

EXACTLY k -TO-1 MAPS BETWEEN GRAPHS

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ABSTRACT. Suppose k is a positive integer, G and H are graphs, and f is a k -to-1 correspondence from a vertex set of G onto a vertex set of H . Conditions on the adjacency matrices are given that are necessary and sufficient for f to extend to a continuous k -to-1 map from G onto H .

I. INTRODUCTION

A k -to-1 function, like a k -to-1 correspondence, is one such that each point in the image has exactly k inverse points. The word “map” means that the function is continuous. Given a positive integer k and two graphs G and H , we study the question of whether or not there is a k -to-1 map from G onto H . For instance, a 3-to-1 map from G_1 (in Figure 1) onto H_1 can be easily seen by collapsing the two triples of arcs to two arcs and then identifying the three original junction points. On the other hand it is not so straightforward to see that there is a 4-to-1 map from G_2 onto H_2 , and that there is no 3-to-1 map from G_2 onto H_2 .

In §IV (Theorem 1) we classify exactly which pairs of graphs admit such functions, for a given positive integer k . For a given k -to-1 correspondence f from a vertex set of G onto a vertex set of H we give necessary and sufficient conditions on the associated adjacency matrices (defined in §II) for f to extend to a k -to-1 map from G onto H . These conditions are simplified in §V for the special case $k = 2$. One problem with our classification is that many vertex sets must be checked, although for a fixed k -to-1 correspondence between two vertex sets for G and H , it is straightforward and quick (polynomial time) to verify the conditions given in Theorem 1. For instance, the vertex set of H_2 must contain the two junction points, and so, for $k = 3$, at least four points of order two in G_2 must be added to the two junction points of G_2 for a suitable vertex set for G_2 . The entries in the adjacency matrices change with different choices of these four points. Theorem 3, in §VI, states that only finitely many vertex sets for G and H need be considered, essentially by showing that there is no new information gained if a point of order two in H is added to the vertex set of H and k inverse points of order two in G are added to the vertex set of G . (See Lemma 4 in §VI.) Nevertheless, for most pairs of

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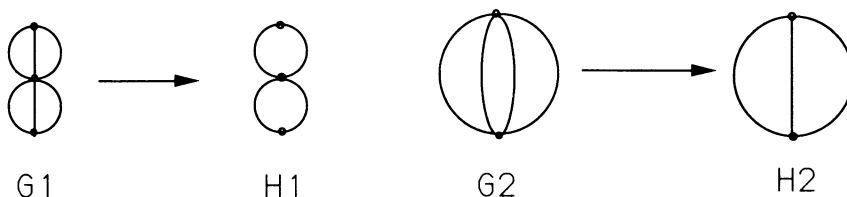


FIGURE 1

graphs, a computer would be needed to check out all of the possibilities. When $k = 1$, the problem of deciding whether there is a k -to-1 map from G onto H is the problem of deciding whether the two graphs are homeomorphic. Using more usual graph theory terminology, this is known as the Graph Isomorphism problem. The computational complexity of Graph Isomorphism is not known and is at present one of the major unsolved problems in its area, and it seems likely that the computational complexity of the problem of deciding whether, for a given $k > 1$ and for two given graphs G and H , there is a k -to-1 map from G to H , will be equally difficult to resolve.

For many graphs, there are quick ways to rule out the existence of a k -to-1 map from one to the other. In [5], the first author established necessary and sufficient conditions for the existence of a k -to-1 finitely discontinuous function from one graph G onto another graph H in terms of their Euler numbers (the Euler number of graph G , $\text{Euler}(G)$, is the number of edges minus the number of vertices, well known to be independent of the vertex set used). Namely, there is a k -to-1 finitely discontinuous function from G onto H iff

- (1) $\text{Euler}(G) \leq k \text{ Euler}(H)$ if $k \geq 2$, and
- (2) $\text{Euler}(G) = 2 \cdot \text{Euler}(H)$ if $k = 2$.

A quick check of the Euler numbers may indicate that there is no finitely discontinuous function from G onto H and so of course there is no continuous k -to-1 function. On the other hand, no information is gained if the two graphs satisfy the Euler condition. This is what happens in the example above, since $\text{Euler}(G_2) = 2$ and $\text{Euler}(H_2) = 1$ and (1) is true for $k = 3$ and 4.

Another quick way to rule out the existence of a k -to-1 map for some pairs of graphs is to find a junction point or endpoint of one of the graphs whose order is incompatible with the orders of the points of the other graph. (This order compatibility is, in fact, property (1) of the necessary and sufficient conditions in Theorem 1.) For instance, if G has a point of order greater than 6 and each point of H has order 2 (H is a circle) then there is no 3-to-1 map from G onto H . Or, if H has an endpoint e (a point of order one), G has only three endpoints and G has no isolated point (a point of order zero), then there cannot be a k -to-1 map from G onto H for any $k > 3$ since the orders of the inverse points of e cannot exceed k . Notice that this type of order incompatibility is not present in the examples G_2 and H_2 .

