TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 332, Number 1, July 1992

STABILITY FOR AN INVERSE PROBLEM IN POTENTIAL THEORY

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ABSTRACT. Let D be a subdomain of a bounded domain Ω in $\mathbb{R}^n$. The conductivity coefficient of D is a positive constant $k \neq 1$ and the conductivity of $\Omega \backslash D$ is equal to 1. For a given current density g on $\partial\Omega$, we compute the resulting potential u and denote by f the value of u on $\partial\Omega$. The general inverse problem is to estimate the location of D from the known measurements of the voltage f. If D_h is a family of domains for which the Hausdorff distance $d(D, D_h)$ equal to $O(h)$ (h small), then the corresponding measurements f_h are $O(h)$ close to f. This paper is concerned with proving the inverse, that is, $d(D, D_h) \leq \frac{1}{c}\|f_h - f\|$, $c > 0$; the domains D and D_h are assumed to be piecewise smooth. If $n \geq 3$, we assume in proving the above result, that $D_h \supset D$ (or $D_h \subset D$) for all small h. For $n = 2$ this monotonicity condition is dropped, provided g is appropriately chosen. The above stability estimate provides quantitative information on the location of D_h by means of f_h.

1. INTRODUCTION

For any two domains D_1, D_2 in $\mathbb{R}^n$ denote by $d(D_1, D_2)$ the Hausdorff distance between them. Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ and let D and D_h (for any $0 < h < h_0$, h_0 small) be subdomains of Ω with closure in Ω such that

$$c_1 h \leq d(D, D_h) \leq c_2 h,$$

where c_1, c_2 are positive constants. Set

$$a_D = \begin{cases} k & \text{in } D, \\ 1 & \text{in } \Omega\backslash D, \end{cases} \qquad a_{D_h} = \begin{cases} k & \text{in } D_h, \\ 1 & \text{in } \Omega\backslash D_h, \end{cases}$$

where k is a positive constant, $k \neq 1$. Consider the Neumann problems

$$\operatorname{div}(a_D \nabla u_D) = 0 \quad \text{in } \Omega, \tag{1.1}$$
$$\frac{\partial u_D}{\partial \nu} = g \quad \text{on } \partial\Omega, \qquad \int_\Omega u_D = 0$$

and

$$\operatorname{div}(a_{D_h} \nabla u_{D_h}) = 0 \quad \text{in } \Omega, \tag{1.2}$$
$$\frac{\partial u_{D_h}}{\partial \nu} = g \quad \text{on } \partial\Omega, \qquad \int_\Omega u_{D_h} = 0,$$

Received by the editors April 9, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 86A20; Secondary 58G10, 35J25.

The second author's research is partially supported by the National Science Foundation grant DMS-86-12880.

where g is a given function, satisfying

$$g \in L^2(\partial\Omega), \quad g \neq 0, \qquad \int_{\partial\Omega} g = 0, \tag{1.3}$$

and set

$$f = u_D|_{\partial\Omega}, \qquad f_h = u_{D_h}|_{\partial\Omega}. \tag{1.4}$$

We are interested in establishing a local stability estimate of the form

$$d(D, D_h) \leq C|f - f_h|_{L^1(\Gamma)} \tag{1.5}$$

where Γ is a nonempty open subset of $\partial\Omega$, and h is sufficiently small. Such an estimate means that the mapping

$$D \to \mathscr{F}(D) \equiv u_D|_\Gamma$$

has nonzero "derivative."

We shall refer to the case

$$D_h \supset D \quad \text{for all } h \text{ (or } D_h \subset D \text{ for all } h) \tag{1.6}$$

as the *monotone case.* When the assumption (1.6) is dropped, we speak of the *nonmonotone case.*

We shall always assume that ∂D is piecewise smooth, and that ∂D_h has the representation

$$\partial D_h\colon x = f(s) + h\sigma_h(s)\nu(s) \quad \text{a.e.} \tag{1.7}$$

where $\nu(s)$ is the normal to ∂D, wherever it exists, and $|\sigma_h(s)| \leq C$; s is an $(n-1)$-dimensional local parameter. Notice that ∂D is given by $x = f(s)$.

Bellout and Friedman [1] established (1.5) in the monotone case, provided ∂D and ∂D_h are in $C^{2,\alpha}$ (uniformly in h); their proof actually requires only $C^{1,1}$ smoothness. An earlier proof of (1.5) for $n = 2$, due to Friedman and Gustafsson [5], also required the same smoothness.

For $n = 2$ Bellout and Friedman [1] have established (1.5) for the nonmonotone case provided ∂D is analytic and certain finite number of "orthogonality" conditions are satisfied; it is however not easy to verify such conditions even, for instance, if the D_h are translates of D.

In §4 we shall extend the stability result (1.5) of Bellout and Friedman to the monotone case when ∂D is only piecewise $C^{1,1}$; the proof requires some new ideas and technical estimates which are developed in §§2, 3. Our interest in the piecewise $C^{1,1}$ case and in particular in polyhedra stems from a recent uniqueness theorem due to Friedman and Isakov [6]. They proved that if D and D' are any convex polyhedra in Ω such that the solution u_D of (1.1) and the corresponding solution $u_{D'}$ for D' satisfy: $u_D = u_{D'}$ on an open nonempty portion Γ of $\partial\Omega$, then $D = D'$. They needed to assume that either Ω is a half-space or D and D' are not "too close" to $\partial\Omega$. They also established (1.5), but only when Ω is a half-space and under some severe restrictions on D.

In §5 we consider the case $n = 2$ and ∂D analytic, but drop the monotonicity assumption (1.6). We establish the stability estimate (1.5) for appropriately chosen function g.

Finally in §6 we extend the results of §5 to the case where D is a convex polygon or, more generally, piecewise analytic.

2. The behavior of ∇u near a vertex of ∂D

Throughout this paper we assume that Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with $C^{1,\alpha}$ boundary.

Let D be a subdomain of Ω with $\overline{D} \subset \Omega$ and let

$$a(x) = \begin{cases} k & \text{if } x \in D, \\ 1 & \text{if } x \in \Omega \backslash D, \end{cases}$$

where k is a positive number $\neq 1$. Consider the diffraction problem

$$\operatorname{div}(a \nabla u) = f \quad \text{in } \Omega, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega, \qquad \int_\Omega u = 0, \tag{2.2}$$

where g satisfies

$$g \in L^2(\partial\Omega), \qquad g \neq 0, \tag{2.3}$$

$$\int_{\partial\Omega} g = \int_\Omega f. \tag{2.4}$$

We shall be interested in the behavior of ∇u near a point $x_0 \in \partial D$ where ∂D is not smooth. For simplicity we first consider the case when $n = 2$, $x_0 = 0$ and, for some ball $B_{r_0} \equiv B_{r_0}(x_0)$,

$$\begin{gathered} \partial D \cap \overline{B}_{r_0} \quad \text{consists of line segments} \\ l_1 = \{(r, \theta); 0 \leq r \leq r_0, \theta = 0\}, \quad \text{and} \\ l_2 = \{(r, \theta); 0 \leq r \leq r_0, \theta = \beta\}, \qquad 0 < \beta < \pi. \end{gathered} \tag{2.5}$$

Consider first the case $f = 0$ and set $u^e = u_{\Omega\backslash D}$, $u^i = u_D$.

Lemma 2.1. *The following expansion holds for* $0 < r < r_1$ $(r_1 = r_0/2)$:

$$\begin{aligned} u^e &= u^e(0) + \sum_{j=1}^\infty r^{\gamma_j}(A_j^e \cos \gamma_j\theta + B_j^e \sin \gamma_j\theta), \\ u^i &= u^i(0) + \sum_{j=1}^\infty r^{\gamma_j}(A_j^i \cos \gamma_j\theta + B_j^i \sin \gamma_j\theta); \end{aligned} \tag{2.6}$$

the series are convergent with their first derivatives, absolutely and uniformly for $0 < r \leq r_1$. *Here, the sequence* γ_j *is monotone increasing,*

$$0 < c_1 \leq \gamma_j / j \leq c_2 < \infty \quad \textit{for all } j, \tag{2.7}$$

and

$$\gamma_1 > \tfrac{1}{2}. \tag{2.8}$$

Proof. Denote by S^1 the unit circle and define on S^1

$$a(\theta) = \begin{cases} k & \text{if } 0 \leq \theta \leq \beta, \\ 1 & \text{if } \beta < \theta < 2\pi. \end{cases}$$

Introduce the function spaces $L_a^2(S^1)$, $H_a^1(S^1)$ with norms

$$\|v\|_{L_a^2(S^1)} = \left\{\int_0^{2\pi} a|v(\theta)|^2\,d\theta\right\}^{1/2},$$

$$\|v\|_{H_a^1(S^1)} = \left\{\int_0^{2\pi} av_\theta^2(\theta)\,d\theta + \int_0^{2\pi} av^2(\theta)\,d\theta\right\}^{1/2}.$$

Set

$$\mathscr{L}v = \frac{1}{a}\frac{\partial}{\partial\theta}\left(a\frac{\partial}{\partial\theta}v\right); \tag{2.9}$$

$\mathscr{L}$ is an unbounded, selfadjoint, positive elliptic operator with dense domain in $L_a^2(S^1)$, and $(\mathscr{L}+1)^{-1}$ is compact. Hence the spectrum of $\mathscr{L}$ consists of positive eigenvalues γ_j^2 $(\gamma_j > 0)$. We denote a corresponding (complete) orthonormal sequence by $\{v_j\}$; it is a basis for $L_a^2(S^1)$.

