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SUBSEQUENCE ERGODIC THEOREMS FOR L^p CONTRACTIONS

ROGER L. JONES, JAMES OLSEN AND MATE WIERDL

ABSTRACT. In this paper certain subsequence ergodic theorems which have previously been known in the case of measure preserving point transformations, or Dunford Schwartz operators, are extended to operators which are positive contractions on L^p for p fixed.

1. INTRODUCTION

Let $(\Omega, \mathscr{F}, \mu)$ be a sigma-finite measure space, T a linear operator from $L^p = L^p(\Omega, \mathscr{F}, \mu)$ to itself, for some fixed p, $1 \le p \le \infty$, and $\{n_k\}_{k=0}^{\infty}$ an increasing sequence of positive integers. In this paper we will be concerned with the almost sure convergence of the averages

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k}(f)(\omega) \tag{1.1}$$

for $f \in L^p$. Such "subsequence ergodic theorems" have been studied by many authors including Baxter and Olsen [2], Bellow and Losert [3, 4], Bourgain [5–8], Jones [11], Jones and Olsen [12], Wierdl [17], as well as others.

For special sequences $\{n_k\}_{k=1}^{\infty}$ that have zero density, Bourgain [5–8], and Wierdl [17] obtain the existence of the limit for averages of the form (1.1) for all T of the form $Tf(\omega) = f(\tau\omega)$ where τ is a measure preserving point transformation. In particular Bourgain obtains this convergence for $n_k = k^2$, for all $f \in L^2$ [5], $f \in L^p$, $p > (1+\sqrt{5})/2$ [7], and $p > 1$ [8]. He also obtained this convergence for the case of $n_k =$ the kth prime, and $p > (1+\sqrt{3})/2$ [6]. (This last result was generalized by Wierdl [17] to L^p, $p > 1$.) In [12] these are generalized by weakening the assumption that T is induced by a measure preserving point transformation to the assumption that T is a contraction on $L^p(\Omega)$ for all $p \ge 1$. In this paper we further weaken the assumption on T. In particular, this paper includes the case when T is a positive contraction on $L^p(\Omega)$, for p fixed.

In §2 of this paper we prove a transfer lemma which allows us to transfer certain inequalities from the integers to the general operator ergodic theory setting. We then study a maximal inequality on the integers. This maximal inequality is used to prove a variational inequality of the type used by Bourgain to obtain

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convergence for a dense subset of $L^p(\Omega)$. Section 3 uses the results of §2 to prove a.e. convergence of averages along certain classes of subsequences for a general class of operators. It is interesting to note that for positive operators, the variational inequality is already enough to establish almost everywhere convergence for all $f \in L^p(\Omega)$, and it is not necessary to appeal to Banach's principle. Also in §3 we use the variational inequality to obtain almost everywhere convergence results for certain Lamperti operators that are not necessarily positive.

The idea to use inequalities on the integers to prove inequalities in ergodic theory was contained in Calderón [9]. A special case of our Theorem 2.1 was contained in the paper by de la Torre [10].

Throughout the paper we will use the following notation. For each positive real number x, A_x will denote the averaging operator defined by

$$A_x = \frac{1}{x}\sum_{j=1}^{[x]} \delta_{n_j}$$

where $\{n_j\}$ is a sequence of integers, and

$$\delta_n(k) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

For ρ a fixed real number, $1 < \rho < 2$, and s a positive integer, define the operator H_s by $H_s = A_{\rho^s}$.

Let φ and ψ denote elements of $l^p(Z)$. Define the convolution operator $\varphi * \psi(j)$ by

$$\varphi * \psi = \sum_{k=-\infty}^{\infty} \overline{\varphi}(k)\psi(j+k).$$

Questions regarding operators on $L^p(\Omega)$ will lead to questions involving convolution operators on $l^p(Z)$. The solution to these problems on $l^p(Z)$ will in turn lead to the solution of the problem on $L^p(\Omega)$.

Definition. An operator T defined on $L^p(\Omega)$ will be called *Lamperti* if it has the property that whenever f and g have disjoint supports,

$$\text{(a) } Tf \text{ and } Tg \text{ have disjoint supports} \tag{1.2}$$

and

$$\text{(b) } T(f+g) = Tf + Tg. \tag{1.3}$$

Note that (1.3) is weaker than linearity, but for bounded operators, (1.3) together with (1.2) does imply linearity. Such linear operators were studied by Lamperti [14], Kan [13], Olsen [16] and others.

Definition. We will say that the operator T is a *quasi-isometry* if there exists an increasing sequence of positive integers $\{L_n\}_{n=1}^{\infty}$ and constants c_1 and c_2 such that for every $f \in L^p(\Omega)$ we have

$$c_1\|f\|^p_{L^p(\Omega)} \le \frac{1}{L_n}\sum_{k=0}^{L_n-1} \|T^k f\|^p_{L^p(\Omega)} \le c_2\|f\|^p_{L^p(\Omega)}. \tag{1.4}$$

Throughout the paper, C will denote a constant, but will not necessarily be the same constant from one occurrence to the next.

Definition. The operator T is said to be *power bounded* if and only if there exists a constant c such that $\|T^n\|_p \leq c$ for all $n \geq 0$.

2

We first prove a transfer theorem which will allow us to "transfer" the necessary inequalities from operators on $l^p(Z)$ to operators on $L^p(\Omega)$. This transfer theorem is general enough to allow us to transfer an $l^p(Z)$ version of the variational inequality used by Bourgain in [5, 6, 7]. He used the variational inequality to establish convergence for a dense subset of $L^2(\Omega)$ in the case of operators induced by measure preserving point transformations. We use the same inequality to transfer the result on the integers to the more general operators on $L^p(\Omega)$ considered in this paper.

