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FINITE DETERMINATION ON ALGEBRAIC SETS

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ABSTRACT. The concept of finite relative determination was introduced by Porto and Loibel [P-L] in 1978 and it deals with subspaces of $\mathbf{R}^n$. In this paper we generalize this concept for algebraic sets, and relate it with finite determination on the right. We finish with an observation between Łojasiewicz ideals and finite relative determination.

INTRODUCTION

We shall denote by $\mathscr{E}(n)$ the $\mathbf{R}$-algebra of germs of differentiable maps and, by $\mathfrak{m}(n)$ its maximal ideal, and by $\mathbf{R}[x]$ the $\mathbf{R}$-algebra of polynomials with coefficients in $\mathbf{R}$. If f is a germ, $j^m f(0)$ will denote the Taylor expansion up to degree m of f around the origin, and $\langle df \rangle$ will denote the ideal of $\mathscr{E}(n)$ generated by $\partial f/\partial x_j$, the partial derivatives of f. If $j^q(n, 1)$ denotes the space of q-jets, then $\pi_q\colon \mathscr{E}(n) \to j^q(n, 1)$ is the canonical map which assigns $j^q f(0)$ to each f.

Let S be a germ of a subset of $\mathbf{R}^n$ containing the origin and J the ideal of germs which vanish at S. Let G_S be the subgroup of diffeomorphisms which are the identity on S. Let f and g be germs such that $j^k g(0) = j^k f(0)$ and $f - g \in J$. We want to give necessary and sufficient conditions to show that g is in the G_S orbit of f.

The works of Mather [M] and Porto-Loibel [P-L] solve the case for S the set of zeros of the ideals $\langle x_1, \dots, x_n \rangle$ and $\langle x_1, \dots, x_s \rangle$ respectively. In this work we solve the case for more general algebraic sets (Theorem 16), for example, the set $x^2 - y^3 = 0$. We also give two theorems (Theorems 19 and 20) relating finite determinacy on the right and finite determinacy with respect to G_S for a particular algebraic set S which generalizes Theorem 1.10 of [P-L]. We finish with a theorem relating Łojasiewicz's ideals and finite relative determination.

Let S be a germ of a subset of $\mathbf{R}^n$ containing the origin and J the ideal of germs that are zero in S. We consider $\mathscr{L}$, the ideal of germs of vector fields whose coordinates belong to J. If ϕ_t is a one-parameter group germ for X in $\mathscr{L}$, then ϕ_t restricted to S is Id, the identity map.

Let G_S be the group of germs of diffeomorphisms of $\mathbf{R}^n$ such that the identity is restricted to S.

Theorem 0. *The tangent space of G_S at the identity is $\mathscr{L}$.*

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Proof. Let $X \in \mathscr{L}$ and consider ϕ_t, the one-parameter group of X; it is clear that $\phi_0 = \mathrm{Id}$, $\phi_t \in G_S$, and $\frac{\partial}{\partial t}\phi_t(x) = X \circ \phi_t(x)$; if we set $t = 0$ we get $\frac{\partial}{\partial t}\phi_t(x)|_{t=0} = X(x) \in T_{\mathrm{Id}}G_S$.

Conversely, given $v \in T_{\mathrm{Id}}G_S$, there exists $\gamma\colon I \to G_S$ with $\gamma(0) = \mathrm{Id}$ and $\dot\gamma(0) = v$. Since $\gamma(t) \in G_S$ it follows that $\gamma(t)(x) = x\ \forall x \in S$. Then $\dot\gamma(0)(x) = 0\ \forall x \in S$ and v is zero in S.

Definition 1. Let $f \in \mathfrak{m}(n)$. We say f is k-determined relative to G_S if given g such that $j^k f(0) = j^k g(0)$ and $f - g \in J$, there exists $\phi \in G_S$ such that $g = f \circ \phi$.

We state without proof:

Theorem 2. *Let $t_0 \in \mathbf{R}$ be fixed, let f and g be in $\mathfrak{m}(n)$ with $f|_S = g|_S$, and let $F\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ be given by $F(x, t) := F_t(x) = (1-t)f(x) + tg(x)$. Then the following assertions are equivalent*:

(A) *There exists a germ $H\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ such that*

(1) $H(x, t) = x;\ t \sim t_0,\ x \sim 0,\ x \in S,$

(2) $H_{t_0} = \mathrm{Id},$

(3) $F_t \circ H_t = F_{t_0};\ t \sim t_0,$

where $\sim$ means near t_0.

(B) *There exists a germ $h\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ such that*

(I) $\sum_{i=1}^n \frac{\partial F}{\partial x_i}(x, t)h_i(x, t) + \frac{\partial F}{\partial t}(x, t) = 0;\ t \sim t_0,$

(II) $h_i(x, t) = 0;\ t \sim t_0,\ x \sim 0,\ x \in S,$

where $h = (h_1, h_2, \dots, h_n)$. □

Observation. Let

$$(3') \qquad \sum_{i=1}^n \frac{\partial F}{\partial x_i}(h(t, x), t)\frac{\partial h_i}{\partial t}(x, t) + \frac{\partial F}{\partial t}(h(t, x), t) = 0,$$

where $H = (h_1, \dots, h_n)$. Then (1), (2), (3) are equivalent to (1), (2), (3$'$).

