TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 331, Number 2, June 1992

THE BERGMAN PROJECTION ON HARTOGS DOMAINS IN $\mathbf{C}^2$

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ABSTRACT. Estimates in L^2 Sobolev norms are proved for the Bergman projection in certain smooth bounded Hartogs domains in $\mathbf{C}^2$. In particular, (1) if the domain is pseudoconvex and "nonwormlike" (the normal vector does not wind on a critical set in the boundary), then the Bergman projection is regular; and (2) Barrett's counterexample domains with irregular Bergman projection nevertheless admit a priori estimates.

In this paper we study the Bergman projection (and, in the pseudoconvex case, the $\bar{\partial}$-Neumann operator) on certain Hartogs domains in $\mathbf{C}^2$. Our attention is drawn to these domains because of a number of important counterexample domains that are Hartogs domains (see [15, 16, 1, 4, 23, 20]).

The so-called "worm domains" constructed by Diederich and Fornæss in [16] are smooth bounded pseudoconvex Hartogs domains in $\mathbf{C}^2$ that provide counterexamples to a number of questions in several complex variables. Recently Kiselman showed in [20] that for certain nonsmooth pseudoconvex worm domains in $\mathbf{C}^2$ the Bergman projection is not regular. Since his argument is sensitive to perturbations of the domain, such as smoothing of corners, it remains an open question whether the Bergman projection is regular for the smooth worm domains. As the counterexample properties of the worm domains are rooted in the winding of the normal on the critical annulus, it is natural to ask whether the Bergman projection and the $\bar{\partial}$-Neumann operator are regular when similar behavior of the normal is excluded. This is indeed the case.

We prove in §1 that all smooth bounded pseudoconvex Hartogs domains in $\mathbf{C}^2$ that are nowhere wormlike (for the precise definition see §1 below) have regular Bergman projection. The Bergman projection for such a domain Ω preserves the space $C^\infty(\overline{\Omega})$ of functions smooth up to the boundary, and moreover maps the Sobolev space $W^k(\Omega)$ continuously into itself for positive k. (Here $W^k(\Omega)$ denotes the space of functions with square-integrable derivatives through order k.) The simplest example of a nowhere wormlike Hartogs domain is a complete one; we have shown previously in [10] that a smooth bounded complete Hartogs domain in $\mathbf{C}^2$ (pseudoconvex or not) has regular Bergman projection. (For some positive regularity results in higher dimensions, see [9].) Here we adapt the technique of [10] to incomplete nowhere worm-

Received by the editors January 30, 1990 and, in revised form, June 20, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32H10; Secondary 32A07, 32F20.

Key words and phrases. Bergman projection, $\bar{\partial}$-Neumann operator, Hartogs domain, worm domain, a priori estimate.

like pseudoconvex domains. A portion of our technique, combined with results from [5, 16], also yields regularity of the Bergman projection on pseudoconvex domains in $\mathbf{C}^2$ (not necessarily Hartogs) whose weakly pseudoconvex boundary points are exactly an analytic disk. Through [8, proof of Catlin's Lemma] or [11, formula (4)], all our estimates for the Bergman projection yield corresponding estimates for the $\overline{\partial}$-Neumann operator.

Barrett constructed in [1] a family of smooth bounded nonpseudoconvex Hartogs domains in $\mathbf{C}^2$ with irregular Bergman projection. In these domains the Bergman projection does not even map all smooth, compactly supported functions into $W^1(\Omega)$. Surprisingly, for these domains the Bergman projection nonetheless satisfies a priori estimates in Sobolev norms. We demonstrate this in §2.

We thank Alan Noell for useful remarks.

0. Preliminaries

We fix the following notation for the remainder of the paper: Ω is a smooth bounded domain in $\mathbf{C}^2$ (exception: we state the lemma below for $\mathbf{C}^n$) with smooth defining function ρ. Most of the time, Ω will also be a Hartogs domain with symmetry plane $\{w = 0\}$. This means that when (z, w) is in Ω, so is $(z, e^{it}w)$ for every real t. We denote the argument of w by θ. The Bergman projection P is the orthogonal projection from the space $L^2(\Omega)$ of square-integrable functions in Ω onto the closed subspace of square-integrable holomorphic functions in Ω. Angle brackets $\langle\ ,\ \rangle$ denote the inner product in $L^2(\Omega)$, and $\|\ \|_k$ denotes the norm in the Sobolev space $W^k(\Omega)$. When writing inequalities, we employ the convention that C and C_k denote constants that may change their identity at each occurrence.

In the proofs of Theorems 1 and 2 we will use the following technical, but standard, integration by parts lemma.

