ISOMETRIES OF CSL ALGEBRAS

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ABSTRACT. We show that every Jordan isomorphism of CSL algebras, whose restriction to the diagonal of the algebra is a selfadjoint map, is the sum of an isomorphism and an anti-isomorphism.

It follows that every surjective linear isometry of CSL algebras is the sum of an isomorphism and an anti-isomorphism, followed by a unitary multiplication.

1. INTRODUCTION

In [K], R. Kadison proved that every linear isometry of one C^* -algebra onto another is given by a Jordan $*$ -isomorphism followed by a unitary multiplication. (Here a linear map φ from a C^* -algebra $\mathcal{B}_1$ into a C^* -algebra $\mathcal{B}_2$ is called a *Jordan $*$ -isomorphism* if it is one-to-one, surjective, $\varphi(x^*) = \varphi(x)^*$ and $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ for all $x, y \in \mathcal{B}_1$.) Kadison also proved that a Jordan $*$ -isomorphism from a von Neumann algebra $\mathcal{B}_1$ onto a von Neumann algebra $\mathcal{B}_2$ can be decomposed into the sum of a $*$ -isomorphism and of a $*$ -anti-isomorphism by a central projection.

For nonselfadjoint algebras the following general result was proved in [AS].

Theorem 1.1. *Let $\mathcal{U} \subseteq B(H)$ and $\mathcal{B} \subseteq B(K)$ be unital norm closed subalgebras, and let $\varphi: \mathcal{U} \rightarrow \mathcal{B}$ be a surjective linear isometry. Then*

- (1) $\varphi(\mathcal{U} \cap \mathcal{U}^*) = \mathcal{B} \cap \mathcal{B}^*$.
 - (2) $\varphi(xy^*z + zy^*x) = \varphi(x)\varphi(y)^*\varphi(z) + \varphi(z)\varphi(y)^*\varphi(x)$ for every x, z in $\mathcal{U}$ and y in $\mathcal{U} \cap \mathcal{U}^*$.
 - (3) $U = \varphi(I)$ is a unitary operator in $\mathcal{B} \cap \mathcal{B}^*$.
- If, moreover, $\varphi(I) = I$, then*
- (4) $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$, $x, y \in \mathcal{U}$.
 - (5) $\varphi(x^*) = \varphi(x)^*$ for $x \in \mathcal{U} \cap \mathcal{U}^*$.

Note that (2) shows that φ preserves the partial triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$, $y \in \mathcal{U} \cap \mathcal{U}^*$, $x, z \in \mathcal{U}$. We shall therefore refer to a surjective, one-to-one linear map that satisfies (1)–(3) as a *partial triple $*$ -isomorphism*.

Also, a surjective one-to-one linear map $\varphi: \mathcal{U} \rightarrow \mathcal{B}$ will be called *Jordan partial $*$ -isomorphism* if it maps I to I and both it and its inverse satisfy

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properties (4) and (5) above. It follows from Theorem 1.1 that every isometry between unital normed closed operator algebras is given by a Jordan partial *-isomorphism followed by a unitary multiplication (thus extending Kadison's result to nonselfadjoint algebras).

The main result of the present paper (Theorem 2.15) is that, when $\mathcal{U}$ and $\mathcal{B}$ are reflexive operator algebras with commutative subspace lattices (called CSL algebras), then every partial Jordan *-isomorphism can be decomposed into the sum of an isomorphism and an anti-isomorphism by a projection in the center of $\mathcal{U}$.

In particular, every isometry from CSL algebra onto another is such a sum followed by a unitary multiplication. For completely distributive CSL algebras a more concrete result was proved by R. Moore and T. Trent in [MT2].