If

$$\mathscr{L}v + \gamma_j^2 v = 0, \tag{2.10}$$

then

$$v'' + \gamma_j^2 v = 0 \text{ on } 0 < \theta < \beta \text{ and on } \beta < \theta < 2\pi, \tag{2.11}$$

so that

$$\begin{aligned} v_j = v_j^i &= M_j^i \cos\gamma_j\theta + N_j^i \sin\gamma_j\theta, \qquad & 0<\theta<\beta, \\ v_j = v_j^e &= M_j^e \cos\gamma_j\theta + N_j^e \sin\gamma_j\theta, \qquad & \beta<\theta<2\pi; \end{aligned}$$

in addition, the diffraction (or transmission) conditions

$$\left.\begin{aligned} v_j^i &= v_j^e \\ kv_{j,\theta}^i &= v_{j,\theta}^e \end{aligned}\right\} \quad \text{at } \theta = 0,\ \theta = \beta \tag{2.12}$$

must be satisfied, as well as the condition

$$\int_0^{2\pi} v_j^2\,d\theta = 1. \tag{2.13}$$

For every $r \in (0, r_0)$ we have the expansion

$$u(r,\theta) = u(0) + \sum_{j=1}^\infty h_j(r)v_j(\theta) \quad \text{in } L^2(S^1) \tag{2.14}$$

and

$$h_j(r) = \int_0^{2\pi} auv_j\,d\theta.$$

The function u satisfies, for $0 < r < r_0$,

$$a\left(\frac{1}{r}(ru_r)_r\right) + \frac{1}{r^2}(au_\theta)_\theta = 0. \tag{2.15}$$

Multiplying (2.15) by $av_j(\theta)$ and integrating over S^1 we get, after using (2.10),

$$\frac{1}{r}\left[r\left(\int_{S^1} auv_j\,d\theta\right)_r\right]_r - \frac{\gamma_j^2}{r^2}\int_{S^1} auv_j = 0. \tag{2.16}$$

Hence $h_j(r)$ satisfies

$$\frac{1}{r}(rh_{j,r})_r - \frac{\gamma_j^2}{r^2}h_j = 0, \tag{2.17}$$

so that

$$h_j(v) = C_j r^{\gamma_j} + D_j r^{-\gamma_j}. \tag{2.18}$$

Since $u_r \in L^2(B_{r_0})$ we have

$$\int_0^{r_0} [(uv_j)_r]^2 r\,dr \le M < \infty, \tag{2.19}$$

i.e.,

$$\int_0^{r_0} |h_j'(r)|^2 r\,dr \le M. \tag{2.20}$$

It follows that $D_j = 0$ and, consequently, from (2.14),

$$u(r, \theta) = u(0) + \sum C_j r^{\gamma_j} v_j(\theta) \tag{2.21}$$

and

$$\int_{S^1} au^2(r, \theta)\,d\theta = u^2(0) + \sum C_j^2 r^{2\gamma_j} \qquad (0 < r \le r_0).$$

Since $u_r \in L_a^2(S^1)$ for $r = r_0$, we actually even have

$$\int_{S^1} au_r^2(r, \theta)\,d\theta = \sum C_j^2 \gamma_j^2 r^{2(\gamma_j - 1)} < \infty$$

for $r = r_0$, so that

$$\sum_{j=1}^{\infty} C_j^2 \gamma_j^2 r_0^{2\gamma_j} < \infty. \tag{2.22}$$

We next estimate the γ_j. We can write

$$v_j^e = \operatorname{Re}\{a_j e^{i\gamma_j\theta}\}, \quad v_j^i = \operatorname{Re}\{b_j e^{i\gamma_j\theta}\}.$$

The refraction conditions at $\theta = 0$ and $\theta = \beta$ then become (for $a = a_j$, $b = b_j$, $\gamma = \gamma_j$)

$$a + \overline{a} = be^{i\gamma 2\pi} + \overline{b}e^{-i\gamma 2\pi}, \tag{2.23}$$

$$ae^{i\gamma\beta} + \overline{a}e^{-i\gamma\beta} = be^{i\gamma\beta} + \overline{b}e^{-i\gamma\beta} \tag{2.24}$$

and

$$k(a - \overline{a}) = be^{i\gamma 2\pi} - \overline{b}e^{-i\gamma 2\pi}, \tag{2.25}$$

$$k(ae^{i\gamma\beta} - \overline{a}e^{-i\gamma\beta}) = be^{i\gamma\beta} - \overline{b}e^{-i\gamma\beta}. \tag{2.26}$$

Taking k times (2.23) and adding to (2.25), we get

$$2ka = (k+1)be^{i\gamma 2\pi} + (k-1)\overline{b}e^{-i\gamma 2\pi}. \tag{2.27}$$

Similarly, taking k times (2.24) and adding to (2.26), we get

$$2ka = (k+1)b + (k-1)\overline{b}e^{-2i\gamma\beta}. \tag{2.28}$$

Comparing (2.27) with (2.28) we find that

$$(k+1)b(e^{i\gamma 2\pi}-1)=(k-1)\overline{b}(e^{-2i\gamma\beta}-e^{-2i\gamma\pi}). \tag{2.29}$$

We need to consider two cases

Case (i). $e^{2i\gamma\pi}\neq 1$.

Then

$$\frac{b}{\overline{b}}=\frac{k-1}{k+1}\frac{e^{-2i\gamma\beta}-e^{-2i\gamma\pi}}{e^{2i\gamma\pi}-1}.$$

Since $|b/\overline{b}|=1$, we conclude that

$$\frac{|e^{-2i\gamma\beta}-e^{-2i\gamma\pi}|}{|e^{2i\gamma\pi}-1|}=\left|\frac{k+1}{k-1}\right|\equiv A>1, \tag{2.30}$$

or

$$\sin\gamma(\pi-\beta)=A\sin\gamma\pi \tag{2.31}$$

and it is easy to see that this equation has an infinite sequence of solutions γ_j satisfying (2.7). We claim that the smallest one, γ_1, satisfies $\gamma_1>\frac{1}{2}$. Indeed, if $\gamma_1\leq\frac{1}{2}$ then $2\pi\gamma_1<\pi$ and $0<2\gamma_1\beta<2\pi\gamma_1<\pi$. But then

$$|e^{2i\gamma_1\pi}-e^{2i\gamma_1\beta}|<|e^{2i\gamma_1\pi}-1|,$$

a contradiction to (2.30).

Case (ii). $e^{2i\gamma\pi}=1$.

Then $\gamma=\gamma_j=n$ for some integer n, and from (2.29) we see that $\beta\gamma/\pi$ is also an integer; consequently

$$\beta=\frac{q}{m}\pi,\quad q \text{ and } m \text{ are relatively prime positive integers.} \tag{2.32}$$

We easily see that all the additional solutions γ, in this case, are multiples of m. Thus the asserted expansion (2.14) still holds, but one has to include the additional sequence of multiples of m into the sequence of the γ_j's.

Finally, using (2.22) it is easily seen that the series expansion of $u(r,\theta)$ and its gradient are absolutely uniformly convergent for $0<r\leq r_0/2$.

We shall now extend Lemma 2.1 to the case $f\not\equiv 0$, assuming that $f\in L^{4/3}(\Omega)$.

Set

$$f_j(r)=\int_0^{2\pi}f(r,\theta)v_j(\theta)\,d\theta.$$

Then formally

$$u(r,\theta)=u(0)+\sum C_j r^{\gamma_j}v_j(\theta)+\sum e_j(r)v_j(\theta) \tag{2.33}$$

where

$$e_j(r)=\frac{r^{\gamma_j}}{2\gamma_j}\int_{r_0/2}^{r}f_j(s)s^{1-\gamma_j}\,ds-\frac{r^{-\gamma_j}}{2\gamma_j}\int_0^r f_j(s)s^{1+\gamma_j}\,ds \tag{2.34}$$

is a solution of

$$\frac{1}{r}(rh')'-\frac{\gamma_j^2}{r^2}h=f_j(r);$$

the fact that $D_j=0$ follows by using (2.19) as before, noting that $e_j=O(r^2)$, $e_j'=O(r)$. Observe that the first integral on the right-hand side of (2.34) is from $r_0/2$ to r (the integral from 0 to r will not converge if $\gamma_j \geq 2$). We have,

(2.35)

$$\left|\int_0^r f_j(s)s^{1+\gamma_j}\,ds\right| = \left|\int_0^r\int_0^{2\pi} f(s,\theta)v_j(\theta)s^{1/p}s^{1+\gamma_j}s^{1-1/p}\,d\theta\,ds\right|$$
$$\leq \left(\int_0^r\int_0^{2\pi}|f|^p s\,d\theta\,ds\right)^{1/p}\left(\int_0^r\int_0^{2\pi}s^{1+(1+\gamma_j)q}|v_j(\theta)|^q\right)^{1/q}\quad \left(q=\frac{p}{p-1}\right)$$
$$\leq \frac{Cr^{1+\gamma_j+1/q}}{\gamma_j^{1/q}}\left(\int|v_j(\theta)|^q\right)^{1/q},$$

if $p=\frac{4}{3}$, $q=4$ (since $f\in L^{4/3}$). Noting that by Sobolev's imbedding [4, p. 27],

$$|v_j|_{L^3}\leq C(|v_j'|_{L^2})^a(|v_j|_{L^2})^{1-a},\qquad a=\tfrac{1}{4},$$

and

$$|v_j|_{L^2}=1,\qquad |v_j'|_{L^2}\leq C\gamma_j|v_j|_{L^2}=C\gamma_j,$$

we get

$$|v_j|_{L^3}\leq C\gamma_j^{1/4}.$$

Substituting this into (2.35), we get

$$\left|\int_0^r f_j(s)s^{1+\gamma_j}\,ds\right|\leq\frac{Cr^{1+\gamma_j+1/4}}{\gamma_j^{1/4}}.$$

A similar estimate holds for the second integral in (2.34). Hence

$$\sum|e_j(r)v_j(\theta)|\leq\sum\frac{cr^2}{\gamma_j^{1+1/4}}\leq Cr^2, \tag{2.36}$$

by (2.7).