Definition 3. Let I be an ideal of $\mathbf{R}[x]$, the ring of polynomials in $x_1, \dots, x_n$ variables, let $z(I) = \{x \in \mathbf{R}^n | f(x) = 0\ \forall f \in I\}$, and suppose $0 \in z(I)$. Then $\widehat{I} = \{f \in \mathscr{E}(n) | f|_{z(I)} \equiv 0\}$. We say I is radical if $I = \widehat{I}$.

Some examples. (1) If $I = \langle x_1, \dots, x_s\rangle$, then $\widehat{I}$ is generated by $\{x_1, \dots, x_s\}$.

(2) If $I = \langle x_i x_j\rangle_{1\le i\le s}^{n-t+1\le j\le n}$ with $s + t \le n$, then $\widehat{I}$ is generated by $\{x_i x_j\}_{1\le i\le s}^{n-t+1\le j\le n}$.

(3) If $I = \langle x_1x_2, x_1x_3, x_2x_3\rangle$, $n = 3$, then $\widehat{I} = I$.

In (3) it is clear that $z(I) = \mathbf{R}\times\{0\}\times\{0\} \cup \{0\}\times\mathbf{R}\times\{0\} \cup \{0\}\times\{0\}\times\mathbf{R}$. By Hadamard's lemma we get for f in $\widehat{I}$:

$$f(x_1, x_2, x_3) = x_1x_2g_{12} + x_1x_3g_{13} + x_2x_3g_{13} + x_1^2g_{11} + x_2^2g_{22} + x_3^2g_{33}.$$

Now

$$\begin{aligned} f(x_1, 0, 0) \equiv 0 &\Leftrightarrow g_{11}(x_1, 0, 0) \equiv 0, \\ f(0, x_2, 0) \equiv 0 &\Leftrightarrow g_{22}(0, x_2, 0) \equiv 0, \\ f(0, 0, x_3) \equiv 0 &\Leftrightarrow g_{33}(0, 0, x_3) \equiv 0. \end{aligned}$$

Then

$$\begin{aligned} x_1^2 g_{11}(x_1, x_2, x_3) &= x_1^2(g_{11}(x_1, x_2, x_3) - g_{11}(x_1, 0, 0)) \\ &= x_1^2(x_2 h_1(x_1, x_2, x_3) + x_3 h_2(x_1, x_2, x_3)), \end{aligned}$$

hence $x_1^2 g_{11} \in \langle x_1x_2, x_1x_3, x_2x_3 \rangle$. Using the same argument we get that $x_2^2 g_{22}$ and $x_3^2 g_{33}$ belong to $\langle x_1x_2, x_1x_3, x_2x_3 \rangle$. □

Theorem 4 [P]. *Let $f \in \mathfrak{m}(n)^\infty$. Then there exists $g \in \mathfrak{m}(n)^\infty$ and $h \in \mathfrak{m}(n)^\infty$ with $g(x) > 0$ for $x \neq 0$ such that*

$$f = gh. \quad \square$$

Corollary 5. *If I is an ideal of $\mathbf{R}[x]$ then $\widehat{I} \cap \mathfrak{m}(n)^\infty = \widehat{I}\mathfrak{m}(n)^\infty$.*

Proof. One contention is obvious. For the other let $f \in \mathfrak{m}(n)^\infty \cap \widehat{I}$; then $f = gh$ as in the previous lemma. Since $f|_S \equiv 0$ we get $h|_S \equiv 0$ and hence $f \in \mathfrak{m}(n)^\infty \widehat{I}$, where $S = z(I)$. □

Theorem 6 (Artin-Rees). *Let $A = \mathbf{R}[[x]]$, $x = (x_1, \dots, x_n)$, be the formal power series ring, with M its maximal ideal and I an ideal of A. Then there exists k such that for $m \geq k$*

$$I \cap \mathbf{M}^m = \mathbf{M}^{m-k}(I \cap \mathbf{M}^k). \quad \square$$

We denote the minimal k with such property by $\mathscr{A}(I)$.

Examples. (1) If $I = \langle x_1, \dots, x_s \rangle$, then $\mathscr{A}(I) = 1$.

(2) If $I = \langle x_i x_j \rangle_{1 \leq i \leq s}^{n-t+1 \leq j \leq n}$, $s + t \leq n$, then $\mathscr{A}(I) = 2$.

(3) If $I = \langle x_1^2 - x_2^3 \rangle$, then $\mathscr{A}(I) = 2$.

Consider $\pi \colon \mathscr{E}(n) \to \mathbf{R}[[x]]$, the canonical Taylor series map, and let I be an ideal in $\mathbf{R}[[x]]$ generated by polynomials. From Theorem 6 we get for $m \geq k$,

$$I \cap \mathfrak{m}^m = \mathfrak{m}^{m-k}(I \cap \mathfrak{m}^k) + \mathfrak{m}^\infty \cap I,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathscr{E}(n)$ and I is now viewed as an ideal in $\mathscr{E}(n)$.