Lemma. *Let Ω be a smooth bounded domain in $\mathbf{C}^n$. Let ψ be a function in $C^\infty(\overline{\Omega})$, and let U be an open subset of Ω containing $\Omega \cap \operatorname{supp}\psi$. Fix a positive integer k.*

(1) *There exists a constant C such that for every function h in $W^k(\Omega)$ that is holomorphic in U and every function f (not necessarily holomorphic) in $W^k(\Omega)$,*

$$|\langle \nabla^j f, \psi \nabla^{k-j} h\rangle| \le C\|f\|_k\|h\|_0$$

when $0 \le j \le k$.

(2) *Let X be a vector field of type $(1, 0)$ with coefficients in $C^\infty(\overline{U})$. Suppose that $X\rho \neq 0$ on $b\Omega \cap \operatorname{supp}\psi$, and let b be a function in $C^\infty(\overline{\Omega})$ that equals $\overline{X}\rho/X\rho$ near $b\Omega \cap \operatorname{supp}\psi$. There exists a constant C such that for all functions g and h in $W^k(\Omega)$ that are holomorphic in U,*

$$|\langle X^k g, \psi h\rangle| \le |\langle g, \psi b^k X^k h\rangle| + C\|g\|_{j-1}\|h\|_{k-j}$$

when $1 \le j \le k$.

Proof. By the Cauchy-Riemann equations, an arbitrary derivative of a holomorphic function can be rewritten as a derivative that is tangential at the boundary of Ω. Hence one can integrate $k - j$ derivatives by parts without boundary terms and then apply the Cauchy-Schwarz inequality to obtain (1).

Since $|b| = 1$ near $b\Omega \cap \operatorname{supp} \psi$, the L^2 norm of a holomorphic function controls its derivatives on a compact set, and $\overline{X} - bX$ is tangential at the boundary on the support of ψ, we have, in view of part (1),

$$\begin{aligned} |\langle X^k g, \psi h\rangle| &\le |\langle b^k X^k g, \psi b^k h\rangle| + C\|g\|_0\|h\|_0 \\ &\le |\langle (\overline{X} - bX)^k g, \psi b^k h\rangle| + C\|g\|_{j-1}\|h\|_{k-j} \\ &\le |\langle g, \psi b^k (\overline{bX} - X)^k h\rangle| + C\|g\|_{j-1}\|h\|_{k-j} \\ &\le |\langle g, \psi b^k X^k h\rangle| + C\|g\|_{j-1}\|h\|_{k-j}. \end{aligned}$$

This proves (2).

1. Nowhere wormlike pseudoconvex Hartogs domains

It is enough to study regularity of the Bergman projection near the boundary of Ω. Since the Hartogs domain Ω is invariant under rotations in θ (which are isometries of $L^2(\Omega)$ that preserve the holomorphic subspace), the Bergman projection commutes with the derivative $\partial/\partial\theta$. Since derivatives transverse to the complex tangent space control all derivatives of holomorphic functions (see, for instance, [24, Lemma 5.3]), it follows that the Bergman projection is regular in Sobolev norms away from the part of the boundary where $w(\partial\rho/\partial w) = 0$, or what is the same, where $\partial\rho/\partial w = 0$. (For boundary points in the symmetry plane, the normal to $b\Omega$ is invariant under rotations in θ, so $\partial\rho/\partial w = 0$ at these points.) Therefore our attention focuses on the set

$$S := \left\{(z, w) \in b\Omega : \frac{\partial\rho}{\partial w}(z, w) = 0\right\}.$$

For each fixed z in $\mathbf{C}$, let S_z denote the intersection of S with the complex line $\{(z, t) : t \in \mathbf{C}\}$. Since Ω is Hartogs, each slice S_z has circular symmetry and is therefore a (possibly empty) union of circles and annuli and possibly a disk or a point. On S_z, the unit normal to $b\Omega$ has the form $(n(z, |w|), 0)$. To rule out the winding of n exhibited by worm domains, we wish to impose the condition that for each z the function n is a constant function of w on each connected component of S_z. We say that Ω is *nowhere wormlike* when this condition holds.

Theorem 1. *Let Ω be a smooth bounded pseudoconvex Hartogs domain in $\mathbf{C}^2$ that is nowhere wormlike. Then*

(1) *the Bergman projection is continuous on the Sobolev space $W^s(\Omega)$ for every positive real s (and a fortiori is continuous on $C^\infty(\overline{\Omega})$), and*
(2) *the $\overline{\partial}$-Neumann operator is continuous on the space of $(0, 1)$-forms with coefficients in $W^s(\Omega)$ for every positive real s (and a fortiori is continuous on $(0, 1)$-forms with coefficients in $C^\infty(\overline{\Omega})$).*

Example 1. A Hartogs domain is called complete if the condition $(z, w) \in \Omega$ implies $(z, \lambda w) \in \Omega$ for every complex number λ of modulus less than one. It is easy to see that a complete Hartogs domain is nowhere wormlike.