From (2.33), (2.36) we deduce that the series

$$\sum_j C_j r^{\gamma_j}v_j(\theta) \tag{2.37}$$

is convergent in $L^2(S^1)$ and therefore

$$\sum C_j^2 r^{2\gamma_j}<\infty,\qquad 0<r<r_0.$$

This implies the absolute uniform convergence of the series (2.37) for $0<r\leq r_0/2$; in particular,

$$|u(r,\theta)-u(0)|\leq Cr^{\gamma_1},\qquad \gamma_1>\tfrac{1}{2}. \tag{2.38}$$

We now consider the function

$$v_\lambda(x)=u(\lambda x)-u(0)$$

for λ small and x in $B_*=\{\frac{1}{4}<|x|<4\}$. Let $B_0=\{\frac{1}{2}<|x|<\frac{7}{2}\}$. Clearly

$$\operatorname{div}(a\nabla v_\lambda)=\lambda^2 f(\lambda x),\qquad |v_\lambda|\leq C\lambda^{\gamma_1}.$$

Let l_λ be any line in $\widetilde{B} = \{1 < |x| < 3\}$ with endpoints on $\partial\widetilde{B}$. Then, by the trace imbedding (of $H^{1/2}(B) \to L^2(l_\lambda)$), Sobolev's imbedding [7, p. 27] and L^p elliptic estimates,

$$\left\{\int_{l_\lambda} |\nabla v_\lambda|^2\,dx\right\}^{1/2} \le C|v_\lambda|_{W^{2,4/3}(B_0)} \le C\int_{B_*} |\lambda^2 f(\lambda x)|^{4/3}\,dx + C_1\lambda^{\gamma_1}.$$

Making the substitution $\lambda x = y$ we find that

$$\int_l |\nabla u|^2 \le C\left(\int_{\lambda/4<|x|<\lambda} |f|^{4/3}\right)^{3/2} + C\lambda^{\gamma_1 - 1/2},$$

where l is the image of l_λ; l is any interval connecting a point on $\{r = 1/\lambda\}$ to a point on $\{r = 3/\lambda\}$. By varying λ, taking for instance $\lambda = 3^{-j}$, we deduce that

$$\int_l |\nabla u|^2 \le C\varepsilon(|l|), \qquad \varepsilon(t) \downarrow 0 \text{ if } t \downarrow 0 \tag{2.39}$$

where l is an interval in $\{r < \varepsilon_0\}$ and $|l| =$ length of l.

If D is a polygon, then by applying the above estimate near each vertex of ∂D we arrive at the following result:

Lemma 2.2. *Suppose D is a polygon and $f \in L^{4/3}(\Omega)$. Then the solution of the refraction problem* (2.1), (2.2) *satisfies*:

For any family of intervals which are the intersection of straight lines parallel to one of the sides of D and Ω_0, a compact subset of Ω,

$$\|\nabla u\|_{L^2(l)} \le C\|f\|_{L^{4/3}(\Omega)} \tag{2.40}$$

where C is a constant depending only on D, Ω_0 and g; furthermore, for any vertex a of ∂D,

$$\|\nabla u\|_{L^2(l\cap B(a,r))} \le C\varepsilon(|l|), \qquad \varepsilon(t) \downarrow 0 \textit{ if } t \downarrow 0. \tag{2.41}$$

Extension of this result to piecewise smooth domain in any number of dimension will be discussed in §4.

3. An auxiliary estimate

Let D be a polygon in $\mathbb{R}^2$ with edges $\Gamma_1, \Gamma_2, \dots, \Gamma_N$ and vertices $s_1, s_2, \dots, s_N$ such that $s_j = \overline{\Gamma}_j \cap \overline{\Gamma}_{j+1}$, $\Gamma_{N+1} = \Gamma_1$. Let D_h $(0 < h \le h_0)$ be a family of polygons with edges $\Gamma_1(h), \Gamma_2(h), \dots, \Gamma_N(h)$ and vertices $s_1(h), \dots, s_N(h)$ such that $s_j(h) = \overline{\Gamma_j(h)} \cap \overline{\Gamma_{j+1}(h)}$, $\Gamma_{N+1}(h) = \Gamma_1(h)$. We assume that

$$\begin{gathered} D \subset D_h, \qquad \overline{D}_h \subset \Omega \quad \text{for } 0 < h \le h_0, \\ c_1 h \le \sum_{j=1}^N |s_j(h) - s_j| \le c_2 h \qquad (0 < c_1 < c_2 < \infty). \end{gathered} \tag{3.1}$$

Set

$$a = \begin{cases} k & \text{in } D, \\ 1 & \text{in } \Omega\backslash D, \end{cases} \qquad a_h = \begin{cases} k & \text{in } D_h \\ 1 & \text{in } \Omega\backslash D_h \end{cases} \quad (k > 0,\ k \ne 1),$$

and consider the diffraction problems

$$\operatorname{div}(a\nabla u) = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega, \qquad \int_\Omega u = 0 \tag{3.3}$$

and

$$\operatorname{div}(a_h\nabla u_h) = 0 \quad \text{in } \Omega, \tag{3.4}$$

$$\frac{\partial u_h}{\partial \nu} = g \quad \text{on } \partial\Omega, \qquad \int_\Omega u_h = 0 \tag{35}$$

where g satisfies

$$g \in L^2(\Omega), \ g \neq 0, \qquad \int_{\partial\Omega} g = 0. \tag{3.6}$$

We are interested in estimating the "quotient difference"

$$U_h = \frac{u_h - u}{h}. \tag{3.7}$$

Lemma 3.1. *For any* $0 < \varepsilon < 2$ *there is a constant* C *such that*

$$\int_\Omega |U_h|^{2+\varepsilon} \le C \quad \forall 0 < h \le h_0. \tag{3.8}$$

For $\varepsilon = 0$ and $\partial D \in C^{1,1}$ this was proven by Bellout and Friedman [1].

Proof. Multiplying the difference of the equations (3.2), (3.4) by a function v in $H^1(\Omega)$ and integrating over Ω, we easily get

$$\int_\Omega a_h \nabla U_h \cdot \nabla v + \frac{k-1}{h}\int_{D_h\setminus D} \nabla u \cdot \nabla v = 0 \quad \forall v \in H^1(\Omega). \tag{3.9}$$

We introduce the solution w_h to the diffraction problem

$$\operatorname{div}(a_h\nabla w_h) = U_h \quad \text{in } \Omega, \tag{3.10}$$

$$\frac{\partial w_h}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega w_h = 0; \tag{3.11}$$

since $\int_\Omega U_h = 0$, this problem does in fact have a unique solution.

Multiplying (3.10) by U_h and integrating over Ω, we get

$$-\int_\Omega a_h \nabla w_h \cdot \nabla U_h = \int_\Omega U_h^2. \tag{3.12}$$

Substituting $v = w_h$ in (3.9) and adding the result to (3.12), we find that

$$\frac{k-1}{h}\int_{D_h\setminus D} \nabla u \cdot \nabla w_h = \int_\Omega U_h^2,$$

so that, by Cauchy's inequality,

$$\int_\Omega U_h^2 \le \frac{|k-1|}{h}\left\{\int_{D_h\setminus D} |\nabla u|^2\right\}^{1/2} \left\{\int_{D_h\setminus D} |\nabla w_h|^2\right\}^{1/2}. \tag{3.13}$$

By extending Γ_j at s_{j+1} as a line segment until it meets ∂D_h, for $1 \le j \le N$, we get a "triangulation" of $D_h \backslash D$ into N quadrangles $Q_j(h)$, each bounded by the extended Γ_{j-1}, Γ_j and portions of $\Gamma_j(h), \Gamma_{j+1}(h)$. Each $Q_j(h)$ can be traced by a family of intervals $l_j(\lambda, h)$ parallel to Γ_j at distance λ from Γ, where λ varies in some interval $0 \le \lambda \le h_j$, $h_j \le Ch$. Hence

$$\begin{aligned}\int_{D_h\backslash D} |\nabla u|^2 &\le \sum_{j=1}^N \int_{Q_j(h)} |\nabla u|^2 = \sum_{j=1}^N \int_0^{h_j} d\lambda \int_{l_j(\lambda, h)} |\nabla u|^2 \\ &\le \sum_{j=1}^N \int_0^{h_j} C\, d\lambda, \quad \text{by Lemma 2.2.}\end{aligned}$$

It follows that

$$\int_{D_h\backslash D} |\nabla u|^2 \le Ch. \tag{3.14}$$

Similarly

$$\int_{D_h\backslash D} |\nabla w_h|^2 \le \sum_{j=1}^N \int_0^{h_j} d\lambda \int_{l_j(\lambda, h)} |\nabla w_h|^2,$$

and

$$\int_{l_j(\lambda, h)} |\nabla w_h|^2 \le C \int_\Omega |U_h|^2,$$

by Lemma 2.2 applied to $w_h/\{\int_\Omega |U_h|^2\}^{1/2}$; hence

$$\int_{D_h\backslash D} |\nabla w_h|^2 \le Ch \int_\Omega U_h^2. \tag{3.15}$$

Substituting the estimates (3.14), (3.15) into the right-hand side of (3.13), we conclude that

$$\int_\Omega U_h^2 \le C\left\{\int_\Omega U_h^2\right\}^{1/2},$$

i.e.,

$$\int_\Omega U_h^2 \le C. \tag{3.16}$$

Having proved (3.8) for $\varepsilon = 0$ we proceed to prove it for ε positive and small. For this purpose we introduce another auxiliary function w_h defined as the solution to

$$\operatorname{div}(a_h \nabla w_h) = |U_h|^\varepsilon U_h - A \equiv F_\varepsilon \quad \text{in } \Omega \tag{3.17}$$

with the same conditions (3.11) as before; the constant A is chosen so that F_ε satisfies the compatibility condition $\int_\Omega F_\varepsilon = 0$, that is

$$A = \frac{1}{|\Omega|} \int_\Omega |U_h|^\varepsilon U_h.$$

From (3.16) it follows that

$$\int_\Omega |F_\varepsilon|^{4/3} \le C,$$

if

$$\frac{4}{3} = \frac{2}{1+\varepsilon}, \quad \text{i.e.,} \quad \text{if } \varepsilon = \frac{1}{2}.$$

We can then apply Lemma 2.2 and deduce that

$$\int_{l_j(\lambda, h)} |\nabla w_h|^2 \le C,$$

for any line $l_j(\lambda, h)$ and, consequently,

$$\int_{D_h|D} |\nabla w_h|^2 \le Ch. \tag{3.18}$$

Next we multiply (3.17) by U_h and integrate over Ω. Since $\int_\Omega U_h = 0$, we obtain

$$-\int_\Omega a_h \nabla w_h \cdot \nabla U_h = \int_\Omega |U_h|^{2+\varepsilon}. \tag{3.19}$$

Substituting $v = w_h$ in (3.9) and adding to (3.19), we find that

$$\int_\Omega |U_h|^{2+\varepsilon} = \frac{k-1}{h} \int_{D_h \setminus D} \nabla u \cdot \nabla w_h, \tag{3.20}$$

and using the estimates (3.14), (3.18), we get

$$\int_\Omega |U_h|^{2+\varepsilon} \le C \tag{3.21}$$

where C is a constant independent of h, or (since $\varepsilon = 1/2$)

$$\int_\Omega |U_h|^{2+1/2} \le C,$$

which is an improvement of (3.16). More generally, assuming that (3.8) holds for $\varepsilon = \varepsilon_m$ the above proof shows that (3.8) will then hold for $\varepsilon = \varepsilon_{m+1}$ where

$$\frac{4}{3} = \frac{2+\varepsilon_m}{1+\varepsilon_{m+1}},$$

and since $\varepsilon_m \uparrow 2$ if $m \uparrow \infty$, the lemma follows.

