taqqu [F2], an important tool used was “the power counting conditions” (used by physicists in quantum field theory), which ensure the convergence of integrals of the form

$$(1.1) \quad \int_{[0,1]^C} \frac{dy_1 \cdots dy_C}{x_1^{z_1} \cdots x_E^{z_E}},$$

where x_e , $e = 1, \dots, E$, are linear combinations with integer coefficients of y_1, \dots, y_C , taken modulo 1, and z_e are positive numbers. Let

$$(x_1, \dots, x_E) = (y_1, \dots, y_C)M,$$

where M is the $C \times E$ matrix of the dependence. It was known that the integral (1.1) converges if for any set of columns A of the matrix M , the power counting conditions

$$(P.C.) \quad \sum_{e \in A} z_e \leq \text{rank}(A), \quad \forall A \subset \{1, \dots, E\}$$

are satisfied.

In this paper, and the paper [AB], we improve on the previous work in two respects.

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(a) We showed in [AB] that under the conditions (P.C.), a Hölder type inequality holds for integrals of the form (1.1). This in turns leads here to some interesting asymptotic results related to a theorem of Szegő, presented in §I, A and B.

(b) We note the existence in matroid theory of various formulae for the function $r(A)$ in the R.H.S. of (1.2), which simplify the analysis of the conditions (P.C.). An example is formula (1.13) in §I, C, applicable when the matroid is induced by a graph.

Using the method of cumulants, in conjunction with the previously mentioned tools, we establish in §II quite complicated central limit theorems for sums of Wick products of Gaussian random variables.

The proofs of the results in §I are given in §III.

In the companion paper [AF], we establish similar results for non-Gaussian random variables, unfortunately under an assumption whose applicability is difficult to check.

I. GENERALIZED SZEGŐ THEOREMS

A. Sums with dependent indices. Let $f^{(e)}(x)$, $e = 1, \dots, E$, be E functions on the torus $[0, 1]$, extended periodically to the whole line, with $f^{(e)} \in L_{p_e}$, $e = 1, \dots, E$. Let $\hat{f}_k^{(e)}$ denotes the Fourier coefficients of $f^{(e)}(x)$, i.e.:

$$(1.2) \quad \hat{f}_k^{(e)} = \int_0^1 e^{2\pi i k x} f^{(e)}(x) dx, \quad e = 1, \dots, E.$$

Definition. A sum with dependent indices is a sum of the form

$$(1.3) \quad S_n = S_n(M, f^{(e)}, e = 1, \dots, E) = \sum_{j_1, \dots, j_V=1}^n \hat{f}_{i_1}^{(1)} \dots \hat{f}_{i_E}^{(E)},$$

where M is a $V \times E$ matrix with integer entries and

$$(i_1, \dots, i_E) = (j_1, \dots, j_V)M.$$

The nullity of the map xM will be denoted by μ ($\mu = V - \text{rank}(M)$). Throughout the paper, we make the assumption

(A) for any row r in M , $\text{rank}(M) = \text{rank}(M \setminus r)$.
Assumption (A) implies $\mu \geq 1$.

Let now M^* be an integer matrix representing the matroid dual to the matroid of M , i.e., a matrix having the same number of columns E as M , and such that the rows of M^* form a basis in the subspace orthogonal to the rows of M . The number of rows C of M^* is thus $C = E - (V - \mu)$.

(Our notation is inspired by the important particular case in which M is the incidence matrix of a graph G with V vertices and E edges, C is the maximal number of independent cycles in G , μ is the number of connected components of G , and the formula above is Euler's formula.)

For any set $A \subset \{1, \dots, E\}$ let $r(A)$ ($r^*(A)$) denote the rank of the set of columns of M (M^*) indexed by A . Let also $z_e = (p_e)^{-1}$ be the reciprocals of the integrability factors p_e . It turns out that the order of magnitude of the sums $S_n(M)$, denoted by $\alpha_M(\mathbf{z})$, depends on $\mathbf{z} = (z_1, \dots, z_E)$ and on the rank function r^* .

Theorem 1. *If the matrix M satisfies assumption (A), then*

$$(1.4) \quad |S_n(M, f^{(e)}, e = 1, \dots, E)| \leq c_M n^{\alpha_M(\mathbf{z})} \prod_{e=1}^E \|f^{(e)}\|_{p_e},$$

where c_M is a constant, and

$$(1.5) \quad \alpha_M(\mathbf{z}) = \mu + \max_{A \subset \{1, \dots, E\}} \left[\sum_{e \in A} z_e - r^*(A) \right].$$

B. The generalized Szegő theorem. It follows from (1.5) that

$$(1.6) \quad \mu \leq \alpha_M(\mathbf{z}) \leq V$$

and $\alpha_M(\mathbf{z}) = \mu$ iff $\sum_A z_e \leq r^*(A)$, $\forall A \subset \{1, \dots, E\}$.

