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A RESTRICTION THEOREM FOR MODULES HAVING A SPHERICAL SUBMODULE

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ABSTRACT. We introduce the following notion: a finite dimensional representation V of a complex reductive algebraic group G is called *spherical of rank one* if the generic stabilizer M is reductive, the pair (G, M) is spherical and $\dim V^M = 1$. Let U be another finite dimensional representation of G; we denote by $S'(U)$ $(S'(U)^G)$ the ring of polynomial functions on U (the ring of G-invariant polynomial functions on U). We characterize the image of $S'(U \oplus V)^G$ under the restriction map into $S'(U \oplus V^M)$ as the $W = N_G(M)/M$ invariants in the Rees ring associated to an ascending filtration of $S'(U)^M$. Furthermore, under some additional hypothesis, we give an isomorphism between the graded ring associated to that filtration and $S'(U)^P$, where P is the stabilizer of an unstable point whose G-orbit has maximal dimension.

I. INTRODUCTION

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ be a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding complexification. Let θ be the associated Cartan involution. Also let $\mathfrak{a}_{\mathbb{R}}$ be a maximal abelian subspace of $\mathfrak{p}_{\mathbb{R}}$ and let $\mathfrak{a}$ be its complexification. Now let K be the analytic subgroup of the adjoint group of $\mathfrak{g}$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Also let M be the centralizer of $\mathfrak{a}$ in K and let W be the Weyl group associated to $(\mathfrak{g}, \mathfrak{a})$, i.e., $W = N_K(M)/M$, where $N_L(S)$ is the notation for the normalizer in L of S.

If V is any finite-dimensional complex vector space, let $S'(V)$ be the ring of all polynomial functions on V. The well-known Chevalley Restriction Theorem states that the restriction homomorphism $S'(\mathfrak{p}) \to S'(\mathfrak{a})$ maps $S'(\mathfrak{p})^K$ isomorphically onto $S'(\mathfrak{a})^W$. (Here V^L denotes the submodule of an L-module V consisting of all L-invariants.) This theorem was generalized by Luna and Richardson [LR]:

Let G be a reductive complex algebraic group acting linearly (and morphically) on a finite-dimensional vector space U and assume that (U, G) has generically closed orbits; i.e., the union of all closed orbits contains a nonempty Zariski open subset of U. Pick any $x \in U$ such that the orbit Gx is closed and G^x is conjugated to G^y for all y in an open neighborhood of x. The conjugacy class of the isotropy subgroup $M = G^x$ is called a principal isotropy class. The generalization of the Chevalley Restriction Theorem given in [LR] states that the restriction map $S'(U) \to S'(U^M)$ maps $S'(U)^G$ isomorphically

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onto $S'(U^M)^W$, where $W = N_G(M)/M$. (A word of caution: this is not a generalization strictu senso because $\mathfrak{a} \neq \mathfrak{p}^M$ in general. The Chevalley Restriction Theorem mentioned in [LR] is the version "$\mathfrak{g}_{\mathbb{R}}$ of type II", in Cartan's terminology).

However, although the Chevalley Restriction Theorem and its generalization are quite powerful tools, they have a restricted field of applications: indeed, almost every representation of a semisimple group has trivial principal isotropy class (see [AP]).

Now let G act linearly on another finite-dimensional vector space N. Then the restriction map $S'(N \oplus U) \to S'(N \oplus U^M)$ induces a monomorphism $S'(N \oplus U)^G \to S'(N \oplus U^M)^{N_G(M)}$ whose image seems to be very difficult to characterize. A first step in this direction was given by Tirao in [T] for the following case: $G = K$, $U = \mathfrak{p}$, $N = \mathfrak{k}$ and $\dim \mathfrak{a} = 1$. The proof given there is geometric and uses a one-parameter subgroup suitably chosen. Another proof can be found in [A1] and one of the aims of this article is to present a generalization of this fact, inspired by this second proof.

We make the following additional hypothesis on (U, G): (a) The pair (G, M) is a spherical pair (sometimes called a Gelfand pair); (b) $\dim U^M = 1$. Then we say that (U, G) is a *spherical representation of rank one.*

In this case, the image of the restriction map for any N is characterized, as in Tirao's case, by

$$\left(\bigoplus_{n \in \mathbb{N}_0} \left(\bigoplus_{\gamma \in \Gamma_n} S'(N)_\gamma^M \otimes S'_n(U^M)\right)\right)^W .$$

Here S'_n is the subspace of all homogeneous polynomials of degree n and

$$\Gamma_n = \{\gamma \in G^\wedge : \gamma^M \neq 0,\, m(\gamma) \leq n\},$$

where $m(\gamma)$ is the degree of homogeneity of γ^* in the harmonic polynomials in U.

A further generalization is found if (U, G) is a "product" of spherical representations of rank one (see also [A2]). Moreover,

$$C_n = \bigoplus_{\gamma \in G^\wedge \,:\, m(\gamma) \leq n} S'(N)_\gamma^M ,$$

defines a filtration of $C = S'(N)^M$.

The second purpose of this article is to characterize the graded ring associated to this filtration as the ring of P-invariants in $S'(N)$ for a suitable subgroup P of G. In the case: $G = K$, $U = \mathfrak{p}$, $N = \mathfrak{k}$, $\dim \mathfrak{a} = 1$ and P is the isotropy subgroup of a principal nilpotent element in $\mathfrak{p}$, this was obtained by Tirao (unpublished) using ideas in the spirit of the proof given in [T]. The proof depends on the existence of a suitable $z \in \mathfrak{k}$. Our proof, however, avoids this and it is available for the general case under some additional conditions. We also remark that one can not expect in general that P contains a maximal unipotent subgroup of G; for example, this is false when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, 1)$, although it is true when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(n, 1)$ or $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(n, 1)$.

This theorem in some sense reduces the study of $S'(U \oplus N)^G$ to the study of $S'(N)^P$. For example, $S'(U \oplus N)^G$ is a polynomial ring if and only if $S'(N)^P$ is a polynomial ring.

Finally, we give some applications; for example, we compute explicitly a presentation of the ring $S'(\mathfrak{g})^K$ when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(2, 1)$.

II. Preliminaries

As usual, $L^\wedge$ denotes the set of equivalence classes of finite dimensional irreducible representations of an algebraic reductive complex linear group L. We will identify $\tau \in L^\wedge$ with the space on which L acts. We will exploit the following well-known version of the

Schur Lemma. *For* $\tau, \lambda \in L^\wedge$, $\dim(\tau \otimes \lambda)^L = 1$ *if* $\tau = \lambda^*$, 0 *otherwise.*

If E is any L-module, and $\tau \in L^\wedge$, we denote by E_τ the isotypic component of type τ.

Let us recall briefly the notion of a spherical pair. Let G be a reductive connected algebraic group and H a closed subgroup, over an algebraically closed field of characteristic zero. (G, H) is a *spherical pair* if it satisfies one of the following equivalent conditions:

(i) H has an open orbit in the flag variety of G.

(ii) H has a finite number of orbits in the flag variety of G.

(iii) Let Z be an algebraic G-variety and let $z \in Z^H$; then G has a finite number of orbits in the closure of Gz.

(iv) Let χ be a one-dimensional representation of H and let X be the representation of G induced by χ (i.e., the space of global sections of the associated line bundle over G/H). Then for every $\gamma \in G^\wedge$, $\dim \operatorname{Hom}_G(\gamma, X) \leq 1$.

(See [BLV] for the history of this result.) It follows from Frobenius reciprocity that (iv) can be also stated as follows: for every $\gamma \in G^\wedge$,

$$\dim \operatorname{Hom}_H(\chi, \gamma) \leq 1,$$

viewing γ as an H-module. In particular, taking χ trivial, $\dim \gamma^H \leq 1$ for all $\gamma \in G^\wedge$. On the other hand this implies the above conditions, whenever H is reductive (see [VK]).

Let G now be a reductive complex linear algebraic group, U a finite-dimensional G-module. Recall that (U, G) is cofree if $S'(U)$ is a $S'(U)^G$-free module. In such a case, $S'(U)^G$ is a polynomial ring (i.e., (U, G) is coregular) and

$$S'(U) = S'(U)^G \otimes H$$

where $\otimes$ is given by multiplication and H is a homogeneous G-submodule of $S'(U)$.

Let M be a principal isotropy group of (U, G) (i.e., the stabilizer of a point in a closed orbit whose conjugacy class is minimal) and let $A = U^M$. Put $W = N_G(M)/M$. We know from [LR] that $S'(U)^G \simeq S'(A)^W$, via the restriction homomorphism.

Lemma 1 (See also [Po]). *Assume that* (U, G) *has generically closed orbits and that* $\dim A = 1$. *Then* (U, G) *is cofree and* W *is a finite cyclic group.*

Proof. Clearly, $\dim A/W$ cannot be zero. (The unique closed orbit would be open.) Thus $\dim A/W = 1$. By [LR] (see the proof of Theorem 4.2) (A, W) has generically closed orbits. So we have

$$\dim A/W = 1 = \dim A - \dim W.$$

Hence W is finite. It is "contained" in $\mathrm{GL}(1, \mathbb{C})$; so it is cyclic and (U, G) is coregular. Now $\operatorname{codim} \pi^{-1}(\zeta) \leq 1$ for every $\zeta \in U/G$; it cannot be 0, so (U, G) is cofree. (See for example, [Sch, §4.3].) $\square$

Definition. We say that (U, G) is *a spherical representation of rank one* if it has generically closed orbits, $\dim A = 1$, and for all $\rho \in G^\wedge$, $\dim \rho^M \leq 1$, where M is a principal isotropy group of (U, G). In particular, (U, G) is irreducible.

Remark 1. There exists a pair (V, L), where L is a simple connected algebraic group, having generically closed orbits and such that:

(i) If M is in the principal isotropy class, V^M is a line but

(ii) (L, M) is not a spherical pair.

Indeed, take $L = A_6$, V the irreducible representation of highest weight φ_3. From [E, Table 1] we know that the generic stabilizer is of type G_2 and hence (see for example [Po2] or [LV]) (V, L) has generically closed orbits. Moreover, $\dim V^M = 1$ [E, Table 1]. One the other hand, (A_6, G_2) is not a spherical pair, as follows from Kramer's table [VK]. In fact, the intersection of Elashvili's and Kramer's tables gives us all the spherical representations of rank one of simple groups.