4. Stability in the monotone case

For simplicity we begin with the case where $n = 2$ and D, D_h are polygonal domains as in §3, satisfying (3.1), and D is convex.

Set

$$f_h = u_h|_{\partial\Omega}, \qquad f = u|_{\partial\Omega}, \tag{4.1}$$

and let Γ be a nonempty open subset of $\partial\Omega$.

Theorem 4.1. *Under the foregoing assumptions*

$$\liminf_{h\to 0} \int_\Gamma \frac{|f_h - f|}{h} > 0. \tag{4.2}$$

This means that

$$d(D_h, D) \le C \int_\Gamma |f_h - f|, \tag{4.3}$$

where the constant C may depend on the family $\{\sigma_h\}$. We note that the reverse inequality

$$\int_\Gamma |f_h - f| \le Cd(D_h, D) \tag{4.4}$$

can easily be established.

From (3.9) we get, by integration by parts,

$$\begin{aligned} &-\int_\Omega a_h U_h \Delta v + (k-1)\int_{\partial D_h} U_h \frac{\partial v}{\partial \nu_e} + \int_{\partial\Omega} \frac{f_h - f}{h}\frac{\partial v}{\partial n} \\ &\quad + \frac{k-1}{h}\int_{D_h\backslash D} \nabla u \cdot \nabla v = 0 \quad \forall v \in H^2(\Omega) \end{aligned} \tag{4.5}$$

where ν_e is the exterior normal to ∂D_h and n is the exterior normal to $\partial\Omega$.

Suppose (4.2) is not true, i.e., for a sequence $h \to 0$,

$$\int_\Gamma \frac{|f_h - f|}{h} \to 0. \tag{4.6}$$

Since $\Delta U_h = 0$ in $\Omega\backslash\overline{D}_h$ and in D_h, U_h is uniformly bounded in $L^2(\Omega)$ (by Lemma 3.1), we may assume that

$$U_h \to U \text{ uniformly in compact subsets of } \overline{\Omega}\backslash\partial D. \tag{4.7}$$

Since further $U = 0$ on Γ (by (4.6)) and $\partial U/\partial n = 0$ on $\partial\Omega$ we have, by unique continuation of harmonic functions,

$$U = 0 \quad \text{in } \Omega\backslash\overline{D}; \tag{4.8}$$

also

$$\Delta U = 0 \quad \text{in } D. \tag{4.9}$$

Consequently

$$-\int_\Omega a_h U_h \Delta v \to -\int_D kU\Delta v \quad \text{as } h \to 0. \tag{4.10}$$

We next prove that

$$\int_{\partial D_h} U_h \frac{\partial v}{\partial \nu_e} \to 0 \quad \text{as } h \to 0. \tag{4.11}$$

Since ∂D_h consists of N edges $\Gamma_j(h)$, it suffices to prove that

$$\int_{\Gamma_j(h)} U_h \frac{\partial v}{\partial \nu_e} \to 0 \tag{4.12}$$

for each j.

Let σ_h be a line segment containing $\overline{\Omega_j(h)}$ in its interior and let

$$F_h(x) = \begin{cases} \partial v(x)/\partial \nu_e & \text{for } x \in \Gamma_j(h), \\ 0 & \text{for } x \in \sigma_h\backslash\Gamma_j(h). \end{cases}$$

Since F_h is piecewise smooth, it belongs to $W_p^{1-1/p}(\sigma_h)$ for any $p = 2 - \delta$, $\delta > 0$ [7, p. 45]. Hence, by the trace theorem [7, p. 37] there exists a function

z_h defined in semicircle S_h in $\Omega \backslash D_h$ with diameter σ_h such that

$$\|z_h\|_{W^{2,p}(S_h)} \leq C, \tag{4.13}$$

$$z_h = 0 \quad \text{on } \sigma_h, \tag{4.14}$$

$$\frac{\partial z_h}{\partial \nu_e} = F_h \quad \text{on } \sigma_h, \tag{4.15}$$

and

$$z_h \text{ vanishes in a neighborhood of } \partial S_h \backslash \sigma_h.$$

It follows that

$$\int_{\Gamma_j(h)} U_h \frac{\partial v}{\partial \nu_e} = \int_{S_h} U_h \Delta z_h. \tag{4.16}$$

As $h \to 0$, $\Gamma_j(h) \to \Gamma_j$ and $\sigma_h \to \sigma$, $S_h \to S$. By regularity of v, $\partial v/\partial \nu_e$ on σ_h converges in $W_p^{1-1/p}$-norm (when the independent variable is properly normalized so as to vary in the same interval σ, say). By the continuity of lifts (see [7, p. 37]) we then have that $z_h \to z$ in $W_{\text{loc}}^{2,p}(S)$. Recalling Lemma 3.1 we conclude that

$$\int_{S_h} U_h \Delta z_h \to \int_S U \Delta z,$$

and the right-hand side is equal to zero by (4.8). This completes the proof of (4.11).

Next we observe that, by (4.7),

$$\int_{\partial\Omega} \frac{f_h - f}{h} \frac{\partial v}{\partial n} = \int_{\partial\Omega} U_h \frac{\partial v}{\partial n} \to \int_{\partial\Omega} U \frac{\partial v}{\partial n} = 0. \tag{4.17}$$

We finally evaluate the last integral on the left-hand side of (4.5). Let T_j be the intersection of $D_h \backslash D$ with a square of side δ centered at the vertex s_j of ∂D. We can trace T_j by two families of intervals $l_{1j}(\lambda)$, $l_{2j}(\lambda)$, where the $l_{1j}(\lambda)$ are parallel to Γ_j at distance λ and the $l_{2j}(\lambda)$ are parallel to Γ_{j+1} at distance λ. Using (2.4) we get

$$\begin{aligned} &\left| \frac{k-1}{h} \int_{T_j} \nabla u \cdot \nabla v \right| \\ &\quad \leq \frac{C}{h} \left[\int d\lambda \left\{ \int_{l_{ij}(\lambda)} |\nabla u|^2 \right\}^{1/2} + \int d\lambda \left\{ \int_{l_{2j}(x)} |\nabla u|^2 \right\}^{1/2} \right] \\ &\quad \leq \frac{C}{h} h = C\varepsilon(\delta) \to 0, \qquad \varepsilon(\delta) \to 0 \text{ if } \delta \to 0. \end{aligned} \tag{4.18}$$

The set $D_h \backslash (D \cup (\bigcup T_j))$ is a disjoint union of rectangles $Q_{j,\delta}$ with two sides nearly parallel at distance $c(h)h$ $(\max c(h) = c_0 > 0)$ and the other two sides lying near s_j and s_{j+1}. Since u is smooth in $\overline{\Omega} \backslash D$ except at the set of points $s_1, \ldots, s_N$, we deduce that

$$\frac{1}{h} \int_{Q_{j,\delta}} \nabla u \cdot \nabla v \to \int_{\Gamma_{j,\delta}} \tilde{\sigma} \nabla u^e \cdot \nabla v, \qquad \tilde{\sigma} \geq 0,$$

where $\Gamma_{j,\delta} \subset \Gamma_j$, and the right-hand side converges to

$$\int_{\Gamma_j} \tilde{\sigma} \nabla u^e \cdot \nabla v$$

as $\delta \to 0$. Combining this with (4.18) it follows that

$$\frac{k-1}{h} \int_{D_h \setminus D} \nabla u \cdot \nabla v \to (k-1) \int_{\partial D} \tilde{\sigma} \nabla u^e \cdot \nabla v \quad \forall v \in H^2(\Omega). \tag{4.19}$$

Notice that $\tilde{\sigma}$ is actually a linear function on each edge Γ_j, and

$$\tilde{\sigma} \geq 0, \tilde{\sigma} \not\equiv 0 \quad \text{on } \partial D. \tag{4.20}$$

We now take $h \to 0$ in (4.5) and use (4.19), (4.17), (4.11) and (4.10); we obtain

$$k \int_D U \Delta v = (k-1) \int_{\partial D} \tilde{\sigma} \nabla u^e \cdot \nabla v \tag{4.21}$$

for any $v \in H^2(\Omega)$.

Let V_ε be an ε-neighborhood of D. If $v \in H^2(V_\varepsilon)$ then we can modify it outside $V_{\varepsilon/2}$ so as to obtain a function $\tilde{v}$ in $H^2(\Omega)$. Since (4.21) is valid for $\tilde{v}$, it is also valid for v. Thus (4.21) holds for any $v \in H^2(V_\varepsilon)$.