Theorem 2.1 (The transfer principle). *Let $k_{n,s}$, $n = 1, 2, \dots, N$; $s = 1, 2, \dots$; be functions in $l^1(\mathbb{Z}^+)$ and let $p > 1$. Suppose that there is a constant C so that for any $\varphi : \mathbb{Z}^+ \to \mathbb{R}$, φ with finite support, we have*

$$\sum_{n=1}^{N} \left\| \sup_s |k_{n,s} * \varphi| \right\|_{l^p(\mathbb{Z}^+)}^p \leq C \|\varphi\|_{l^p(\mathbb{Z}^+)}^p . \tag{2.1}$$

Then if T is a Lamperti operator and a quasi-isometry on $L^p(\Omega)$, or if T is any positive contraction of $L^p(\Omega, \Sigma, m)$, then for any $f \in L^p(\Omega)$ the operators $K_{n,s}$ defined by

$$K_{n,s} f(\omega) = \sum_{m=0}^{N} k_{n,s}(m) T^m(f)(\omega) \tag{2.2}$$

satisfy

$$\sum_{n=1}^{N} \left\| \sup_s |K_{n,s} f| \right\|_{L^p(\Omega)}^p \leq C \|f\|_{L^p(\Omega)}^p . \tag{2.3}$$

Before we begin the proof we prove some lemmas which will be needed in the proof. The first lemma is well known but is included here for completeness.

Lemma 2.2. *For T a positive isometry in L^p, or T a positive Lamperti operator on $L^p(\Omega)$, we have for any sequence $f_1, f_2, \dots, f_n$ of functions in $L^p(\Omega)$,*

$$\left| T\left(\max_i |f_i| \right) \right| = \max_i |T f_i| .$$

Proof. The case where T is a positive Lamperti operator includes the case of positive isometries. To see this, note that for $p > 1$, $f \geq 0$, $g \geq 0$, we have $\|f + g\|_p^p = \|f\|_p^p + \|g\|_p^p$ if and only if f and g have disjoint supports. For T a positive isometry, and f and g having disjoint supports, we have $\|Tf + Tg\|_p^p = \|T(f+g)\|_p^p = \|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p = \|Tf\|_p^p + \|Tg\|_p^p$, thus property (1.2) is satisfied. Property (1.3) is obvious for positive isometries.

For T a positive Lamperti operator, we have $|Tf_i| = T|f_i|$. To see this write $f_i = f_i^+ - f_i^-$, where f_i^+ and f_i^- denote the positive and negative parts of f_i. Since these have disjoint supports, and T maps functions with disjoint supports to functions with disjoint supports, then using the fact that T

is positive, we have $|Tf_i| = Tf_i^+ + Tf_i^- = T|f_i|$. To see that T commutes with the "max" operator, we write

$$\max_i |f_i| = \sum_{i=1}^{n} \chi_{E_i} |f_i|$$

where E_i is the set where the maximum value is achieved for the first time, using the function $|f_i|$. Since the nonnegative functions $\chi_{E_i}|f_i|$ have disjoint supports, by property (1.3) we have $T(\max_i |f_i|) = \sum_{i=1}^n T(\chi_{E_i}|f_i|)$, and by property (1.2) there exist pairwise disjoint sets B_i, $i = 1, 2, \dots, n$, such that

$$T\left(\max_i |f_i|\right) = \sum_{i=1}^{n} \chi_{B_i} T(\chi_{E_i}|f_i|).$$

However, using positivity, the right-hand side is dominated by $\sum_{i=1}^n \chi_{B_i} T(|f_i|)$. Using the fact that the B_i are pairwise disjoint, this sum is dominated by $\max T|f_i|$ which is equal to $\max |Tf_i|$. To see the reverse inequality, simply note that $|Tf_i| = T|f_i| \le T(\max_i |f_i|)$ for each i, and then take the maximum over i on each side. □

We now extend this result to operators which are not necessarily positive.

Lemma 2.3. *For T a positive isometry or T a Lamperti operator on $L^p(\Omega)$, we have for any sequence $f_1, f_2, \dots, f_n$ of functions in $L^p(\Omega)$, and any $j \ge 0$, that*

$$\left|T^j\left(\max_i |f_i|\right)\right| = \max_i |T^j f_i|.$$

Further, if the operator $S(f)$ is defined by $S(f) = |Tf^+| - |Tf^-|$ then S^j is a positive Lamperti operator for each $j \ge 1$, and satisfies $|S^j f| = |T^j f|$.

Proof. Note that since the support of Tf^+ and Tf^- are disjoint, we have by the definition of S, $|Tf^+| = (Sf)^+$ and $|Tf^-| = (Sf)^-$, which implies that the operator S satisfies $|Tf| = |Sf|$.

If f and g have disjoint supports, then $(f+g)^+ = f^+ + g^+$, and $(f+g)^- = f^- + g^-$. Hence for such f and g, we have

$$\begin{aligned} S(f+g) &= |T(f+g)^+| - |T(f+g)^-| \\ &= |Tf^+| + |Tg^+| - |Tf^-| - |Tg^-| \\ &= Sf + Sg. \end{aligned}$$

Thus the operator S satisfies property (1.3). It also obviously satisfies property (1.2). As a consequence S is a positive Lamperti operator. We now use our result for positive Lamperti operators to conclude that $S(\max_i |f_i|) = \max_i |Sf_i|$. Since $|Sg| = |Tg|$ we also have $|T(\max_i |f_i|)| = \max_i |Tf_i|$.