In an earlier paper, [4], the authors established exactly which trees admit continuous 3-to-1 functions onto the circle. Requiring that G be a tree (a connected graph with no cycles), that H be the circle, and that $k = 3$ is of course a very special case, but the description of valid trees (the G 's) is concrete. In this paper we do not restrict G , H or k but there are no geometric descrip-

tions here of exactly which G , H and k work and which do not. Curiously enough, this previous geometric result required global considerations (we had to construct an arc in G that contained each point of order four that also mapped onto the image H) whereas the algebraic results of the present paper require only local information (for each vertex p in H we need only inspect the edges emanating from p and the edges in G emanating from $f^{-1}(p)$). It might also be mentioned that continuous k -to-1 maps on graphs are studied from a topological viewpoint in [1, 2, and 3], but the problem of classifying the G , H and k that work is not addressed in these papers.

II. DEFINITIONS

A few words about our terminology, which, at least for graph theorists, is nonstandard. An “arc” is a compact topological space homeomorphic to the real unit interval $[0, 1]$. A “graph” is the (not necessarily connected) union of a finite number of isolated points and a finite number of arcs (called “edges”) each two of which either do not intersect or else intersect in one or both common endpoints. Each endpoint and each isolated point is called a “vertex” or “node.” The “order” of a vertex is the number of edges the vertex is in (this is usually known to graph theorists as the “degree” or “valency” of the vertex). The order of any point which is not a vertex is two. There are many ways to decompose a graph into vertices and edges since extra vertices of order two can be added, but the order of a point is independent of the vertex set used. Note that this definition precludes any loops in the vertex-edge decomposition of a graph.

Now suppose that G and H are graphs with vertex sets N and M respectively and suppose that f is a k -to-1 correspondence from N onto M . We wish to define two key matrices that depend on G , H , N , M , and f . First, the matrix A is the usual “adjacency” matrix for H indexed by $M \times M$ such that the entry $A(p, q)$, for vertices p and q in M , is defined to be the number of edges in H with endpoints p and q . The matrix B is the “inverse adjacency” matrix for G , H and f , also indexed by $M \times M$, such that the entry $B(p, q)$, for vertices p and q in M , is defined to be the number of edges in G with one endpoint in $f^{-1}(p)$ and the other endpoint in $f^{-1}(q)$. Note that $A(p, p)$ is zero and $B(p, p)$ is the number of edges in G with both endpoints in $f^{-1}(p)$, and note that both A and B are symmetric. In addition, it will be useful to name the sets of edges whose cardinalities are the entries of A and B . Accordingly, let “ $a(p, q)$ ” denote the set of edges in H with endpoints p and q , and let “ $b(p, q)$ ” denote the set of edges in G with one endpoint in $f^{-1}(p)$ and the other endpoint in $f^{-1}(q)$. For each vertex p in M let “ $O(p)$ ” denote the order of p in H as defined in the previous paragraph, and let “ $O(f^{-1}(p))$ ” denote the sum of the orders of the vertices in $f^{-1}(p)$ in G . Note that for each vertex p in M , $O(p)$ is just the row sum of the p th row of the matrix A , and $O(f^{-1}(p))$ is the sum of the off diagonal entries of the p th row of B plus twice the diagonal entry $B(p, p)$. $B(p, p)$ is counted twice since each edge between points of $f^{-1}(p)$ is counted twice for $O(f^{-1}(p))$.

III. THREE LEMMAS

Lemma 1 describes a classic, perhaps folklore, map used in many papers on k -to-1 maps.

Lemma 1. *Suppose m is a positive odd integer. Then there is a continuous map from $[0, 1]$ onto $[0, 1]$ such that:*

- (1) *1 is the only number that maps to 1,*
- (2) *exactly $(1/2)(m + 1)$ numbers map to 0 and*
- (3) *if x is in $(0, 1)$ then exactly m numbers map to x .*

Proof. First, as carefully described in [5], there is a simple piecewise linear map h from a closed interval onto itself that is exactly m -to-1 on the interior of the interval and is $(1/2)(m + 1)$ -to-1 at both endpoints. Its graph has the shape of a slanted “ N ” if m is 3 and has the shape of a generalized slanted “ N ” with more hills and valleys if $m > 3$. Now to construct f , divide $[0, 1)$ into infinitely many subintervals $[0, 1/2]$, $[1/2, 3/4]$, $[3/4, 7/8]$, ... and construct a copy of the map h from each subinterval onto itself. The union of these copies of h plus the one ordered pair $(1, 1)$ is f . \square

Corollary. *If m and n are odd positive integers, then there is a continuous map from $[0, 1]$ onto itself that is 1-to-1 at both 0 and 1, is m -to-1 on $(0, 1/2)$, is n -to-1 on $(1/2, 1)$, and is $(1/2)(m + n)$ -to-1 at $1/2$.*

Proof. Piece together two maps similar to the one constructed in the proof of Lemma 1, one from $[0, 1/2]$ onto itself with $(1/2)(m + 1)$ numbers mapped to $1/2$, the other from $[1/2, 1]$ onto itself with $(1/2)(n + 1)$ numbers mapped to $1/2$. \square

Lemma 2. *Suppose f is a continuous finite-to-1 function from $[0, 1]$ into $[0, 1]$, E is the set of points in $[0, 1]$ whose inverse set has even cardinality, and U is the set of points whose inverse has odd cardinality. Then, if $f(0) = 0$ and $f(1) = 1$ then E is not uncountable, and if $f(0) = 0 = f(1)$ then U is not uncountable.*

Proof. For each positive integer k , let $M(k)$ be the set of points x in $[0, 1]$ such that if $f(a) = f(b) = x$ then $|a - b| > 1/k$. Since f is continuous and finite-to-1, each point on the graph of f is either a local maximum, a local minimum, or a “crossing point,” i.e. in some neighborhood of the point, the point is higher than all of the points on one side of it and lower than all of the points on the other side of it. This is a direct result of the intermediate value theorem for continuous functions and the fact that each domain point c has a neighborhood in which it is the only point that maps to $f(c)$.