It turns out that when $\alpha_M(\mathbf{z}) = \mu$, the inequality (1.4) is in fact tight, and a Szegő type theorem holds for the sums $S_n(M)$. Let

$$(1.7) \quad I = I(M^*, f^{(e)}, e = 1, \dots, E) := \int_{[0, 1]^C} \prod_{e=1}^E f^{(e)}(x_e) \prod_{c=1}^C dy_c,$$

where $(x_1, \dots, x_e) = (y_1, \dots, y_c)M^*$, each x_e being reduced modulo $[0, 1]$.

Theorem 2. *Let M satisfy assumption (A), and let $f^{(e)}(x) \in L_{p_e}$, $e = 1, \dots, E$, where L_p denotes the closure of trigonometric polynomials in the L_p sense, i.e.:*

$$(1.8) \quad L_p = \begin{cases} L_p & \text{if } p \neq \infty, \\ C & \text{if } p = \infty. \end{cases}$$

If $\alpha_M(\bar{\mathbf{z}}) = \mu$, or, equivalently,

$$(1.9) \quad \sum_{e \in A} z_e \leq r^*(A), \quad \forall A \subset \{1, \dots, E\},$$

then,

$$(1.10) \quad \frac{1}{n^\mu} S_n(M) \rightarrow c_M I(M^*)$$

(the constant c_M is defined in 2.8).

Note. The conditions (1.9) imply that the R.H.S. of (1.10) is well defined and a continuous multilinear functional, by the “generalized Hölder inequality” (see Theorem 1 of [AB]).

The method used in proving Theorem 2 yields also

Corollary 1. *If $\alpha_M \geq \mu$ and $f^{(e)} \in t_{p_e}$, for $e = 1, \dots, E$, then*

$$(1.12) \quad S_m(M) = o(n^{\alpha_M}).$$

C. Graph sums. An interesting special case of a sum with dependent indices is that in which the dependency matrix M is associated with a graph. Let $G = (\mathcal{V}, \mathcal{E})$ be a directed graph with V vertices, E edges, and μ components. For any $v \in \mathcal{V}$ and $e \in \mathcal{E}$, let

$$M_{v,e} = \begin{cases} 0 & \text{if } v \notin e, \\ 1 & \text{if } v \text{ is the end point of } e, \\ -1 & \text{if } v \text{ is the start point of } e. \end{cases}$$

In this case, we will denote the sums (1.3) by $S_n(G)$, and call them “graph sums.” Since this M represents the cycle matroid of the graph (see Bixby [B, p. 350], or Welsh [W, pp. 171–172]) its dual is the bond matroid. Let $c(A)$ denote the number of components in the graph (\mathcal{G}, A) it is known that the rank function of the bond matroid is given by

$$r^*(A) = |A| - c(\mathcal{G} \setminus A) + \mu,$$

where $c(\mathcal{G} \setminus A)$ denotes the number of components left in G after the edges indexed by A have been removed. (This formula follows for example from the formulas

$$r^*(A) = |A| - r(M) + r(M \setminus A)$$

and $r(A) = V - c(A)$, of [W, p. 35, (2.1.5) and p. 29, (1.10.5)].)

Formula (1.5) becomes then in this case

$$(1.13) \quad \alpha_G = \max_{A \subset \{1, \dots, E\}} \left[c(\mathcal{G} \setminus A) - \sum_{e \in A} (1 - z_e) \right].$$

It is basically the use of this formula that makes the deriving of our C.L.T.’s in the next section simpler than the similar results of previous authors. One can think of formula (1.13) as of a “game of breaking the graph:” removing an edge e “costs” $1 - z_e$, and breaking a new component brings a “benefit” of 1; α_G is then the maximal profit possible.

We point out now the particular form Theorems 1, 2 and Corollary 1 take in the case of graph sums.