We have previously shown in [10] that completeness (even without pseudoconvexity) implies regularity of the Bergman projection. If Ω intersects the plane $\{w = 0\}$, then pseudoconvexity forces Ω to be complete, so the interesting new domains to which Theorem 1 applies do not intersect the plane

$\{w = 0\}$. (In this case the closure $\overline{\Omega}$ also does not intersect $\{w = 0\}$, in view of [4, Lemma 1].) Examples 2–4 provide large classes of incomplete Hartogs domains that satisfy the assumptions of Theorem 1.

Example 2. If $\{(z, |w|) \in \mathbf{R}^3 : (z, w) \in \Omega\}$ is a convex subset of $\mathbf{R}^3$, then Ω is nowhere wormlike. If Ω is in addition pseudoconvex, then Theorem 1 applies.

Example 3. On the components of S_z that are circles, the normal is constant by rotation invariance. Therefore those pseudoconvex Hartogs domains for which none of the slices S_z contain annuli (that is, for which the planar sets S_z all have empty interior) satisfy the assumptions of Theorem 1.

Example 4. If the Hartogs domain Ω admits a defining function ρ that is plurisubharmonic in the interior of Ω, then Ω is pseudoconvex and nowhere wormlike. Indeed, the plurisubharmonicity implies that $\partial\rho/\partial z$ is a holomorphic function of w in the interior of each planar set S_z (see [19, pp. 113–114]). By rotation invariance, the argument of $\partial\rho/\partial z$ is independent of θ. Hence $\partial\rho/\partial z$ is constant on each annulus contained in S_z, and so Ω is nowhere wormlike (in view of the observation in Example 3). We note that there exist nowhere wormlike pseudoconvex Hartogs domains not admitting a defining function that is plurisubharmonic inside (see [18, 6]).

It is interesting to compare this special case of Theorem 1 with the result of Bonami and Charpentier from [13] (see also [14]) that if Ω (not necessarily Hartogs) admits a defining function that is plurisubharmonic in the interior, then the Bergman projection is at least minimally regular: namely, continuous on the space $W^{1/2}(\Omega)$.

Proof of Theorem 1. The two parts of the theorem are equivalent (see the proof of Catlin's lemma in [8]). A more general equivalence of regularity for the Bergman projection and the $\overline{\partial}$-Neumann operator is demonstrated in [11]. We will now prove part (1) of the theorem.

By interpolation theory, it will suffice to prove this when k is a positive integer. The first step is to prove an estimate of the form $\|Pf\|_{W^k(\Omega)} \le C_k\|f\|_{W^k(\Omega)}$, with C_k independent of f, under the assumption that f and Pf are in $C^\infty(\overline{\Omega})$. This is a so-called a priori estimate. We proceed by induction on k, the case $k = 0$ holding by definition of the Bergman projection. We will omit some of the details of the proof, since it is similar to the proof in [10, §1].

As noted above, the rotational symmetry implies that $\|(1-\phi)Pf\|_k \le C_k\|f\|_k$ when ϕ is a smooth cutoff function that equals 1 in a neighborhood of the critical set S. It remains to estimate the norm of Pf near the set S, that is, to estimate $\|\phi X^k Pf\|_0$ when X is a vector field transverse to the boundary in a neighborhood of S. Suppose we can find such a field of type $(1, 0)$ with coefficients holomorphic in a neighborhood of S that is a close approximation on S to the normal to the boundary. More precisely, suppose that $4k\,|\arg X\rho| < \pi$ in a neighborhood of S. (This is Barrett's condition (A_k) from [2, p. 334].) Take ϕ to be supported in this neighborhood, and take b in $C^\infty(\overline{\Omega})$ equal to $\overline{X}\rho/X\rho$ on the support of ϕ. Then $|\arg b^k| < \pi/2$ on the support of ϕ, so there is a positive number δ and a positive number α less than one such that $|b^k - \delta| \le \alpha$ on the support of ϕ.