To complete the proof of the first part of the lemma, note that if T is a Lamperti operator, so is T^j for any $j \ge 0$. Using what was proved above for any Lamperti operator T, we can conclude that the same result holds for the Lamperti operator T^j.

For the last part of the lemma we use induction. The case $j = 1$ is true by the above argument. Let $j \ge 2$, and assume that the result is true for $j - 1$.

Since S is a positive Lamperti operator, we have $|Sf| = S|f|$. We can write

$$\begin{aligned}|T^j f^+| - |T^j f^-| &= |T(T^{j-1}f^+)| - |T(T^{j-1}f^-)| \\ &= |S(T^{j-1}f^+)| - |S(T^{j-1}f^-)| \\ &= S|T^{j-1}f^+| - S|T^{j-1}f^-| \\ &= S(|T^{j-1}f^+| - |T^{j-1}f^-|).\end{aligned}$$

However, using the induction hypothesis, the last expression is just $S(S^{j-1}f) = S^j f$. Thus we have $S^j f = |T^j f^+| - |T^j f^-|$. Take the absolute value of each side to complete the proof. □

Proof of Theorem 1. We will first prove the theorem with a number of special restrictions, and then later show how these restrictions can be removed. Assume that the $k_{n,s}$, $n = 1, 2, \dots, N$ and $s = 1, 2, \dots, S$; are supported in $[0, M]$ for some integer M, and that T is a positive isometry of $L^p(\Omega)$.

Fix a large integer L and define

$$\varphi_\omega(j) = \begin{cases} T^j(f)(\omega) & \text{if } 0 \le j \le L+M, \\ 0 & \text{if } j > L+M. \end{cases}$$

We have for each fixed n, and $0 \le j \le L$,

$$\begin{aligned}\max_{1\le s\le S} |T^j K_{n,s} f(\omega)| &= \max_{1\le s\le S} \left| T^j \sum_{m=0}^{M} k_{n,s}(m) T^m(f)(\omega) \right| \\ &= \max_{1\le s\le S} \left| \sum_{m=0}^{M} k_{n,s}(m) T^{j+m}(f)(\omega) \right| \\ &= \max_{1\le s\le S} |k_{n,s} * \varphi_\omega(j)|.\end{aligned}$$

Raising both sides to the pth power, and summing over j and n, we get

$$\begin{aligned}\sum_{n=1}^{N} \sum_{j=0}^{L} \max_{1\le s\le S} |T^j K_{n,s} f(\omega)|^p &= \sum_{n=1}^{N} \sum_{j=0}^{L} \max_{1\le s\le S} |k_{n,s} * \varphi_\omega(j)|^p \\ &\le C\|\varphi_\omega\|^p_{l^p(Z^+)} = C \sum_{j=0}^{L+M} |T^j(f)(\omega)|^p.\end{aligned}$$

The inequality in the above being true by assumption. Integrating both sides of this inequality, we have

$$\int \sum_{j=0}^{L} \sum_{n=1}^{N} \left| \left(\max_{1\le s\le S} |T^j K_{n,s} f| \right)(\omega) \right|^p d\mu \le C \int \sum_{j=0}^{L+M} |T^j(f)(\omega)|^p \, d\mu.$$

Using Lemma 2.3, this is equivalent to

$$\int \sum_{j=0}^{L} \sum_{n=1}^{N} \left| T^j \left(\max_{1\le s\le S} |K_{n,s} f| \right)(\omega) \right|^p d\mu \le C \int \sum_{j=0}^{L+M} |T^j(f)(\omega)|^p \, d\mu.$$

Interchanging the integration and the finite summations, we have

$$\sum_{j=0}^{L} \sum_{n=1}^{N} \left\| T^j \left(\max_{1\le s\le S} |K_{n,s} f| \right) \right\|^p_{L^p(\Omega)} \le C \sum_{j=0}^{L+M} \|T^j f\|^p_{L^p(\Omega)}. \tag{2.4}$$

Using the fact that T is an isometry, this is equivalent to

$$\sum_{n=1}^{N} \left\| \max_{1\le s\le S} |K_{n,s}f| \right\|_{L^p(\Omega)}^p \le \frac{L+M}{L} C\|f\|_{L^p(\Omega)}^p . \tag{2.5}$$

We now let $L \to \infty$, to obtain

$$\sum_{n=1}^{N} \left\| \max_{1\le s\le S} |K_{n,s}f| \right\|_{L^p(\Omega)}^p \le C\|f\|_{L^p(\Omega)}^p . \tag{2.6}$$

To prove the theorem in the case T is a Lamperti operator (not necessarily positive) which is also a quasi-isometry, note that everything in the above argument holds except the equivalence of (2.4) and (2.5). However, using the fact that T is a quasi-isometry, and selecting L to be an element of the sequence $\{L_n\}_{n=1}^{\infty}$ from the quasi-isometry property, we still obtain (2.6) (with a different C).

To extend the case of positive isometries to positive contractions on $L^p(\Omega)$ we use the fact that for T a positive contraction there exists a positive isometric embedding D of $L^p(\Omega)$ into a larger space $L^p(\widetilde{\Omega})$, a positive isometry Q on $L^p(\widetilde{\Omega})$, and a conditional expectation operator E such that $DT = EQD$, and more generally $DT^n = EQ^nD$. (See the important paper by Akcoglu and Sucheston [1] for this.) Now we have that

$$\begin{aligned} D(K_{n,s}f)(\omega) &= D\left(\sum_{j=1}^{\infty} k_{n,s}(j)T^jf\right)(\omega) = \sum_{j=1}^{\infty} k_{n,s}(j)D(T^jf)(\omega) \\ &= \sum_{j=1}^{\infty} k_{n,s}(j)EQ^j(Df)(\omega) = E\left(\sum_{j=1}^{\infty} k_{n,s}(j)Q^j(Df)(\omega)\right) \\ &= E(\widetilde{K}_{n,s}(Df)) \end{aligned}$$