Remark 2. From Lemma 1, (U, G) is cofree. Let H be as above; then the multiplicity of ρ in H is ≤ 1. (See [Sch, §4.3].)

Definition. We will say that (U, G) is *a spherical representation* (of rank s) if $U = U_1 \oplus \cdots \oplus U_s$, $G = G_1 \times \cdots \times G_s$, each U_i is a G_i-spherical module of rank one and G acts on U via

$$(k_1, \dots, k_s)(v_1, \dots, v_s) = (k_1 v_1, \dots, k_s v_s).$$

Note that it has generically closed orbits.

III. The restriction theorem

Now assume that (U, G) is a spherical representation of rank one, $S'(U) = S'(U)^G \otimes H$ and let $\Gamma = \{\rho \in G^\wedge : \rho^M \neq 0\}$. Then $H = \bigoplus_{\rho \in \Gamma} H_\rho$. Moreover, H_ρ is irreducible and homogeneous. Clearly, $\rho \in \Gamma \Rightarrow \rho^* \in \Gamma$. So we put

$$m(\gamma) = \text{degree of homogeneity of } H_{\gamma^*}, \qquad \gamma \in \Gamma.$$

Now let $(U, G) = (U_1, G_1) \oplus \cdots \oplus (U_s, G_s)$ be a spherical representation, where (U_i, G_i) are spherical representations of rank one. We introduce the following notation:

$$\begin{aligned} A &= A_1 \oplus \cdots \oplus A_s, & \Gamma &= \Gamma_1 \times \cdots \times \Gamma_s, \\ H &= H_1 \otimes \cdots \otimes H_s, & M &= M_1 \times \cdots \times M_s, \end{aligned}$$

where M_i, Γ_i, A_i, H_i correspond to (U_i, G_i).

Recall now that $G^\wedge$ identifies with $G_1^\wedge \times \cdots \times G_s^\wedge$. Thus $\gamma \in \Gamma$ if and only if γ appears in H; and in such a case, it does with multiplicity one. Moreover, let us consider the $\mathbb{N}_0^s$-grading in $S'(U)$ given by the decomposition $U = U_1 \oplus \cdots \oplus U_s$. Therefore, H is an $\mathbb{N}_0^s$-graded G-submodule of $S'(U)$ and for $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_s \in \Gamma$, γ^* appears only in $H_{m(\gamma)}$, where $m(\gamma) = (m(\gamma_1), \dots, m(\gamma_s)) \in \mathbb{N}_0^s$ and $m(\gamma_i)$ corresponds to (U_i, G_i), γ_i.

Clearly, M is a principal isotropy group of (U, G) and $A = U^M$. Let $W = N_G(M)/M$. We consider the order in $\mathbb{N}_0^s$ given by $(a_1, \ldots, a_s) \leq (b_1, \ldots, b_s)$ iff $a_i \leq b_i$ for all $i = 1, \ldots, s$ and we set

$$\Gamma_r = \{\gamma \in \Gamma : m(\gamma) \leq r\},$$

for every $r \in \mathbb{N}_0^s$.

Theorem 1. *Keep the notations and the hypothesis as above. Let N be a finite-dimensional G-module; then the restriction from $N \oplus U$ to $N \oplus A$ induces an isomorphism σ of $S'(N \oplus U)^G$ onto*

$$\left(\bigoplus_{r\in\mathbb{N}_0^s}\left(\bigoplus_{\gamma\in\Gamma_r} S'(N)_\gamma^M \otimes S'_r(A)\right)\right)^W.$$

Proof. The injectivity follows from [LR], Lemma 3.5.

If E_1, E_2 are finite dimensional L-modules, the latter trivial, then $(E_1 \otimes E_2)^L = E_1^L \otimes E_2$. Thus

$$\begin{aligned}
S'(N \oplus U)^G &= (S'(N) \otimes S'(U))^G \\
&= (S'(N) \otimes S'(U)^G \otimes H)^G \\
&= S'(U)^G \otimes (S'(N) \otimes H)^G \\
&= S'(U)^G \otimes \left(\bigoplus_{\lambda\in\Gamma} (S'(N)_\lambda \otimes H_{\lambda^*})^G\right) \\
&\xrightarrow{\sigma} S'(A)^W \otimes \left(\bigoplus_{\lambda\in\Gamma} \sigma(S'(N)_\lambda \otimes H_{\lambda^*})^G\right).
\end{aligned}$$

Let us look more closely at $(S'(N)_\lambda \otimes H_{\lambda^*})^G$. In general, if $S'(N)_\lambda = \bigoplus_i T_i$, $T_i \simeq \lambda$, then $(S'(N)_\lambda \otimes H_{\lambda^*})^G = \bigoplus_i (T_i \otimes H_{\lambda^*})^G$. But σ is 1-1 and

$$\dim(T_i \otimes H_{\lambda^*})^G = \dim(T_i)^M = 1.$$

So

$$\sigma(S'(N)_\lambda \otimes H_{\lambda^*})^G = S'(N)_\lambda^M \otimes S'_{m(\lambda)}(A).$$

Hence

$$\sigma(S'(N \oplus U)^G) = S'(A)^W \otimes \left(\bigoplus_{\lambda\in\Gamma} S'(N)_\lambda^M \otimes S'_{m(\lambda)}(A)\right).$$

Therefore, we have

$$\begin{aligned}
\sigma(S'(N \oplus U)^G) &= \left(\bigoplus_{n\in\mathbb{N}_0^s} S'_n(A) \otimes \left(\bigoplus_{\lambda\in\Gamma} S'(N)_\lambda^M \otimes S'_{m(\lambda)}(A)\right)\right)^W \\
&= \left(\bigoplus_{n\in\mathbb{N}_0^s}\left(\bigoplus_{\lambda\in\Gamma_n} S'(N)_\lambda^M \otimes S'_n(A)\right)\right)^W. \quad \square
\end{aligned}$$

Remark 3. The theorem remains true if we replace N by any affine variety on which G acts.

IV. The ascending filtration

Now let (U, G) be a spherical representation of rank one, X an irreducible G-variety and M a principal isotropy group of (U, G). Let $C = \mathbb{C}[X]^M$ and set

$$C^e = \bigoplus_{\lambda\in\Gamma,\, m(\lambda)=e} \mathbb{C}[X]^M_\lambda, \qquad C_d = \bigoplus_{e\le d} C^e.$$

Theorem 2. $C_0 \subseteq C_1 \subseteq \cdots \subseteq C_j \subseteq \cdots$ *is a filtration of* C.

Proof. It suffices to show: if $\lambda, \gamma \in G^\wedge$, $f \in U_d \subseteq \mathbb{C}[X]$, $U_d \simeq \lambda$, $m(\lambda) = d$, $g \in U_b \subseteq \mathbb{C}[X]$, $U_b \simeq \gamma$, $m(\gamma) = b$, $f, g \in \mathbb{C}[X]^M$, then $f \cdot g \in C_{b+d}$.

We have an epimorphism of G-modules $\gamma \otimes \lambda \to U_b \cdot U_d$. Let

$$\gamma\otimes\lambda = \delta_1 \oplus \cdots \oplus \delta_t, \qquad \delta_i \in G^\wedge.$$

Let us decompose $f \otimes g = h_1 + \cdots + h_s$, $s \le t$, $h_i \in \delta_i - 0$, reordering the index set if necessary. We only need to check that $m(\delta_i) \le b + d$, for all $i \le s$. We have the following well-known isomorphism of G-modules:

$$\mathbb{C}[G/M] \simeq \bigoplus_{\gamma\in G^\wedge} \gamma^M \otimes \gamma^*.$$

From the inclusions given by f and g $\lambda^* \hookrightarrow \lambda^M \otimes \lambda^*$ and $\gamma^* \hookrightarrow \gamma^M \otimes \gamma^*$ we have a morphism of G-modules

$$\lambda^*\otimes\gamma^* \to (\lambda^M\otimes\lambda^*)(\gamma^M\otimes\gamma^*) \subseteq \bigoplus_{i=1,\dots,t} \delta_i^M \otimes \delta_i^*.$$

Now we claim that $(\lambda^M\otimes\lambda^*)(\gamma^M\otimes\gamma^*) \supseteq \bigoplus_{i=1,\dots,s} \delta_i^M \otimes \delta_i^*$. Indeed, if $c_i \in \delta_i^*$, there exist $\alpha_j \in \lambda^*$, $\beta_j \in \gamma^*$ such that $\sum c_i = \sum \alpha_j \otimes \beta_j$ in $\lambda^*\otimes\gamma^*$. Thus for every $x \in G$:

$$\begin{aligned}\left\langle \sum xc_i, \sum h_i \right\rangle &= \left\langle \sum x\alpha_j \otimes x\beta_j, f\otimes g\right\rangle \\ &= \left\langle \sum x\alpha_j, f\right\rangle \cdot \left\langle \sum x\beta_j, g\right\rangle.\end{aligned}$$

That is, $\sum h_i \otimes c_i = (\sum f \otimes \alpha_j)\cdot(\sum g\otimes\beta_j)$.

Now, for any $v \in U$ such that Gv is closed and $G^v = M$, we have the following diagram of G-modules

$$\begin{tikzcd} \lambda^*\otimes\gamma^* \arrow[rr] \arrow[dr] & & H_{\lambda^*}\cdot H_{\gamma^*} \arrow[rr, hook] & & S'(U) \arrow[dl, "\text{Restriction}"] \\ & \mathbb{C}[G/M] \arrow[rr] & & \mathbb{C}[Gv] & \end{tikzcd}$$

which is clearly commutative. Hence the homogeneous module, of degree $b+d$, $H_{\lambda^*}\cdot H_{\gamma^*}$ contains an irreducible G-module of type δ_i^* for each $i = 1, \dots, s$; as $S'(U)_{\delta_i^*} = S'(U)^G \cdot H_{\delta_i^*}$, we conclude that $m(\delta_i) \le b + d$. □

To generalize the preceding, we need the concept of a Δ-filtration; it is surely well known, but as we do not know a good reference, we shall introduce it, along with some formal generalities.

Let $(\Delta, +, <)$ be a monoid equipped with a partial order $<$, compatible with $+$, a p.o.m. for short. For example, let A be a noetherian commutative k-algebra, k a field; $\mathscr{S}(A)=\{W \subseteq A\colon W$ is a k-vector space$\}$ then $(\mathscr{S}(A), \cdot, \subseteq)$ is a p.o.m.