Theorem 3. *Let $S_n(G)$ be the graph sums associated to a graph G , let μ denote the number of components of the graph, let $C = E - (V - \mu)$ denote the maximal number of independent cycles of G and let M^* be a $C \times E$ matrix defined as follows: select a maximal set \mathcal{C} of C independent cycles in G , assign them arbitrary orientations and let, for $c = 1, \dots, C$ and $e = 1, \dots, E$,*

$$(M^*)_{c,e} = \begin{cases} 0 & \text{if } e \neq c, \\ 1 & \text{if } e \in c, \text{ and their orientations coincide,} \\ -1 & \text{if } e \in c, \text{ and they have opposite orientations.} \end{cases}$$

Then

$$(a) \quad |S_n(G)| \leq c_G n^{\alpha_G} \prod_{e=1}^E \|f^{(e)}\|_{p_e}.$$

$$(b) \quad \text{If, moreover, } f^{(e)} \in L_{p_e}, \text{ for } e = 1, \dots, E, \text{ and } \forall A \subset \mathcal{C},$$

$$(1.14) \quad \sum_{e \in A} (1 - z_e) \geq c(G \setminus A) - \mu,$$

we have

$$(1.15) \quad \frac{1}{n^\mu} S_n(G) \rightarrow I(G) := \int_{[0,1]^C} \prod_{e=1}^E f^{(e)}(x_e) \prod_{c=1}^C dy_c,$$

where $\mathbf{x} = \mathbf{y} M^*$.

$$(c) \quad \text{If } \alpha_G > \mu, \text{ then } S_n(G) = o(n^{\alpha_G}).$$

Note. Theorem 3(b) was first obtained in [AB]. Two particular cases of it were already well known: (a) when the graph G is a cycle, and $f^{(1)} = \dots = f^{(E)}$,

Theorem 3(b) reduces to a well-known result of Szegö (improved in [A]) on the trace of a product of Toeplitz matrices. (b) When $E = 2$, Theorem 3(b) reduces to the classical Parseval relation (see Katznelson [K, p. 35]).

II. THE CENTRAL LIMIT THEOREM

Theorem 3 is a convenient tool for establishing central limit theorems by the method of cumulants.

Corollary 2. *Let T_n be a sequence of zero mean random variables, with cumulants of all orders, for which*

$$(2.1) \quad \text{cum}_s(T_n) = \sum_{G \in \mathcal{G}_s} S_n(G),$$

where $S_n(G)$ are graph sums, and the summation runs over all G in a certain family of connected graphs, \mathcal{G}_s . Then, if

$$(2.2) \quad \alpha_G \leq s/2, \quad \forall G \in \mathcal{G}_s,$$

the central limit theorem $T_n/\sigma\sqrt{n} \xrightarrow{d} N(0, 1)$ holds, with $\sigma^2 = \sum_{G \in \mathcal{G}_2} I(G)$.

Proof. Here, $\mu = 1$, $\forall G \in \mathcal{G}_s$. Note now that $\text{cum}_s(T_n/\sigma\sqrt{n}) \xrightarrow[n \rightarrow \infty]{} \delta_2(s)$, by applying Theorem 3(b) when $s = 2$, and 3(c) when $s \geq 3$.

Corollary 2 may reduce the establishing of very complicated C.L.T's to some simple "graph breaking" problems. We consider now two such specific situations.

Let X_i be a ϕ mean, stationary Gaussian sequence, with $EX_\phi^2 = 1$, and spectral function $f(x)$ (i.e., $EX_\phi X_k = \int e^{2\pi i k x} f(x) dx$), with $f(x) \in L_{p_1}$, let $a(x) \in L_{p_2}$, with $a(x)$ even and let \hat{a}_k denote its Fourier coefficients. Let $z_i := (p_i)^{-1}$, for $i = 1, 2$.

We will present conditions on the z_i which imply the C.L.T. For the following two types of sums:

$$(2.3a) \quad T_n = \sum_{j=1}^n :X_j^{(m)}:$$

$$(2.3b) \quad T_n = \sum_{j,k=1}^n a_{j-k} :X_j^{(m)}, X_k^{(l)}:$$

where $:X_j^{(m)}:$, $:X_j^{(m)}, X_k^{(l)}:$ denote respectively the m th Wick power of X_j and the Wick product of X_j m times and X_k l times. for a definition of the Wick products, see [G2]; below, however, we will need only to use the fact that the cumulants of Wick products can be conveniently expanded by means of the diagram formula (see [G2, Theorem 4(IV)]).

Theorem 4. (a) *If T_n is defined by (2.3a), and $z_1 \leq 1 - 1/m$, then*

$$T_n/\sqrt{n} \rightarrow N(0, \sigma^2),$$

with $\sigma^2 = \sum_{G \in \mathcal{G}_2} I(G)$.

Here, \mathcal{G}_s is the family of connected undirected graphs with s vertices each having degree m .