where $\widetilde{K}_{n,s}$ is an operator associated with the positive isometry Q. Since D is a positive isometry, by Lemma 2.2 it commutes with the max operator and the absolute value operator. Consequently we have

$$\begin{aligned} \sum_{n=1}^{N} \left\| \max_{1\le s\le S} |K_{n,s}f| \right\|_{L^p(\Omega)}^p &= \sum_{n=1}^{N} \left\| D\left(\max_{1\le s\le S} |K_{n,s}f| \right) \right\|_{L^p(\widetilde{\Omega})}^p \\ &= \sum_{n=1}^{N} \left\| \max_{1\le s\le S} |D(K_{n,s}f)| \right\|_{L^p(\widetilde{\Omega})}^p = \sum_{n=1}^{N} \left\| \max_{1\le s\le S} |E(\widetilde{K}_{n,s}Df)| \right\|_{L^p(\widetilde{\Omega})}^p \\ &\le \sum_{n=1}^{N} \left\| \max_{1\le s\le S} E|\widetilde{K}_{n,s}Df| \right\|_{L^p(\widetilde{\Omega})}^p \le \sum_{n=1}^{N} \left\| E\left(\max_{s} |\widetilde{K}_{n,s}Df \right) \right\|_{L^p(\widetilde{\Omega})}^p \\ &\le \sum_{n=1}^{N} \left\| \max_{1\le s\le S} |\widetilde{K}_{n,s}Df \right\|_{L^p(\widetilde{\Omega})}^p \le C\|Df\|_{L^p(\widetilde{\Omega})}^p = C\|f\|_{L^p(\Omega)}^p . \end{aligned}$$

To remove the restriction of finite support for the functions $k_{n,s}$, write $k_{n,s} = g_{n,s} + b_{n,s}$ where $g_{n,s}(j) = k_{n,s}(j)$ if $0 \le j \le M$, and $g_{n,s}(j) = 0$ for $j > M$.

We first see that the sequence of functions $\{g_{n,s}\}$ satisfy the hypothesis of the theorem. We have

$$\begin{aligned}
&\left(\sum_{n=1}^{N}\left\|\max_{1\le s\le S}|g_{n,s}*\varphi|\right\|_{l^p(\mathbb{Z}^+)}^{p}\right)^{1/p}\\
&\quad=\left(\sum_{n=1}^{N}\left\|\max_{1\le s\le S}|k_{n,s}*\varphi-b_{n,s}*\varphi|\right\|_{l^p(\mathbb{Z}^+)}^{p}\right)^{1/p}\\
&\quad\le\left(\sum_{n=1}^{N}\left\|\max_{1\le s\le S}|k_{n,s}*\varphi|+\max_{1\le s\le S}|b_{n,s}*\varphi|\right\|_{l^p(\mathbb{Z}^+)}^{p}\right)^{1/p}\\
&\quad\le\left(\sum_{n=1}^{N}\left[\left\|\max_{1\le s\le S}|k_{n,s}*\varphi|\right\|_{l^p(\mathbb{Z}^+)}+\left\|\max_{1\le s\le S}|b_{n,s}*\varphi|\right\|_{l^p(\mathbb{Z}^+)}\right]^{p}\right)^{1/p}\\
&\quad\le\left(\sum_{n=1}^{N}\left\|\max_{1\le s\le S}|k_{n,s}*\varphi|\right\|_{l^p(\mathbb{Z}^+)}^{p}\right)^{1/p}+\left(\sum_{n=1}^{N}\left\|\max_{1\le s\le S}|b_{n,s}*\varphi|\right\|_{l^p(\mathbb{Z}^+)}^{p}\right)^{1/p}\\
&\quad=\mathrm{I}+\mathrm{II}.
\end{aligned}$$

By assumption, we have that (I) is dominated by $C^{1/p}\|\varphi\|_{l^p(\mathbb{Z}^+)}$. We will show that $\{g_{n,s}\}$ can be selected so that (II) is dominated by $\varepsilon\|\varphi\|_{l^p(\mathbb{Z}^+)}$. To see this we first replace the maximum in (II) by the larger sum. Thus

$$\begin{aligned}
(\mathrm{II})&\le\left(\sum_{n=1}^{N}\left\|\left(\sum_{s=1}^{S}|b_{n,s}*\varphi|^p\right)^{1/p}\right\|_{l^p(\mathbb{Z}^+)}^{p}\right)^{1/p}\\
&\le\left(\sum_{n=1}^{N}\sum_{m=M}^{\infty}\sum_{s=1}^{S}|b_{n,s}*\varphi(m)|^p\right)^{1/p}\\
&\le\left(\sum_{n=1}^{N}\sum_{s=1}^{S}\sum_{m=M}^{\infty}|b_{n,s}*\varphi(m)|^p\right)^{1/p}\\
&\le\left(\sum_{n=1}^{N}\sum_{s=1}^{S}\|b_{n,s}*\varphi\|_{l^p(\mathbb{Z}^+)}^{p}\right)^{1/p}\\
&\le\left(\sum_{n=1}^{N}\sum_{s=1}^{S}\|b_{n,s}\|_{l^1(\mathbb{Z}^+)}^{p}\right)^{1/p}\|\varphi\|_{l^p(\mathbb{Z}^+)}.
\end{aligned}$$