In such case, a Δ-ascending filtration (resp., descending filtration) in A is a morphism of p.o.m. sets $\Delta \xrightarrow{F} \mathscr{S}(A)$ (resp., $\Delta^0 \xrightarrow{F} \mathscr{S}(A)$, where Δ^0 is Δ with the order reversed) such that $F(a)F(b) \subseteq F(a+b)$ for all a, $b \in \Delta$ (resp., $F(a)F(b) \supseteq F(a+b)$) and if 0 is the identity of Δ, then $k \subseteq F(0)$. Here we shall only consider the case of ascending filtrations.

So let A be as above, equipped with a Δ-filtration, and put $A_d = F(d)$, $d \in \Delta$. Clearly, A_0 is a k-subalgebra of A, and each A_d is an A_0-module. Now put

$$A_{(d)} = A_d \Big/ \left(\sum_{e<d} A_e \right) \quad \text{and} \quad \Delta\text{-gr}(A) = \bigoplus_{d \in \Delta} A_{(d)}.$$

The bilinear forms $A_{(d)} \times A_{(b)} \to A_{(d+b)}$ are well defined and extend to Δ-gr(A), giving it a Δ-graded k-algebra structure.

Examples. (i) Let $\Delta = \mathbb{N}_0^s$, $C = S'(N)^M$ be as in Theorem 1. For $r \in \mathbb{N}_0^s$ put $C_r = \bigoplus_{\gamma \in \Gamma_r} S'(W)_\gamma^M$. In fact, this gives a $\mathbb{N}_0^s$-filtration in C, as follows easily from Theorem 2.

Moreover, we have

Lemma 2. *Let* $D = \bigoplus_{r \in \mathbb{N}_0^s} (\bigoplus_{\gamma \in \Gamma, m(\gamma) \leq r} S'(N)_\gamma^M \otimes S_r'(A))$ *and let* J *be the ideal of* D *generated by* A^*. *Then*

$$D/J \simeq \Delta\text{-gr}(C).$$

Proof. Left to the reader. □

(ii) As in the classical case, if $A = \bigoplus_{d \in \Delta} A_{(d)}$ is a Δ-graded k-algebra, then $A_d = \sum_{e \leq d} A_{(e)}$ induces a filtration in A; its Δ-graded algebra is again A.

(iii) Let $\Delta = \mathbb{N}_0^s$, $t_1, \dots, t_N \in \Delta$; we construct a Δ-grading in the polynomial ring $B = k[X_1, \dots, X_N]$ putting for $d \in \Delta$

$$B_{(d)} = \left\langle \left\{ X_1^{h_1} \cdots X_N^{h_N} \colon \sum_i h_i t_i = d \right\} \right\rangle.$$

Now let A be a noetherian commutative k-algebra equipped with a $\mathbb{N}_0^s$-filtration. The following result will be useful later.

Lemma 3. *Let* $x_i \in A_{t_i}$ $(i = 1, \dots, N)$, ξ_i *its image in* $A_{(t_i)}$.

(A) *If the* ξ_i *are* k*-algebraically independent in* Δ-gr(A), *then the* x_i *are algebraically independent in* A.

(B) *If the* ξ_i *generate* Δ-gr(A) *as a* k*-algebra and* $A = \bigcup_{d \in \mathbb{N}_0^s} A_d$, *then the* x_i *generate* A.

Proof (As in [Bo, p. 39]). Consider $B \xrightarrow{\phi} A$, $X_i \mapsto x_i$. Clearly, $\phi(B_d) \subseteq A_d$ and hence we have $B \xrightarrow{\psi} \Delta$-gr$(A)$.

(A) ψ is 1-1; thus $\psi\colon B_d / \sum_{e<d} B_e \to A_d / \sum_{e<d} A_e$ is 1-1. Hence $\phi^{-1}(0) \subseteq B_0$. But $\psi\colon B_{(0)} \to A_{(0)}$ is $\phi\colon B_0 \to A_0$ thus ϕ is 1-1.

(B) Left to the reader. □

Proposition 1. *Let* (U, G) *be a spherical representation,* N *a finite-dimensional* G*-module,* M *a principal isotropy subgroup of* (U, G), $A = U^M$, $W = N_G(M)/M$, $C = S'(N)^M$ *with the filtration introduced above. Let*

$$D = \bigoplus_{n \in \mathbb{N}_0^s} \left(\bigoplus_{\gamma \in \Gamma, d(\gamma) \leq n} S'(N)_\gamma^M \otimes S'_n(A) \right)$$

Consider the following statements:

(a) $S'(U \oplus N)^G = D^W$ *is a polynomial ring*;

(b) D *is a polynomial ring*;

(c) $\operatorname{gr} C$ *is a polynomial ring*;

(d) $S'(N)^M$ *is a polynomial ring*;

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d)

Proof. (b) $\Leftrightarrow$ (c): Let $H_1, \dots, H_s$ be elements of a basis of A^*. As they form a regular sequence in $C \otimes S'(A)$, they do in D. On the other hand, D (and hence, $\operatorname{gr} C$) inherits the usual graded structure of $S'(N) \otimes S'(A)$. With respect to it, let D_+ (resp., $(\operatorname{gr} C)_+$) be the maximal homogeneous ideal of D (resp., of $\operatorname{gr} C$). Thanks to Lemma 2 we have the following exact sequence

$$0 \to \langle H_1, \dots, H_s \rangle \to D_+/(D_+)^2 \to (\operatorname{gr} C)_+/((\operatorname{gr} C)_+)^2 \to 0.$$

Moreover, D is regular iff Krull dim $D = \dim_{\mathbb{C}} D_+/(D_+)^2$ (idem for $\operatorname{gr} C$). Let J be the ideal of D generated by A^*; as it is generated by a regular sequence, $\operatorname{ht}(J) = s$; since D is an integral $\mathbb{C}$-algebra of finite type

$$\operatorname{Krull}\dim(\operatorname{gr} C) + \operatorname{ht}(J) = \operatorname{Krull}\dim D.$$

But

$$\begin{aligned} \operatorname{Krull}\dim(\operatorname{gr} C) + \operatorname{ht}(J) &\leq \dim_{\mathbb{C}}(\operatorname{gr} C)_+/((\operatorname{gr} C)_+)^2 + s \\ &= \dim_{\mathbb{C}} D_+/(D_+)^2 \geq \operatorname{Krull}\dim D, \end{aligned}$$

and the announced equivalence follows easily.

(a) $\Leftrightarrow$ (b) From the proof of (b) $\Leftrightarrow$ (c) it follows that there exist $j(1), \dots, j(s) \in \mathbb{N}_0^s$, $f_i \in E(N, M, j(i))$ for each i, such that

$$D = \mathbb{C}[f_i H^{j(i)}, J_1, \dots, H_s],$$

where if $j \in \mathbb{N}_0^s$, $H^j = \prod_k H_k^{j(k)}$. But the $f_i H^{j(i)}$ are W-invariants (see the proof of Theorem 1); hence

$$D^W = \mathbb{C}[f_i H^{j(i)}, H_1^{\nu(1)}, \dots, H_s^{\nu(s)}],$$

for some integers $\nu(i)$.

(c) $\Rightarrow$ (d) follows from Lemma 3. $\square$

Remark 4. The implication (a) $\Rightarrow$ (d) is a particular case of the Luna Slice Etale Theorem application to Invariant Theory, see [KPV]. (b) $\Leftrightarrow$ (c) is a standard fact in commutative algebra; however it provides the interesting implication (c) $\Rightarrow$ (a). A word of caution: it does not follow from the proof of (a) $\Leftrightarrow$ (b) that D is a polynomial ring over $\mathbb{C}[f_i H^{j(i)}]$; see [A2].

V. The characterization of the associated graded ring

Now let (U, G) be a spherical representation, $\Delta = \mathbb{N}_0^s$, $\Delta\text{-}\operatorname{gr} S'(N)^M = \Delta\text{-}\operatorname{gr} C$ as in example (i). We shall make the following hypothesis:

(T) There exist a closed subgroup P of G and for each $\gamma \in \Gamma$, a "natural" isomorphism of vector spaces

$$\gamma^M \xrightarrow{\phi} \gamma^P, \qquad \phi = \phi(\gamma),$$

satisfying: for each finite-dimensional G-module N, the bijective isomorphism

$$\Delta\text{-}\operatorname{gr} C \to S'(N)^P,$$

given by the $\phi(\gamma)$ is actually an isomorphism of k-algebras.

Now let $(U_1, G_1), \dots, (U_j, G_j)$ be spherical representations and let $(U, G) = (U_1, G_1) \oplus \cdots \oplus (U_j, G_j)$, $M = M_1 \times \cdots \times M_j$, $P = P_1 \times \cdots \times P_j$, etc. Let us assume that (U_i, G_i) satisfies (T) for $i = 1, \dots, j$. If $\gamma \in \Gamma$, $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_j$, we define

$$\gamma^M \xrightarrow{\phi} \gamma^P, \qquad \phi = \phi(\gamma) = \phi(\gamma_1) \otimes \cdots \otimes \phi(\gamma_j).$$

Then we have

Lemma 4. (U, G), P, $\{\phi(\gamma) : \gamma \in \Gamma\}$ *satisfies* (T).

Proof. It suffices to treat the case $j = 2$. If s_i is the rank of (U_i, G_i), $s = s_1 + s_2$ is the rank of (U, G). Thus if $t \in \mathbb{N}_0^s$, we shall denote $t = (t_1, t_2)$ in the obvious way. Let N be a finite dimensional G-module. We introduce the following notation:

$$E(N, M, t) = \bigoplus_{\gamma \in \Gamma,\, m(\gamma) = t} S'(N)^M_\gamma.$$

If $t = (t_1, t_2)$ observe that $E(N, M, t) \subseteq E(N, M_i, t_i)$ when we consider N as G_i-module via the inclusion in the i-factor. Moreover $\gamma_1^{M_1} \otimes \gamma_2^{M_2} \to \gamma_1^{P_1} \otimes \gamma_2^{P_2}$ is given by the composition $(\mathrm{Id} \otimes \phi(\gamma_2)) \circ (\phi(\gamma_1) \otimes \mathrm{Id})$. Let us denote $\phi_1 = (\phi(\gamma_1) \otimes \mathrm{Id})$, and similarly ϕ_2.