Applying the lemma from §0 gives

$$\|\phi X^k Pf\|_0^2 \leq |\langle Pf, \phi^2 b^k X^{2k} Pf\rangle| + C_k \|Pf\|_{k-1} \|Pf\|_k.$$

Adding and subtracting δ in the right-hand factor of the inner product and using the lemma a second time gives

$$\|\phi X^k Pf\|_0^2 \leq \delta |\langle Pf, \phi^2 X^{2k} Pf\rangle| + |\langle b^k \phi X^k Pf, (b^k - \delta)\phi X^k Pf\rangle| + C_k \|Pf\|_{k-1}\|Pf\|_k.$$

We replace the Bergman projection in the left-hand factor of the first inner product with the identity minus $\overline{\partial}^* N\overline{\partial}$, where N is the $\overline{\partial}$-Neumann operator. Moving the $\overline{\partial}^*$ to the other side of the inner product as $\overline{\partial}$ and applying the lemma to the term with the identity gives

$$|\langle Pf, \phi^2 X^{2k} Pf\rangle| \leq |\langle N\overline{\partial} f, (X^{2k} Pf)(\overline{\partial}\phi^2)\rangle| + C_k \|f\|_k \|Pf\|_k,$$

since $X^{2k} Pf$ is holomorphic in Ω on the support of ϕ. Now the inner product on the right-hand side is $O(\|f\|_k \|Pf\|_k)$ because the support of $\overline{\partial}\phi^2$ is contained in the set where the rotations are transverse. (The operator $N\overline{\partial}$ is regular on this set; see [9, Proof of Theorem 2].) Consequently,

$$\|\phi X^k Pf\|_0^2 \leq \alpha \|\phi X^k Pf\|_0^2 + C_k(\|f\|_k + \|Pf\|_{k-1})\|Pf\|_k.$$

The first term on the right-hand side can be absorbed into the left-hand side. Invoking the induction hypothesis, and recalling that $\|(1-\phi)Pf\|_k$ is under control, we obtain

$$\|Pf\|_k^2 \leq C_k \|Pf\|_k \|f\|_k,$$

from which the desired a priori estimate follows.

Thus the proof of the a priori estimate will be complete if we show how to find a holomorphic vector field in a neighborhood of S that on S approximates the normal to $b\Omega$ as closely as we wish to prescribe. To do this, we first associate to Ω a Riemann surface R as in [4, §4] by identifying points (z, w_1) and (z, w_2) if both are in Ω and if (z, η) is in Ω whenever $|\eta|$ is between $|w_1|$ and $|w_2|$. The projection $\pi: R \to \mathbf{C}$ taking the equivalence class $[(z, w)]$ to z is a local biholomorphism, and the map $p: \Omega \to R$ taking (z, w) to $[(z, w)]$ is holomorphic.

By [4, Proposition 1], there is an extension $(\widetilde{R}, \tilde{\pi})$ of (R, π) such that $\widetilde{R}\backslash\overline{R}$ has no component relatively compact in $\widetilde{R}$. In principle $\widetilde{R}$ could have branch points on the boundary of R (in $\widetilde{R}$). However, this does not happen in our situation of a nowhere wormlike Ω because, in view of [4, proof of Lemma 2], there would correspond to such a branch point a component of some S_z on which the normal does wind a definite amount. Moreover, since Ω is both pseudoconvex and nonwormlike, no slice S_z can have a component of the type called "interior" in [4, bottom of p. 67] (compare [10, §1]). In this special situation it is implicit in [4] that there is an extension $\tilde{p}$ of p that maps a neighborhood of $\overline{\Omega}$ holomorphically into $\widetilde{R}$ and that maps the critical set S onto the boundary of R in $\widetilde{R}$.

We apply an approximation theorem on Riemann surfaces. The hypothesis that $n(z, |w|)$ is constant on each component of each set S_z means that a continuous function h on the boundary of R in $\widetilde{R}$ is well defined by the formula $h(\tilde{p}(z, w)) = n(z, |w|)$ for (z, w) in S. Choose a point (z_0, w_0) in Ω such that S_{z_0} is empty. The complement of the boundary of R in $\widetilde{R}\backslash\{[z_0, w_0]\}$ has no relatively compact component, so by Bishop's theorem (see [22, Theorem 1.4]), there exists a holomorphic function g on $\widetilde{R}\backslash\{[z_0, w_0]\}$ that approximates h as closely as desired on the boundary of R. The field $X := (g(\tilde{p}(z, w)), 0)$ then closely approximates the normal to the boundary of Ω at points of S, and it is holomorphic in a neighborhood of S.