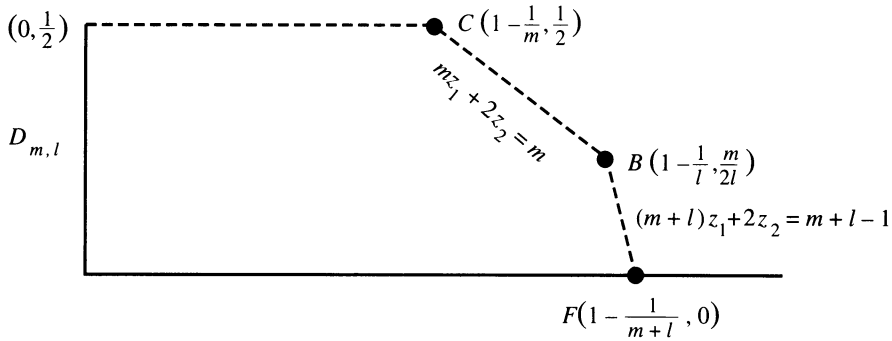


FIGURE 1

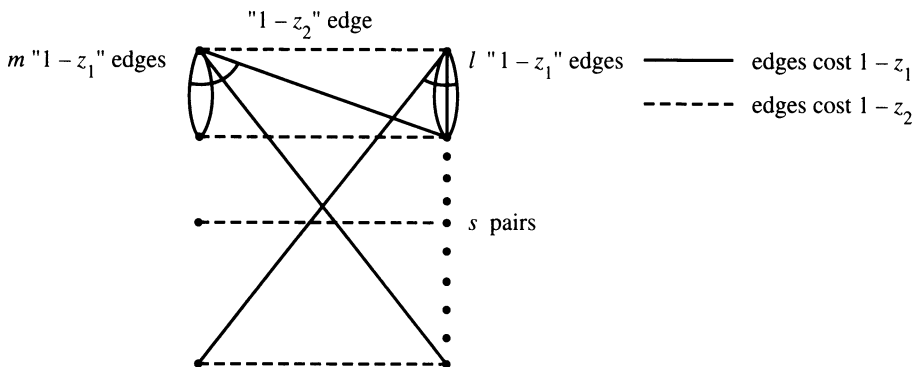


FIGURE 2

(b) If T_n is defined by (2.3b), $2 \leq m \leq l$, and $(z_1, z_2) \in D_{m,l}$ where

$$(2.4) \quad D_{m,l} = \{(z_1, z_2) \in [0, 1]^2 : z_2 \leq 1/2, \\ (m+l)z_1 + 2z_2 \leq m+l-1, mz_1 + 2z_2 \leq m\},$$

then $T_n/\sqrt{n} \rightarrow N(0, \sigma^2)$, with $\sigma^2 = \sum_{G \in \mathcal{G}_s} I(G)$ (see Figure 1). Here, \mathcal{G}_s is the family of all connected undirected graphs formed of s "horizontal" pairs of vertices, each horizontal pair being connected by an edge with "price" $1 - z_2$, and $(m+1)s/2$ "nonhorizontal" edges (i.e., which cannot connect two left vertices of the same pair), with price $1 - z_1$, and arranged such that the left vertices of each pair have all degree $m+1$, and the right vertices have degree $l+1$ (see Figure 2).

Proof. Both (a) and (b) follow from Corollary 2.

One has to show first that (2.1) holds, where \mathcal{G}_s is the corresponding family of graphs. We show now this in case (a) (case (b) requires only minor modifications). By the multilinearity of cumulants,

$$\begin{aligned} \text{cum}_s(T_n) &= \text{cum} \left(\sum_{j_1=1}^n X_{j_1}^{(m)} : , \dots, \sum_{j_s=1}^n X_{j_s}^{(m)} : \right) \\ &= \sum_{\mathbf{j} \in \{1, \dots, n\}} \text{cum}_s(X_{j_1}^{(m)} : , \dots, X_{j_s}^{(m)}). \end{aligned}$$

By the diagram formula (see Theorem 4(IV) of [G2])

$$(2.5) \quad \text{cum}(: X_{j_1}^{(m)} : , \dots , : X_{j_s}^{(m)} :) = \sum_{P=\{(j_{k_1}, j_{l_1}), \dots, (j_{k_{ms/2}}, j_{l_{ms/2}})\} \in P} r_{j_{k_1}-j_{l_1}} \cdots r_{j_{k_{ms/2}}-j_{l_{ms/2}}}$$

where \mathcal{P} is the set of partitions in pairs of the table

$$\begin{array}{c} j_1, \dots, j_1 \quad (m \text{ times}), \\ \vdots \\ j_s, \dots, j_s \quad (m \text{ times}), \end{array}$$

such that each pair connects two distinct rows, and no subset of pairs has as union a subset of rows strictly included in $\{1, \dots, s\}$.