Because each of the $k_{n,s}$ is in $l^1(\mathbb{Z}^+)$, we can select M so large that $\|b_{n,s}\|_{l^1(\mathbb{Z}^+)}^p < \varepsilon^p/NS$, and thus (II) is dominated by $\varepsilon\|\varphi\|_{l^p(\mathbb{Z}^+)}$ as desired. We now have

$$\left(\sum_{n=1}^{N}\left\|\max_{1\le s\le S}|g_{n,s}*\varphi|\right\|_{l^p(\mathbb{Z}^+)}^{p}\right)^{1/p}\le(C^{1/p}+\varepsilon)\|\varphi\|_{l^p(\mathbb{Z}^+)}.$$

Define $G_{n,s}f(\omega) = \sum_{m=0}^{\infty} g_{n,s}(m)T^m(f)(\omega)$, and

$$B_{n,s}f(\omega) = \sum_{m=0}^{\infty} g_{n,s}(m)T^m(f)(\omega).$$

Because the $\{g_{n,s}\}$ satisfy the hypothesis of the original argument, we have

$$\sum_{n=1}^{N} \left\| \sup_s |G_{n,s}f| \right\|_{L^p(\Omega)}^p \le (C^{1/p} + \varepsilon)^p \|f\|_{L^p(\Omega)}^p.$$

We also have

$$\sum_{n=1}^{N} \left\| \sup_s |B_{n,s}f| \right\|_{L^p(\Omega)}^p \le \sum_{n=1}^{N} \left\| \sum_{s=1}^{S} |B_{n,s}f| \right\|_{L^p(\Omega)}^p$$
$$\le \sum_{n=1}^{N} \sum_{s=1}^{S} \| |B_{n,s}f| \|_{L^p(\Omega)}^p.$$

However,

$$\| |B_{n,s}f| \|_{L^p(\Omega)} = \left[\int_\Omega \left| \sum_{m=1}^{\infty} b_{n,s}(m)T^m f(\omega) \right|^p d\omega \right]^{1/p}$$
$$\le \sum_{m=1}^{\infty} |b_{n,s}(m)| \left[\int_\Omega |T^m f(\omega)|^p \, d\omega \right]^{1/p}$$
$$\le \|b_{n,s}\|_{l^1(\mathbb{Z}^+)} C \|f\|_{L^p(\Omega)}.$$

Consequently, we have

$$\sum_{n=1}^{N} \sum_{s=1}^{S} \| |B_{n,s}f| \|_{L^p(\Omega)}^p \le \sum_{n=1}^{N} \sum_{s=1}^{S} \|b_{n,s}\|_{l^1(\mathbb{Z}^+)}^p C \|f\|_{L^p(\Omega)}^p \le C\varepsilon \|f\|_{L^p(\Omega)}^p.$$

The desired inequality is now obtained from these results and an application of Minkowski's inequality as follows:

$$\left[\sum_{n=1}^{N} \left\| \sup_s |K_{n,s}f| \right\|_{L^p(\Omega)}^p \right]^{1/p}$$
$$\le \left[\sum_{n=1}^{N} \left\| \sup_s |G_{n,s}f| + \sup_s |B_{n,s}| \right\|_{L^p(\Omega)}^p \right]^{1/p}$$
$$\le \left[\sum_{n=1}^{N} \left\| \sup_s |G_{n,s}f| \right\|_{L^p(\Omega)}^p \right]^{1/p} + \left[\sum_{n=1}^{N} \left\| \sup_s |B_{n,s}f| \right\|_{L^p(\Omega)}^p \right]^{1/p}$$
$$\le [(C^p + \varepsilon)\|f\|_{L^p(\Omega)}^p]^{1/p} + \varepsilon \|f\|_{L^p(\Omega)}.$$

Since we can take ε as small as we want, we obtain the inequality with the original constant.

To complete the proof let S increase to infinity. □

Remark. An analysis of what is needed for the proof to work shows that in fact we can take T to be a power bounded Lamperti operator with the property

$$\limsup \frac{1}{n}(\|f\|_p^p + \|Tf\|_p^p + \cdots + \|T^{n-1}f\|_p^p) > \delta^p \|f\|_p^p .$$

This condition appears in Kan [13], and can be used in place of the quasi-isometry condition in the later theorems.

Theorem 2.4. *Let $\{g_s\}_{s=1}^{\infty}$ be a sequence of functions in $l^1(\mathbb{Z}^+)$. If $\{s_n\}$ is an increasing sequence of positive integers such that*

$$\sum_{n=1}^{N} \left\| \max_{s_n < s \le s_{n+1}} |(g_s - g_{s_n}) * \varphi| \right\|_{l^2(\mathbb{Z}^+)}^2 \le b(N) \|\varphi\|_{l^2(\mathbb{Z}^+)}^2 \tag{2.7}$$

where $b(N) = o(N)$, and

$$\left\| \max_s |g_s * \varphi| \right\|_{l^p(\mathbb{Z}^+)}^p \le c_p^p \|\varphi\|_{l^p(\mathbb{Z}^+)}^p \tag{2.8}$$

for some p satisfying $1 < p < 2$, then for any r satisfying $p < r < 2$ we have

$$\sum_{n=1}^{N} \left\| \max_{s_n < s \le s_{n+1}} |(g_s - g_{s_n}) * \varphi| \right\|_{l^r(\mathbb{Z}^+)}^r \le c(p, r) B(N)^{r/2} \|\varphi\|_{l^r(\mathbb{Z}^+)}^r \tag{2.9}$$

where $c(p, r)$ is a constant that depends on only p and r, and $B(N) = o(N)$.
Further, if $r > 2$ and we have $\sup_s \|g_s\|_{l^1(\mathbb{Z}^+)} \le M$, then

$$\sum_{n=1}^{N} \left\| \max_{s_n < s \le s_{n+1}} |(g_s - g_{s_n}) * \varphi| \right\|_{l^r(\mathbb{Z}^+)}^r \le c(r, M) B(N) \|\varphi\|_{l^r(\mathbb{Z}^+)}^r \tag{2.10}$$

where $c(r, M)$ is a constant that depends only on r and M, and $B(N) = o(N)$.