We have now seen that the existence of a suitable vector field X implies an a priori estimate for the Bergman projection, and we have seen how to construct the field. It remains to convert the a priori estimate to a genuine estimate on $W^k(\Omega)$. To do this, exhaust Ω by the domains $\Omega_\varepsilon := \{(z, w) : \operatorname{dist}[(z, w), b\Omega] > \varepsilon(|z|^2 + |w|^2)\}$, which for all sufficiently small positive ε are smooth bounded strictly pseudoconvex Hartogs domains. If the field X approximates the normal to $b\Omega$ within say δ in a neighborhood V of S, then the same field X approximates the normal to $b\Omega_\varepsilon$ within 2δ on $V \cap b\Omega_\varepsilon$, uniformly in ε for sufficiently small positive ε. On $b\Omega_\varepsilon\backslash V$, the rotations in w are transverse to the boundary, the angle with the normal being bounded away from zero uniformly in ε (for ε sufficiently small). Accordingly, the above argument applies to the approximating domains uniformly in ε. Since the Bergman projection P_ε for the strictly pseudoconvex domain Ω_ε is of course regular, and $C^\infty(\overline{\Omega}_\varepsilon)$ is dense in $W^k(\Omega_\varepsilon)$, we get the (genuine!) estimate $\|P_\varepsilon f\|_{W^k(\Omega_\varepsilon)} \leq C_k\|f\|_k$ for $f \in W^k(\Omega)$, with C_k independent of ε. Passing to the limit as ε goes to zero gives the required conclusion for the Bergman projection of Ω via a slight extension of Ramadanov's convergence theorem [21] for the Bergman kernels. This completes the proof of Theorem 1.

It is interesting to consider what happens if one attempts to apply the above method of proof to the worm domains denoted Ω_r by Diederich and Fornæss. In these domains, the critical set S contains an annulus, which consists of (exactly) the weakly pseudoconvex boundary points. It is easy to see that the normal direction to the boundary cannot be arbitrarily well approximated on this annulus by holomorphic fields (compare the discussion in [19, pp. 113–114]). When $r \geq e^{\pi/2}$, it is even impossible to find a holomorphic field that is transverse to the boundary on this annulus (see [5, Theorem 4.6]).

However, since the normal to the boundary of Ω_r winds on the critical annulus by $2\log r$, a constant field approximates the normal to high accuracy when r is close to 1. This means that when $r < e^{\pi/2}$, one can obtain estimates near the critical annulus for the Bergman projection in W^k up to a certain level k that depends on r. The Bergman projection is locally regular on the remaining (strictly pseudoconvex) part of the boundary, so just as in [8, Example 3] one obtains some regularity for worm domains that wind only a little. The precise amount of regularity is determined by Barrett's condition (A_k).

Proposition 1. *Let k be a positive integer, or $k = 1/2$, and suppose that $1 < r < \exp(\pi/4k)$. The Bergman projection on the Diederich-Fornæss worm domain Ω_r is continuous on the Sobolev space $W^s(\Omega_r)$ when $0 \leq s \leq k$. Also, the $\overline{\partial}$-Neumann operator is continuous on the space of $(0, 1)$-forms with coefficients in $W^s(\Omega_r)$ when $0 \leq s \leq k$.*

Sketch of the proof. The initial step in the argument for $k = 1/2$ is different from the case of integral k in [8, Example 3]. One uses that the $W^{1/2}(\Omega)$ norm of a holomorphic function is equivalent to the $L^2(b\Omega)$ norm. Now if ϕ is a cutoff function localized near the critical annulus, and X is a constant vector field that is transverse to the boundary of Ω near this annulus, then

$$\begin{aligned}\int_{b\Omega} \phi|Pf|^2\,d\sigma &\le C\left|\int_{b\Omega} \phi|Pf|^2|\nabla\rho|^{-1}(X\rho)\,d\sigma\right| \\ &\le C(|\langle\phi XPf\,,\,Pf\rangle| + \|Pf\|_0^2).\end{aligned}$$

As before, we reduce to a known estimation off the critical annulus (and hence on the strictly pseudoconvex part of $b\Omega$) by replacing P in the right-hand side of the inner product by $I - \overline{\partial}^* N\overline{\partial}$ and integrating by parts.

For all k, the continuity of the $\overline{\partial}$-Neumann operator follows from that of the Bergman projection by [11, formula (4)] in view of the regularity of the $\overline{\partial}$-Neumann operator in (top) degree 2 [17, p. 63].

The fact that the worm domains have a Bergman projection and a $\overline{\partial}$-Neumann operator regular up to some level in the Sobolev scale has been known for some time; apparently this was first realized by David Catlin (via a different argument). We remark that the worm domains to which the proposition applies are the less pathological ones that do admit a basis of Stein neighborhoods (see [5]).