Now let $f \in E(N, M, t)$, $g \in E(N, M, r)$, $f \cdot g = h_0 + \cdots + h_{t+r}$ with $h_l \in E(N, M, l)$. Certainly, if $h_l \neq 0$, $l \leq t + r$. That is $l_i \leq t_i + r_i$, $i = 1, 2$. Hence $\phi_1(f)\phi_1(g) = \sum_{j \,:\, j_1 = t_1 + r_1} \phi_1(h_j)$, looking at N as G_1-module.

Then $\phi_1(f) \in E(N, M_2, t_2)$, $\phi_1(g) \in E(N, M_2, r_2)$, and

$$\phi_2(\phi_1(f)) \cdot \phi_2(\phi_1(g)) = \phi_2(\phi_1(h_{t+r})),$$

i.e., $\phi(f) \cdot \phi(g) = \phi(h_{t+r})$. □

Now let (U, G) be a spherical representation and let $G' \xrightarrow{f} G$ be a finite covering. Then we may also consider the spherical representation (U, G'). Furthermore, if there exists a subgroup P of G satisfying the hypothesis (T), then the inverse image P' of P in G' satisfies (T) for (U, G'). (Alternatively, replace N by the G-variety $N/\operatorname{Ker} f$; it corresponds to $S'(N)^{\operatorname{Ker} f} = \bigoplus_{\gamma \in G^\wedge} S'(N)_\gamma$.)

We shall give a characterization, for $\gamma \in G^\wedge$, of $m(\gamma)$. We begin showing that every spherical representation of rank one is visible, i.e., the unstable cone has only a finite number of orbits, if G is connected.

Theorem 3 [Se, 6.2]. *If G is a connected reductive linear group and Y is an irreducible affine G-variety such that $\mathbb{C}[Y]$ contains each $\gamma \in G^\wedge$ at most once, then G has only a finite number of orbits in Y.*

Lemma 5. *Let (U, L) be an irreducible representation of a connected algebraic reductive group L such that $S'(U)^L$ is a polynomial ring generated by a single* *invariant, say* J. *Let* $\pi: U \to U/L$ *be the application associated to the inclusion, let* $\mathfrak{N} = \mathfrak{N}(U, L) = \pi^{-1}(\pi(0))$ *be the unstable cone. Then* $\mathfrak{N}$ *is irreducible and the ideal associated to* $\mathfrak{N}$ *is* $S'(U)J$.

Proof. Note that $\mathfrak{N}$ is the zero set of J. If L is semisimple, a standard argument shows that J is irreducible in $S'(U)$: let $J = p_1 \cdots p_s$ be the factorization of J in primes; then for each i there exists a character χ_i of L such that for every $k \in L$, $k \cdot p_i = \chi_i(k) p_i$, but L has no nontrivial characters. But in our case, as (U, L) is irreducible, the "nonsemisimple" part of L acts on U by a single character χ; hence $\chi_i = \chi^{\deg p_i}$. It follows that the image of χ is finite, but L is connected. □

Proposition 2. *Let* (U, G) *be a spherical representation of rank one,* G *connected. Then* $\mathfrak{N}(U, G)$ *contains only a finite number of orbits. Moreover, for each* $x \in \mathfrak{N}$, (G, G^x) *is a spherical pair.*

Proof. We want to apply the above-quoted theorem of Servedio to $Y = \mathfrak{N}$. We only need to observe that the restriction $S'(U) \to \mathbb{C}[\mathfrak{N}]$ induces an epimorphism $H \to \mathbb{C}[\mathfrak{N}]$. (Recall the decomposition $S'(U) = S'(U)^G \otimes H$.) The second assertion follows from [VK, Corollary 3]. □

Remark 5. The last result reduces drastically the list of candidates of possible spherical representations; see [Kc]. We now generalize an argument of Kostant:

Let (U, L) be a cofree representation of an algebraic reductive group L, π as in Lemma 5. Let H be a homogeneous subspace of $S'(U)$ such that $S'(U) = S'(U)^L \otimes H$. We set $F(\gamma) = \operatorname{Hom}_L(\gamma, H)$, for $\gamma \in L^\wedge$ and we fix γ such that $F(\gamma^*) \neq 0$. Let $d_1, \dots, d_m$ be the degrees of homogeneity of γ^* in H; $m = \dim F(\gamma^*)$. We shall assume that the ideal $S'(U)^L_+ \cdot S'(U)$ is prime and that there exists an $x \in \mathfrak{N}(U, L)$ such that the closure of the orbit $L \cdot x$ is $\mathfrak{N}$. Now let $P = L^x$, P' =normalizer of $\mathbb{C}x$ in L; P' acts in $\mathbb{C}x$ via a character χ. Finally, for any $y \in U$, $\sigma \in F(\gamma^*)$, $\alpha \in \gamma^*$, let $\beta_y: F(\gamma^*) \to \gamma^{G^y}$ be given by

$$\langle \beta_y(\sigma), \alpha \rangle = \sigma(\alpha)(y).$$

Proposition 3. β_x *is one-to-one; moreover,* P' *acts in* $\operatorname{Im} \beta_x$ *decomposing it in* P'*-submodules of dimension one and the associated characters are precisely of the form* $\chi^{d_1}, \dots, \chi^{d_m}$. *Finally, the* d_i*'s are determined by this fact.*

Proof. The injectivity of β_x and the following identity, from which the second assertion is deduced, can be found for example in [K]:

$$\text{for all } a \in G: \qquad a \cdot \beta_x(\sigma) = \beta_{a \cdot x}(\sigma).$$

Now $L \cdot x$ is open in $\mathfrak{N}$ and $\mathbb{C}_x \subseteq \mathfrak{N}$; thus $P' \cdot x = L \cdot x \cap \mathbb{C}x$ is a nonempty open subset of $\mathbb{C}x$ and hence the image of χ cannot be finite. □

If (U, G) is a spherical representation of rank one and $x \in \mathfrak{N}(U, G)$ is such that Lx is dense in $\mathfrak{N}$, β_x is an isomorphism of vector spaces for all $\gamma \in \Gamma$, because (G, P) is a spherical pair. Clearly $\{\gamma \in \Gamma: \gamma^P \neq 0\} \supseteq \Gamma$; so let us assume that the equality holds. This is true if for example, the codimension in $\mathfrak{N}$ of $(\mathfrak{N} - Gx)$ is ≥ 2 (see [K]: $\mathfrak{N}$ is normal since it is an irreducible hypersurface). For those spherical representations of simple groups, one can check that the condition is fulfilled in view of [Po3]. On the other hand, β_y is also an isomorphism for every $y \in U^M - 0$. Fix such a y. We define $\phi = \phi(\gamma)$ by making commutative the following diagram:

$$\begin{tikzcd} \gamma^M \arrow[rr, "\phi"] & & \gamma^P \\ & F(\gamma^*) \arrow[ul, "\beta_y"] \arrow[ur, "\beta_x"'] & \end{tikzcd}.$$

Remark 6. Let us assume that the codimension in $\mathfrak{N}$ of $(\mathfrak{N} - Gx)$ is ≥ 2. Then the equality $\{\rho \in G^\wedge : \rho^M \neq 0\} = \{\rho \in G^\wedge : \rho^P \neq 0\}$ can be viewed as a generalization of the well-known Cartan-Helgason Theorem. Indeed, when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(n, 1)$ the claimed equality is a consequence of Cartan-Helgason.

Theorem 4. *The isomorphisms $\{\phi(\gamma) : \gamma \in \Gamma\}$ give rise to an isomorphism of algebras between* $\mathrm{gr}(\mathbb{C}[X]^M)$ *and* $\mathbb{C}[X]^P$, *for every G-variety X.*

Proof. We shall denote by $h^\sharp$ the image of $h \in \tau^M$ by ϕ. Let $f \in \gamma^M$, $g \in \lambda^M$. We have

$$\gamma \otimes \lambda = \delta_1 \oplus \cdots \oplus \delta_t, \qquad \delta_i \in G^\wedge.$$

Let us decompose $f \otimes g = h_1 + \cdots + h_r + \cdots + h_s$, $r \leq s \leq t$, $h_i \in \delta_i - 0$. Reordering the index set if necessary, we may assume $m(\delta_i) = m(\gamma) + m(\lambda)$ if $i \leq r$ and $m(\delta_i) < m(\gamma) + m(\lambda)$ if $r < i \leq s$. We only need to check that $f^\sharp \otimes g^\sharp = h_1^\sharp + \cdots + h_r^\sharp$. Let $\mathscr{A} \in F(\gamma^*)$, $\mathscr{B} \in F(\lambda^*)$, $\mathscr{C}_i \in F(\delta_i^*)$ corresponding to f, g, h_i. This means, for example, that if $u \in \gamma^*$

$$\langle f, u \rangle = \mathscr{A}(u)(y), \qquad \langle f^\sharp, u \rangle = \mathscr{A}(u)(x).$$

So let $u \in \gamma^*$, $v \in \lambda^*$; with the above identification, there exist $w_i \in \delta_i^*$ such that

$$u \otimes v = w_1 + \cdots + w_t.$$

Hence

$$\langle f \otimes g, u \otimes v \rangle = \langle h_1, w_1 \rangle + \cdots + \langle h_r, w_r \rangle + \cdots + \langle h_s, w_s \rangle,$$

i.e.,

$$\mathscr{A}(u)(y)\mathscr{B}(v)(y) = \mathscr{C}_1(w_1)(y) + \cdots + \mathscr{C}_r(w_r)(y) + \cdots + \mathscr{C}_s(w_s)(y).$$