Corollary 7. (1) *For $I = \langle x_1, \dots, x_s \rangle$ we get $I \cap \mathfrak{m}^{l+1} = \mathfrak{m}^l I + \mathfrak{m}^\infty \cap I \; \forall l$.*

(2) *For $I = \langle x_i x_j \rangle_{1 \leq i \leq s}^{n-t+1 \leq j \leq n}$ we get $I \cap \mathfrak{m}^{l+2} = \mathfrak{m}^l I + \mathfrak{m}^\infty \cap I \; \forall l$, where $s + t \leq n$.*

(3) *For $I = \langle x_1^2 - x_2^3 \rangle$ we get $I \cap \mathfrak{m}^{l+2} = \mathfrak{m}^l I + \mathfrak{m}^\infty \cap I \; \forall l$.* □

Lemma 8. *In each of the above cases $I = \widehat{I}$ and hence $\mathfrak{m}^\infty \cap I = \mathfrak{m}^\infty I \subset \mathfrak{m}^l I$. Then we get the following equalities*:

(1) $I \cap \mathfrak{m}^{l+1} = I\mathfrak{m}^l \; \forall l$.

(2) $I \cap \mathfrak{m}^{l+2} = I\mathfrak{m}^l \; \forall l$.

(3) $I \cap \mathfrak{m}^{l+2} = I\mathfrak{m}^l \; \forall l$.

Proof. The first two cases are easy consequences of Hadamard's lemma. For the third case ($n = 2$) let $\phi(x, y) = (x, x^2 - y^3)$. Then by the Malgrange Preparation Theorem for $f \in \mathfrak{m}(2)$ we get

$$f(x, y) = h_0(x, x^2 - y^3) + y h_1(x, x^2 - y^3) + y^2 h_2(x, x^2 - y^3).$$

If $S = \{(x, y) | x^2 - y^3 = 0\}$ and $f|_S \equiv 0$ we get $0 = h_0(x, 0) + yh_1(x, 0) + y^2 h_2(x, 0)$ if $x^2 - y^3 = 0$, hence $0 = h_0(x^3, 0) + x^2 h_1(x^3, 0) + x^4 h_2(x^3, 0)$ and $\pi(h_0(x, 0)) = \pi(h_1(x, 0)) = \pi(h_2(x, 0)) = 0$. Then

$$\begin{aligned} f &= (h_0(x, x^2 - y^3) - h_0(x, 0)) + (h_1(x, x^2 - y^3) - h_1(x, 0))y \\ &\quad + (h_2(x, x^2 - y^3) - h_2(x, 0))y^2 + \eta(x) \\ &= (x^2 - y^3) g(x, y) + \eta(x), \qquad \eta \in \mathfrak{m}(1)^\infty. \end{aligned}$$

Finally, since $f|_S \equiv 0 \Rightarrow \eta(x) \equiv 0$ for $x^2 - y^3 = 0$, it follows that $\eta \equiv 0$ and $\widehat{J} \subset J$. The other contention is obvious. □

Proposition 9. *Let I be an ideal of $\mathbf{R}[x]$ and consider I as an ideal of $\mathscr{E}(n)$. Hence $I = \widehat{I}$ if and only if $\pi(I) = \pi(\widehat{I})$ and $\widehat{I}$ is finitely generated in $\mathscr{E}(n)$.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Our equality is equivalent to $I + \mathfrak{m}(n)^\infty = \widehat{I} + \mathfrak{m}(n)^\infty$; if we intersect with $\widehat{I}$ we get $I + \mathfrak{m}(n)^\infty \cap \widehat{I} = \widehat{I}$. Since $\mathfrak{m}(n)^\infty \cap \widehat{I} = \mathfrak{m}(n)^\infty \widehat{I}$ and $\widehat{I}$ is a finitely generated $\mathscr{E}(n)$-module, by Nakayama's lemma we get $I = \widehat{I}$. □

Observation. By Theorem 2 of [K], $\widehat{I}$ is a finitely generated ideal if and only if $z(I)$ is a coherent algebraic set.

Lemma 10. *Let $I = \langle f_1, \dots, f_s \rangle$ be polynomials in $\mathbf{R}[[x]]$, let $S = z(I)$ be their common zeros, and suppose I is radical. Consider $\widetilde{I} = \langle \tilde{f}_1, \dots, \tilde{f}_s, t \rangle$ in $\mathbf{R}[[x, t]]$, where $\tilde{f}_i(x_1, \dots, x_n, t) = f_i(x_1, \dots, x_n)$. Then $\hat{\tilde{I}} = \widetilde{I}$ and $\mathscr{A}(I) = \mathscr{A}(\widetilde{I})$.*