In order to state their result we shall first set some notation and terminology. A Hilbert space H would be assumed to be separable, an operator on H would be assumed to be bounded and a projection is assumed to be orthogonal. A lattice $\mathcal{L}$ of projections is a strongly closed collection of projections that is closed under the usual lattice operations $\vee$ and $\wedge$ and contains 0 and I . In this paper we will deal only with commutative lattices (in which the projections commute pairwise). Such a lattice is called a *CSL*. A *nest* is a linearly ordered lattice. If $\mathcal{L}$ is a lattice we write $\text{Alg } \mathcal{L}$ for the collection of operators in $B(H)$ which leave invariant the ranges of all of the projections in $\mathcal{L}$, i.e.

$$\text{Alg } \mathcal{L} = \{T \in B(H) : (I - N)TN = 0, N \in \mathcal{L}\}.$$

$\text{Alg } \mathcal{L}$ is a weakly closed subalgebra of $B(H)$, containing I . If $\mathcal{A}$ is a subalgebra of $B(H)$ we write

$\mathcal{L}at_{\mathcal{A}} = \{N : N \text{ is a projection, } (I - N)TN = 0 \text{ for all } T \in \mathcal{A}\}$. If $\mathcal{L}$ is commutative, then $\mathcal{L}at(\text{Alg } \mathcal{L}) = \mathcal{L}$ [A, Theorem 1.6.3]. For a projection E we write $E^\perp = I - E$ and for a lattice $\mathcal{L}$ we write $\mathcal{L}^\perp = \{N^\perp : N \in \mathcal{L}\}$. Note also that $\text{Alg } (\mathcal{L}^\perp) = (\text{Alg } \mathcal{L})^*$ and $(\text{Alg } \mathcal{L})^* \cap \text{Alg } \mathcal{L} = \mathcal{L}'$.

A CSL $\mathcal{L}$ is said to be *completely distributive* if it satisfies a certain lattice-theoretic condition (see [D, Chapter 23]). An alternative characterization of completely distributive CSL was proved by Laurie and Longstaff [LL]. They showed that $\mathcal{L}$ is completely distributive if and only if the linear span of the rank one operators in $\text{Alg } \mathcal{L}$ is σ -weakly dense in $\text{Alg } \mathcal{L}$.

The main result of [MT2] is the following.

Theorem 1.2 [MT2, Theorem 2.1]. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be completely distributive CSL's on a Hilbert space H . Let $\theta : \text{Alg } \mathcal{L}_1 \rightarrow \text{Alg } \mathcal{L}_2$ be a linear surjective isometry. Let $U = \theta(I)$ and write $\varphi(T) = U^*\theta(T)$. Then there exist projections $E_1 \in \mathcal{L}_1 \cap \mathcal{L}_1^\perp$, $E_2 \in \mathcal{L}_2 \cap \mathcal{L}_2^\perp$ and an involution J such that*

(1) *φ restricted to $(\text{Alg } \mathcal{L}_1)E_1$, is implemented by a unitary operator V (i.e. $\varphi(T) = V^*TV$) such that $N \mapsto V^*NV$ is an order isomorphism of $\mathcal{L}_1E_1$ onto $\mathcal{L}_2E_2$.*

(2) *For φ , restricted to $(\text{Alg } \mathcal{L}_1)E_1^\perp$, there is a unitary operator W such that $\varphi(T) = W^*JTJW$ and the map $N \mapsto W^*JNJW$ is an order isomorphism from $\mathcal{L}_1E_1^\perp$ onto $\mathcal{L}_2^\perp E_2^\perp$.*

The proof of this result in [MT2] uses heavily the fact that there are many rank one operators in $\text{Alg } \mathcal{L}$. In this paper we deal with general CSL lattices and, in the general case, $\text{Alg } \mathcal{L}$ might contain no rank one operators. Therefore

the methods are completely different. As we remark at the end of the paper, our main result (Theorem 2.15) can be combined with [DP, Theorem 2.1] to yield an alternative proof for Theorem 1.2.

For the special case where $\mathcal{L}_1$ and $\mathcal{L}_2$ are nests Theorem 1.2 was proved, independently (and using different methods), in [AS and MT1]. In this case either $E_1 = 0$ or $E_1 = I$ (as $\mathcal{L} \cap \mathcal{L}^\perp = \{0, I\}$).