Let us denote by J a homogeneous generator of $S'(U)^G$. As $\mathscr{C}_i(w_i) \in H$ for all i, there exist integers $d_{r+1}, \ldots, d_s$ such that $J^{d_i}\mathscr{C}_i(w_i)$ is homogeneous of degree $m(\gamma) + m(\lambda)$. Put $j_i = (J^{d_i}(y))^{-1}$. Hence $\mathscr{A}(u)\mathscr{B}(v)$ and

$$\mathscr{C}_1(w_1) + \cdots + \mathscr{C}_r(w_r) + j_{r+1}J^{d_{r+1}}\mathscr{C}_{r+1}(w_{r+1}) \cdots + j_s J^{d_s}\mathscr{C}_s(w_s)$$

are homogeneous polynomials which agree on y, hence on $k \cdot y$ for all $k \in G$; if they agree on z, they do on tz for all $t \in \mathbb{C}^\times$. It follows that they agree on the whole of U. In particular, as $x \in \mathfrak{N} = \{z \in U : J(z) = 0\}$, we have

$$\mathscr{A}(u)(x)\mathscr{B}(v)(x) = \mathscr{C}_1(w_1)(x) + \cdots + \mathscr{C}_r(w_r)(x),$$

i.e.,

$$\langle f^\sharp \otimes g^\sharp, u \otimes v \rangle = \langle h_1^\sharp, w_1 \rangle + \cdots + \langle h_r^\sharp, w_r \rangle. \quad \square$$

Thanks to Lemma 4, Theorem 4 generalizes to spherical representations of arbitrary rank, provided that the hypothesis discussed above is fulfilled for each factor of G. Theorem 4 combined with Proposition 1 gives:

Theorem 5. *Let (U, G) be a spherical representation, N a finite-dimensional G-module, $x \in \mathfrak{N}(U, G)$ such that the closure of the orbit $G \cdot x$ is $\mathfrak{N}$ and the codimension in $\mathfrak{N}$ of $(\mathfrak{N} - Gx)$ is ≥ 2. Now let $P = G^x$. Then $S'(U \oplus N)^G$ is a polynomial ring if and only if $S'(N)^P$ is a polynomial ring.* □

VI. Some examples

Let us retain the notation of the introduction. As a first application, we have:

Theorem 6. *$S'(\mathfrak{g})^K$ is a polynomial ring if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$.*

Proof. We observe that Theorems 1, 2, 4, and 5 apply if (U, G) is $(\mathfrak{p}, K)$ and A is a one-dimensional Cartan subspace, even if $A \neq \mathfrak{p}^M$ with the following slight modification:

$$\operatorname{gr}(\mathbb{C}[X]^M) \simeq \mathbb{C}[X]^P_\Gamma = \oplus_{\gamma \in \Gamma} \mathbb{C}[X]^P_\gamma .$$

Now if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(n, 1)$ then we may assume that $K = SO(n, \mathbb{C})$; it is known that $\mathfrak{p}$ is the natural representation. Let $E \in \mathfrak{p}$ be a highest weight vector, with respect to a fixed Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ of $\mathfrak{k}$. We claim that $E \in \mathfrak{N}(\mathfrak{p}, K)$ and that KE has maximal dimension. The first is clear: $0 = \lim_{t \to 0, t \in \mathbb{C}^*} \Lambda(t) E$, where ψ is the highest weight of $\mathfrak{p}$ and Λ is the one parameter subgroup dual to ψ. It is known that $\dim \mathfrak{N}(\mathfrak{p}, K) = \dim \mathfrak{p} - \dim \mathfrak{p}/K = \dim \mathfrak{p} - 1$ so for the second it suffices to prove that $\dim KE = \dim \mathfrak{p} - 1$, that is

$$\dim K - \dim K^E = \dim \mathfrak{p} - 1 ,$$

which is equivalent in our case to

$$\dim K^E = (n^2 - n)/2 - n + 1 ,$$

or even to $\dim \mathfrak{k}^E = (n^2 - n)/2 - n + 1$, which is very easy to verify. So let us put $P = K^E$. Clearly $P \supseteq N$ where N is the maximal unipotent subgroup corresponding to $\mathfrak{n}$. Let us denote by $V(\tau)$ the irreducible $\mathfrak{k}$-module of highest weight $\tau \in \mathfrak{h}^*$. We claim that

$$\Gamma = \{\gamma \in K^\wedge : \gamma = V(j\psi),\ j \in \mathbb{N}_0\} ,$$

and that if $\gamma \in \Gamma$ then $\gamma^P = \gamma^N$. This second statement is clearly true. Let Ψ be the character of $\operatorname{Ad} \mathfrak{h}$ corresponding to ψ, i.e., $\Psi = \exp \psi$. Clearly $P \supseteq \operatorname{Ker} \Psi$; if $f \in S'(\mathfrak{p})^N_\psi$ then $f^j \in S'(\mathfrak{p})^N_{j\psi}$ and hence $\Gamma \supseteq \{\gamma \in K^\wedge : \gamma = V(j\psi),\ j \in \mathbb{N}_0\}$. But if $\gamma \in \Gamma$ has highest weight ξ and $\Xi = \exp \xi$, then γ^N is stabilized by $\operatorname{Ker} \Xi \cdot \operatorname{Ker} \Psi$ and the other inclusion follows. Thus

$$S'(\mathfrak{k})^P = \bigoplus_{\gamma \in \Gamma} S'(\mathfrak{k})^N_\gamma = \bigoplus_{j \geq 0} S'(\mathfrak{k})^N_{j\psi} = \bigoplus_{j \geq 0} S'(\mathfrak{k})^N_{2j\psi} ,$$

because we know from [K] that an irreducible representation arises in the coordinate ring of the adjoint representation if and only if its highest weight lives in the root lattice. Now a theorem of Levstein (see Theorem 7 below) guarantees that $S'(\mathfrak{k})^P$ is a polynomial ring; we conclude from Theorem 5 that $S'(\mathfrak{g})^K$ is a polynomial ring too.

On the other hand, if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(n, 1)$ then $\mathfrak{k} = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$ and it is well known that $\mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{p}_+$ where $\mathfrak{sl}(n, \mathbb{C})$ (resp., $\mathbb{C}$) acts in $\mathfrak{p}_+$ via the natural representation (respectively, via a nontrivial character) and $\mathfrak{p}_-$ is the dual of $\mathfrak{p}_+$. In fact, one can choose a realization as follows: $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$,

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : A \in \mathfrak{gl}(n, \mathbb{C}), \operatorname{tr} A + a = 0 \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} : u \in \mathbb{C}^{n\times 1}, v \in \mathbb{C}^{1\times n} \right\}.$$

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be the Borel subalgebra of $\mathfrak{k}$ of upper triangular matrices in $\mathfrak{k}$, where $\mathfrak{h}$ are the diagonal matrices in $\mathfrak{k}$, and let $E_+ \in \mathfrak{p}_+$ (resp., $E_- \in \mathfrak{p}_-$) be given by $u = e_1$, $v = 0$ (resp., $u = 0$, $v = e_n$). We claim that $E = E_+ + E_- \in \mathfrak{N}(\mathfrak{p}, K)$ and that KE has maximal dimension. The first statement is easy and the second will follow from

$$\dim K^E = n^2 - 2n + 1,$$

or even from

$$\dim \mathfrak{k}^E = n^2 - 2n + 1.$$

But

$$\begin{aligned} \mathfrak{k}^E &= \{Z \in \mathfrak{k} \colon ZE_+ + ZE_- = 0\} \\ &= \{Z = Z_1 + Z_2 \in \mathfrak{k} \colon Z_1E_+ + Z_2E_+ = Z_1E_- - Z_2E_- = 0\} \\ &= \left\{ \begin{pmatrix} z & {}^tu & w & 0 \\ 0 & A & v & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix} \in \mathfrak{k} \colon A \in \mathfrak{gl}(n-2, \mathbb{C}), z \in \mathbb{C} \right\}, \end{aligned}$$

which has the necessary dimension. Thus

$$S'(\mathfrak{k})^P = S'([\mathfrak{k}, \mathfrak{k}])^P_{\Gamma'} \otimes S'(\mathfrak{c}),$$

where $\mathfrak{c}$ is the one-dimensional center and $\Gamma' = \{\gamma \in \mathrm{PSL}(n, \mathbb{C})^\wedge \colon \gamma \otimes \mathrm{id} \in \Gamma\}$. Let ψ (resp., ψ^*) be the dominant weight of the natural representation (resp., of its dual). We claim that

$$\Gamma' = \{V(j(\psi + \psi^*)), j \in \mathbb{N}_0\},$$

and that if $\gamma \in \Gamma'$ then $\gamma^P = \gamma^N$. In fact, the second assertion is easy and we can show that $V(j(\psi + \psi^*)) \in \Gamma'$ by induction on j. The other inclusion follows as above; again, Levstein's result and Theorem 5 guarantee that $S'(\mathfrak{g})^K$ is a polynomial ring. $\square$

Theorem 7. *Let $\mathfrak{l}$ be a classical simple complex Lie algebra, $\mathfrak{b}$ a Borel subalgebra and let ψ (resp., ψ^*) be the highest weight (with respect to $\mathfrak{b}$) of the natural representation of $\mathfrak{l}$ (resp., of its dual). Let $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}] \oplus \operatorname{Ker}(\psi + \psi^*)$. Then $S'(\mathfrak{l})^{\mathfrak{u}}$ is a polynomial ring.*

Proof. See [L]. $\square$

Remark 7. It was shown in [A3] that these are the only cases for which $S'(\mathfrak{g})^K$ is regular. Moreover, Theorem 6 was proved in [C] by geometric considerations; and the coregularity of $(\mathfrak{g}, K)$ when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(n, 1)$ (resp., $\mathfrak{su}(n, 1)$) is also proved in [AG, B, Sch] (resp., [J]).

In order to get a second application let us recall the following:

Theorem 8. *$S'(\mathfrak{p})^P$ is a polynomial ring if $\mathfrak{g}_{\mathbb{R}}$ is classical of rank one.*

Proof. See [BT, Theorem 3.14]. □

Theorem 5 says that this is equivalent to

Theorem 9. *$S'(\mathfrak{p} \oplus \mathfrak{p})^K$ is a polynomial ring if $\mathfrak{g}_{\mathbb{R}}$ is classical of rank one.*

Remark 8. Theorem 9 follows from classical invariant theory if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$; so in this case Theorem 8 can be deduced from Theorem 5. On the other hand, Theorem 9 seems to be new if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, 1)$. More explicitly, let V_n be the natural representation of $\mathfrak{sp}(n, \mathbb{C})$; then

$$S'((V_n \otimes V_1) \oplus (V_n \otimes V_1))^{\mathrm{SP}(n, \mathbb{C}) \times \mathrm{SP}(1, \mathbb{C})}$$

is regular.