Note now that the set of partitions \mathcal{P} can be put in a 1-to-1 correspondence with the set of connected unoriented graphs \mathcal{G}_s with s vertices of degree m (one for each index j_k , $k = 1, \dots, s$), by representing each partition pair (j_k, j_l) as an edge between the vertices associated to j_k, j_l . Finally, note that summing \mathbf{j} in each term in the R.H.S. of (2.5) corresponding to fixed P yields a sum of the form (1.3), with the matrix M being the incidence matrix of the graph G , arbitrarily oriented (since r_k (and \hat{a}_k in case (b)) is an even sequence, $S_n(G)$ does not depend on the orientation of the edges).

To end the proof, it remains now only to check that (2.2) holds. In case (b), this is done in Lemma 1. We proceed now to show that (2.2) holds in case (a). Since the function $\alpha_G(z_1)$ is increasing in z_1 , it will be enough to consider the “worst” case $z_1 = 1 - 1/m$.

Let the “profit” of a set of edges A be

$$p(A) = c(\mathcal{E} \setminus A) - \sum_{e \in A} (1 - z_e).$$

To find $\alpha_G = \max p(A)$, it is enough to find a set A which achieves the maximum profit, and which is also maximal with respect to inclusion. We will call such a set A a maximal optimal breaking, M.O.B.

We will show now that when the cost of breaking an edge, $1 - z_1$, equals $1/m$, the “total breaking” is the unique M.O.B. Indeed, suppose that after applying a M.O.B., one vertex would be still connected to some others. Cut now all the edges around this vertex. At a cost of no more than $m \times 1/m$, we increase the profit by the least 1, contradicting thus that we had a M.O.B.

Finally, note that the profit of the M.O.B. at $z_1 = 1 - 1/m$ is $p(\mathcal{E}) = s - ms/2$, $1/m = s/2$, and thus

$$\alpha_G(1 - 1/m) = p(\mathcal{E}) = s/2, \quad \forall G \in \mathcal{G}_s,$$

establishing thereby (2.2) and Theorem 4 in the case (a).

Note. Theorem 4(a) was obtained by Breuer and Major [BM]. We included it here to illustrate the fact that our method reduces the quite involved initial proof to the very simple “graph breaking” problem above. The “graph breaking” problem in the more complicated case (b) can be also handled in a similar way.

Lemma 1. Let \mathcal{G}_s be the family of graphs of Theorem 4(b). Then $\forall (z_1, z_2) \in \mathcal{D}_{m,l}$ (defined in (2.4)), and $\forall G \in \mathcal{G}_s$, $s \geq 2$, we have

$$(2.6) \quad \alpha_G(z_1, z_2) \leq s/2.$$

Proof. Since $\alpha_G(\mathbf{z})$ is convex and increasing in \mathbf{z} , it is enough to establish (2.6) for the 3 extremal points of $\mathcal{D}_{m,l}$:

$$F(1 - 1/m + 1, 0), \quad B(1 - 1/l, m/2l), \quad C(1 - 1/m, 1/2).$$

For the point F , one notes that since removing z_2 edges costs 1, there can be no profit in removing them. This in effect fuses together each horizontal pair into a single point, of degree $m + l$, each edge having a cost of $1/(m + 1)$, and so we are back in the case of Theorem 4(a), and thus

$$\alpha_G(F) = s - \frac{(m + 1)s}{2}(1 - z_1) = \frac{s}{2}.$$

For the point B , we note first that the total breaking $A = \mathcal{E}$ achieves a profit of $s/2$:

$$(2.7) \quad p(\mathcal{E}) = 2s - \frac{(m + 1)s}{2} \frac{1}{l} - s \left(1 - \frac{m}{2l}\right) = \frac{s}{2}.$$

Next, we show that \mathcal{E} is the (only) M.O.B. at this cost, i.e. there after the removal of a M.O.B. there can be no component left which is not a singleton. Indeed, note first that a nonsingleton component cannot contain only one of the vertices of a “horizontal” pair, since by further disconnecting this vertex we could increase the benefit by 1 at a cost of at most l/l , thereby increasing the profit. Thus, the nonsingleton component has to be formed of a set of K “horizontal” pairs, for some k . However, just like in (2.7), by totally breaking the component one would increase the profit by $k/2 - 1$, and thus we cannot have $k \geq 2$. Since clearly a M.O.B. cannot leave a one pair component unbroken, it follows that the only components left can be singletons, and $\text{M.O.B.} = \mathcal{E}$. Hence, $\alpha_G(B) = p(\mathcal{E}) = s/2$.