The function $u^i = u|_D$ is smooth in D and therefore for any $x_0 \in D$ and $0 < \lambda < 1$, the function

$$v_\lambda(x) \equiv u^i(x_0 + \lambda(x - x_0)) \quad \text{is in } H^2(V_\varepsilon) \tag{4.22}$$

for some $\varepsilon > 0$. Substituting $v = v_\lambda$ into (4.21), we get

$$\int_{\partial D} \tilde{\sigma}(x) \nabla u^e(x) \cdot \nabla u^i(x_0 + \lambda(x - x_0)) = 0.$$

Letting $\lambda \uparrow 1$ and using Lemma 2.2, we easily conclude that

$$\int_{\partial D} \tilde{\sigma}(x) \nabla u^e(x) \cdot \nabla u^i(x) = 0. \tag{4.23}$$

Since finally

$$k \frac{\partial u^i}{\partial \nu_e} = \frac{\partial u^e}{\partial \nu_e}, \qquad \frac{\partial u^i}{\partial \tau} = \frac{\partial u^e}{\partial \tau} \quad \text{on } \partial D$$

where τ is the tangential direction, it follows that

$$\int_{\partial D} \tilde{\sigma} |\nabla u^i|^2 = 0.$$

Recalling (4.20) we deduce that $\nabla u^i = 0$ on some arc on ∂D and hence, by harmonic continuation, $u = \text{const}$ in Ω. This implies, in particular, that $g = \partial u^e / \partial n \equiv 0$, which is a contradiction.

As we shall see below, Theorem 4.1 can be extended to general piecewise smooth domains D, D_h.

Definition 4.1. If each $\Gamma_j(h)$ is $C^{1,1}$ curve (instead of a line segment) with $C^{1,1}$-norm bounded independently of h, and if the angles $\beta_j(h)$ at $s_j(h)$ satisfy

$$0 < c_1 \leq \beta_j(h) \leq c_2 < 2\pi \quad \forall j$$

then we say that D_h is uniformly piecewise $C^{1,1}$. Similarly we define "D is piecewise $C^{1,1}$."

We shall need the following assumptions:

(A_1) D_h are uniformly piecewise $C^{1,1}$ and D is piecewise $C^{1,1}$; further, D is strongly starshaped with the respect to the origin in the sense that $\overline{D} \subset \mu D$ for any $\mu > 1$.

(A_2) The vertices $s_j(h)$ of D_h and s_j of D are such that

$$|s_j(h) - s_j| \le Ch.$$

(A_3) The representation (1.7) holds outside some $\delta(h)$-neighborhood of $\{s_1, \dots, s_N\}$ where $\delta(h) \to 0$ if $h \to 0$; further,

$$\sigma_h(s) \to \sigma(s) \not\equiv 0 \quad \text{as } h \to 0$$

uniformly outside any δ_0-neighborhood of $\{s_1, \dots, s_N\}$.

(A_4) $D_h \supset D$ or $D_h \subset D$ $\forall 0 < h \le h_0$.

Theorem 4.2. *Under the assumptions* (A_1)–(A_4), *the stability property* (4.2) *holds.*

The proof is similar to the proof of Theorem 4.1. The main difference occurs in the estimates near a vertex. Here we first perform a local diffeomorphism so as to make D locally a sector, and then proceed as before, with minor changes.

We now proceed to the case of dimension $n \ge 2$.

Definition 4.2. Let D be a domain in $\mathbb{R}^n$ $(n \ge 2)$. We shall say that D is piecewise $C^{1,1}$ if for any $x_0 \in \partial D$ there exists a polyhedron D_* in $\mathbb{R}^n$, a point $x_* \in \partial D_*$ and a $C^{1,1}$ diffeomorphism G_{x_0} from a ball $B(x_0, \delta)$ onto a ball $B(x_*, \delta_*)$ such that

$$G_{x_0}(B(x_0, \delta) \cap D) = B(x_*, \delta_*) \cap D_*.$$

Definition 4.3. A family of domains D_h $(0 < h \le h_0)$ is said to be uniformly piecewise $C^{1,1}$ if in Definition 4.2 δ can be chosen independently of h, and the diffeomorphism $G_{x_0} \equiv G_{x_0,h}$ have $C^{1,1}$ norms bounded independently of h.

Let D be a piecewise $C^{1,1}$ bounded domain in $\mathbb{R}^n$ $(n \ge 2)$, and let D_h be bounded domains in $\mathbb{R}^n$, uniformly piecewise $C^{1,1}$. Assume that ∂D_h is given by

$$\partial D_h\colon x = x_0 + h\sigma_h(x_0)\nu(x_0) \tag{4.24}$$

outside $\delta(h)$-neighborhood of the set S of points of ∂D where ∂D is not $C^{1,1}$; here $\nu(x_0)$ is the outward normal, $\delta(h) \to 0$ if $h \to 0$, and

$$\begin{gathered} |\sigma_h(x_0)| + |\nabla_{x_0}\sigma_h(x_0)| \le C, \\ \sigma_h(x_0) \to \sigma(x_0) \not\equiv 0 \quad \text{as } h \to 0. \end{gathered} \tag{4.25}$$

Remark 4.1. As shown in [1], if the D_h are obtained from D by affine transformations, then (4.25) is valid.

Theorem 4.4. *Under the foregoing assumptions, if D is strongly star-shaped with respect to the origin and* (A_4) *is satisfied then the stability property* (4.2) *holds.*

The proof is similar to the proof of Theorems 4.1 and 4.2. In fact, once we can prove it for the case where D, D_h are polyhedra, the proof for the general case follows by using the same estimates after performing local diffeomorphism about points of the set S.

In proving the theorem for polyhedra D, D_h, the main new effort is in extending Lemma 2.2 (upon which Lemma 3.1 depends). Here we can probably again apply eigenfunction expansion to ζu where ζ is a cut-off function. We shall not attempt to carry it out since a proof of Lemma 2.2, which is valid in fact for any Lipschitz domain D (f is assumed to belong to $L^{2n/(n+1)}$), was recently given by Escauriaza and Fabes [2]. We note however that Lemma 2.1 (used in the proof of Lemma 2.2) will be needed in §6; it is mainly for this reason that we have included in this paper our original proofs of Lemmas 2.1 and 2.2.

We finally remark that if $n \geq 3$ we only need to use (3.8) for $\varepsilon = 0$. Indeed, for $n = 2$ (3.8) with $\varepsilon > 0$ was used only in establishing (4.11). In the present case of $n \geq 3$, the trace theorem [7, p. 37] allows $p = 2$ in (4.13), (4.14) and (4.15); thus (4.11) follows by using (3.8) with $\varepsilon = 0$.

Remark 4.2. The star-shaped assumption on D was used only in order to establish (4.22) for any $0 < \lambda < 1$. If D is in $C^{1,1}$ then the star-shaped assumption may be dropped since $u^i(x)$ is in $H^2(D)$ and can therefore be extended into a function in $H^2(V_\varepsilon)$.

5. The nonmonotone case

From now on we drop the monotonicity assumption (1.6) but assume that

(5.1) there exists a diffeomorphism $y = x + \phi_h(x)$ of Ω onto Ω which maps D onto D_h and satisfies
$$|\nabla_x \phi_h(x)| \leq Ah \quad (A \text{ constant}).$$

Lemma 5.1. *Let D_h be uniformly piecewise $C^{1,1}$ domain and let D be such that* (4.24) *and* (4.25) *hold. Assume also that* (5.1) *is satisfied. If the stability property* (4.2) *is not satisfied then*

$$k\int_D U\Delta v = (k-1)\int_{\partial D} \tilde{\sigma}\nabla\tilde{u}\cdot\nabla v \quad \forall v \in H^2(\Omega) \tag{5.2}$$

holds, where

$$\Delta\tilde{u} = \begin{cases} \nabla u^e & \text{if } \tilde{\sigma}(x) > 0, \\ \nabla u^i & \text{if } \tilde{\sigma}(x) < 0, \end{cases} \tag{5.3}$$

and $\tilde{\sigma}(x)$ is a continuous function, $\tilde{\sigma}(x) \not\equiv 0$ and

$$\operatorname{sgn}\tilde{\sigma}(x) = \operatorname{sgn}\sigma(x). \tag{5.4}$$

The proof is similar to the proof of (4.21) for polygonal domains in the monotone case; for $C^{1,1}$ domain the theorem was already proved in [1]. The main difference in the proof for the piecewise $C^{1,1}$ case occurs in establishing (4.19); it is here that the assumption (5.1) is needed (cf. [1], following the proof of Lemma 3.3).

Corollary 5.2. *If $\sigma(x) = 0$ on a nonempty open subset of $\partial\Omega$ then the stability property* (4.2) *holds.*

Indeed, this follows from the proof of Lemma 5.1 in precisely the same way as Corollary 3.4 of [1] which dealt with the case where D and D_h are $C^{1,1}$ domains.

In the remaining part of this section we assume that

$$n = 2 \quad \text{and} \quad \partial D \text{ is analytic.} \tag{5.5}$$

This implies that u^e is analytic on ∂D. We shall prove that, for appropriately chosen g, the stability property (4.2) holds.

In addition to (5.5) we shall assume that

$$\partial D \text{ is strongly star-shaped with respect to the origin,} \tag{5.6}$$

and that

$$\sigma(x) \text{ changes sign along } \partial D \text{ only a finite number of times.} \tag{5.7}$$

The assumption (5.6) is made so that one may apply (5.2) to a function as in (4.22) $(0 < \lambda < 1)$ and thus deduce, as $\lambda \uparrow 1$, that if the stability property (4.2) does not hold then

$$\int_{\partial D} \tilde{\sigma} \nabla \tilde{u} \cdot \nabla v = 0 \quad \text{if } \Delta v = 0 \text{ in } D \text{ and } \nabla v \in L^1(\partial D). \tag{5.8}$$

One can actually easily verify the condition (5.1) when (5.5) and (5.6) hold.