We will prove only one of the two conclusions. Suppose $f(0) = 0$ and $f(1) = 1$. For each positive integer k , let $E(k) = E \cap M(k)$. We will show $E(k)$ is finite to complete this case. Suppose, on the contrary, that $E(k)$ is infinite. For each number b in $E(k)$ there is a point (a, b) on the graph of f that is either a local maximum or is a local minimum since the intermediate value theorem precludes the graph of f intersecting the line $y = b$ exactly at an even number of crossing points. Hence we may assume that for infinitely many b in $E(k)$ there is a local maximum on $y = b$. (The argument for infinitely many minima is similar.) Now let A denote the set of numbers, a , in $[0, 1]$ such that (a, b) is a local maximum for f and b is in $E(k)$. Then A is infinite and there must be two numbers a and a' in A such that $|a' - a| < 1/k$. One of $b = f(a)$ and $b' = f(a')$ is larger; say $b < b'$. Note that b cannot equal b' since the definition of $M(k)$ prevents two inverse points,

here a and a' , from being within $1/k$ of each other. But $b < b'$ contradicts the intermediate value theorem since (a', b') is above the line $y = b$, some point of the graph between (a, b) and (a', b') is below $y = b$ (since (a, b) is a local maximum) and no point between (a, b) and (a', b') is on the line $y = b$ (from the definition of $M(k)$). \square

Lemma 3. *Let f be a k -to-1 map from a graph G to a graph H , and let p be a point of order 1 in H . Let $f^{-1}(p)$ contain x points of order 1 and y points of order 0. Then $(1/2)(k - x) \leq y \leq k - 1$.*

Proof. Since p has nonzero order, there are clearly at most $k - 1$ points of order 0 in $f^{-1}(p)$, so $y \leq k - 1$. Suppose $f^{-1}(p)$ contains z points whose order is greater than one. Then $k = x + y + z$. Since p has order 1, there is a point u close to p which has at least $2z + x$ points mapped to it. Hence $k \geq 2z + x = 2(k - x - y) + x = 2k - x - 2y$ which implies $(1/2)(k - x) \leq y$. \square

IV. THEOREM 1

As promised in the Introduction, Theorem 1 classifies the graphs G and H and positive integers k for which there exist continuous k -to-1 maps from G onto H . Condition (3) for even k is somewhat messy; nevertheless for a given k -to-1 correspondence between vertex sets for G and H , it is routine to decide whether or not the three conditions hold. The main problems with this classification are (1) there are many vertex sets to check (although Theorem 3 reduces this to a finite number), and (2) for fixed vertex sets, there are many possible k -to-1 correspondences.

Theorem 1. *Suppose G and H are graphs and f is a k -to-1 correspondence from a vertex set of G onto a vertex set of H . Then f extends to a k -to-1 map from G onto H iff f , the adjacency matrix A , and the inverse adjacency matrix B satisfy:*

- (1) $k \cdot O(p) \geq O(f^{-1}(p))$ for each vertex p in H ,
- (2) each off-diagonal entry of $kA - B$ is even and nonnegative, and
- (3) if k is odd then each entry of $B - A$ is nonnegative; and if k is even then, for each vertex p of H ,

$$B(p, p) \geq \sum_{q \neq p} \max\{A(p, q) - (1/2)B(p, q), 0\}.$$

Example. Before proving Theorem 1, we will demonstrate its usefulness on the two simple graphs G_2 and H_2 described in the introduction. Recall that the two shortcut methods (of determining that no 3-to-1 map exists from G_2 onto H_2) fail for these two graphs. (See Figure 2.)

First note that each point of G_2 has even order, so whatever the vertex sets for G_2 and H_2 are, if p is a vertex in H_2 then $O(f^{-1}(p))$ is even. But $O(f^{-1}(p))$ is the sum of the off-diagonal entries of the p th row of B plus twice $B(p, p)$, so the sum of the off-diagonal entries of the p th row of B is even. There are, then, an even number of odd off-diagonal entries on the p th row of B . Now suppose p is a vertex of H_2 of odd order, either a or b . Since the p th row sum of A is the order of p and the diagonal entry is zero, there are an odd number of odd off-diagonal entries in the p th row of A . Thus

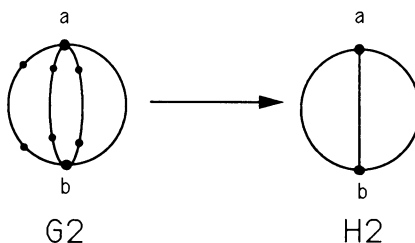


FIGURE 2

some off-diagonal entry of the p th row of $3A - B$ is odd, contrary to condition (2) in Theorem 1. (This argument is the basis of Corollaries 1.1 and 1.2.)