Proof. It is clear that $z(\widetilde{I}) = S \times \{0\}$. Let

$$\phi \colon \{g \in \mathscr{E}(n+1) | g|_{S \times \{0\}} = 0\} \to \{f \in \mathscr{E}(n) | f|_S = 0\} \times \langle t \rangle \mathscr{E}(n+1)$$

be given by $\phi(g) = (g(x_1, \dots, x_n, 0),\ g - g(x_1, \dots, x_n, 0))$. This map is clearly an isomorphism and hence

$$\hat{\tilde{I}} \simeq \widehat{I} \times \langle t \rangle \mathscr{E}(n+1) = I \times \langle t \rangle \mathscr{E}(n+1).$$

Similarly, $\widetilde{I} = \langle \tilde{f}_1, \dots, \tilde{f}_s, t \rangle \simeq I \times \langle t \rangle \mathscr{E}(n+1)$. Then $\hat{\tilde{I}} = \widetilde{I}$ and using Theorem 6 with $\mathfrak{m}(n+1)$ instead of $\mathfrak{m}(n)$ we have $\widetilde{I} \cap \mathfrak{m}(n+1)^m = \mathfrak{m}(n+1)^{m-k}(\widetilde{I} \cap \mathfrak{m}(n+1)^k)$. □

Theorem 11. *Let I be a radical ideal. If $\mathfrak{m}(n)^m \cap I \subset I\langle df \rangle$ and $I \cap \mathfrak{m}(n)^k$ is finitely generated, where $\mathscr{A}(I) = k$, then f is m-determined relative to G_S, where $S = z(I)$.*

Proof. Let $t_0 \in \mathbf{R}$ be fixed, g a germ with $g|_S \equiv f|_S$, and $j^m f(0) = j^m g(0)$. Consider the map $F \colon (\mathbf{R}^n \times \mathbf{R}, (0, t_0)) \to \mathbf{R}$ given by $F(x, t) = F_t(x) = (1-t) f(x) + t g(x)$.

We will show that F_t is G_S-equivalent to F_{t_0} if $t \sim t_0$.

By Theorem 2 it is enough to find $h \colon (\mathbf{R}^n \times \mathbf{R}, 0 \times t_0) \to \mathbf{R}^n$ such that

(I) $\sum_{i=1}^n \frac{\partial F}{\partial x_i}(x, t) h_i(x, t) + \frac{\partial F}{\partial t}(x, t) = 0$,

(II) $h_i(x, t) = 0$ for $t \sim t_0$, $x \sim 0$ in S.

Let $N = \{\omega \in \mathscr{E}(n+1) | \omega|_{S \times \{t_0\}} = 0$ and $j^{m-1}\omega_t(0) = 0,\ t \sim t_0\}$ and $K = \{\sum_{i=1}^n \frac{\partial F}{\partial x_i}(x, t) h_i(x, t) | h_i$ as in II$\}$.

By Lemma 10, N is a finitely generated module.

If we can show that $N \subset K$, we have $\partial F/\partial t = g - f \in K$ and we obtain conditions (I) and (II).

Letting $h \in N$, we can write $h(x, t) = h(x, t) - h(x, t_0) + h(x, t_0)$. It is clear that $h(x, t) - h(x, t_0) \in \mathfrak{m}(n+1)N$. On the other hand, $h(x, t_0) \in \mathfrak{m}(n)^m \cap I \subset I\langle df\rangle$; then

$$h(x, t_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\eta_i(x), \qquad \eta_i \in I\ \forall i.$$

By hypothesis,

$$g - f \in \mathfrak{m}(n)^{m+1} \cap I = \mathfrak{m}(n)^{m+1-k}(\mathfrak{m}(n)^k \cap I),$$

hence $(\partial g/\partial x_i - \partial f/\partial x_i)(x)\eta_i(x) \in N$ and

$$h(x, t_0) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x, t)\eta_i(x) - t\sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\right)(x)\eta_i(x)$$

is an element of $K + \mathfrak{m}(n+1)N$. Thus, $N \subset K + \mathfrak{m}(n+1)N$, which by Nakayama's lemma implies $N \subset K$. $\square$

Notation. Let $z \in J_0^q(n, 1)$ be the space of q-jets which send $\overline{0}$ to 0, and f a representative of z. Let

$$J_0^q(f, S, n) = \{j^q g(0) | g - f \in J\},$$

and let $\overline{\pi}_q\colon f + J \to J_0^q(f, S, n)$ and $\overline{\pi}_q\colon J \to J_0^q(0, S, n) := J_S^q(n)$ be the restrictions of the canonical map $\pi_q\colon \mathscr{E}(n) \to J^q(n)$.

Finally, let $G_S^q = \{j^q h(0) | h \in G_S\}$ and zG_S^q be the orbit of z.

Proposition 12. *Let I be the ideal of* $\mathbf{R}[x]$. *If $\overline{0} \in S = z(I)$ then G_S^q is a Lie group.*

Proof. We shall show that

$$G_S^q = \{j^q(\mathrm{Id} + (h_1, \dots, h_n)) | h_i \in \widehat{I})\} \cap G^q,$$

where $G^q = G^q_{\{\overline{0}\}}$.