The proof of Proposition 1 we have just sketched differs slightly from the proof of Theorem 1: the set S in Theorem 1 is not the set of weakly pseudoconvex points. The common feature of the two proofs is that off a certain "critical set" one knows estimates (by the transversality of the rotations in Theorem 1, by strict pseudoconvexity in Proposition 1), and on the critical set S one can find a vector field of type $(1, 0)$ transverse to $b\Omega$, with coefficients holomorphic in a neighborhood of S, and with $|\arg X\rho|$ small on S. This can be done in other situations as well. We briefly digress from the context of Hartogs domains to prove the following result, which says that the Bergman projection and the $\overline{\partial}$-Neumann operator are regular if the set of weakly pseudoconvex boundary points is precisely an analytic disk.

Proposition 2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbf{C}^2$ (not necessarily Hartogs). Suppose there exists a holomorphic embedding $\varphi\colon D \hookrightarrow b\Omega$ of the unit disk D into the boundary of Ω that extends to a diffeomorphism between $\overline{D}$ and $\overline{\varphi(D)}$, and suppose $\overline{\varphi(D)}$ coincides with the set of weakly pseudoconvex boundary points of Ω. Then*

(1) *the Bergman projection is continuous on the Sobolev space $W^s(\Omega)$ for every positive real s (and a fortiori is continuous on $C^\infty(\overline{\Omega})$), and*
(2) *the $\overline{\partial}$-Neumann operator is continuous on the space of $(0, 1)$-forms with coefficients in $W^s(\Omega)$ for every positive real s (and a fortiori is continuous on $(0, 1)$-forms with coefficients in $C^\infty(\overline{\Omega})$).*

By [7], a biholomorphic mapping between smooth bounded pseudoconvex domains extends smoothly to the boundaries if one of the domains has globally regular Bergman projection. Accordingly, Proposition 2 implies that the property of the weakly pseudoconvex points being exactly an analytic disk (in the above sense) is a biholomorphic invariant.

Proof of Proposition 2. As observed in the proof of Theorem 1, the two parts are equivalent; we prove part (1). We need to produce a vector field X of type $(1, 0)$ with coefficients holomorphic in a neighborhood of $\overline{\varphi(D)}$ that is transverse to the boundary and such that $|\arg X\rho|$ is as small as we wish (near $\overline{\varphi(D)}$). By [5, Remark 2.5 and the proof of Theorem 1.1 (p. 13)], there is a biholomorphic mapping g from the unit ball B in $\mathbf{C}^2$ onto a smooth bounded strictly pseudoconvex domain such that the restriction of g to $D \times \{0\}$ agrees with φ. In view of [16, Claim, p. 290], the restriction to $D \times \{0\}$ of the unit complex normal to the hypersurface $g^{-1}(b\Omega \cap g(B))$ has the form $(0, e^{i\Theta})$, where Θ is a harmonic function in the unit disk D that is smooth on $\overline{D}$. Multiplying by $e^{\widetilde{\Theta}}$, where $\widetilde{\Theta}$ is a harmonic conjugate of Θ in the disk, gives a (still normal) field that is holomorphic in B. Pulling this field back via g^{-1} and approximating it by fields holomorphic in a neighborhood of the (strictly pseudoconvex) domain $g(B)$ gives the required field X.

The a priori estimate now follows as indicated above. To obtain a genuine estimate, fix a neighborhood of $\overline{\varphi(D)}$ and approximate Ω from inside by strictly pseudoconvex domains whose Levi forms are uniformly bounded away from zero outside this neighborhood. By using the same field X for the approximating domains, one obtains uniform estimates and completes the proof of Proposition 2 the same way as the proof of Theorem 1 above.

2. A priori estimates on Barrett's domains

The proof of Theorem 1 had two steps. The first step was to show that if both f and Pf are in $C^\infty(\overline{\Omega})$, then $\|Pf\|_k \le C_k\|f\|_k$. This is a so-called a priori estimate: it holds assuming that Pf is known to be smooth a priori. The second step was to remove the a priori assumption from the estimate. There is a wide-spread philosophical belief that it should be possible to accomplish the second step (by some technical argument) whenever some a priori estimate can be established. This philosophy is wrong: we show in this section that Barrett's nonpseudoconvex counterexample domains from [1] do admit a priori estimates in $W^k(\Omega)$ for every natural number k even though they have irregular Bergman projections.

We will say that P satisfies an a priori estimate in $W^k(\Omega)$ if there exists a constant C_k (independent of f) such that $\|Pf\|_k \le C_k\|f\|_k$ whenever f and Pf belong to $W^k(\Omega)$. The existence of an a priori estimate in $W^k(\Omega)$ can be interpreted in the following way. View the Bergman projection P as an unbounded operator on the Sobolev space $W^k(\Omega)$ with domain equal to $\{f \in W^k(\Omega) : Pf \in W^k(\Omega)\}$. Its graph is closed because P is by definition continuous in the L^2 topology. By the closed graph theorem, an a priori estimate in $W^k(\Omega)$ is therefore equivalent to the domain of P being closed in $W^k(\Omega)$. In this case, P is a bounded operator on its domain.