On the other hand, the situation is different for $k = 4$. Let $f(a') = a$ and $f(b') = b$, and map the three upper unnamed vertices in the drawing to a and the lower three unnamed vertices to b . Then the A and B matrices are as follows:

B	a	b
a	3	4
b	4	3

A	a	b
a	0	3
b	3	0

Property (1) is verified by checking that 4 times each row sum of A is not less than the corresponding row sum for B with the diagonal element added in twice (so $4 \cdot 3 \geq (3+4) + 3$). Both off-diagonal elements of $4 \cdot A - B$ are 8, and 8 is both even and nonnegative; and $B(b, b) = B(a, a) = 3$ is not less than $\max\{1, 0\}$. Hence the three properties hold and there is a 4-to-1 map from $G2$ onto $H2$. The structure of this 4-to-1 map is described in the "sufficiency" part of the proof of Theorem 1.

Observation. Note that if $k = 1$, then B is the adjacency matrix for G as well as the inverse adjacency matrix for f , and note that $B(p, p) = 0$ for each vertex p in H since there are no loops in G . Hence every entry of $A - B$ is nonnegative and so is every entry of $B - A$ if properties (2) and (3) are true. Thus for $k = 1$, Theorem 1 reduces to the unsurprising statement that two graphs are homeomorphic iff they have equal adjacency matrices.

Proof of Theorem 1.

I. Proof of sufficiency. Assume that G , H , and f satisfy the hypothesis and (1), (2), and (3).

We first establish a useful fact true for each vertex of H whether k is even or odd:

$$\begin{aligned}
 B(p, p) &= \left(\frac{1}{2}\right) \left[O(f^{-1}(p)) - \sum_{q \neq p} B(p, q) \right] \\
 &\leq \left(\frac{1}{2}\right) \left[k \cdot O(p) - \sum_{q \neq p} B(p, q) \right] \\
 &= \left(\frac{1}{2}\right) \cdot \sum_{q \neq p} [k \cdot A(p, q) - B(p, q)].
 \end{aligned}$$

Therefore,

$$(i) \quad \sum_{q \neq p} [k \cdot A(p, q) - B(p, q)] \geq 2 \cdot B(p, p).$$

The first equality follows from the fact that each edge in $b(p, p)$ is counted twice in $O(f^{-1}(p))$, the inequality follows from property (1) and the next equality follows from the fact that H has no loops. The sets $a(p, q)$ and $b(p, q)$ are defined in the §II. The k -to-1 continuous map is now constructed as follows.

Case 1. k is odd. 1. The edges in $b(p, q)$ are mapped each 1-to-1 onto the edges in $a(p, q)$, extending f , in such a way that each edge in $a(p, q)$ has an odd number ($\leq k$) of edges in $b(p, q)$ mapped onto it. It follows from property (3) that there are enough edges in $b(p, q)$ for each edge in $a(p, q)$ to get at least one; it follows from the fact that the entries of $kA - B$ are nonnegative that there are not too many edges in $b(p, q)$ for this; and it follows from the fact that the entries of $kA - B$ are even that $A(p, q)$ and $B(p, q)$ have the same parity and hence that an odd number of edges in $b(p, q)$ can be allotted to each edge in $a(p, q)$.

2. We now wish to map the edges in $b(p, p)$ onto various of the edges in H with one endpoint p . If the edge in $a(p, q)$ has only m edges in $b(p, q)$ mapped onto it from Step 1 then it has a $k - m$ "shortfall" at this point. The sum of these shortfalls, as q ranges over the vertices of H , is represented by the left side of reference inequality (i). Hence the edges in $b(p, p)$ can be mapped at most 2-to-1 (as described next) onto various edges with one endpoint p without any point having more than k inverse points. To describe the map wanted, let X be an edge in $b(p, p)$ and Y an edge in H with one endpoint p . Simply fold X in half mapping its "midpoint" to the "midpoint" of Y , and continuing to map both endpoints to p . For each edge, any interior point may be chosen to be its "midpoint."

3. The map constructed maps G continuously onto H , extends f , and is at most k -to-1. In addition, each edge Y in H has an odd number of edges from G that map 1-to-1 onto it and Y may or may not have some edges from G that are "folded" onto half of Y as described in Step 2. We will now correct the map so that it is exactly k -to-1. Suppose Y is an edge in H from p to q with n edges mapped 1-to-1 into it, with m edges folded onto its p half and with j edges folded onto its q half. The function is k -to-1 at the endpoints of Y , $(n + 2m)$ -to-1 at points in its p half, $(n + m + j)$ -to-1 at its "midpoint," and $(n + 2j)$ -to-1 at points in its q half. Let X denote one of the n edges in $b(p, q)$ that maps 1-to-1 onto Y . Since k and n are odd, it follows that $k - (n + 2m - 1)$ and $k - (n + 2j - 1)$ are both odd, and so, by using the corollary to Lemma 1 on X to Y , the map on X can be changed to one that is still 1-to-1 at the endpoints of Y , that is $(k - (n + 2m - 1))$ -to-1 on the p side of Y , that is $(k - (n + m + j - 1))$ -to-1 at the midpoint of Y , and that is $(k - (n + 2j - 1))$ -to-1 on the q half of Y . The careful reader can verify that the revised map is now exactly k -to-1 on Y .

Case 2. k is even. 1. Suppose p and q are two vertices in H . Pairs of edges in $b(p, q)$ are mapped each 1-to-1 onto the edges in $a(p, q)$ in such a way that (i) as many as possible of the edges in $a(p, q)$ get at least one pair of edges in $b(p, q)$ mapped to them, and (ii) no edge in $a(p, q)$ gets more than k edges

mapped to it. By property (2) and the fact that k is even, each $B(p, q)$ is even, and so, since $kA - B$ is nonnegative, all of the edges in $b(p, q)$ can be used.