Let $\sigma = j^q\phi \in G_S^q$, where $\phi = (\phi_1, \dots, \phi_n)$; then $\phi|_S = \mathrm{Id}$. If we write $\phi = \mathrm{Id} + (\phi - \mathrm{Id})$, we clearly have that $h_i = \phi_i - x_i \in \widehat{I}$. The other contention is obvious.

Hence G_S^q is a closed subgroup of the Lie group G^q. $\square$

Observation. $T_{\mathrm{Id}}G_S^q = j^q(\widehat{I} \times \cdots \times \widehat{I})$.

Lemma 13. $\overline{\pi}_q^{-1}(T_z zG_S^q) = \widehat{I}\langle df\rangle + \widehat{I} \cap \mathfrak{m}(n)^{q+1}$.

Proof. Let $\beta \in T_{\mathrm{Id}}G_S^q$ be a tangent vector, $\beta = j^q\beta'$. For $t \in R$ we define $\delta_t = \mathrm{Id} + t\beta'$. If we consider $\pi_q \circ \delta_t\colon (-\varepsilon, \varepsilon) \to G_S^q$, then $\beta = \frac{\partial}{\partial t}(\pi_q \circ \delta_t)|_{t=0}$. On the other hand,

$$\frac{\partial}{\partial t}(z \cdot (\pi_q \circ \delta_t))|_{t=0} = \frac{\partial}{\partial t}(\pi_q(f \circ \delta_t))|_{t=0} = \pi_q\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\beta_i'\right),$$

where $\beta_i' \in \widehat{I}$.

Then $T_z zG_S^q = \overline{\pi}_q(\langle df\rangle\widehat{I})$ and hence $\overline{\pi}_q^{-1}(T_z zG_S^q) = \langle df\rangle\widehat{I} + \mathfrak{m}(n)^{q+1}\cap\widehat{I}$. □

Lemma 14. *Let* $q \geq 0$ *and* $z \in J_0^q(n, 1)$ *such that* $z = j^q f$, *and let* $l \leq q$. *If* z *is* l*-determined, then*

$$\widehat{I}\cap\mathfrak{m}(n)^{l+1} \subset \widehat{I}\langle df\rangle + \mathfrak{m}(n)^{q+1}\cap\widehat{I}.$$

Proof. Let $A = \{z' \in J_S^q(n) | \pi_{q,l}(z') = \pi_{q,l}(z)\}$ where $\pi_{q,l}\colon J_0^q(n) \to J_0^l(n)$ is the canonical projection. Since A is an affine space, it follows that $T_zA = \overline{\pi}_q(\widehat{I}\cap\mathfrak{m}(n)^{l+1})$. By hypothesis we have $A \subset zG_S^l$, hence $T_zA \subset T_z zG_S^l$ and $\overline{\pi}_q(\widehat{I}\cap\mathfrak{m}(n)^{l+1}) \subseteq \overline{\pi}_q(\langle df\rangle\widehat{I})$. As before we get $\widehat{I}\cap\mathfrak{m}(n)^{l+1} \subseteq \widehat{I}\langle df\rangle + \mathfrak{m}^{q+1}(n)\cap\widehat{I}$. □

Theorem 15. *Let* f *be an* m*-determined germ relative to* G_S, *where* $S = z(I)$, I *radical, and* $k = \mathscr{A}(I)$. *Then*

$$I\cap\mathfrak{m}(n)^{m+1} \subset I\langle df\rangle \quad \text{for } m \geq k.$$

Proof. Since f is $\mathfrak{m}$-determined relative to G_S, $\overline{\pi}_{m+1}f$ is m-determined relative to G_S^{m+1} and, using Lemma 14 with $k = m$ and $q = m+1$, we obtain

$$\widehat{I}\cap\mathfrak{m}(n)^{m+1} \subset \widehat{I}\langle df\rangle + \mathfrak{m}(n)^{m+2}\cap\widehat{I};$$

but $\mathfrak{m}(n)^{m+2}\cap\widehat{I} = \mathfrak{m}(n)(\mathfrak{m}(n)^{m+1}\cap\widehat{I})$ and by Nakayama's lemma we obtain

$$\widehat{I}\cap\mathfrak{m}(n)^{m+1} \subseteq \widehat{I}\langle df\rangle. \quad \square$$

Joining Theorems 11 and 15 we obtain

Theorem 16. *Let* $f \in \mathfrak{m}(n)$, $I = \langle f_1, \dots, f_s\rangle$ *be a radical ideal in* $R[x]$, *and* S *be the set of common zeros. Suppose* $\mathscr{A}(I) = k$ *and* $\widehat{I}\cap\mathfrak{m}(n)^k$ *is finitely generated. Then* f *is finitely determined relative to* G_S *if and only if there exists* l *such that* $\mathfrak{m}(n)^l\cap I \subset I\langle df\rangle$. □