We now recall the construction of Barrett's counterexample domains from [1]. These have the form $\Omega := \{(z, w) \in \mathbf{C}^2 : 1 < |w| < 6,\ |z - c(|w|)| > r_1(|w|),$ and $|z| < r_2(|w|)\}$, where r_1, r_2, and c are certain real-valued functions. Thus Ω has circular symmetry in the variable w, and each slice for fixed w is a disk centered at the origin with a second disk of variable center removed. The following conditions are imposed on the functions used to define Ω:

(1) on the interval $2 \le x \le 5$, the function r_1 equals 1 and the function r_2 equals 4;
(2) $r_1(1) = r_2(1) = r_1(6) = r_2(6) = 3$;
(3) $c(x) = 0$ when $1 \le x \le 2$ and when $5 \le x \le 6$;
(4) $c(x) = (x-3)^{2j} - 1$ for x near 3 and $c(x) = -(x-4)^{2j} + 1$ for x near 4, where j is a positive integer;
(5) the functions r_1, r_2, and c are monotonic on the intervals where they have not yet been specified, and they are chosen to make Ω smooth.

Thus the circles $(0, 3e^{i\theta})$ and $(0, 4e^{i\theta})$ are contained in the boundary of Ω.

Theorem 2. *Let Ω be one of Barrett's domains, recalled above. Then, even though the Bergman projection P for Ω is irregular, it satisfies an a priori estimate in $W^k(\Omega)$ for each positive integer k. More precisely, there exists a constant C_k such that if $Pf \in W^k(\Omega)$, then the estimate $\|Pf\|_k \le C_k\|f\|_k$ holds.*

We reemphasize that there exists a smooth compactly supported function φ in Ω such that $P\varphi \notin W^k(\Omega)$.

Proof of Theorem 2. We claim the following.

(1) When $k \ge 1$, every holomorphic function in $W^k(\Omega)$ extends holomorphically at least to the set $\{(z, w) : 1 < |w| < 6 \text{ and } |z| < r_2(|w|)\}$.
(2) When $k \ge 1$, every holomorphic function in $W^k(\Omega)$ can be approximated in the norm of $W^k(\Omega)$ by polynomials in z, w, and w^{-1}.

Notice the stark contrast of (2) to the nonapproximation result in [1] for $k = 0$.

Let us accept the claim for the moment and proceed with the proof of the theorem. Suppose then that $Pf \in W^k(\Omega)$, and we wish to estimate $\|Pf\|_k$ in terms of $\|f\|_k$. As observed in §1, this estimate is immediate near boundary points at which the rotations in w act transversely to the complex tangent space, which includes in particular points for which $|z| = 3$. Because Pf extends holomorphically to $\{(z, w): 1 < |w| < 6 \text{ and } |z| < r_2(|w|)\}$, the maximum principle and Cauchy's estimates imply that the k-norm of Pf on the set where $|z| < 3 - \delta$ is dominated by the supremum of Pf on a compact subset of Ω, and hence by $\|Pf\|_0$, which by definition does not exceed $\|f\|_0$. The remaining boundary points are those for which simultaneously $|z| > 3$ and $\partial\rho/\partial w = 0$. Call this critical set S.

At points of S, the holomorphic field $X := z(\partial/\partial z)$ is normal. No approximation is required, and essentially the argument from [8, Example 3] gives the required estimates. For the reader's convenience, we sketch the argument. We may normalize the defining function ρ so that $X\rho = 1$ on S. If ϕ is a smooth cutoff function that is equal to 1 in a neighborhood of the critical set S, and $\{g_j\}$ is a sequence of polynomials in z, w, and w^{-1} approximating Pf in $W^k(\Omega)$, then the lemma from §0 gives

$$|\langle \phi X^k Pf, \phi X^k g_j\rangle| \le |\langle Pf, \phi^2 b^k X^{2k} g_j\rangle| + C_k\|Pf\|_{k-1}\|g_j\|_k,$$

or, after adding and subtracting 1 on the right-hand side of the inner product,

$$|\langle \phi X^k Pf, \phi X^k g_j\rangle| \le |\langle Pf, X^{2k} g_j\rangle| + |\langle Pf, (\phi^2 b^k - 1)X^{2k} g_j\rangle| + C_k\|Pf\|_{k-1}\|g_j\|_k.$$