2. JORDAN PARTIAL *-ISOMORPHISMS

We now fix two CSL's $\mathcal{L}$ and $\mathcal{L}_1$ and $\varphi: \text{Alg } \mathcal{L} \rightarrow \text{Alg } \mathcal{L}_1$, a *Jordan partial *-isomorphism*; i.e. φ is linear, one-to-one, surjective, $\varphi(I) = I$, $\varphi(x^*) = \varphi(x)^*$ for $x \in \text{Alg } \mathcal{L} \cap (\text{Alg } \mathcal{L})^*$, $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ for $x, y \in \text{Alg } \mathcal{L}$ and its inverse also satisfies these properties. We have the following:

- (I) $\varphi(\text{Alg } \mathcal{L} \cap (\text{Alg } \mathcal{L})^*) = \text{Alg } \mathcal{L}_1 \cap (\text{Alg } \mathcal{L}_1)^*$; i.e. $\varphi(\mathcal{L}') = \mathcal{L}'_1$.
- (II) As φ preserves commutativity, $\varphi(\mathcal{L}'') = \mathcal{L}''_1$. Thus φ , restricted to $\mathcal{L}''$ (which is an abelian von Neumann algebra), is a *-isomorphism. In particular, for every collection of projections $\{E_\alpha\} \subseteq \mathcal{L}''$, $\varphi(\bigvee E_\alpha) = \bigvee \varphi(E_\alpha)$ and $\varphi(\bigwedge E_\alpha) = \bigwedge \varphi(E_\alpha)$.
- (III) For all $R, S, T \in \text{Alg } \mathcal{L}$ we have $\varphi(RST + TSR) = \varphi(R)\varphi(S)\varphi(T) + \varphi(T)\varphi(S)\varphi(R)$ [AS, Corollary 2.11].

We write $\mathcal{A}$ for $\text{Alg } \mathcal{L}$ and for $\mathcal{A}_1$ for $\text{Alg } \mathcal{L}_1$. For a subset $S \subseteq H$ we write $[S]$ for the closed linear subspace spanned by S .

Lemma 2.1. *The following are equivalent for a projection $E \in \mathcal{L}''$.*

- (1) $E = E_1 - E_2$ for some $E_i \in \mathcal{L}$, $i = 1, 2$.
- (2) For every T, S in $\text{alg } \mathcal{L}$, $ETESE = ETSE$.
- (3) For every T, S in $\text{alg } \mathcal{L}$,

$$ETESE + ESETE = ETSE + ESTE.$$

Proof. (1) $\Rightarrow$ (2). For every T, S in $\text{alg } \mathcal{L}$ and $E = E_1 - E_2$, $E_1, E_2 \in \mathcal{L}$ we have

$$ET = E_1(1 - E_2)T = E_1(1 - E_2)T(1 - E_2)$$

and $SE = SE_1(1 - E_2) = E_1SE$; hence $ETSE = ETESE$. (2) $\Rightarrow$ (3) is obvious.

(2) $\Rightarrow$ (1). Assume (2) holds. Write E_1 for the orthogonal projection onto $[\mathcal{A}E(H)]$. Then $E_1 \in \mathcal{L}$. It is left to show that $E_1 - E \in \mathcal{L}$. Fix $T \in \mathcal{A}$ and x in $(E_1 - E)(H)$. Suppose $x = SEy$ for some $S \in \mathcal{A}$ (and $EX = 0$). Clearly $Tx = TSEy \in E_1(H)$ and also $ETx = ETSEy = ETESEy = ETEx = 0$. Hence $Tx \in (E_1 - E)(H)$. As vectors of the form SEy are dense in $(E_1 - E)(H)$ we see that $Tx \in (E_1 - E)(H)$ for all $x \in (E_1 - E)(H)$.

(3) $\Rightarrow$ (2). Let $\{E_i\}_{i=1}^\infty$ be a countable subset of $\mathcal{L}$ that is strongly dense in $\mathcal{L}$ and such that $E_1 = 0$, $E_2 = I$. As in [A, Proof Theorem 2.2.3] we fix $n \geq 1$ and for each n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i = +1$ or -1 we define $E^\alpha = E_1^{\alpha_1} E_1^{\alpha_2} \dots E_n^{\alpha_n}$ where $E_i^1 = E_i$ and $E_i^{-1} = I - E_i$.