Observation. Let I be the ideal of $\mathscr{E}(n)$ and suppose $\pi(I\cap\mathfrak{m}(n)^k)$ is generated by $\{h_1, \dots, h_s\}$. We let $f_i \in \mathscr{E}(n)$ be such that $\pi(f_i) = h_i$ for $1 \leq i \leq s$ and we write $f_i = g_i + \xi_i$, where $g_i \in I\cap\mathfrak{m}(n)^k$ and $\xi_i \in \mathfrak{m}(n)^\infty$. Then we have

$$(*) \qquad I\cap\mathfrak{m}(n)^k = \langle g_1, \dots, g_s\rangle + \mathfrak{m}(n)^\infty\cap I.$$

Theorem 17. *If* $I = \widehat{I}$, *the following three assertions are equivalent*:

(1) $I\cap\mathfrak{m}(n)^k$ *is a finitely generated ideal of* $\mathscr{E}(n)$,

(2) $I\cap\mathfrak{m}(n)^k = \langle g_1, \dots, g_s\rangle$,

(3) $\langle g_1, \dots, g_s\rangle \supset I\cap\mathfrak{m}(n)^\infty$.

Proof. (1) $\Rightarrow$ (2). Since $I\mathfrak{m}(n)^\infty = (I\cap\mathfrak{m}(n)^\infty)\mathfrak{m}(n)^\infty$, we have $I\cap\mathfrak{m}(n)^k = \langle g_1, \dots, g_s\rangle + \mathfrak{m}(n)^\infty I \supset \langle g_1, \dots, g_s\rangle + \mathfrak{m}(n)^\infty(I\cap\mathfrak{m}(n)^k) \supset \langle g_1, \dots, g_s\rangle + \mathfrak{m}(n)^\infty(I\cap\mathfrak{m}(n)^\infty) = \langle g_1, \dots, g_s\rangle + \mathfrak{m}(n)^\infty I$.

Then $I\cap\mathfrak{m}(n)^k = \langle g_1, \dots, g_s\rangle + \mathfrak{m}(n)^\infty(I\cap\mathfrak{m}(n)^k)$ and by Nakayama's lemma we get $I\cap\mathfrak{m}(n)^k = \langle g_1, \dots, g_s\rangle$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). From $(*)$ we get $I\cap\mathfrak{m}(n)^k = \langle g_1, \dots, g_s\rangle$. □

Lemma 18. *Let* $\{p_j = x_1^{i_j^1}\cdots x_n^{i_j^n}\}_{j=1}^s$ *be monomials with* $0 \leq i_1^k \leq 1, \dots, 0 \leq i_n^k \leq 1$ *and* $\sum_{k=1}^n i_j^k = \alpha > 0$. *Then*

(1) $I = \widehat{I}$,
(2) $\mathfrak{m}(n)^m \cap I = \mathfrak{m}(n)^{m-\alpha} I \quad \forall m \geq \alpha$,
where I *is the ideal generated by the polynomials* p_j. □

Theorem 19. *Let* f *be a germ finitely determined on the right,* $\{p_j\}_{j=1}^s$ *monomials as in the previous lemma, and* $S = z(I)$. *Then* f *is finitely determined relative to* G_S.

Proof. We know there exists an l such that $\mathfrak{m}(n)^l \subset \langle df \rangle$, hence $\mathfrak{m}(n)^l I \subset I\langle df \rangle$. If we set $l = m - \alpha$, we will have $\mathfrak{m}(n)^m \cap I \subset I\langle df \rangle$. Applying Theorem 11 we finish. □

Theorem 20. *Let* $I = \langle p_j \rangle_{j=1}^s$, p_j *monomials of degree* α *as in Lemma* 19. *Suppose that* f *is finitely determined relative to* G_S, *where* $S = z(I)$, *and that*

$$W_i = \overline{\left(\bigcap_{\substack{j=1 \\ j \neq 1}}^{k} z(p_j) z(I) - z(p_i) \right)} \supset \bigcap_{j=1}^{k} z(p_j) = z(I) \quad \forall_i .$$

Then f *is finitely determined on the right.*

Proof. We know there exists an m such that $\mathfrak{m}(n)^m \cap I \subset I\langle df \rangle$.

Let $x \in \mathfrak{m}(n)^{2(m-\alpha)}$ and put $x = yy'$ with y, y' in $\mathfrak{m}(n)^{m-\alpha}$. Then

$$yp_i \in \mathfrak{m}(n)^{m-\alpha} I = \mathfrak{m}(n)^m \cap I \subset I\langle df \rangle,$$

hence

$$\begin{aligned} yp_i &= \sum_{i=1}^{s} \frac{\partial f}{\partial x_j} h_j \quad (\text{where } h_j \in I = \langle p_1, \dots, p_s \rangle) \\ &= \sum_{j=1}^{s} \sum_{k=1}^{s} h_k^j p_k \frac{\partial f}{\partial x_j} = p_i \sum_{j=1}^{s} h_i^j \frac{\partial f}{\partial x_j} + \sum_{j=1}^{s} \sum_{k \neq i} h_k^j p_k \frac{\partial f}{\partial x_j} . \end{aligned}$$