We may assume by induction that $\|Pf\|_{k-1} \leq C_{k-1}\|f\|_{k-1}$. Since $\phi^2 b^k = 1$ on S, and the norm of Pf away from S is under control, the right-hand side is at most

$$\varepsilon\|Pf\|_k\|g_j\|_k + |\langle Pf, X^{2k} g_j\rangle| + C_k\|f\|_k\|g_j\|_k$$

for an arbitrary positive ε. Because X has holomorphic coefficients, the middle term is

$$|\langle f, X^{2k} g_j\rangle| \leq C_k\|f\|_k\|g_j\|_k.$$

Accordingly,

$$|\langle \phi X^k Pf, \phi X^k g_j\rangle| \leq \varepsilon\|Pf\|_k\|g_j\|_k + C_k\|f\|_k\|g_j\|_k,$$

where the constant depends on ε and k, but not on j. By fixing ε sufficiently small (depending on k) and passing to the limit as $j \to \infty$, we conclude that $\|Pf\|_k \leq C_k\|f\|_k$. Thus the proof of Theorem 2 is reduced to verification of the claim above.

To verify (1), let h be a holomorphic function in $W^1(\Omega)$. Observe that when z is fixed and $|z| > 2$, the slice $\{w \in \mathbf{C} : (z, w) \in \Omega\}$ is an annulus, in which h has a Laurent expansion $h(z, w) = \sum_{n=-\infty}^{\infty} a_n(z) w^n$. The coefficients a_n are given by the integrals

$$a_n(z) = \frac{w^{-n}}{2\pi}\int_0^{2\pi} h\left(z, we^{i\theta}\right) e^{-in\theta}\, d\theta.$$

Here w can be any point contained in the slice, and consequently the a_n continue holomorphically from the annulus $\{2 < |z| < 4\}$ to the annulus $\{0 < |z| < 4\}$.

Even more is true. Differentiating under the integral sign and applying Fubini's theorem implies that the a_n are in $W^1(\Omega)$, and in particular in $L^4(\Omega)$ by the Sobolev embedding theorem. By [1, Lemma 3], the $a_n(z)$ extend holomorphically to the disk $\{|z| < 4\}$.

Our goal is to show that the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z) w^n$ converges when $|z| < 3$ and $1 < |w| < 6$. By the classical lemma of Hartogs, the sum of the series will be holomorphic. Consider the sum of the terms for which n is positive. The upper semicontinuous regularization of $\limsup_{n\to\infty} |a_n(z)|^{1/n}$ in the disk $\{|z| < 4\}$ is a subharmonic function (in view of the above integral formula and the maximum principle, the sequence $\{|a_n|^{1/n}\}$ is locally uniformly bounded) that is at most equal to $1/6$ when $|z| = 3$. By the maximum principle, it does not exceed $1/6$ in the disk $\{|z| \leq 3\}$. Consequently the series $\sum_{n=0}^{\infty} a_n(z) w^n$ converges for $|z| < 3$ and $|w| < 6$. After the inversion $w \mapsto w^{-1}$, the same argument shows that $\sum_{n=-\infty}^{-1} a_n(z) w^n$ converges for $|z| < 3$ and $|w| > 1$. This proves part (1) of the claim.

To prove (2), consider for small positive δ the Reinhardt domain $R := \Omega \cup \{(z, w) : 1+\delta < |w| < 6-\delta \text{ and } |z| < 3\}$. By claim (1), every holomorphic function in $W^k(\Omega)$ extends to R, and the argument used at the beginning of the proof of Theorem 2 shows that this extension is in $W^k(R)$. The extension has a series expansion in powers of z, w, and w^{-1} that converges in $W^k(R)$ (since the monomials are orthogonal in W^k of approximating Reinhardt subdomains), and since $\Omega \subset R$, part (2) of the claim is proved. This completes the proof of Theorem 2.

We remark that we know no smooth bounded domains that fail to admit a priori estimates for the Bergman projection.

Note added June 1990. Using techniques related to the ones we develop in this paper, we recently showed [12] that the Bergman projection and the $\overline{\partial}$-Neumann operator are regular in W^s for all positive s in every smooth bounded domain in $\mathbf{C}^n$ (not necessarily Hartogs) admitting a defining function that is plurisubharmonic on the boundary. This result contains the above-mentioned theorem of Bonami and Charpentier.

Note added November 1990. Recently Barrett [3] has shown that the Bergman projections for the worm domains Ω_r are not continuous on the Sobolev space $W^k(\Omega_r)$ when $k \geq \pi/(2\log r)$. Continuity in the space $C^\infty(\overline{\Omega_r})$ remains open.

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