We conclude this section by giving an explicit presentation of the ring $S'(\mathfrak{g})^K$ when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(2, 1)$. The case $\mathfrak{sp}(1, 1) \simeq \mathfrak{so}(4, 1)$ is covered by Theorem 6; it turns out (at least as far as we know) that $\mathfrak{sp}(2, 1)$ is the first *nonregular* $(\mathfrak{g}, K)$ of rank one computed in the literature. The strategy is as follows: first we compute $S'(\mathfrak{k})^P$, where P is the isotropy subgroup of a nilpotent element in $\mathfrak{p}$ whose orbit has maximal dimension. It turns out that it is a hypersurface; from Theorem 4 we can conclude that $S'(\mathfrak{k})^M$ is also a hypersurface. Then we compute the generators and the relation of $S'(\mathfrak{k})^M$. Using this information and Theorem 1, we give the generators and relation of the image by the restriction morphism of $S'(\mathfrak{g})^K$. The details of these last two computations were presented in [A2].

Let us fix some notation. We have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{sp}(3, \mathbb{C}) \\ &= \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} = Z : Z_i \in \mathbb{C}^{3\times 3},\ Z_1 = -{}^tZ_4,\ Z_2,\ Z_3 \text{ symmetric} \right\}, \end{aligned}$$

$\mathfrak{k} = \{Z \in \mathfrak{g} : Z_1 = \left(\begin{smallmatrix} A & 0 \\ 0 & \alpha \end{smallmatrix}\right),\ Z_2 = \left(\begin{smallmatrix} B & 0 \\ 0 & \beta \end{smallmatrix}\right),\ Z_3 = \left(\begin{smallmatrix} C & 0 \\ 0 & \gamma \end{smallmatrix}\right),\ A, B, C \in \mathbb{C}^{2\times 2},\ \alpha, \beta, \gamma \in \mathbb{C}\} \simeq \mathfrak{k}_1 \times \mathfrak{k}_2$ where $\mathfrak{k}_1 = \mathfrak{sp}(2, \mathbb{C})$, $\mathfrak{k}_2 = \mathfrak{sl}(2, \mathbb{C})$. We denote the entries of A, B, C as $a_1, \dots, b_1, b_2, b_3, c_1$, etc. where

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}, \qquad C = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}.$$

We will think of the a_i, b_j, etc. as elements of $(\mathfrak{k}_1)^*$. We have

$$\begin{aligned} K = \Bigg\{ \left(\begin{pmatrix} T & Q \\ Z & V \end{pmatrix}, \begin{pmatrix} t & y \\ v & z \end{pmatrix} \right) &: T, Q, V, Z \in \mathbb{C}^{2\times 2} \cdot t, y, v, z \in \mathbb{C} \\ &\text{s.t. } {}^tTV - {}^tZQ = I,\ {}^tTZ - {}^tZT = 0 = {}^tQV - {}^tVQ,\ tz - yv = 1 \Bigg\}. \end{aligned}$$

As a $\mathfrak{k}_1 \times \mathfrak{k}_2$-module, $\mathfrak{p}$ is then isomorphic to $\lambda_1(\mathfrak{k}_1) \otimes \lambda_1(\mathfrak{k}_2)$, where λ_1 means in each case the natural representation on $\mathbb{C}^4$ or $\mathbb{C}^2$. (This is in general true for $\mathfrak{sp}(n, 1)$, where $\mathfrak{k}_1 = \mathfrak{sp}(n, \mathbb{C})$, etc.). We shall denote $\mathfrak{p} = \{\left(\begin{smallmatrix} u & x \\ r & w \end{smallmatrix}\right)\} = \mathbb{C}^{4\times 2}$, where the action is given by

$$(x, y) \cdot P = xP - Py \qquad (x \in \mathfrak{k}_1,\ y \in \mathfrak{k}_2,\ P \in \mathfrak{p}).$$

Furthermore, if $X \in \mathfrak{p}$ is given by $u = w = e_1$, $r = x = 0$, then we can choose $\mathfrak{a} = \mathbb{C} \cdot X$ and hence M is the connected subgroup of K corresponding to

$$\mathfrak{m} = \left\{ \left(\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & a & 0 & b \\ \gamma & 0 & -\alpha & 0 \\ 0 & c & 0 & -a \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \right) \in \mathfrak{k} \colon \alpha, \beta, \gamma, a, b, c \in \mathbb{C} \right\},$$

$\mathfrak{m} = \mathfrak{m}_1 \times \mathfrak{m}_2$, where $m_i \simeq \mathfrak{sl}(2, \mathbb{C})$, $i = 1, 2$, in an evident way.

As an M-module, $\mathfrak{k}_1$ is isomorphic to $\operatorname{Ad}\mathfrak{m}_1 + \operatorname{Ad}\mathfrak{m}_2 + \mathfrak{p}^\sim$, where

$$\mathfrak{p}^\sim = \{X \in \mathfrak{k}_1 \colon a_1 = a_4 = b_1 = b_3 = c_1 = c_3 = 0\}.$$

Remark 9. $(\mathfrak{k}_1, M) = (\mathfrak{g}, K)$ for $\mathfrak{sp}(1, 1) \simeq \mathfrak{so}(4, 1)$; as we noted above, $S'(\mathfrak{g})^K$ is a polynomial ring in 4 variables.

We shall also fix a Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_1 \times \mathfrak{t}_2 \subseteq \mathfrak{k}$, $\mathfrak{t}_i \subseteq \mathfrak{k}_i$ are the diagonal matrices.

Finally, let $Y \in \mathfrak{p}$ be given by $u = e_1$, $x = e_2$, $w = r = 0$. Let

$$\gamma_t = \left(\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \in K,$$

with $T = tI$, $t \in \mathbb{C}$. Then $\gamma_t \cdot Y$ is given by $u = te_1$, $x = te_2$, $w = r = 0$ and this shows that Y is nilpotent.

The isotropy subgroup of Y in K is

$$P = \left\{ \left(\begin{pmatrix} T & Q \\ 0 & V \end{pmatrix}, \begin{pmatrix} t & y \\ v & z \end{pmatrix} \right) : T = \begin{pmatrix} t & y \\ v & z \end{pmatrix}, Q = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \right.$$
$$\left. V = \begin{pmatrix} z & -v \\ -y & t \end{pmatrix}, tg - ve = zf - yh, tz - yv = 1 \right\}.$$

As $\dim KY = \dim K - \dim P$ is maximal, we conclude that Y is principal nilpotent. It is easy to see that $P = RH$, where

$$R = \left\{ \left(\begin{pmatrix} T & 0 \\ 0 & V \end{pmatrix}, \begin{pmatrix} t & y \\ v & z \end{pmatrix} \right) : T = \begin{pmatrix} t & y \\ v & z \end{pmatrix}, \right.$$
$$\left. V = \begin{pmatrix} z & -v \\ -y & t \end{pmatrix}, tz - yv = 1 \right\},$$

and

$$H = \left\{ \left(\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) : Q = \begin{pmatrix} e & f \\ f & h \end{pmatrix} \right\}.$$

In other words, H is the (abelian, three-dimensional) unipotent radical of P and R (a copy of $\mathrm{SL}(2, \mathbb{C})$) is a Levi factor.

Our first step is to compute $S'(\mathfrak{k})^H \simeq S'(\mathfrak{k}_1)^H \otimes S'(\mathfrak{t}_2)$. The action of H in $\mathfrak{k}_1$ is of course given by

$$\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix} = \begin{pmatrix} A + QC & B - Q{}^tA - AQ - QCQ \\ C & -CQ - {}^tA \end{pmatrix}.$$

Thus $\left(\begin{smallmatrix} I & -Q \\ 0 & I \end{smallmatrix}\right)$ acts as follows in $S'(\mathfrak{k}_1)$ (if $Q = \left(\begin{smallmatrix} e & f \\ f & h \end{smallmatrix}\right)$):

$$\begin{aligned}
&a_1 \mapsto a_1 + ec_1 + fc_2\,,\\
&a_2 \mapsto a_2 + ec_2 + fc_3\,,\\
&a_3 \mapsto a_3 + fc_1 + hc_2\,,\\
&a_4 \mapsto a_4 + fc_2 + hc_3\,,\\
&c_i \mapsto c_i\,,\ i = 1\,,2\,,3\,,\\
&b_1 \mapsto b_1 - 2(ea_1 + fa_2) - (e^2c_1 + 2efc_2 + f^2c_3)\,,\\
&b_2 \mapsto b_2 - (fa_1 + ha_2 + ea_3 + fa_4) - (efc_1 + (f^2 + eh)c_2 + fhc_3)\,,\\
&b_3 \mapsto b_3 - 2(fa_3 + ha_4) - (f^2c_1 + 2fhc_2 + h^2c_3).
\end{aligned}$$

Let us introduce the following polynomials in $\mathfrak{k}_1$:

$$\begin{aligned}
\Delta &= c_2^2 - c_1c_3\,,\\
\vartheta_1 &= a_4c_2 + a_2c_1 - a_1c_2 - a_3c_3\,,\\
\vartheta_2 &= b_1\Delta - a_1^2c_3 + 2a_1a_2c_2 - a_2^2c_1\,,\\
\vartheta_3 &= \Delta(b_2c_2 + a_2a_3) - (a_2c_1 - a_1c_2)(a_4c_2 - a_3c_3)\,,\\
\vartheta_4 &= \Delta(b_3c_2^2 - a_3^2c_3 + 2a_3a_4c_2)\\
&\quad + c_1[(a_2c_1 - a_1c_2)^2 + 2(a_2c_1 - a_1c_2)(a_4c_2 - a_3c_3)]\,,\\
\vartheta_5 &= \Delta(b_2c_1c_3 - a_1a_4c_2 + a_2a_4c_1 + a_1a_3c_3)\\
&\quad - c_2(a_2c_1 - a_1c_2)(a_4c_2 - a_3c_3)\,,\\
\vartheta_6 &= \Delta(b_3c_3 + a_4^2) + (a_2c_1 - a_1c_2)^2\\
&\quad + 2(a_2c_1 - a_1c_2)(a_4c_2 - a_3c_3)\,,\\
\vartheta_7 &= \Delta b_2 + a_2a_3(c_2 - c_3) + a_1a_4(c_2 - c_1)\,,\\
\vartheta_8 &= \Delta b_3 - (a_3^2c_3 - 2a_3a_4c_2 + a_4^2c_1)\,,\\
\vartheta_9 &= \det = \Delta(b_2^2 - b_1b_3) + b_1(a_3^2c_3 - 2a_3a_4c_2 + a_4^2c_1)\\
&\quad + 2b_2(a_1a_4c_2 + a_2a_3c_2 - a_2a_4c_1 - a_1a_3c_3)\\
&\quad + b_3(a_2^2c_1 + a_1^2c_3 - 2a_1a_2c_2) + (a_1a_4 - a_2a_3)^2\,,\\
\vartheta_{10} &= b_1c_1 + 2b_2c_2 + b_3c_3 + a_1^2 + 2a_2a_3 + a_4^2.
\end{aligned}$$