As α runs over all n -tuples E^α runs over the atoms of the Boolean algebra generated by $\{E_1, \dots, E_n\}$. Also, if $\alpha \neq \beta$, $E^\alpha E^\beta = 0$. We assume here that $ETESE + ESETE = ETSE + ESTE$ for all $T, S \in \mathcal{A}$. Apply this to $E^\alpha T$ and SE^β for some $\alpha \neq \beta$ to get $EE^\alpha TSE^\beta E = EE^\alpha TSESE^\beta E$ (as $SE^\beta E^\alpha T = SE^\beta EE^\alpha T = 0$). Write $K = ETSE - ETESE$. Then $E^\alpha KE^\beta = 0$

for all $\alpha \neq \beta$. As $\sum E^\alpha = I$ we get

$$K = \sum_{\alpha} E^\alpha K E^\alpha \quad \text{and} \quad K \in \{E_1, \dots, E_n\}'.$$

We have, for $1 \leq i \leq n$,

$$KE_i = ET(1 - E)SEE_i = ET(1 - E)E_iSE_iE$$

and

$$\begin{aligned} K(I - E_i) &= (I - E_i)K(I - E_i) = (1 - E_i)ET(I - E)SE(I - E_i) \\ &= ET(I - E)(I - E_i)S(I - E_i)E. \end{aligned}$$

Using this repeatedly we get

$$KE^\alpha = ET(1 - E)E^\alpha SE^\alpha E$$

for all α . Hence $K = \sum KE^\alpha = ET(1 - E)(\sum_{\alpha} E^\alpha SE^\alpha)E$. Write $S_n = \sum_{\alpha} E^\alpha SE^\alpha$. Then $S_n \in \{E_1, \dots, E_n\}'$ and $\|S_n\| \leq \|S\|$. Hence there is a weakly convergent subsequence $S_{n_k} \rightarrow S_0$. Clearly $S_0 \in \mathcal{L}'$ and thus $K = ET(1 - E)S_0E$. As $S_0 \in \mathcal{L}'$, $S_0E = ES_0$ and $K = 0$. This proves (2). $\square$

Corollary 2.2. *If $E = E_1 - E_2$, $E_i \in \mathcal{L}$, then there are projections Q_1, Q_2 , in $\mathcal{L}_1$, such that $\varphi(E) = Q_1 - Q_2$.*

Proof. With E as above we have

$$ETESE + ESETE = ETSE + ESTE$$

for every $T, S \in \mathcal{A}$. Since φ preserves Jordan products and $\varphi(ERE) = \varphi(E)\varphi(R)\varphi(E)$ for every $R \in \mathcal{A}$, we have

$$\begin{aligned} \varphi(ETESE + ESETE) &= \varphi(ETE)\varphi(ESE) + \varphi(ESE)\varphi(ETE) \\ &= \varphi(E)\varphi(T)\varphi(E)\varphi(S)\varphi(E) + \varphi(E)\varphi(S)\varphi(E)\varphi(T)\varphi(E). \end{aligned}$$

But this is equal to

$$\begin{aligned} \varphi(ETSE + ESTE) &= \varphi(E)\varphi(TS + ST)\varphi(E) \\ &= \varphi(E)\varphi(S)\varphi(T)\varphi(E) + \varphi(E)\varphi(T)\varphi(S)\varphi(E). \end{aligned}$$

As φ is surjective we can use Lemma 2.1 to complete the proof. $\square$

A projection E in $\mathcal{L}''$ that can be written as $E_1 - E_2$ for some (not necessarily unique) E_1 and E_2 in $\mathcal{L}$ is called an *interval*.

Now fix $N \in \mathcal{L}$. Write $E = \varphi(N)$. By Corollary 2.2, $E = E_1 - E_2$ for some $E_1, E_2 \in \mathcal{L}_1$. (Note that the choice of E_1 and E_2 is not unique. We choose one possible pair.) Applying Corollary 2.2 to φ^{-1} we can find intervals Q and P such that $\varphi(Q) = I - E_1$ and $\varphi(P) = E_2$. As $E_2 + (I - E_1) = I - E$, $P + Q = I - N$. Also, as $E_2(I - E_1) = 0$, $PQ = 0$.

Lemma 2.3. *With N fixed in $\mathcal{L}$ and E, E_1, E_2, P, Q as above we have*

$$\varphi(NTP) = E_2\varphi(T)E \quad \text{and} \quad \varphi(NTQ) = E\varphi(T)(I - E_1)$$

for all $T \in \mathcal{A}$.