Proof. We will first consider the case of $p < 2$, and $p < r < 2$. Then using (2.8) we have

$$\begin{aligned} \left\| \max_{s_n < s \le s_{n+1}} |(g_s - g_{s_n}) * \varphi| \right\|_{l^p(\mathbb{Z}^+)} &\le \| \max |g_s * \varphi| \, \|_{l^p(\mathbb{Z}^+)} + \|g_{s_n} * \varphi\|_{l^p(\mathbb{Z}^+)} \\ &\le c_p \|\varphi\|_{l^p(\mathbb{Z}^+)} + \|\varphi\|_{l^p(\mathbb{Z}^+)} = b_p \|\varphi\|_{l^p(\mathbb{Z}^+)} \end{aligned}$$

where $b_p = c_p + 1$. Raising both sides to the pth power and summing, we obtain

$$\sum_{n=1}^{N} \left\| \max_{s_n < s \le s_{n+1}} |(g_s - g_{s_n}) * \varphi| \right\|_{l^p(\mathbb{Z}^+)}^p \le b_p^p N \|\varphi\|_{l^p(\mathbb{Z}^+)}^p . \tag{2.11}$$

We want to interpolate between this inequality and (2.7). Let $V = \mathbb{Z}^+ \times \{1, 2, \dots, N\}$. Define the sublinear operator $U \colon l^p(\mathbb{Z}^+) \to l^p(V)$ by

$$U(\varphi)(j, n) = \max_{s_n < s \le s_{n+1}} |(g_s - g_{s_n}) * \varphi(j)| .$$

Then we can write (2.11) as $\|U\varphi\|_{l^p(V)} \le b_p N^{1/p} \|\varphi\|_{l^p(\mathbb{Z}^+)}$ and (2.7) as

$$\|U\varphi\|_{l^2(V)} \le b(N)^{1/2} \|\varphi\|_{l^2(\mathbb{Z}^+)} .$$

Interpolating between these two inequalities, we have for $1/r = (1-t)/p + t/2$, (hence $0 < t < 1$)

$$\|U\varphi\|_{l^r(V)} \le C(b_p N^{1/p})^{1-t} b(N)^{t/2} \|\varphi\|_{l^r(\mathbb{Z}^+)}$$

which is (2.9) rewritten in terms of the operator U, and C depends only on p and r. Eliminating t in the above inequality, raising both sides to the rth power, and modifying C, we have

$$\|U\varphi\|_{l^r(V)} \le CN(b(N)/N)^{(r-p)/(2-p)} \|\varphi\|_{l^r(\mathbb{Z}^+)}.$$

Since $b(N)$ is $o(N)$, $b(N)/N = o(1)$. We have a positive exponent because $r > p$ and $2 > p$. Thus $(b(N)/N)^{(r-p)/(2-p)} = o(1)$ and we have (2.9).

The case of $p > 2$ is handled in a similar way. We have the estimate

$$\left\|\max_s |g_s * \varphi|\right\|_{l^\infty(\mathbb{Z}^+)} \le M\|\varphi\|_{l^\infty(\mathbb{Z}^+)}.$$

We interpolate between this and the $l^2(Z^+)$ inequality (2.7) above to obtain (2.10).

Theorem 2.5. *Let $\{g_s\}_{s=1}^\infty$ denote a sequence of functions in $l^1(\mathbb{Z}^+)$. Let $\{s_n\}$ denote an increasing sequence of positive integers such that*

$$\sum_{n=1}^N \left\| \max_{s_n < s \le s_{n+1}} |(g_s - g_{s_n}) * \varphi| \right\|_{l^p(\mathbb{Z}^+)}^p \le o(N)\|\varphi\|_{l^p(\mathbb{Z}^+)}^p \tag{2.12}$$

then for T a Lamperti operator which is also a quasi-isometry, or T a positive contraction on $L^p(\Omega)$, we have

$$\sum_{n=1}^N \left\| \max_{s_n < s \le s_{n+1}} |G_s f - G_{s_n} f| \right\|_{L^p(\Omega)}^p \le o(N)\|f\|_{L^p(\Omega)}^p, \tag{2.13}$$

where $G_s f(\omega) = \sum_{m=1}^\infty g_s(m) T^m f(\omega)$.

Proof. For each fixed N we use Theorem 2.1 with $k_{n,s} = g_s - g_{s_n}$ and $C = C(N)$ is the constant which appears in Theorem 2.1. In the hypothesis above, this constant is assumed to be $o(N)$. □

3

In this section we use the results from the previous section to obtain a.s. convergence for averages (1.1) associated with positive contractions on $L^p(\Omega)$ and certain special subsequences for which the variational inequality on the integers can be obtained. In particular, we show how to use the variational inequality (2.13) to establish almost sure convergence. Recall that for $\rho > 1$, $H_s f(\omega) = A_{\rho^s} f(\omega)$, where $A_l f(\omega) = \frac{1}{l}\sum_{j=1}^l T^{n_j} f(\omega)$. Let h_s denote the $l^1(\mathbb{Z}^+)$ function associated with the operator H_s.