For the point C , the M.O.B. might depend on which particular graph in $\mathcal{G}_{s,m,l}$ we break. However, by an analysis similar with that of point B , one can show that a M.O.B. has to break isolated all the points on the left side of the graph. Let then c_k be the number of components of k points on the right side of the graph, and r the number of edges connecting two points on the right side, left after the removal of a M.O.B. We have to show

$$\alpha_G(C) = s + \sum_{i=1}^{\infty} c_i - \left[\frac{(m + l)s}{2} - r \right] \frac{1}{m} - \frac{s}{2} \leq \frac{s}{2},$$

or

$$\sum_{i=1}^{\infty} c_i \leq \frac{s}{2} + \left[\frac{ls}{2} - r \right] \frac{1}{m}.$$

Since

$$\frac{c_1}{2} + \sum_{i=2}^{\infty} c_i \leq \frac{\sum_{i=1}^{\infty} i c_i}{2} = \frac{s}{2},$$

it is enough to show that

$$(2.8) \quad \frac{m c_1}{2} \leq \left(\frac{ls}{2} - r \right).$$

Let now b , d denote the number of edges in the initial graph with both ends in the right side, and with ends on different sides, respectively. Thus, $d + 2b = ls$. Also, let $b' = b - r$ the number of edges with both ends on the right side which have been removed by the M.O.B. We have

$$lc_1 \leq d + 2b' = d + 2(b - r) = ls - 2r,$$

and thus (2.8) follows. \square

III. PROOFS OF RESULTS IN §I

Proof of Theorem 1. We will need the following integral representation of $S_n(M)$:

$$(3.1) \quad S_n(M) = \int_{[0,1]^E} \prod_{v=1}^V \Delta_n(v_n) \prod_{e=1}^E f^{(e)}(x_e) dx_e,$$

where $\Delta_n(x) = \sum_{k=1}^n e^{2\pi i k x}$, and $u_v = \sum_e M_{v,e} x_e$: as in Lemma 1 of [AB], (3.1) is obtained by plugging $\int e^{2\pi i k x_e} f^{(e)}(x_e) dx_e$ instead of $f_k^{(e)}$ in (1.3).

Let us note now that

$$(3.2) \quad \|\Delta_n\|_{s^{-1}} \leq k(s)n^{1-s}, \quad \forall s \in [0, 1),$$

where the constant $k(s)$ increases with s and explodes at $s = 1$.

To prove Theorem 1, we apply now to the integral representation (3.1) the generalized Hölder inequality (Theorem 1) of [AB]. We get

$$(3.3) \quad \begin{aligned} |S_n(M)| &\leq \prod_{v=1}^V \|\Delta_n\|_{1/(s_v)} \prod_{e=1}^E \|f^{(e)}\|_{p_e}, \\ &\leq \prod_{v=1}^V k(s_v)n^{\sum_{v=1}^V (1-s_v)} \prod_{e=1}^E \|f^{(e)}\|_{p_e}, \end{aligned}$$

$\forall s_1, \dots, s_V \in [0, 1)$ so that the (P.C.) conditions for $(z_e, e = 1, \dots, E, s_v, v = 1, \dots, V)$ and $(x_e, e = 1, \dots, E, u_v, v = 1, \dots, V)$ are satisfied.

Theorem 1 is then an immediate corollary of (3.3) and of

Lemma 2.

$$\alpha_M = \min \sum_{v=1}^V (1 - s_v),$$

where the minimum is taken over all $s_1, \dots, s_V \in [0, 1)$ so that $(z_e, e = 1, \dots, E, s_v, v = 1, \dots, V)$ and $(x_e, e = 1, \dots, E, u_v, v = 1, \dots, V)$ satisfy the (P.C.) conditions. Furthermore, the minimizing (s_1, \dots, s_V) can be chosen such that $s_i \leq 1 - 1/V, \forall i$.

Proof of Lemma 2. Consider the matrix T representing the matroid $(x_e, e \in \{1, \dots, E\}, u_v, v \in \{1, \dots, V\})$:

$$T = (I_E, M^t),$$

when I_n denotes the identity matrix over \mathbf{R}^n . We have to find

$$(3.4) \quad \min \sum_{v=1}^V (1 - s_v), \text{ under the (P.C.) constraints} \\ \left\{ \begin{array}{l} \sum s_v \leq r(A, B) - \sum_{e \in A} z_e, \quad \forall B \subset \{1, \dots, V\}, \quad \forall A \subset \{1, \dots, E\}, \\ s_v \geq 0, \quad v = 1, \dots, V. \end{array} \right.$$

Here, $r(A, B)$ denotes the rank of the corresponding columns in the matrix T . Letting

$$d_B = \min_{A \subset \{1, \dots, E\}} \left[r(A, B) - \sum_{e \in A} z_e \right],$$

the constraints can be further written as

$$(3.5) \quad \left\{ \begin{array}{l} \sum_{v \in B} s_v \geq d_B, \quad \forall B \subset \{1, \dots, V\}, \\ s_v \geq 0, \quad \forall v. \end{array} \right.$$