Assumption (5.7) implies that

$$\begin{gathered} \{\tilde{\sigma} > 0\} = \bigcup_{j=1}^{M} I_j^+, \\ \{\tilde{\sigma} < 0\} = \bigcup_{j=1}^{M'} I_j^- \quad \text{where } I_j^+,\ I_k^- \text{ are disjoint arcs on } \partial D. \end{gathered} \tag{5.9}$$

By (5.3),

$$\nabla \tilde{u} = \begin{cases} \nabla u^e & \text{on } \bigcup_{j=1}^{M} I_j^+, \\ \nabla u^i & \text{on } \bigcup_{j=1}^{M'} I_j^-. \end{cases} \tag{5.10}$$

In view of Corollary 5.2, we may assume from now on that $M' = M$ and the union of the $\overline{I_j^+}$, $\overline{I_k^-}$ is all of ∂D.

Lemma 5.3. *If for any C^1 function h there exists a solution w to*

$$\begin{gathered} \Delta w = 0 \quad \textit{in } D, \\ \nabla \tilde{u} \cdot \nabla w = h \quad \textit{on } \partial D, \\ \nabla w \in L^1(\partial D), \end{gathered} \tag{5.11}$$

then the stability property (4.2) *holds.*

Indeed, using (5.8) we conclude that

$$\int_{\partial D} \tilde{\sigma} h = 0;$$

since h is arbitrary, $\tilde{\sigma} = 0$ which is a contradiction.

Remark 5.1. It is actually sufficient to solve (5.11) just for $h = \tilde{\sigma}$, but $\tilde{\sigma}$ may not be C^1.

In order to establish (5.11) we shall rely on the index theory for the Riemann-Hilbert problem as exposed in [8].

We recall (see [8, §40, (40.8)]) that for a continuous vector field $V = a + ib$ on $\overline{\Omega}$ and a smooth curve $\Gamma \subset \overline{\Omega}$ which is the boundary of a subdomain in Ω one defines the index of V with respect to Γ by

$$\kappa(V;\Gamma) = \frac{1}{\pi}[\arg(a - ib)]_\Gamma \tag{5.12}$$

provided $V \neq 0$ on Γ.

It is well known that the index is homotopic invariant, i.e., if $V(\theta)$ is a family of such vector fields continuous in θ, $0 \le \theta \le 1$, then

$$\kappa(V(0);\Gamma) = \kappa(V(1);\Gamma) \tag{5.13}$$

provided $V(\theta) \neq 0$ on Γ for all θ. The definition of the index of $V = a + ib$ with respect to Γ can be extended to the case where V may vanish or have finite number of jump-discontinuities at points $c_1, \dots, c_N$ on Γ. Setting $G = (a - ib)/(a + ib)$ one defines (see [8, §93, p. 273]),

$$\kappa(V;\Gamma) = \frac{1}{2\pi}[\arg G]_\Gamma \tag{5.14}$$

provided the limits $G(c_i \pm 0)$ exist, where the passage from $G(c_i + 0)$ to $G(c_i - 0)$ is selected as [8, §85]. If $V(\theta)$ varies continuously with θ and each $V(\theta)$ vanishes or has jump-discontinuities only at $c_1, \dots, c_N$ then (5.13) is still valid provided the limits $G(c_i \pm 0)$ exist for all $0 \le \theta \le 1$.

Consider the example of

$$V_0(\theta) = \left(1 - \frac{(k-1)\theta}{k}\right) u_N^e \vec{N} + u_\tau^e \vec{\tau} \qquad (0 \le \theta \le 1) \tag{5.15}$$

where $\vec{N}$ is the outward unit normal and $\vec{\tau}$ is unit tangent (in the counter-clockwise direction) along ∂D. From the diffraction conditions

$$u^e = u^i, \qquad \frac{\partial u^e}{\partial N} = k\frac{\partial u^i}{\partial N} \tag{5.16}$$

we see that, for any $z \in \partial D$,

$$\nabla u^e(z) \neq 0 \quad \text{if and only if} \quad \nabla u^i(z) \neq 0.$$

Since $V_0(\theta)$ is a homotopy from ∇u^e to ∇u^i, we conclude that

$$\kappa(\nabla u^e;\partial D) = \kappa(\nabla u^i;\partial D)) \quad \text{if } \nabla u^e \neq 0 \text{ on } \partial D.$$

Consider next the vector field

$$V_1(\theta) = \begin{cases} V_0(\theta) & \text{on } \partial D^+ \equiv \bigcup_j I_j^+, \\ \nabla u^i & \text{on } \partial D^- \equiv \bigcup_j I_j^-. \end{cases}$$

Clearly $V_1(0) = \nabla \tilde{u}$ and $V_1(1) = \nabla u^i$. Notice that $V_1(\theta)$ has a finite number of jump discontinuities along ∂D, i.e., at the endpoints of the I_j^-. As explained above the invariance formula (5.13) is still valid, so that

$$\kappa(\nabla u^i;\partial D) = \kappa(\nabla \tilde{u};\partial D) \quad \text{provided } \nabla u^i \neq 0 \text{ on } \partial D. \tag{5.18}$$

The Riemann-Hilbert problem in D is concerned with finding a holomorphic function ϕ in D, continuous in $\overline{D}$, such that

$$a \operatorname{Re} \phi + b \operatorname{Im} \phi = c \quad \text{on } \partial D;$$

here D is a C^1 domain and a, b, c are piecewise continuous with a finite number of discontinuities $z_1, \dots, z_N$, and their derivatives are bounded in each arc $z_j z_{j+1}$ $(z_{N+1} = z_1)$. By [8, §93], if $a^2 + b^2 > 0$ and the index of $V = a + ib$ with respect to ∂D is ≥ -1 then for any c there exists a solution ϕ and [8, (93.1)],

$$|\phi(z)| \leq \frac{C}{|z - z_j|^\alpha} \qquad (0 < \alpha < 1)$$

for z near z_j.

We note that the Riemann-Hilbert problem for holomorphic function $\phi(z)$ is equivalent to the problem

$$\begin{aligned} \Delta v &= 0 \quad \text{in } D, \\ a v_x + b v_y &= c \quad \text{on } \partial D, \end{aligned}$$

for $v = \operatorname{Re} \int \phi(z)\, dz$. We therefore conclude:

Lemma 5.4. *If $\kappa(\nabla \tilde{u}; \partial D) \geq -1$ then for any piecewise C^1 function h there exists a solution to* (5.11).

The fact that $\nabla w \in L^1(\partial D)$ follows from the estimate of $\phi(z)$ near z_j, where z_j are the points of discontinuity of V. Recalling (5.18) we have thus reduced the proof of the stability property (4.2) to showing that

$$\nabla u^i \neq 0 \quad \text{on } \partial D \tag{5.19}$$

and

$$\kappa(\nabla u^i; \partial D) \geq -1. \tag{5.20}$$

Since u^i is analytic in $\overline{D}$, it has analytic extension into a neighborhood N^+ of $\overline{D}$; we denote it by u^i and note that u^i is harmonic in N^+. Similarly u^e has analytic (and harmonic) extension into an Ω-neighborhood N^- of $\Omega \backslash D$.

We shall now make a special choice of g as follows:

Definition of g. Let $z = z(t)$ be a $C^{1,\alpha}$ parametrization of $\partial\Omega$ $(0 \leq t \leq 2\pi)$ and let $f(z(t))$ be a $C^{1,\alpha}$ function such that $f(z(t))$ has a unique maximum at $t = 0$, a unique minimum at some point $t = t_0$, and

$$\begin{aligned} \frac{d}{dt} f(z(t)) &< 0 \quad \text{if } 0 < t < t_0, \\ \frac{d}{dt} f(z(t)) &> 0 \quad \text{if } t_0 < t < 2\pi. \end{aligned}$$

Let u be the solution of the diffraction problem

$$\operatorname{div}(a \nabla u) = 0 \quad \text{in } \Omega, \tag{5.21}$$

$$u = f \quad \text{on } \partial\Omega, \tag{5.22}$$

and set

$$g = \frac{\partial u}{\partial \nu}. \tag{5.23}$$

We shall prove

Lemma 5.5. *For the special choice of* g *in* (5.23), (5.19) *holds and*

$$\kappa(\nabla u^i\,;\,\partial D) = 0\,. \tag{5.24}$$

Since (5.24) implies (5.20), we deduce

Theorem 5.6. *Under the assumptions* (5.5)–(5.7), *the stability property* (4.2) *holds.*

Proof of Lemma 5.5. From the transmission conditions (5.16) one can easily show that u cannot take minimum or maximum at points on ∂D. Therefore u attains its maximum at $z(0)$ and its minimum at $z(t_0)$ and, by the maximum principle, $\partial u/\partial \nu \neq 0$ at these two points. At all other points of $\partial\Omega$ we also have $\partial u/\partial\tau \neq 0$ (by the choice of f). Consequently $\nabla u^e \neq 0$ on $\partial\Omega$. Since further the tangential components of ∇u^e have the same sign on $(0\,,\,t_0)$ and (the reverse sign) on $(t_0\,,\,2\pi)$, it can be seen that

$$\kappa(\nabla u^e\,;\,\partial\Omega) = 0\,. \tag{5.25}$$

The vector field ∇u^i has a finite number of zeros $z_1\,,\,\dots\,,\,z_m$ in $\overline{D}$ and similarly (since $\nabla u^e \neq 0$ in an Ω-neighborhood of $\partial\Omega$) the vector field ∇u^e has a finite number of zeros in $z_{m+1}\,,\,\dots\,,\,z_\sigma$ in $\Omega\backslash D$. On ∂D, ∇u^i and ∇u^e have common zeros (if any); we denote them by $z_{l+1}\,,\,z_{l+2}\,,\,\dots\,,\,z_m$.