Theorem 3.1. *Let $\{n_i\}$ be a subsequence of the integers, and T a positive contraction on $L^p(\Omega)$. If for each increasing sequence $\{s_n\}$ of positive integers, and for each ρ with $1 < \rho < 2$, we have*

$$\sum_{n=1}^N \left\| \max_{s_n < s \le s_{n+1}} |(h_s - h_{s_n}) * \varphi| \right\|_{l^p(\mathbb{Z}^+)}^p \le o(N)\|\varphi\|_{l^p(\mathbb{Z}^+)}^p \tag{3.1}$$

then the averages $\frac{1}{N}\sum_{i=1}^N T^{n_i}(f)(\omega)$ converge a.e. for all $f \in L^p(\Omega)$.

Remark. It is easy to see from the proof that it is enough to only consider sequences $\{s_n\}_{n=1}^{\infty}$ which are rapidly increasing. In some cases this makes it easier to establish inequality (3.1).

Proof. Assume $f \geq 0$. (If not apply the argument to the positive and negative parts separately.) We first establish the almost everywhere convergence of $H_s f(\omega) = A_{\rho^s} f(\omega)$, $s = 1, 2, 3, \ldots$. Assume that for some f with $\|f\|_{L^p(\Omega)} = 1$, there is a set $B \subset \Omega$ such that $H_s f(\omega)$ does not converge for all $\omega \in B$, with $\mu(B) > 0$. Then there is a set $B_1 \subset B$ such that

$$\limsup H_s f(\omega) - \liminf H_s f(\omega) \geq \eta > 0$$

for all $\omega \in B_1$, and $\mu(B_1) > 0$. It then follows that there is an infinite sequence of integers $s_1, s_2, \ldots$ with $s_{i+1} \geq s_i$ and such that

$$\begin{aligned} &\left\| \max_{s_n < s \leq s_{n+1}} |H_s f - H_{s_n} f| \right\|_{L^p(\Omega)}^p \\ &\quad \geq \int_{B_1} \max_{s_n < s \leq s_{n+1}} |H_s f(\omega) - H_{s_n} f(\omega)| \, d\mu > (\tfrac{1}{2}\eta)^p \tfrac{1}{2}\mu(B_1). \end{aligned}$$

Consequently, we have

$$\sum_{n=1}^{N} \left\| \max_{s_n < s \leq s_{n+1}} |H_s f - H_{s_n} f| \right\|_{L^p(\Omega)}^p \geq N \left(\frac{1}{2}\eta\right)^p \frac{1}{2}\mu(B_1) \|f\|_{L^p(\Omega)}^p .$$

However, by the hypothesis, and Theorem 2.5 we know

$$\sum_{n=1}^{N} \left\| \max_{s_n < s \leq s_{n+1}} |H_s f - H_{s_n} f| \right\|_{L^p(\Omega)}^p = o(N) \|f\|_{L^p(\Omega)}^p ,$$

a contradiction, since $N(\frac{1}{2}\eta)^p \frac{1}{2}\mu(B_1)$ is not $o(N)$. Thus we know that $\lim_{s \to \infty} A_{\rho^s} f(\omega)$ exists for almost every ω.

Now take a sequence of $\{\rho_i\}_{i=1}^{\infty}$ which converge to 1, and such that $\{\rho_i^n\}_{n=1}^{\infty} \subset \{\rho_j^n\}_{n=1}^{\infty}$ whenever $\rho_i > \rho_j$. (The sequence $\rho_j = 2^{(1/2^j)}$ will do.) Thus the limit using ρ_j must be the same as that obtained using ρ_i. Call this limit $\mathrm{Pf}(\omega)$. Now fix a ρ from this sequence, and consider $A_x f(\omega) = \frac{1}{x} \sum_{j=1}^{[x]} T^{n_j} f(\omega)$. Let $\varepsilon > 0$ be given. Select r so large that for all $s > r$ we have $\mathrm{Pf}(\omega) - \varepsilon < A_{\rho^s} f(\omega) < \mathrm{Pf}(\omega) + \varepsilon$. Let x be larger than ρ^r and select s such that $\rho^s \leq x < \rho^{s+1}$. Then using the fact that T is positive, and $f \geq 0$, we have

$$\begin{aligned} A_x f(\omega) &= \frac{\rho^{s+1}}{x} \frac{1}{\rho^{s+1}} \sum_{j=1}^{[x]} T^{n_j} f(\omega) \\ &\leq \frac{\rho^{s+1}}{\rho^s} \frac{1}{\rho^{s+1}} \sum_{j=1}^{[\rho^{s+1}]} T^{n_j} f(\omega) \leq \rho(\mathrm{Pf}(\omega) + \varepsilon). \end{aligned}$$

By the same argument, we also have $A_x f(\omega) \geq \frac{1}{\rho}(\mathrm{Pf}(\omega) - \varepsilon)$. Because $\varepsilon > 0$ is arbitrary, and we can take a sequence of such ρ which converge to 1, the result follows. □

Remark. Note that we have obtained convergence for all $f \in L^p(\Omega)$, directly, and have not had to appeal to Banach's principle.

Corollary 3.2. *If $n_k = k^2$, or $n_k =$ the kth prime, then the averages*

$$\frac{1}{N}\sum_{j=1}^{N} T^{n_j} f(\omega)$$

converge almost everywhere for all $f \in L^p(\Omega)$ whenever T is a positive contraction on $L^p(\Omega)$.

Proof. For the sequence $n_k = k^2$, Bourgain [5, 8] has proven (2.7) with $b(N) = o(N)$, and (2.8). The result follows by applying these and the above theorems. In the case $n_k =$ the kth prime, Bourgain [6] proved (2.7) and (2.8) for $p > (1+\sqrt{3})/2$. Wierdl [17] extended this to prove the maximal inequality (2.8) for each $p > 1$. This result is also contained in Bourgain [8]. □

Corollary 3.3. *Let $n_k = k^2$, or $n_k =$ the kth prime, and T a Lamperti operator on $L^p(\Omega)$, which is a quasi-isometry on $L^p(\Omega)$, then the averages*

$$\frac{1}{N}\sum_{j=1}^{N} T^{n_j} f(\omega) \tag{3.2}$$

converge for almost every ω for all $f \in L^p(\Omega)$.