Using the constraint for $\mathcal{V} := \{1, \dots, V\}$, we get

$$(3.6) \quad \min \sum_{v=1}^V (1 - s_v) \geq V - d_{\mathcal{V}}.$$

The proof proceeds as follows:

(a) We find a point satisfying (3.5), where the constraints $\sum_{v=1}^V s_v = d_{\mathcal{V}}$ holds, proving thereby that (3.6) holds with equality:

$$(3.7) \quad \min \sum_{v=1}^V (1 - s_v) = V - d_{\mathcal{V}}.$$

(b) We show furthermore that the point above can be chosen such that $s_v \leq 1 = 1/V$, $\forall v$.

(c) We show that $r(A, \mathcal{V}) - \mu + r^*(A)$, which together with (3.7) leads to the desired formula

$$\min \sum_{v=1}^V (1 - s_v) = \mu + \max_{A \subset \{1, \dots, E\}} \left[\sum_{e \in A} z_e - r^*(A) \right].$$

(a) Note that the function d_B is submodular, i.e. $d_{B_1 \cup B_2} + d_{B_1 \cap B_2} \leq d_{B_1} + d_{B_2}$, since the rank function $r(A, B)$ is submodular. Thus, the polytope determined by the constraints (3.5) is a polymatroid (see Welsh, 18, 3, Theorem 1); for this type of polytopes, explicit formula for the vertices are available (see [W, 18, 4, Theorem 1]); namely, for any permutation $\sigma = (i_1, i_2, \dots, i_V)$, the formulas $s_{i_1} = d_{\{i_1\}}$, $s_{i_2} = d_{\{i_1, i_2\}} - d_{\{i_1\}}$, \dots , $s_{i_V} = d_{\{i_1, \dots, i_V\}} - d_{\{i_1, \dots, i_{V-1}\}}$ yield a vertex $s^{(\sigma)}$ of the polytope. For this vertex, $\sum_{v=1}^V s_v^{(\sigma)} = d_{\mathcal{V}}$, which establishes (3.6).

(b) Note also that the vertex above satisfies $s_{i_V}^{(\sigma)} = 0$, since by assumption (A),

$$\begin{aligned} d_{\mathcal{V}} &= \min_A \left[(A, \mathcal{V}) - \sum_{e \in A} z_e \right] \\ &= \min_A \left[r(A, \mathcal{V} \setminus i_V) - \sum_{e \in A} z_e \right] = d_{\mathcal{V} \setminus i_V}. \end{aligned}$$

Let now σ denote the cyclic permutation $(2, 3, \dots, V, 1)$, and consider the vertices $s^{(\sigma^i)}$, $i = 1, 2, \dots, V$, where σ^i is the i th iteration of σ . For each of these V points, $\sum_{v=1}^V (1 - s_v^{(\sigma^i)}) = V - d_{\mathcal{V}}$ and $s_i^{(\sigma^i)} = 0$.

Letting now $\mathbf{s} = (1/V) \sum_{v=1}^V s^{(\sigma^i)}$, we obtain a point where the minimum is achieved, and with all coordinates less than $1 - 1/V$.

(c) Consider the general formula $\dim(C \cup B) = \dim(C) + \dim(P_B(C^\perp))$, where C^\perp is the orthogonal complement of C . We apply this to $C = \text{span}\{m_1^t, \dots, m_V^t\}$, where m_1, \dots, m_V are the rows of M , and $B = \text{span}(e_i, i \in A)$, where e_i is the i th unit vector. We get then

$$\begin{aligned} r(A, \{1, \dots, V\}) &= r(\mathcal{V}) + \dim(P_{\text{span}(e_i, i \in A)}(\text{span}(m_1^t, \dots, m_V^t)^\perp)) \\ &= V - \mu + \dim(P_{\text{span}(e_i, i \in A)}(\text{span}(m_1^t, \dots, m_V^t)^\perp)). \end{aligned}$$

The rows of M^* form a basis for $\text{span}(m_1^t, \dots, m_V^t)^\perp$. A moment's thought reveals that the dimension of the projection of the row space of M^* on the span of e_i , $i \in A$, is given by the rank of submatrix of M^* consisting of the columns with indices $i \in A$, yielding thus the formula $r(A, \mathcal{V}) = V - \mu + r^*(A)$.

Proof of Theorem 2. For a given matrix M , the sequence of multilinear functionals

$$T_n(f^{(1)}, \dots, f^{(E)}) := n^{-\mu} S_n(M)$$

is, by Theorem 1, uniformly bounded on $L_{p_1} \times \dots \times L_{p_E}$. In Lemma 3 below we prove that (1.10) holds when $f^{(e)}$, $e = 1, \dots, E$, are the trigonometric functions $f^{(e)}(x) = e^{2\pi i k_e x}$. By multilinearity this can be extended to the case of trigonometric polynomials. These being dense in L_p we can use the uniform boundedness of T_n to obtain (1.10) for all L_p functions.