If we denote $\phi = y - \sum_{j=1}^{s} h_i^j \partial f / \partial x_j$, we get

$$p_i \cdot \phi = \sum_{j=1}^{s} \sum_{j \neq i} h_k^j p_k \left(\frac{\partial f}{\partial x_j} \right) .$$

Hence ϕ vanishes in W_i, and by hypothesis $\phi \in I$. If we denote $\gamma = \sum_{j=1}^{s} h_i^j (\partial f / \partial x_j)$ we obtain $p_i y = p_i(\phi + \gamma)$, so $y = \phi + \gamma$ and

$$x = yy' = \phi y' + \gamma y' .$$

Since $\phi \in I$, $\phi y' \in I\mathfrak{m}(n)^{m-\alpha} = \mathfrak{m}(n)^m \cap I \subset I\langle df \rangle$, and $\gamma y' \in \langle df \rangle$, it follows that $x \in \langle df \rangle$. We have shown that $\mathfrak{m}(n)^{2(m-\alpha)} \subset \langle df \rangle$, therefore f is finitely determined on the right. □

Example. $I = \langle x_1 x_2, x_3 x_4 \rangle$.

Definition 21. Let $f \colon \mathbf{R}^n, 0 \to \mathbf{R}$ be an analytic germ which is finitely determined on the right. Then

$$l(f) = \min\{k \mid \langle df \rangle \supset \mathbf{M}^k \text{ and } \mathbf{M}\langle df \rangle \not\supset \mathbf{M}^k\} .$$

Proposition 22. *Consider* $I = \langle df \rangle$ *in* $\mathbf{R}[[x]]$ *and suppose* $\mathscr{A}(I) = s$. *Then we have* $l(f) = s$.

Proof. From the definition of $l = l(f)$ it is clear that $\mathbf{M}^{l+r} = \langle df \rangle \cap \mathbf{M}^{l+r}$ $\forall r \geq 0$ and $(\langle df \rangle \cap \mathbf{M}^l)\mathbf{M}^r = \mathbf{M}^{l+r}$ $\forall r \geq 0$, hence

$$\langle df \rangle \cap \mathbf{M}^{l+r} = (\langle df \rangle \cap \mathbf{M}^l)\mathbf{M}^r \quad \forall r \geq 0.$$

Thus, $l \geq s$. If $l > s$ we have $(\langle df \rangle \cap \mathbf{M}^s)\mathbf{M}^r = \langle df \rangle \cap \mathbf{M}^{s+r}$ $\forall r \geq 0$.

In particular for $r = 1$ we get $(\langle df \rangle \cap \mathbf{M}^s)\mathbf{M} = \langle df \rangle \cap \mathbf{M}^{s+1}$ and $\mathbf{M}^l \subset \langle df \rangle \cap \mathbf{M}^{s+1} = (\langle df \rangle \cap \mathbf{M}^s)\mathbf{M} \subset \langle df \rangle \mathbf{M}$, but this contention contradicts the choice of $l = l(f)$.

Ideals of Łojasiewicz. Let $C^\infty(\Omega, \mathbf{R})$ be the algebra of smooth functions from an open set Ω in $\mathbf{R}^n$ to $\mathbf{R}$. We let X be a closed subset of $\mathbf{R}^n$.

Definition 23. (1) We say that a function f satisfies a Łojasiewicz inequality for X if for every compact subset K of Ω there exist constants $C > 0$, $\alpha \geq 0$ such that

$$|f(x)| \geq C d(x, X)^\alpha \quad \forall x \in K.$$

(2) An ideal I of $C^\infty(\Omega, \mathbf{R})$ is a Łojasiewicz ideal if there exists a map in I with the Łojasiewicz property for $X = z(I)$, the set of common zeros of I.

(3) $J_k(I)$ is the ideal generated by I and all the $k \times k$ minors of the matrix $(\partial f_i / \partial x_j)$, $1 \leq j \leq k$, $1 \leq j \leq n$, where $f_1, \ldots, f_k$ belong to I.

Proposition 24 (Tougeron). *If* $I = \langle \varphi_1, \ldots, \varphi_p \rangle$ *and* $J_p(I)$ *is a Łojasiewicz ideal, then*

1. I *itself is a Łojasiewicz ideal.*
2. *If* f *is flat on* $z(J_p(I))$ *and* $f|_{z(I)} \equiv 0$, *then* f *belongs to* I. □

Example. $I = \langle x^2 + y^2 \rangle$, $J_1(I) = \langle x, y \rangle$. Hence $z(J_1(I)) = \{\bar{0}\}$ and $\mathfrak{m}(n)^\infty \subset I$. That means that for $f \in \mathfrak{m}(n)^\infty$ there exists g_1 such that $f = (x^2 + y^2)g_1$.

Corollary 25. *If we consider our local case,*

$$I = \langle \varphi_1, \ldots, \varphi_p \rangle \quad \text{and} \quad z(J_p(\varphi_1, \ldots, \varphi_p)) = \{\bar{0}\},$$

where φ_i *are analytic, then*

(1) $\mathfrak{m}(n)^\infty \cap \widehat{I} = \mathfrak{m}(n)^\infty \widehat{I} = \mathfrak{m}(n)^\infty I = \mathfrak{m}(n)^\infty \cap I$,

(2) $\widehat{I}$ *is finitely generated.*

Proof. The first part is a direct consequence of the last proposition.