Proof. From Theorem 2.1 and Theorem 3.1, we see that we have convergence for positive Lamperti operators whenever inequality (3.1) is satisfied. (In Theorem 3.1 we assumed we were working with a positive contraction, but the proof works just as well if the operator is positive, power bounded, and we have inequality (3.1).) Let S be a positive Lamperti operator such that $|S^j f| = |T^j f|$. Such an operator exists by Lemma 2.3. Thus we know there is an operator P such that

$$\frac{1}{x}\sum_{j=1}^{[x]} S^{n_j} f(\omega) \to \mathrm{Pf}(\omega) \quad \text{for a.e. } \omega.$$

We also have the convergence of (3.2) for x of the form ρ_j^s for integer s and $\{\rho_j\}_{j=1}^{\infty}$ a sequence converging to 1. This follows exactly as in the proof of Theorem 3.1. Thus there is a limit operator Q such that for $\rho \in \{\rho_j\}$ we have

$$\frac{1}{\rho^s}\sum_{j=1}^{[\rho^s]} T^{n_j} f(\omega) \to Qf(\omega) \quad \text{for a.e. } \omega.$$

Let $\varepsilon > 0$ be given. Fix $\rho \in \{\rho_j\}_{j=1}^{\infty}$ such that $\rho < 1 + \varepsilon$. Let $s(\varepsilon)$ be such that for all $s > s(\varepsilon)$ we have

$$\left|\frac{1}{\rho^s}\sum_{j=1}^{[\rho^s]} T^{n_j} f(\omega) - Qf(\omega)\right| < \varepsilon.$$

Let $x(\varepsilon)$ be such that for all $x > x(\varepsilon)$ we have

$$\left|\frac{1}{x}\sum_{j=1}^{[x]} S^{n_j}|f|(\omega) - P|f|(\omega)\right| < \varepsilon.$$

Given y with the property that $y > x(\varepsilon)$ and $\rho^s > y > \rho^{s-1}$ with $s > s(\varepsilon)$, we have

$$\begin{aligned}
&\left|\frac{1}{y}\sum_{j=1}^{[y]} T^{n_j} f(\omega) - Qf(\omega)\right| \\
&\quad \le \left|\frac{1}{y}\sum_{j=1}^{[y]} T^{n_j} f(\omega) - \frac{1}{\rho^s}\sum_{j=1}^{[\rho^s]} T^{n_j} f(\omega) + \frac{1}{\rho^s}\sum_{j=1}^{[\rho^s]} T^{n_j} f(\omega) - Qf(\omega)\right| \\
&\quad \le \left|\frac{1}{y}\sum_{j=1}^{[y]} T^{n_j} f(\omega) - \frac{1}{\rho^s}\sum_{j=1}^{[\rho^s]} T^{n_j} f(\omega)\right| + \varepsilon \\
&\quad \le \left(\frac{1}{y} - \frac{1}{\rho^s}\right)\sum_{j=1}^{[y]} |T^{n_j} f(\omega)| + \frac{1}{\rho^s}\sum_{j=[y]+1}^{[\rho^s]} |T^{n_j} f(\omega)| + \varepsilon \\
&\quad \le \left(\frac{1}{y} - \frac{1}{\rho^s}\right)\sum_{j=1}^{[y]} S^{n_j} |f(\omega)| + \frac{1}{\rho^s}\sum_{j=[y]+1}^{[\rho^s]} S^{n_j} |f(\omega)| + \varepsilon \\
&\quad \le \left(\frac{1}{y} - \frac{1}{\rho^s}\right)\sum_{j=1}^{[y]} S^{n_j} |f(\omega)| + \frac{1}{\rho^s}\sum_{j=1}^{[\rho^s]} S^{n_j} |f(\omega)| - \frac{1}{\rho^s}\sum_{j=1}^{[y]} S^{n_j} |f(\omega)| + \varepsilon \\
&\quad \le \frac{1}{y}\sum_{j=1}^{[y]} S^{n_j} |f(\omega)| + \frac{1}{\rho^s}\sum_{j=1}^{[\rho^s]} S^{n_j} |f(\omega)| - 2\frac{y}{\rho^s}\frac{1}{y}\sum_{j=1}^{[y]} S^{n_j} |f(\omega)| + \varepsilon \\
&\quad \le (P|f|(\omega) + \varepsilon) + (P|f|(\omega) + \varepsilon) - 2\frac{y}{\rho^s}(P|f|(\omega) - \varepsilon) + \varepsilon \\
&\quad \le 2P|f|(\omega)\left(1 - \frac{y}{\rho^s}\right) + 2\varepsilon\left(1 + \frac{y}{\rho^s}\right) + \varepsilon \\
&\quad < 2P|f|(\omega)\left(1 - \frac{1}{\rho}\right) + 2\varepsilon(1+\rho) + \varepsilon \le 2\varepsilon P|f|(\omega) + 2\varepsilon(1+2) + \varepsilon .
\end{aligned}$$

Since ε was arbitrary, we are done. □

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Department of Mathematics, DePaul University, Chicago, Illinois 60614
E-mail address: matrlj@depaul.bitnet

Department of Mathematics, North Dakota State University, Fargo, North Dakota 58105

Department of Mathematics, Ohio State University, Columbus, Ohio 43210
Current address: Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599