2. For each edge Y in $a(p, q)$ not yet mapped onto, map one of the edges in $b(p, p)$ onto the p half of Y using a fold as described in Step 2 of Case 1. To see that there are enough edges in $b(p, p)$ to do this for every vertex r in H , note that the number of edges in H with one endpoint p not mapped onto in Step 1 is $\sum_{r \neq p} \max[A(p, r) - (1/2)B(p, r), 0]$ which is no more than $B(p, p)$ by property (3) for even k . Map the rest of the edges in $b(p, p)$, if any, by folds on the p halves of any of the edges in H with one endpoint p , making sure that no point in the image has more than k points mapping to it. Property (1) ensures that there are not too many edges in $b(p, p)$ to do this, since the edges in $b(p, p)$ are counted twice in $O(f^{-1}(p))$. Now do the same on the rest of H .

3. We now have a continuous map from G onto H which extends f and is at most k -to-1. Furthermore, if p and q are vertices of H and Y is an edge in $a(p, q)$ then there is an even number $2n$ (possibly equal to 0) of edges in $b(p, q)$ mapped 1-to-1 onto Y , there are m edges in $b(p, p)$ folded onto the p half of Y and there are j edges in $b(q, q)$ folded onto the q half of Y . Furthermore, if $n = 0$ then each of m and j is positive. The function is k -to-1 at the endpoints of Y , $(2n + 2m)$ -to-1 at points in its p half, $(2n + m + j)$ -to-1 at its midpoint, and $(2n + 2j)$ -to-1 at points in its q half. Let X denote one of the n edges in $b(p, q)$ that maps 1-to-1 onto Y , if there is such an edge. By using the corollary to Lemma 1 on X as before, the map on X can be changed to one that is still 1-to-1 at the endpoints of Y , $(k - (2n + 2m - 1))$ -to-1 on the p side of Y , $(k - (2n + m + j - 1))$ -to-1 at the midpoint of Y , and $(k - (2n + 2j - 1))$ -to-1 on the q half of Y . If there is no such edge X then half of a fold at p and half of a fold at q can be changed using Lemma 1 in a similar way to correct the cardinalities of the point inverses. If the function is thus corrected for each pair of vertices p and q in H , the resulting function satisfies all of the required properties.

II. *Proof of the necessity.* Now suppose that f is a k -to-1 correspondence from the vertex set N of G onto the vertex set M of H that extends to a continuous k -to-1 function from G onto H . We will show that f , the adjacency matrix A and the inverse adjacency matrix B satisfy properties (1), (2) and (3). The extension of f to all of G will also be called f .

First note that because f is continuous and is exactly k -to-1 from the vertices of G to the vertices of H , the image of each edge in $b(p, q)$ is an edge in $a(p, q)$ with only endpoints mapping to endpoints.

Now suppose that Y is an edge in $a(p, q)$. It follows from Lemma 2 that there is a point y of Y close enough to p that (i) if the edge X in $b(p, q)$ maps onto Y then the number of points in X that map to y is an odd number, (ii) if the edge X in $b(p, p)$ maps into Y , then y is in its image and the number of points in X that map to y is even, and (iii) y has no other inverse points; i.e. y is not in the image of any edge from $b(q, q)$. We will see that the three properties of Theorem 1 follow from these observations.

Suppose Y is an edge in $a(p, q)$ and y is one of the special points described above.

(1) Since y has exactly k inverse points in G , the number of edges in $b(p, q)$ with $p \neq q$ that map to Y plus twice the number of edges of G in $b(p, p)$ that map to Y cannot exceed k . Property (1) follows from this fact.

(2) Each entry, $k \cdot A(p, q) - B(p, q)$, off the diagonal of $kA - B$ is nonnegative since otherwise some edge in $a(p, q)$ has more than k edges in $b(p, q)$ mapping to it. To see that the entry $k \cdot A(p, q) - B(p, q)$ is even, two cases must be taken. In the case that k is odd, there must be an odd number of edges in $b(p, q)$ mapping to Y due to the parities of the inverse points of y in the edges of G that map to Y . This implies that the parity of $B(p, q)$ is the same as the parity of $A(p, q)$. Hence $k \cdot A(p, q) - B(p, q)$ is even. In the case k is even, the parities of the sets of inverses of y dictate that an even number of edges in $b(p, q)$ map onto Y which in turn implies that $B(p, q)$ is even. Hence again $k \cdot A(p, q) - B(p, q)$ is even.

(3) Property (3) changes with the parity of k .

For the case where k is odd, property (3) is that each entry of $B - A$ is nonnegative. The diagonal entries of A are zero and all entries of A and of B are nonnegative, so only the off-diagonal entries of $B - A$ need be considered. Suppose for some pair of vertices p and q of H there are more edges in $a(p, q)$ than edges in $b(p, q)$. Then some edge Y in $a(p, q)$ has no inverse points in any edge from $b(p, q)$ and so the set of inverse points of any special point y must be even (a contradiction).