Let L_ε be the Jordan curve formed by the arcs of the $\partial B(z_j\,;\,\varepsilon)$ $(l+1 \leq j \leq m)$ which are contained in $\Omega\backslash D$ and by $\partial D\backslash\bigcup_{j=l+1}^m B(z_j\,;\,\varepsilon)$. We claim

Lemma 5.7. *If* ε *is sufficiently small then*

$$\nabla u^e|_{L_\varepsilon} \textit{ is homotopic to } \nabla u^i|_{L_\varepsilon}\,. \tag{5.26}$$

Proof. If ε is sufficiently small then $\nabla u \neq 0$ in a neighborhood of

$$\Gamma_\varepsilon \equiv \partial D\backslash\bigcup_{l+1}^m B(z_j\,;\,\varepsilon)\,.$$

Let $\nabla u^e = u_N^e\vec{N} + u_\tau^e\vec{\tau}$. Then u_N^e and u_z^e do not vanish simultaneously on Γ_ε. We define, for $0 \leq \theta \leq 1$,

$$V(\theta) = \left(1 - \frac{k-1}{k}\theta\right)u_N^e\vec{N} + u_\tau^e\vec{\tau} \quad \text{on } \Gamma_\varepsilon\,. \tag{5.27}$$

Then $V(0) = \nabla u^e$, $V(1) = \nabla u^i$ and $V(\theta)$ is continuous in θ; moreover, $V(\theta) \neq 0$ on $\overline{\Gamma}_\varepsilon$.

We next wish to define $V(\theta)$ on any arc $\partial B(z_j\,;\,\varepsilon)\backslash\overline{D}$ of L_ε. To do this we introduce a conformal mapping of the lower half-plane onto D which maps 0 into z_j. By analytic continuation, the mapping is conformal in a neighborhood of 0. Since the refraction conditions (5.16) are invariant under conformal mapping, we may assume from the start that $z = 0$ and that D, near $z = 0$, coincides with the lower half-plane. Expanding u^e, u^i near $z = 0$ into series

$$u^e = \sum r^n(a_n^e\cos n\varphi + b_n^e\sin n\varphi)\,,$$
$$u^i = \sum r^n(a_n^i\cos n\varphi + b_n^i\sin n\varphi)$$

and using the refraction conditions, we obtain

$$a_n^e = a_n^i\,, \qquad b_n^e = kb_n^i\,. \tag{5.28}$$

We now define (in the variables (r, φ) of the conformal mapping) $V(\theta) = \nabla u(\theta)$ where

$$u(\theta) = \sum r^n \left(a_n^e \cos n\varphi + \left(1 - \frac{k-1}{k}\theta\right) b_n^e \sin n\varphi\right).$$

Then, by (5.28), $u(0) = u^e$, $u(1) = u^i$ so that

$$V(0) = \nabla u^e, \qquad V(1) = \nabla u^i.$$

Further,

$$\nabla V(\theta) \neq 0 \quad \text{on } \partial B(z_j; \varepsilon)$$

and $V(\theta)$ continuously fits with (5.27) at the two points of $\partial\Gamma_\varepsilon \cap \partial B(z_j; \varepsilon)$.

We have thus constructed a homotopy $V(\theta)$ of ∇u^i along L_ε; this establishes the assertion (5.26).

Completion of the proof of Lemma 5.5. Consider the index

$$\kappa(\zeta) \equiv \kappa(\nabla u; \partial B(\zeta; \varepsilon)) \quad (\varepsilon \text{ small})$$

of ∇u at a zero $k = \zeta$ of ∇u, where $\zeta \in \Omega \backslash \partial D$. Introducing $h = u + iv$ (h holomorphic), we have

$$h'(z) = u_x + iv_x = u_x - iu_y,$$

and

$$\begin{aligned} \pi\kappa(\zeta) &= \pi\kappa(u_x - iu_y; \partial B(\zeta; \varepsilon)) = \mathrm{Var}_{\partial B(\zeta;\varepsilon)} h'(z) \\ &= \mathrm{Var}_{\partial B(\zeta;\varepsilon)} (z-\zeta)^{n-1} = 2\pi(n-1) \end{aligned}$$

where $h(z) = a_0(z-\zeta)^n + \cdots$, $a_0 \neq 0$. Hence

(5.29) $$\kappa(\zeta) = 2 \times \{\text{order of zero of } \nabla u \text{ at } z = \zeta\}.$$

Denote by π_j the order of the zero of ∇u at $z = z_j$. Then, for small $\varepsilon > 0$,

(5.30) $$\kappa(\nabla u^e; \partial\Omega) = \kappa(\nabla u^e; L_\varepsilon) + \sum_{j=m+1}^{\sigma} 2\pi_j.$$

From Lemma 5.7 we also have

(5.31) $$\kappa(\nabla u^e; L_\varepsilon) = \kappa(\nabla u^i; L_\varepsilon).$$

Finally, by (5.29),

(5.32) $$\kappa(\nabla u^i; L_\varepsilon) = \sum_{j=1}^{m} 2\pi_j.$$

Combining (5.30)–(5.32) and recalling (5.25) we deduce that

$$\sum_{j=1}^{\sigma} \pi_j = 0.$$

Hence ∇u has no zeros in $\overline{D}$ and (5.24) holds. This completes the proof of Lemma 5.5 and therefore also of Theorem 5.6.

6. The nonmonotone case with piecewise analytic ∂D

In this section we continue to consider the nonmonotone case for $n = 2$, but assume that ∂D is piecewise analytic with a finite number of vertices

$s_1, \ldots, s_N$. For simplicity we take D and D_h to be polygons, as in §2, with D convex, and

$$c_1 h \le \sum_{j=1}^{N} |s_j(h) - s_j| \le c_2 h \qquad (0 < c_1 < c_2 < \infty);$$

however our results easily extend to any piecewise analytic ∂D.

Our starting point is Lemma 5.1; as in the proof of Theorem 5.6 (recall Remark 5.1) the stability property (4.2) holds if there exists a solution v to

$$\Delta v = 0 \quad \text{in } D, \tag{6.1}$$

$$\nabla \tilde{u} \cdot \nabla v = \tilde{\sigma} \quad \text{on } \partial D, \tag{6.2}$$

such that $\nabla v_\lambda \to \nabla v$ in $L^1(\partial D)$ as $\lambda \uparrow 1$; $v_\lambda(x) = v(x_0 + \lambda(x - x_0))$ for some $x_0 \in D$. The function $\tilde{\sigma}$ is linear on each edge Γ_j.

By Lemma 2.1 it follows that any $\nabla \tilde{u}$ has a finite variation along any arc of ∂D which contains a vertex s_j of ∂D_i; consequently

$$\kappa(\nabla \tilde{u}; \partial D) < \infty. \tag{6.3}$$

In the original theory of Muskhelishvili [8, §§93, 94] the domain D is assumed to be smooth. However, the results remain true if ∂D is piecewise C^1; see [3, Example 8.4]. One can see it by using conformal mapping $z = z(\omega)$ of the unit disc $\{|\omega| < 1\}$ onto D, and applying the original theory in the ω-domain noting that the index of $z'(\omega)$ is zero since $\omega \to z(\omega)$ is conformal (i.e. $z'(\omega)$ does not vanish in the unit disc).

If $\kappa \ge -1$ then (by [8, §93]) there exists a unique solution of (6.1), (6.2) satisfying

$$|\nabla v| \le \frac{C}{|x - c_j|^\alpha}. \tag{6.4}$$

On the other hand, if $\kappa \le -2$ then there are $-\kappa - 1$ solutions of the homogeneous problem, and the solution v of (6.1), (6.2), (6.4) exists if and only if $\tilde{\sigma}$ is orthogonal to these solutions. The orthogonality relations can be written in the form

$$\int_{\partial D} \tilde{\sigma} l_m = 0, \qquad m = 1, \ldots, -\kappa - 1. \tag{6.5}$$

We summarize

Theorem 6.1. (i) *If* $\kappa \ge -1$ *then the stability property* (4.2) *holds*;
(ii) *if* $\kappa \le -2$ *and*

$$\tilde{\sigma} \text{ is not a linear combination of } l_1, \ldots, l_{-\kappa-1} \tag{6.6}$$

then the stability property (4.2) *holds.*

This result for analytic ∂D was proved in [1].

We now wish to estimate the index κ for the special choice of g made in §5.

From the results of [9, p. 201] it follows that the solution u of the diffraction problem (3.2) cannot take local maximum (or local minimum) at a vertex of ∂D. So as in the proof of (5.25):

Lemma 6.2. *For the special choice of* g *in* (5.21)–(5.23),

$$\kappa(\nabla u^e ; \partial\Omega) = 0 . \tag{6.7}$$

From now on we shall work with the special choice of g in (5.21)–(5.23).

Lemma 6.3. $\nabla u(x) \neq 0$ *for* $x \neq s_1, \ldots, s_N$.

Proof. Let ξ_j, η_j be points on ∂D, $\xi_j \in \Gamma_j$ and $\eta_j \in \Gamma_{j+1}$, such that $|\xi_j - s_j|$ and $|\eta_j - s_j|$ are small and $\nabla u^e(\xi_j) \neq 0$ $\nabla u^e(\eta_j) \neq 0$. Let $\widetilde{\Gamma}_\varepsilon$ be C^2 and piecewise analytic curves in $\overline{D}$ which converge to ∂D as $\varepsilon \to 0$ such that $\widetilde{\Gamma}_\varepsilon$ connects ξ_j to η_j by an analytic arc and it is a line-segment between η_j and ξ_{j+1} (this segment lies on ∂D). Denote by $\widetilde{D}_\varepsilon$ the domain bounded by $\widetilde{\Gamma}_\varepsilon$, and let $\tilde{u}_\varepsilon$ denote the solution of the refraction problem corresponding to $\widetilde{\Gamma}_\varepsilon$. Then

(6.8) $\nabla \tilde{u}^i_\varepsilon \to \nabla u^i$ and $\nabla \tilde{u}^e_\varepsilon \to \nabla u^e$ uniformly outside any neighborhood of $\{s_1, \ldots, s_N\}$.