Proposition 4. *$S'(\mathfrak{k}_1)^H$ is generated by the polynomials c_1, c_2, c_3, ϑ_1, ϑ_2, ϑ_7, ϑ_8, ϑ_9, ϑ_{10}.*

Proof. A fastidious computation shows that the ϑ_i's are invariants. (In the course of the proof, we will give some indications of how to get them.) Now for (a_i, b_j, c_k) in a suitable open subset of $\mathfrak{k}_1$, the orbit $H(a_i, b_j, c_k)$ intersects the subspace given by $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ at the point

$$(0, 0, 0, c_2^{-1}\vartheta_1, c_1, c_2, c_3, \Delta^{-1}\vartheta_2, (c_2\Delta)^{-1}\vartheta_3, (c_2^2\Delta)^{-1}\vartheta_4).$$

Similarly, for (a_i, b_j, c_k) in another open subset of $\mathfrak{k}_1$, the orbit $H(a_i, b_j, c_k)$ intersects the subspace given by $a_1 = 0$, $a_2 = 0$, $a_4 = 0$ at the point

$$(0, 0, -c_3^{-1}\vartheta_1, 0, c_1, c_2, c_3, \Delta^{-1}\vartheta_2, (c_1c_3\Delta)^{-1}\vartheta_5, (c_3\Delta)^{-1}\vartheta_6).$$

Thus if $f \in S'(\mathfrak{k}_1)^H$, there exists nonnegative integers i, j, k, ℓ, m such that

$$c_2^i \Delta^j f \in \mathbb{C}[c_1, c_2, c_3, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4],$$
$$c_1^k c_3^\ell \Delta^m f \in \mathbb{C}[c_1, c_2, c_3, \vartheta_1, \vartheta_2, \vartheta_5, \vartheta_6],$$

and hence

$$\Delta^{i+j+k+\ell+m} f \in \mathbb{C}[c_1, c_2, c_3, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6].$$

But, as $\vartheta_3 = c_2\vartheta_7$, $\vartheta_4 = c_1\vartheta_6 + \Delta\vartheta_8$, $\vartheta_5 = c_1c_3\vartheta_7$, $\vartheta_6 + c_1\vartheta_2 + 2c_2\vartheta_7 = \Delta\vartheta_{10}$, we have that for some nonnegative integer n:

$$\Delta^n f \in \mathbb{C}[c_1, c_2, c_3, \vartheta_1, \vartheta_2, \vartheta_7, \vartheta_8, \vartheta_{10}].$$

Let us also remark that

$$(*) \qquad \vartheta_7^2 - \vartheta_2\vartheta_8 = \Delta\vartheta_9, \qquad \vartheta_1^2 + c_1\vartheta_2 + 2c_2\vartheta_7 + c_3\vartheta_8 = \Delta\vartheta_{10}.$$

Now consider the obvious application

$$\mathbb{C}[C_1, C_2, C_3, Y_1, Y_2, Y_7, Y_8, Y_9, Y_{10}]$$
$$\xrightarrow{\Phi} \mathbb{C}[c_1, c_2, c_3, \vartheta_1, \vartheta_2, \vartheta_7, \vartheta_8, \vartheta_9, \vartheta_{10}],$$

(where the C_i's, Y_j''s, are algebraically independent) and let us also introduce

$$\alpha\colon \mathbb{C}[a_i, b_j, c_k] \to \mathbb{C}[\theta, \omega, a_i, b_j]$$
$$c_1 \mapsto \theta^2, c_2 \mapsto \theta\omega, c_3 \mapsto \omega^2, a_i \mapsto a_i, b_j \mapsto b_j.$$

It is not so difficult to see that $\operatorname{Ker}\alpha$ is the principal ideal expanded by Δ. We claim that

$$\operatorname{Ker}(\alpha \circ \Phi) = \langle C_2^2 - C_1C_3, Y_1^2 + c_1Y_2 + 2c_2Y_7 + c_3Y_8, Y_7^2 - Y_2Y_8\rangle.$$

$\supseteq$ is clear. For $\subseteq$, let us introduce the auxiliary variables

$$T_1 = a_4\theta - a_3\omega, \qquad T_2 = -a_2\theta + a_1\omega.$$

Let us observe that θ, ω, T_1, T_2 are linearly independent. Thus $\alpha \circ \Phi$ applies:

$$\begin{array}{lll} C_1 \mapsto \theta^2 & Y_1 \mapsto -\theta T_2 + \omega T_1 & Y_7 \mapsto T_1T_2 \\ C_2 \mapsto \theta\omega & Y_2 \mapsto -T_2^2 \quad Y_8 \mapsto -T_1^2 & \\ C_3 \mapsto \omega^2 & Y_9 \mapsto b_1T_1^2 + 2b_2T_1T_2 + b_3T_2^2 + (a_1a_4 - a_2a_3)^2 & \\ & Y_{10} \mapsto b_1\theta^2 + 2b_2\theta\omega + b_2\omega^2 + a_1^2 + 2a_2a_3 + a_4^2. & \end{array}$$

Let $p \in \operatorname{Ker}(\alpha \circ \Phi)$. As the images of Y_9, Y_{10} are linearly independent, we can assume that $p \in \mathbb{C}[C_1, C_2, C_3, Y_1, Y_2, Y_7, Y_8]$ and even that

$$p = P_1(C_1, C_2, C_3, Y_2, Y_7, Y_8)$$
$$+ P_2(C_1, C_2, C_3, Y_2, Y_7, Y_8)Y_1.$$

But the image of the first summand (resp., the second) is a sum of monomials of total degree in θ, ω even (resp., odd) and hence

$$P_1(C_1, C_2, C_3, Y_2, Y_7, Y_8)$$
$$= P_2(C_1, C_2, C_3, Y_2, Y_7, Y_8) = 0,$$

and the claim follows.

Now we are ready to prove the proposition (i.e., that Φ is surjective). Let $f \in S'(\mathfrak{k}_1)^H$, n a nonnegative integer such that $\Delta^n f = \Phi(g)$ for some $g \in \mathbb{C}[C_1, C_2, C_3, Y_1, Y_2, Y_7, Y_8, Y_9, Y_{10}]$. If $n = 0$ we are done. If not, $g \in \operatorname{Ker}(\alpha \circ \Phi)$ and hence (using $(*)$):

$$\Phi(g) \in \operatorname{Im}\Phi\Delta + \operatorname{Im}\Phi(\vartheta_7^2 - \vartheta_2\vartheta_8) + \operatorname{Im}\Phi(\vartheta_1^2 + c_1\vartheta_2 + 2c_2\vartheta_7 + c_3\vartheta_8),$$

i.e.,

$$\Phi(g) \in \operatorname{Im}\Phi\Delta.$$

Thus $\Delta^{n-1} f \in \operatorname{Im}\Phi$ and the proposition follows. $\square$

Let us observe now that $\vartheta_1, \vartheta_9, \vartheta_{10}$ are R-invariants. Thus

$$\begin{aligned} S'(\mathfrak{k})^P &\simeq (S'(\mathfrak{k}_1)^H \otimes S'(\mathfrak{k}_2))^R \\ &\simeq \mathbb{C}[\vartheta_1, \vartheta_9, \vartheta_{10}] \otimes (\mathbb{C}[c_1, c_2, c_3, \vartheta_2, \vartheta_7, \vartheta_8] \otimes S'(\mathfrak{k}_2))^R. \end{aligned}$$

It is not so difficult to see that $\langle \vartheta_2, \vartheta_7, \vartheta_8 \rangle$ is R-stable. Let us retain the notation of the above proposition. Considering

$$\begin{aligned} &\mathbb{C}[C_1, C_2, C_3, Y_2, Y_7, Y_8] \otimes S'(\mathfrak{k}_2) \\ &\qquad \to \mathbb{C}[c_1, c_2, c_3, \vartheta_2, \vartheta_7, \vartheta_8] \otimes S'(\mathfrak{k}_2), \end{aligned}$$

and the following theorem by Formanek (see [F]):

Theorem 10. *If* Ad *denotes the irreducible* $\mathrm{SL}(2, \mathbb{C})$*-module of dimension* 3, *then* $S'(\mathrm{Ad} \oplus \mathrm{Ad} \oplus \mathrm{Ad})^{\mathrm{SL}(2, \mathbb{C})}$ *is a hypersurface generated by the seven elements* $\operatorname{tr}(X_i X_j)$, $\operatorname{tr}(X_1X_2X_3)$ (X_i *in the* i*-copy*).