For the case where k is even, property (3) is that

$$B(p, p) \geq \sum_{q \neq p} \max[A(p, q) - (1/2)B(p, q), 0]$$

for each vertex p of H . Suppose that q is a vertex of H different from p for which $m = A(p, q) - (1/2)B(p, q)$ is a positive number, Y is an edge in $a(p, q)$ and y is its special point close to p . Since k is even, an even number of edges in $b(p, q)$ must map onto Y , using once again the parities of the inverse sets of y . Hence all of the edges in $b(p, q)$ can map to at most $(1/2)B(p, q)$ edges in $a(p, q)$. Thus the difference, m , from above is the minimum number of edges in $b(p, p)$ needed to map onto the edges of $a(p, q)$ left out. Since $b(p, p)$ must make up this shortfall for all of the vertices q different from p for which $m > 0$, the inequality follows. \square

By the argument used to show that there is no 3-to-1 map from G_2 onto H_2 , we obtain the following corollary to Theorem 1. Note that a graph is "Eulerian" if every point has even order.

Corollary 1.1. *If k is odd, G is an Eulerian graph and H is a graph that is not Eulerian, then there is no k -to-1 map from G onto H .*

More generally, the same argument yields:

Corollary 1.2. *If k is odd and the graph G has fewer points of odd order than the graph H has, then there is no k -to-1 map from G onto H .*

V. THE CASE $k = 2$

Theorem 1 holds for $k = 2$, but for this special case there is a simpler set of conditions on G and H which are equivalent to the existence of a 2-to-1 map from G onto H .

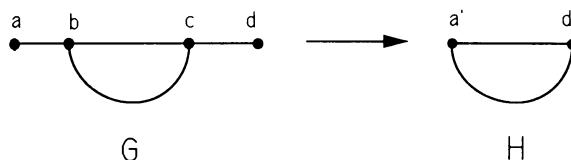


FIGURE 3

Theorem 2. Suppose G and H are graphs. Then there is a 2-to-1 continuous function from G onto H iff there are vertex sets for G and H and an exactly k -to-1 correspondence f from the vertex set of G onto the vertex set of H such that f , the adjacency matrix A , and the inverse adjacency matrix B satisfy:

- (1) $2 \cdot O(p) = O(f^{-1}(p))$ for each vertex p of H , and
- (2) each off-diagonal entry of $2A - B$ is even and nonnegative.

Before proving Theorem 2, we will give some examples. Note that by inspecting the matrices A and B , condition (1) can be easily checked since $O(p)$ is the sum of the elements in the p th row of A , and $O(f^{-1}(p))$ is the sum of the elements in the p th row of B plus the diagonal element $B(p, p)$ (so that $B(p, p)$ is counted twice).

Example 1. Suppose G is the graph pictured at the left and H is a simple circle. We will use Theorem 2 to determine that there are two different 2-to-1 maps from G onto H . (See Figure 3.)

Every point of H has order two so each of the points of order three in G must map the same as one of the endpoints, by condition (1). Thus either $f(a) = f(b)$ and $f(c) = f(d)$, case (i), or $f(a) = f(c)$ and $f(b) = f(d)$, case (ii). In each case, let $a' = f(a)$ and $d' = f(d)$.

We will see in Lemma 4 in the next section that there is no advantage in adding nodes of order two to G or nodes other than a' and d' of order two to H . That is, if any two vertex sets for G and H satisfy properties (1) and (2) of Theorem 2 then so will (a, b, c, d) for G and (a', d') for H .

The adjacency matrices in these two cases are:

			case (i)			or			case (ii)		
A	a'	d'	B	a'	d'				B	a'	d'
a'	0	2	a'	1	2	or			a'	0	4
d'	2	0	d'	2	1				d'	4	0

Both of the B matrices work with the A matrix, since the only off-diagonal element of $2A - B$ is either 2 or 0, and twice the row sum of A is 4 (for both rows), and the row sum for each B matrix, with the diagonal element added twice, is 4. These cases correspond to the following two maps. In the drawing, G is distorted so that the identifications are clear. (See Figure 4.)

Example 2. Now suppose that G and H are the graphs pictured below. We will show using Theorem 2 that there is no 2-to-1 map from G onto H . Note that



FIGURE 4

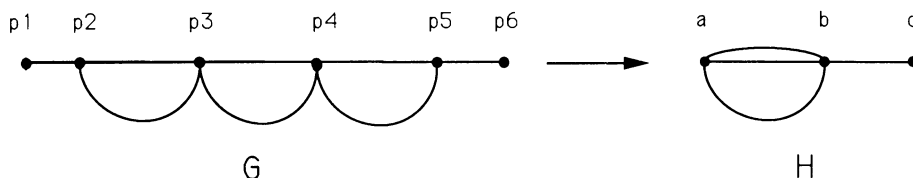


FIGURE 5

the Euler number of G is 2 and the Euler number of H is 1, so the criterion mentioned in the introduction ($2 \cdot \text{Euler}(H) = \text{Euler}(G)$) is satisfied; hence there is a finitely discontinuous 2-to-1 function from G onto H , but this yields no information concerning continuous 2-to-1 maps from G onto H . (See Figure 5.)

From condition (1) alone we can determine that (i) p_3 and p_4 must map to b , since they are the only two points in G whose orders add up to 8, (ii) p_2 and p_5 must map to a , since they are the only two points of G left whose orders add up to 6, and (iii) p_1 and p_6 must map to c , since they are the only two points whose orders add up to 2. Once again, we will see in Lemma 4 that if the two vertex sets pictured do not satisfy the hypothesis of Theorem 2 then no two vertex sets will. The matrices for the map described are:

B	a	b	c
a	0	4	2
b	4	2	0
c	2	0	0

A	a	b	c
a	0	3	0
b	3	0	1
c	0	1	0

Condition (1) holds but the (a, c) entry of $2A - B$ is -2 , so condition (2) does not hold. Hence there is no 2-to-1 continuous map from G onto H .