Observe that $\nabla \tilde{u}_\varepsilon \neq 0$ at ξ_j, η_j for all ε, and that $\nabla \tilde{u}^e_\varepsilon$, $\nabla \tilde{u}^i_\varepsilon$ are analytic across $\widetilde{\Gamma}_\varepsilon \backslash \{\xi_1, \eta_1, \ldots, \xi_N, \eta_N\}$. Therefore $\nabla \tilde{u}^e_\varepsilon$ and $\nabla \tilde{u}^i_\varepsilon$ have only a finite number of zeros. Since also $\kappa(\nabla \tilde{u}_\varepsilon ; \partial\Omega) = 0$, we can repeat an argument used in §5 and deduce that

$$\begin{aligned} &\nabla \tilde{u}^e_\varepsilon(x) \neq 0 \quad \text{in } \overline{\Omega} \backslash \widetilde{\Gamma}_\varepsilon , \\ &\nabla \tilde{u}^i_\varepsilon(x) \neq 0 \quad \text{in } \widetilde{D}_\varepsilon \cup \widetilde{\Gamma}_\varepsilon . \end{aligned} \tag{6.9}$$

We now suppose that $\nabla u^e(x_0) = 0$ for some $x_0 \in \Omega \backslash D$, $x_0 \neq$ vertex. If $x_0 \in \partial D$ then we choose the ξ_j, η_j above so that x_0 lies on one of the line segments of $\widetilde{\Gamma}_\varepsilon$. Then (whether $x_0 \in \partial D$ or $x_0 \notin \partial D$) there exists a small disc $B_\delta(x_0)$ such that u^e and $\tilde{u}^e_\varepsilon$ are analytic in $\overline{B_\delta(x_0)}$ and $\nabla u^e \neq 0$ on $\partial B_\delta(x_0)$. From (6.8), (6.9) we then deduce that

$$0 = \kappa(\nabla \tilde{u}^e_\varepsilon ; x_0) = \kappa(\nabla \tilde{u}^e_\varepsilon ; \partial B_\delta(x_0)) = \kappa(\nabla u^e ; \partial B_\delta(x_0)) = \kappa(\nabla u^e ; x_0)$$

if ε is small enough, which is a contradiction.

Similarly one can prove that $\nabla u^i(x) \neq 0$ if $x \in \overline{D}$, $x \neq$ vertex.

Denote by $\widetilde{\Gamma}^e_\varepsilon$ smooth curves in $\Omega \backslash D$ such that $\widetilde{\Gamma}^e_\varepsilon$ coincides with ∂D outside ε-neighborhood of each s_j, and such that $\widetilde{\Gamma}^e_\varepsilon$ connects a point in Γ_j to a point in Γ_{j+1} by an arc $\sigma^e_{j,\varepsilon}$ which "approximately" lies on $|z - s_j| = \varepsilon$; by "approximately" we mean that we make $\widetilde{\Gamma}^e_\varepsilon$ smooth as it intersects ∂D, by slightly modifying the arc $|z - s_j| = \varepsilon$. Similarly we define curves $\widetilde{\Gamma}^i_\varepsilon$ and the approximate arcs $\sigma^i_{j,\varepsilon}$ in D.

By Lemma 2.1, near s_j,

$$\begin{aligned} \overline{\nabla u^e} &= A^e_j (z - s_j)^{\gamma_j - 1}(1 + o(1)) , \\ \overline{\nabla u^i} &= A^i_j (z - s_j)^{\gamma_j - 1}(1 + o(1)) , \end{aligned} \tag{6.10}$$

where $A^e_j \neq 0$, $A^i_j \neq 0$ and γ_j is a positive number larger than 1/2. Indeed, take for simplicity $j = 1$, $s_j = 0$. By (2.21),

$$u = \operatorname{Re} \sum_{m=1}^{k-1} B_m z^{\gamma_m} + \sum_{m=k}^{\infty} C_m r^{\gamma_m} v_m(\theta) , \qquad B_1 \neq 0; \tag{6.11}$$

k is chosen so that $k > 1$ and $\gamma_k > 3 + \gamma_1$. By (2.13) and (2.11)

$$\int_0^{2\pi} (v_m'(\theta))^2 \le C\gamma_m^2,$$

so that

$$|v_m(\theta)| \le C\gamma_m. \tag{6.12}$$

Therefore, by (2.11),

$$|v_m''(\theta)| \le C\gamma_m^3 \quad \text{if } \theta \ne 0,\ \theta \ne \beta$$

and then also

$$|v_m'(\theta)| \le C\gamma_m^3. \tag{6.13}$$

Using (6.12), (6.13) and (2.22), (2.7), we deduce from (6.11) that

$$\overline{\nabla u} = \operatorname{Re} \sum_{m=1}^{k-1} \gamma_m B_m z^{\gamma_m - 1} + O(r^{\gamma_k - 3}),$$

which implies (6.10).

From Lemma 6.4 and Corollary 6.3 we have

$$\kappa(\nabla u^e; \widetilde{\Gamma}_\varepsilon^e) = 0, \tag{6.14}$$

$$\kappa(\nabla u^i; \widetilde{\Gamma}_\varepsilon^i) = 0. \tag{6.15}$$

Take any vertex s_j and introduce polar coordinates (r, θ) about s_j. We want to evaluate the index of ∇u^i with respect to the boundary of a small domain $\Omega_j \subset D$ such that $\partial\Omega_j$ consists of two line segments

$$l_1 = \{(r, 0), 0 \le r \le \delta_0\}, \qquad l_2 = \{(r, \beta_j), 0 \le r \le \delta_0\}$$

and a circular arc

$$l_3 = \{(\delta_0, \theta), 0 \le \theta \le \beta_j\}.$$

Set $\Sigma_\varepsilon = l_2 \cup l_1 \cup l_3$. By the definition of index in (5.14) and the expansion (6.10) for ∇u^i,

$$\begin{aligned} \kappa(\nabla u^i; \Sigma_\varepsilon) &= \frac{1}{2\pi} \arg \left[\frac{\overline{\nabla u^i}}{\nabla u^i} \right]_{l_3} + \frac{1}{2\pi} \arg \left[\frac{\overline{\nabla u^i}}{\nabla u^i} \right]_{l_2 \cup l_1} \\ &= \frac{(\gamma_j - 1)\beta_j}{\pi} + \frac{1}{\pi} \{\text{jump of } \arg(z - s_j)^{\gamma_j - 1} \\ &\qquad\qquad \text{from } \theta = \beta_j \text{ to } \theta = 0 \text{ at } r = 0\} \\ &= \frac{(\gamma_j - 1)\beta_j}{\pi} - \left\{ \text{nonintegral part of } \frac{(\gamma_j - 1)\beta_j}{\pi} \right\} \\ &= \left[\frac{(\gamma_j - 1)\beta_j}{\pi} \right] \end{aligned} \tag{6.16}$$

where $[x]$ is the integral part of x if $x > 0$ and $[x] = 0$ if $x < 0$ (if $\gamma_j - 1 < 0$ then, since $\gamma_j > \frac{1}{2}$, the right-hand side of (6.16) is nonpositive and smaller than 1 in absolute value; since the index is an integer, it must then be equal to zero).

Repeating the above argument at each vertex s_j and recalling (6.15) we deduce

Lemma 6.5. $\kappa(\nabla u^i; \partial D) \geq 0$.

To compute $\kappa(\nabla\tilde{u}; \partial D)$ we deform ∇u^i into $\nabla\tilde{u}$. Consider first the case where $\nabla\tilde{u}$ is obtained by replacing ∇u^i by ∇u^e on a single closed arc σ. If σ lies inside one edge Γ_j then

$$\kappa(\nabla u^i; \partial D) - \kappa(\nabla\tilde{u}; \partial D) = \frac{\lambda_1}{\pi} + \frac{\lambda_2}{\pi}$$

where λ_k is the difference in the arguments of ∇u^i and ∇u^e at an endpoint of σ. Since the index is an integer whereas $|\lambda_k| < \pi/2$, it follows that

$$\kappa(\nabla u^i; \partial D) - \kappa(\nabla\tilde{u}; \partial D) = 0. \tag{6.17}$$

Suppose next that σ contains a vertex s_j and its endpoints lie inside Γ_j and Γ_{j+1}. Then, by (5.14),

$$\kappa(\nabla u^i; \partial D) - \kappa(\nabla\tilde{u}; \partial D) = \frac{\lambda_1}{\pi} + \frac{\lambda_2}{\pi} + \frac{\Delta}{2\pi} \tag{6.18}$$

where Δ is the difference between

$$B^i e^{2\sqrt{-1}(\gamma_j-1)\beta_j} - B^i e^{2\sqrt{-1}(\gamma_j-1)0} \qquad \left(B^i = \frac{\alpha_i - \sqrt{-1}\beta_i}{\alpha_i + \sqrt{-1}\beta_i},\ \alpha_i + \sqrt{-1}\beta_i = A_j^i\right)$$

and the corresponding expression with B^e (associated with A_j^e). Since the index is an integer and $|(\lambda_1 + \lambda_2)/\pi| < 1$, there is no ambiguity about the choice of the corresponding limits $G(s_j \pm 0)$ if $|A^i - A^e|$ is small; in fact, the correct choice is such that the limits of

$$\frac{1}{B^i}\frac{\overline{\nabla u^i}}{\nabla u^i} \text{ and } \frac{1}{B^e}\frac{\overline{\nabla u^e}}{\nabla u^e} \text{ agree at } s_j \pm 0$$

so that the left-hand side of (6.18) is equal to zero. By continuously deforming A^e we deduce that (6.17) holds in the general case.

Finally, a similar argument shows that (6.17) holds if σ lies in Γ_j and one of its endpoints is a vertex of ∂D.

By deforming ∇u^i step-by-step a finite number of times so as to obtain $\nabla\tilde{u}$, and applying (6.17) at each step, we deduce that (6.17) is valid for general $\nabla\tilde{u}$. Consequently, by Lemma 6.5,

$$\kappa(\nabla\tilde{u}; \partial D) \geq 0. \tag{6.19}$$

This together with Theorem 6.1 implies

Theorem 6.6. *For the special choice of* g *made in* (5.21)–(5.23), *the stability property* (4.2) *holds.*

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