We can conclude

Theorem 11. $S'(\mathfrak{k})^P$ *is a hypersurface generated by* 8 *homogeneous polynomials* $p_1, \dots, p_8$ *with degrees* 2, 2, 2, 2, 2, 4, 4, 5 *respectively, satisfying the relation:*

$$\begin{aligned} &p_8^2 + p_1p_7^2 - p_2(p_3^2 - p_4p_1 + p_1^2)^2 + p_1p_6p_5^2 \\ &\qquad + 4p_1^2p_2p_6 + (p_3^2 - p_4p_1 + p_1^2)p_5p_7 = 0. \end{aligned}$$

Proof. This follows from the explicit description of the generators in Formanek's theorem. Let us remark that a naive application of the quoted theorem will give 10 generators, but it is easy to reduce the number to 8. $\square$

Proposition 5. $S'(\mathfrak{g})^K$ *and* $S'(\mathfrak{k})^M$ *are hypersurfaces.*

Proof. This follows from Theorem 11 (see [A2]). $\square$

Now we give a system of homogeneous generators for $S'(\mathfrak{k})^M$. First of all, $S'(\mathfrak{k})^K = S'(\mathfrak{k}_1)^K \otimes S'(\mathfrak{k}_2)^K$ is a polynomial ring generated by

$$\begin{aligned} f_1 &= \det{}_{\mathfrak{k}_2} = \alpha^2 + \beta\gamma, \\ f_2 &= \det{}_{\mathfrak{k}_1} = \vartheta_9, \\ f_3 &= a_1^2 + a_4^2 + 2a_2a_3 + b_1c_1 + 2b_2c_2 + b_3c_3. \end{aligned}$$

We know from Remark 9 that $S'(\mathfrak{k}_1)^M$ is a polynomial ring of Krull dim 4; indeed, it is generated by f_2, f_3, f_4, f_5, where

$$f_4 = a_4^2 + b_3c_3, \qquad f_5 = a_1^2 + b_1c_1.$$

Obviously, $S'(\mathfrak{k}_2)^M = S'(\mathfrak{k}_2)^K$. So far, we need to find f_6, f_7, f_8, homogeneous of degree 2, 4, 5 in $S'(\mathfrak{k})^M$. This was done in [A2]. We get

$$\begin{aligned}
f_6 &= (\beta c_1 + 2\alpha a_1 + \gamma b_1)/2, \\
f_7 &= \beta(2a_3a_4c_2 + b_3c_2^2 - a_3^2c_3) \\
&\quad + 2\alpha(a_2a_3a_4 - a_4b_2c_2 + a_2b_3c_2 + a_3b_2c_3) \\
&\quad + \gamma(2a_2a_4b_2 + b_2^2c_3 - a_2^2b_3), \\
f_8 &= \beta(a_2a_3a_4c_1 + a_2b_3c_1c_2 + a_1a_3^2c_3 + a_3b_2c_1c_3 \\
&\qquad\qquad - 2a_1a_3a_4c_2 - a_4b_2c_1c_2 - a_1b_3c_2^2) \\
&\quad + \alpha(-2a_2a_4b_2c_1 + a_2^2b_3c_1 - a_3^2b_1c_3 \\
&\qquad\qquad - b_2^2c_1c_3 + 2a_3a_4b_1c_2 + b_1b_3c_2^2) \\
&\quad + \gamma(2a_1a_2a_4b_2 - a_1a_2^2b_3 - a_3b_1b_2c_3 + a_1b_2^2c_3 \\
&\qquad\qquad + a_4b_1b_2c_2 - a_2a_3a_4b_1 - a_2b_1b_3c_2).
\end{aligned}$$

As their images in $S'(\mathfrak{k})_+^M/(S'(\mathfrak{k})_+^M)^2$ are linearly independent, $f_1, \dots, f_8$ form a system of generators for $S'(\mathfrak{k})^M$. The relation was found in [A2]:

Theorem 12. *$S'(\mathfrak{k})^M$ is generated by $f_1, \dots, f_8$; the generating relation is*

$$\begin{aligned}
&f_8^2 - f_1(f_2 - f_4f_5 - \tfrac{1}{4}(f_3 - f_4 - f_5)^2)^2 \\
&\quad - 2(f_2 - f_4f_5 - \tfrac{1}{4}(f_3 - f_4 - f_5)^2)f_6f_7 \\
&\quad + f_1(f_3 - f_4 - f_5)^2f_4f_5 - f_5f_7^2 \\
&\quad - (f_3 - f_4 - f_5)^2f_4f_6^2 = 0.
\end{aligned}$$

Furthermore, in order to get a system of generators of $S'(\mathfrak{g})^K$ we need generators of $S'(\mathfrak{k})^M\varphi_1, \dots, \varphi_8$ such that $\varphi_i \in \bigoplus_{\gamma\,:\,m(\gamma)=d_i} S'(\mathfrak{k})_\gamma^M$ for some d_i. In fact, we can deduce the φ_i's from the f_i's, decomposing the $\mathfrak{k}$-module generated by f_i in irreducible components. We need too the d_i's; all this information is given below (see [A2] for the proofs):

$$\varphi_1 = f_2, \quad \varphi_2 = f_2, \quad \varphi_3 = \tfrac{1}{10}f_3,$$

of course with $d_i = 0$ $(i = 1, 2, 3)$,

$$\begin{aligned}
&\varphi_4 = \tfrac{1}{2}(f_4 - f_5); \quad d_4 = 2, \\
&\varphi_5 = \tfrac{1}{2}(f_4 + f_5 - \tfrac{3}{5}f_3); \quad d_5 = 4, \\
&\varphi_6 = f_6; \quad d_6 = 2, \\
&\varphi_8 = f_8; \quad d_8 = 4, \\
&\varphi_7 = f_7 + \tfrac{1}{2}f_6(f_3 - 4f_4), \quad d_7 = 2.
\end{aligned}$$

Therefore, a system of generators for $S'(\mathfrak{g})^K$ is $\psi_i = \varphi_i \cdot H^{d_i}$, $i = 1, \dots, 8$, and $\psi_9 = H^2$, where H is a generator of $\mathfrak{a}^*$.

Finally, we want to give the relation between the ψ_j's. From Theorem 12 we obtain (after some cumbersome calculations; see [A2]):

Theorem 13. *If $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(2, 1)$ $S'(\mathfrak{g})^K$ is a hypersurface given by* $\psi_8^2 - 4\psi_1\psi_2\psi_5^2 - 100\psi_1\psi_3^2\psi_5^2 - \psi_1\psi_4^4 + 20\psi_1\psi_3\psi_5\psi_4^2 + 2\psi_6\psi_4^2\psi_7 - 4\psi_2\psi_5\psi_6^2 - 75\psi_3^2\psi_5\psi_6^2 - 10\psi_3\psi_4^2\psi_6^2 - 10\psi_3\psi_6\psi_5\psi_7 + \psi_9(4\psi_2\psi_4\psi_6^2 - 25\psi_3\psi_4\psi_6^2 + 10\psi_3\psi_6\psi_4\psi_7 - \psi_4\psi_7^2)$

$$+\psi_9^2(-2\psi_1\psi_2\psi_4^2 - 700\psi_1\psi_3^3\psi_5 + 70\psi_1\psi_3^2\psi_4^2 + 28\psi_1\psi_2\psi_3\psi_5 - 6\psi_3\psi_7^2 - 2\psi_2\psi_6\psi_7 - 14\psi_2\psi_3\psi_6^2 - 10\psi_3^2\psi_6\psi_7 + 200\psi_3^2\psi_6^2) \; +\psi_9^4(-\psi_1\psi_2^2 - 1225\psi_1\psi_3^4 + 74\psi_1\psi_2\psi_3^2) = 0 .$$

References

[AG] O. M. Adamovich and E. O. Golovina, *Simple linear groups having free algebra of invariants*, Selecta Math. Soviet **3** (no. 2) (1983/1984).

[AP] E. M. Andreev and V. L. Popov, *On stationary subgroups of points in general position in the Representation space of a Lie group*, Funct. Anal. Appl. **5** (1971), 1–8.

[A1] N. Andruskiewitsch, *A new proof of Tirao's restriction theorem* Rev. Un. Mat. Argentina (to appear).

[A2] ——, *Computing some rings of invariants* (to appear in the Proceedings of the IX ELAM).

[A3] ——, *On the complicatedness of the pair* $(\mathfrak{g}, K)$, Rev. Mat. Univ. Complut. Madrid **2** (1989).

[B] Abdel-Ilah Benabdallah, *Generateurs de l'algebre* $\mathfrak{U}(G)^K$ *avec* $G = \mathrm{SO}(m)$ *ou* $\mathrm{SO}_0(1, m-1)$ *et* $K = \mathrm{SO}(m-1)$, Bull. Soc. Math. France **111** (1983), 303–326.

[Bo] N. Bourbaki, *Algèbre commutative*, Chapitres 3 et 4, Hermann, Paris, 1967.

[BT] A. Brega and J. A. Tirao, *A property of a distinguished class of K-modules associated to the classical rank one semisimple Lie algebras*, Proceedings of the IX ELAM (to appear).

[BLV] M. Brion, D. Luna, and Th. Vust, *Espaces homogènes spheriques*, Invent. Math. **84** (1986), 617–632.

[C] Allan Cooper, *The classifying ring of groups whose classifying ring is commutative*, Doctoral Thesis, MIT (unpublished).

[E] A. G. Elashvili, *Canonical form and stationary subalgebras of points of general position for simple linear Lie groups*, Funct. Anal. Appl. **6** (1972), 44.

[F] E. Formanek, *Invariants and the ring of generic matrices*, J. Algebra **89** (1984), 178–223.

[J] K. Johnson, *The centralizer of a Lie algebra in an enveloping algebra*, J. Reine Angew. Math. **395** (1989), 196–201.

[Kc] V. G. Kac, *Some remarks on nilpotent orbits*, J. Algebra **64** (1980), 190–213.

[K] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[KPV] V. G. Kac, V. L. Popov, and E. B. Vinberg, *Sur les groupes linéaires algébriques dont l'algèbre des invariantes est libré*, C. R. Acad. Sci. Paris **283** (1976), 875–878.

[L] F. Levstein, *On some rings of invariants that are polynomial rings*, Proceedings of the IX ELAM (to appear).

[LR] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem.* Duke Math. J. **46** (1979), 487–497.

[LV] D. Luna and T. Vust, *Une théorème sur les orbites affines des groupes algébriques semi-simples*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **27** (1973), 527–535.

[Po] V. L. Popov, *Representations with a free module of covariants*, Funct. Anal. Appl. **10** (1977), 242–244.

[Po2] ——, *Stability criteria for the actions of a semisimple group on an algebraic manifold*, Izv. Akad. Nauk SSSR Ser. Mat. **4** (1970), 527.

[Po3] ——, *The classification of representations which are exceptional in the sense of Igusa*, Funct. Anal. Appl. **9** (1976), 348–350.

[Sch] G. W. Schwarz, *Representations of simple Lie groups with regular rings of invariants*, Invent. Math. **49** (1978), 167–191.

[Se] F. J. Servedio, *Homogeneous vector spaces and varieties*, Trans. Amer. Math. Soc. **176** (1973), 421–444.

[T] J. Tirao, *A restriction theorem for semisimple Lie groups of rank one*, Trans. Amer. Math. Soc. **279** (1983), 651–660.

[VK] E. B. Vinberg and B. N. Kimel'feld, *Homogeneous domains on flag manifolds and spherical subgroups*, Funct. Anal. Appl. **12** (1978), 168–174.

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