Observation. If the union of the edges in the graph G is an arc then there is no 2-to-1 map from G onto any graph H . (This is a weak version of theorems in [2 and 6].)

Suppose f is such a map. If a is one of the endpoints of G , then it has order 1 in G and since $O(f^{-1}(f(a)))$ must be even, by condition (1), and since no point of G except an endpoint has odd order, a must map the same as the other endpoint of G , and its image has order 1. Now, every node in G that is not an endpoint has order 2, so $O(f^{-1}(q)) = 4$, for any node q of H different from $f(a)$ and so $O(q) = 2$. Thus we have the impossible graph H with exactly one node of order one and every other node of order two.

Proof of Theorem 2. Suppose f, A, B, G, H satisfy properties (1) and (2) of Theorem 2. Then properties (1) and (2) of Theorem 1 are satisfied and we will show that property (3) of Theorem 1, for $k = 2$, is true. This will imply that there is a 2-to-1 map from G onto H as wanted. So, let p be any vertex in H . Then

$$\begin{aligned}
 B(p, p) &= \left(\frac{1}{2}\right) \left(O(f^{-1}(p)) - \sum_{q \neq p} B(p, q) \right) \\
 &= \left(\frac{1}{2}\right) (2 \cdot O(p) - \sum_{q \neq p} B(p, q)) \\
 &\quad \text{(by property (1) of Theorem 1)} \\
 &= \left(\frac{1}{2}\right) \sum_{q \neq p} (2 \cdot A(p, q) - B(p, q)) \\
 &= \sum_{q \neq p} (A(p, q) - \left(\frac{1}{2}\right) \cdot B(p, q)).
 \end{aligned}$$

But from property (2), each $A(p, q) - (1/2) \cdot B(p, q)$, is nonnegative so it is equal to $\max\{A(p, q) - (1/2) \cdot B(p, q), 0\}$, and condition (3) of Theorem 1 is true.

Now suppose that there is a 2-to-1 map f from G onto H . Then there are vertex sets for G and H such that the restriction of f is a 2-to-1 correspondence between them and such that the adjacency matrix A and the inverse adjacency matrix B satisfy properties (1), (2), and (3) of Theorem 1. Clearly property (2) of Theorem 2 holds, but suppose for some vertex p of H , $2O(p) > O(f^{-1}(p)) = n$. Let p_1 and p_2 denote the two vertices of G that map to p and suppose there are m edges from p_1 and p_2 , i.e. $B(p, p) = m$. There is a neighborhood U about p such that only points from the edges in G with at least one endpoint p_1 or p_2 map into U . It follows from Lemma 2 that each edge in H with endpoint p contains a point q in U such that $f^{-1}(q)$ has both points in one of the m edges from p_1 to p_2 , or has one point in each of two different edges in G , and each of these two edges in G has either p_1 or p_2 as an endpoint but not both. Since the m edges from p_1 to p_2 are counted twice in $O(f^{-1}(p))$ there are $n - 2m$ edges in G with exactly one endpoint either p_1 or p_2 . Also, as was previously observed, each edge in G can only map into one edge in H . Hence at most $(1/2)(n - 2m) + m$ edges in H with one endpoint p are mapped onto. But $O(p) > (1/2)n$ by assumption, so some edge in H is not mapped onto, a contradiction.

VI. THEOREM 3

Given a graph G , every vertex set must contain all points whose order is not two; and, to eliminate loops, some points of order two may be needed in every vertex set. There are however infinitely many vertex sets of larger and larger cardinalities which can be constructed by adding to this minimal set of vertices any number of points of order two. Thus it would seem that there are too many vertex sets, for two graphs G and H , which need to be checked (using Theorem 1) to see if any k -to-1 correspondence between vertex sets satisfy

properties (1), (2), and (3). The purpose of Theorem 3 is show that there is a number m such that only vertex sets for H of cardinality no more than m need be checked (and so the cardinality of relevant vertex sets for G is bounded by km). Furthermore, suppose E is an arc in H whose endpoints have orders other than two, and each interior point of E has order two. If the minimal vertex set of H , which must contain the endpoints of E , is to be augmented by adding j interior points of E , it clearly makes no difference for conditions (1), (2), and (3) which j points are chosen. Thus Theorem 3 reduces the number of vertex sets to be tested to a finite number.

A vertex set for a graph must contain each point whose order is not two, and perhaps some points of order two so that there are no loops. For instance, each vertex set for a circle must contain at least two points of order two. We define a subset V' of vertex set V to be “inessential” if $V \setminus V'$ is still a vertex set.