For the second part let $I = \langle \varphi_1, \ldots, \varphi_p \rangle$. Now $\pi(\widehat{I})$ is finitely generated, hence we have $\pi(\widehat{I}) = \langle h_1, \ldots, h_s \rangle$, $h_i \in \mathbf{R}[[x]]$, $1 \leq i \leq s$. Let $g_i \in \widehat{I}$ with $\pi(g_i) = h_i$, $1 \leq i \leq s$. Therefore $\widehat{I} = \langle g_1, \ldots, g_s \rangle + \widehat{I} \cap \mathfrak{m}(n)^\infty$. We can suppose that $\{\varphi_1, \ldots, \varphi_p\} \subset \{g_1, \ldots, g_s\}$. Since $\widehat{I} \cap \mathfrak{m}(n)^\infty \subset I$ we get $\widehat{I} = \langle g_1, \ldots, g_s \rangle$.

Theorem 26. *Suppose* $I = \langle f_1, \ldots, f_p \rangle$ *is an ideal of analytic maps and that*

(1) $J_p(I)$ *is a Łojasiewicz ideal.*

(2) $z(J_p(I)) = \{\bar{0}\}$.

(3) $\widehat{I} \cap \mathfrak{m}(n)^k$, *where* $k = \mathscr{A}(\pi(\widehat{I}))$ *is finitely generated.*

If $\mathfrak{m}(n)^m \cap \widehat{I} \subset \widehat{I}\langle df \rangle$, *then* f *is m-determined relative to* G_S, *where* $S = z(I)$.

Proof. (1) $\exists k$ with $\pi(\widehat{I}) \cap \mathbf{M}(n)^m = \mathbf{M}(n)^{m-k}(\pi(\widehat{I}) \cap \mathbf{M}^k(n))$.

(2) $\pi^{-1}(\pi(\widehat{I}) \cap \mathbf{M}(n)^m) = \widehat{I} \cap \mathfrak{m}^m(n) + \mathfrak{m}(n)^\infty$.

Let $f \in \pi^{-1}(\pi(\widehat{I}) \cap \mathbf{M}(n)^m)$; then $\pi(f) \in \pi(\widehat{I}) \cap \mathbf{M}(n)^m$. Hence there exists $g \in \widehat{I}$ with $\pi(g) = \pi(f)$ and $\pi(g) \in \mathbf{M}(n)^m$. Hence $g \in \widehat{I} \cap m(n)^m$ and $f \in \widehat{I} \cap m(n)^m + m(n)^\infty$.

Conversely let $g \in \widehat{I} \cap m(n)^m + m(n)^\infty$. Then $\pi(g) \in \pi(\widehat{I}) \cap \pi(m(n)^m) = \pi(\widehat{I}) \cap \mathbf{M}(n)^m$.

(3) $\pi^{-1}(\mathbf{M}(n)^{m-k}(\pi(\widehat{I}) \cap \mathbf{M}(n)^m)) = m(n)^{m-k}(\widehat{I} \cap m(n)^k) + \mathfrak{m}(n)^\infty$. This is done in a similar way to (2).

From (1) we get

$$\widehat{I} \cap m(n)^m + m(n)^\infty = m(n)^{m-k}(\widehat{I} \cap m(n)^k) + m(n)^\infty,$$

and if we intersect each member of the equality with $\widehat{I}$, we get

$$\begin{aligned}\widehat{I} \cap m(n)^m &= m(n)^{m-k}(\widehat{I} \cap m(n)^k) + \widehat{I} \cap m(n)^\infty \\ &= m(n)^{m-k}(\widehat{I} \cap m(n)^k) + \widehat{I} m(n)^\infty \\ &= m(n)^{m-k}(\widehat{I} \cap m(n)^k) \quad \forall m \geq k.\end{aligned}$$

Since $\widehat{I} \cap m(n)^k$ is finitely generated, so is $\widehat{I} \cap m(n)^m$ $\forall m \geq k$. We now proceed as in Theorem 11.

Corollary 27. *Let $f \in m(n)^2$ be a finitely determined analytic map and let I be the ideal generated by f. If $\widehat{I} \cap m(n)^k$ is finitely generated, where k is as in* (3) *of the last theorem, then f is finitely determined relative to G_S, where $S = f^{-1}(0)$.*

Proof. Conditions (1) and (2) of the last theorem are obviously satisfied since there exists $l \in \mathbf{N}$ such that $(x_1^2 + \cdots + x_n^2)^l \in J_1(f)$. Condition (3) is given by hypothesis. Now, since there exists l with $m(n)^l \subset \langle df \rangle$, we get

$$\widehat{I} \cap m(n)^m = m(n)^{m-k}(\widehat{I} \cap m^k(n)) \subset \widehat{I}\langle df \rangle$$

for $m \geq k + l$.

We now use the last theorem to complete the proof. □

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