Lemma 3. Let $f^{(e)}(x) = e^{2\pi i k_e x}$, $e = 1, \dots, E$, where $\mathbf{k} = (k_1, \dots, k_E)$ is a vector of integers. Let $S_n^{(\mathbf{k})}(M)$, $I^{(\mathbf{k})}(M^*)$ denote $S_n(M, f^{(e)})$, $e = 1, \dots, E$) and $I(M^*, f^{(e)})$, $e = 1, \dots, E$) in this case. Then, $\forall \mathbf{k} \in \mathbb{Z}$, we have

$$(3.8) \quad \lim_{n \rightarrow \infty} \frac{1}{n^\mu} S_n^{(\mathbf{k})}(M) = c_M I^{(\mathbf{k})}(M^*),$$

where the constant c_M is defined in (3.11).

Proof. Note that the L.H.S. of (3.8) is

$$\begin{aligned} (3.9) \quad S_n^{(\mathbf{k})}(M) &:= \sum_{j_1, \dots, j_v=1}^n \delta_{k_1}(i_1) \cdots \delta_{k_E}(i_E) \\ &= \text{card}\{\mathbf{j} \in \{1, \dots, n\}^V : \mathbf{j}M = \mathbf{k}\}, \end{aligned}$$

and the R.H.S. of (3.8) is

$$(3.10) \quad \begin{aligned} I^{(\mathbf{k})}(M^*) &:= \int_{[0,1]^C} e^{2\pi i \mathbf{x} \mathbf{k}'} dy \\ &= \int_{[0,1]^C} e^{2\pi i y M^* \mathbf{k}' dy=1} = 1_{\{\mathbf{k} \in R(M)\}}, \end{aligned}$$

where $R(M)$ denotes the subspace generated by the rows of M . (Note also at this point that the R.H.S. of (3.8) is independent of the particular choice of the matrix chosen to represent M^* .)

We consider now three cases:

(a) if $\mathbf{k} \notin R(M)$, then we get 0 in both (3.9) and (3.10).

(b) The case $\mathbf{k} = \mathbf{0}$. Note that since M is an integer matrix, the set K of all integer solutions of $\mathbf{j}M = \mathbf{0}$ is a module of rank μ , contained in the μ -dimensional space $N = \{\mathbf{u}: \mathbf{u}M = \mathbf{0}\}$. We want to determine

$$S_n^{(0)}(M) = \text{card}(K \cap [0, n]^V) = \text{card}\left(\frac{1}{n}K \cap [0, 1]^V\right).$$

Let m_b denote the determinant of a basis of K (= Lebesgue measure on N of a basic cell of K), let m_1 denote the Lebesgue measure on N of $N \cap [0, 1]^V$, and let

$$(3.11) \quad c_M = m_1/m_b.$$

An elementary consideration (tantamount to the definition of the Lebesgue measure) shows that the measures λ_n defined on N by

$$\lambda_n(A) := m_b n^{-\mu} \text{card}\left(\frac{1}{n}K \cap A\right)$$

converge weakly to the Lebesgue measure on N . Thus,

$$(3.12) \quad \frac{S_n^{(0)}(M)}{n^\mu} = \frac{1}{m_b} \lambda_n(N) \xrightarrow{n \rightarrow \infty} c_M.$$

(c) We consider now the last case, when \mathbf{k} satisfies $\mathbf{k} \in R(M)$, but $\mathbf{k} \neq \mathbf{0}$; in this case, (3.12) holds again, since

$$S_{n-p}^{(0)}(M) \leq S_n^{(\mathbf{k})}(M) \leq S_n^{(0)}(M),$$

where p is the maximal coordinate of some fixed preimage \mathbf{k}' of \mathbf{k} (i.e. $\mathbf{k}'M = \mathbf{k}$).

Thus, in all cases, we have

$$\frac{S_n^{(\mathbf{k})}(M)}{n^\mu} \rightarrow c_M I^{(\mathbf{k})}(M^*).$$

Proof of Corollary 1. (1.12) holds when $f^{(e)}$ are trigonometric polynomials (in that case, $\alpha_M = \alpha_M(0, \dots, 0) = \mu$, and by Theorem 2 we have $S_n(M) = O(n^\mu)$). The same approximation argument used in the proof of Theorem 2 allows us then to extend (1.12) to any functions $f^{(e)} \in L_{p_e}$.

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