Lemma 4. *Suppose f is a k -to-1 correspondence from a vertex set N for a graph G onto a vertex set M for a graph H such that (1) properties (1), (2), and (3) of Theorem 1 are satisfied, and (2) there is a vertex y of M such that $\{y\}$ is inessential in M and $f^{-1}(y)$ is inessential in N . Then the restriction of f to $N \setminus f^{-1}(y)$ still satisfies properties (1), (2) and (3) of Theorem 1.*

Proof. Since f satisfies the properties of Theorem 1, f extends to a k -to-1 map from G onto H . Since f restricted to $N \setminus f^{-1}(y)$ also extends (to the same k -to-1 map), it follows from Theorem 1 that f restricted to $N \setminus f^{-1}(y)$ also satisfies properties (1) (2), and (3) from Theorem 1. \square

An upper bound $m = m(G, H, k)$ of the cardinalities of relevant vertex sets for H will now be defined. Let $e(G)$ and $e(H)$ denote the number of endpoints (the points of order one) in G and H respectively; let $i(G)$ and $i(H)$ denote the number of isolated points (the points of order zero) in G and H respectively; let j denote the number of points of order two in any minimal vertex set of G ; and let h denote the cardinality of any minimal vertex set of H . It follows immediately from property (1) of Theorem 1 that each vertex in H of order zero has as its inverse k vertices in G of order zero. Hence $i(G) - k \cdot i(H)$ is the number of “surplus” vertices in G of order zero, vertices available to map to some vertex of H whose order is not zero. Let S denote this number of surplus isolated points in G . The number m is defined to be

$$\begin{cases} h + e(G) - e(H) + S + j & \text{if } S \geq (k-1) \cdot e(H), \\ h + e(G) - k \cdot e(H) + 2S + j & \text{if } S \leq (k-1) \cdot e(H). \end{cases}$$

After the proof of Theorem 3, a simple example is given of two graphs for which this m is the smallest integer for which Theorem 3 holds.

Theorem 3. *Suppose G and H are graphs and k is a positive integer, suppose that M is a vertex set for H with more than $m = m(G, H, k)$ points, and suppose f is a k -to-1 correspondence from some vertex set N for G onto M that satisfies properties (1), (2), and (3) of Theorem 1. Then some proper restriction of f is still a k -to-1 correspondence from a vertex set of G onto a vertex set of H and satisfies the three properties of Theorem 1.*

Proof. Let m , G , H , M , N , and f satisfy the supposition of Theorem 3 where M has m' elements with $m' > m$. We will show that one point y of

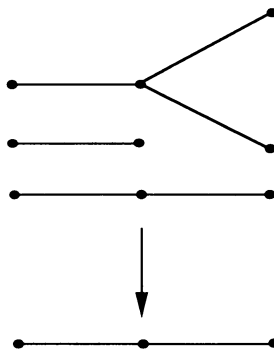


FIGURE 6

M and the k points of $f^{-1}(y)$ in N are inessential and so from Lemma 4 the restriction of f to $N \setminus f^{-1}(y)$ satisfies the conclusion of Theorem 3.

Since h is the size of any minimal vertex set of H , there are $m' - h$ inessential vertices in M .

If a point of order 1 in H has x points of order 1 in G in its inverse and y points of order 0 in G in its inverse, then, by Lemma 3, $(1/2)(k - x) \leq y \leq k - 1$, and from the first inequality, $x + y \geq k - y$. Suppose that the $e(H)$ points of order 1 in H have a total of X points of order 1 in G in their inverses and a total of Y points of order 0 in G in their inverses. Then $Y \leq \min\{S, (k - 1) \cdot e(H)\}$ and

$$\begin{aligned} X + Y &\geq k \cdot e(H) - Y \geq k \cdot e(H) - \min\{S, (k - 1) \cdot e(H)\} \\ &= \begin{cases} e(H) & \text{if } S \geq (k - 1) \cdot e(H), \\ k \cdot e(H) - S & \text{if } S \leq (k - 1) \cdot e(H). \end{cases} \end{aligned}$$

The number of points of order 0 and 1 in G available for inverses for points of order 2 or more in H is $(e(G) - X) + (S - Y)$. It follows from property (1) (or from the argument used in the proof of Lemma 3) that if q has order 2 in H and no inverse point for q has order 0 or 1 in G , then all k of the inverse points of q have order 2 in G . Therefore the number of inessential vertices in M which have inverses made up only of points of order 2 in G is at least $m' - h - (e(G) + S - X - Y)$, and this number is at least

$$\begin{cases} m' - h - e(G) + e(H) - S & \text{if } S \geq (k - 1) \cdot e(H), \\ m' - h - e(G) + k \cdot e(H) - 2S & \text{if } S \leq (k - 1) \cdot e(H). \end{cases}$$

Of these inverse sets containing only order 2 vertices, at most j can fail to be inessential (by the definition of j). Hence, since

$$\begin{cases} m' - h - e(G) + e(H) - S - j & \text{if } S \geq (k - 1) \cdot e(H), \\ m' - h - e(G) + k \cdot e(H) - 2S - j & \text{if } S \leq (k - 1) \cdot e(H), \end{cases}$$

is positive, there is an inessential vertex q of order 2 in M such that $f^{-1}(q)$ is inessential in N . \square

Example. Let G be the graph pictured with three components, and H the graph below G in Figure 6. Then for $k = 3$, $m(G, H, 3)$ is the smallest integer for which Theorem 3 holds.

Proof. By inspection, $m(G, H, 3) = 3$. Since a downward projection from G onto H is a continuous 3-to-1 map, the three conditions of Theorem 1 must hold for the vertex sets illustrated. No smaller vertex set is possible for H since the minimal vertex set for G has eight vertices. Thus nine is the least multiple of three that is at least eight and the vertex set for H must have at least $9/3$ points.

Question. Is there a formula for $m(G, H, k)$ that is never greater than the one given above but is, for same class of graphs G, H and integers k , less than that given above? More generally, what is the best (least) formula for $m(G, H, k)$?

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