Proof. For $T \in \mathcal{A}$ we have $QTN = 0$ (as $N \in \mathcal{L}$ and $Q \leq I - N$) and $(I - E_1)\varphi(T)E = (I - E_1)\varphi(T)E_1E = 0$ (as $E_1 \in \mathcal{L}_1$). Using property (III) of φ (see the beginning of this section) we now have,

$$\begin{aligned} \varphi(NTQ) &= \varphi(NTQ + QTN) = E\varphi(T)(I - E_1) + (I - E_1)\varphi(T)E \\ &= E\varphi(T)(I - E_1). \end{aligned}$$

The proof for NTP is similar and is omitted. $\square$

Lemma 2.4. Write F_1, F_2 for the projection onto $[N\mathcal{A}P(H)]$ and $[N\mathcal{A}Q(H)]$ respectively. Then $\varphi(F_1)$ and $\varphi(F_2)$ are the projections onto $[E\mathcal{A}_1^*E_2(H)]$ and $[E\mathcal{A}_1(I - E_1)(H)]$ respectively.

For every $T \in \mathcal{A}$ we have $F_1NTP = NTP$. Hence $NTP = F_1NTP + NTPF_1$ (as $F_1 \leq N \leq I - P$) and, thus, $E_2\varphi(T)E = \varphi(NTP) = \varphi(F_1)E_2\varphi(T)E + E_2\varphi(T)E\varphi(F_1)$. But $\varphi(F_1)E_2 = \varphi(F_1P) = 0$; hence for every $T \in \mathcal{A}$,

$$E_2\varphi(T)E = E_2\varphi(T)E\varphi(F_1)$$

and for every $S \in \mathcal{A}_1^*$,

$$ESE_2 = \varphi(F_1)ESE_2.$$

Hence $\varphi(F_1)$ is larger or equal to the projection onto $[E\mathcal{A}_1^*E_2(H)]$. If we write F'_1 for the latter projection we have

$$ET^*E_2 = F'_1ET^*E_2$$

for every $T \in \mathcal{A}_1$. We have $E_2TE = (ET^*E_2)^* = (F'_1ET^*E_2)^* = E_2TEF'_1 = E_2TEF'_1 + F'_1E_2TE$ (as $F'_1 \leq E \leq I - E_2$). Hence, for $T \in \mathcal{A}_1$, $N\varphi^{-1}(T)P = \varphi^{-1}(E_2TE) = \varphi^{-1}(E_2TE)\varphi^{-1}(F'_1) + \varphi^{-1}(F'_1)\varphi(E_2TE) = N\varphi^{-1}(T)P\varphi^{-1}(F'_1) + \varphi^{-1}(F'_1)N\varphi^{-1}(T)P = \varphi^{-1}(F'_1)N\varphi^{-1}(T)P$ as $P\varphi^{-1}(F'_1) = \varphi^{-1}(E_2F'_1) = 0$. Hence $\varphi^{-1}(F'_1) \geq F_1$; i.e. $F'_1 \geq \varphi(F_1)$ proving that $\varphi(F_1)$ is equal to F'_1 . The proof for $[N\mathcal{A}Q(H)]$ is similar and is omitted. $\square$

Lemma 2.5. With the notation of Lemma 2.4, $F_1F_2 = 0$.

Proof. For every T, S in $\mathcal{A}$ we have $0 = NTPNSQ + NSQNT P$ as $PN = QN = 0$. Hence

$$\begin{aligned} 0 &= \varphi(NTPNSQ + NSQNT P) = \varphi(NTP)\varphi(NSQ) + \varphi(NSQ)\varphi(NTP) \\ &= E_2\varphi(T)EE\varphi(S)(I - E_1) + E\varphi(S)(I - E_1)E_2\varphi(T)E \\ &= E_2\varphi(T)E\varphi(S)(I - E_1) \end{aligned}$$

as $(I - E_1)E_2 = 0$. Hence $E_2\mathcal{A}_1E\mathcal{A}_1(I - E_1) = \{0\}$. From Lemma 2.4 it follows that $E_2\mathcal{A}_1E\varphi(F_2) = \{0\}$; hence $\varphi(F_2)E\mathcal{A}_1^*E_2 = \{0\}$. Using Lemma 2.4 again we get $\varphi(F_2)\varphi(F_1) = 0$ and, thus, $F_1F_2 = 0$.

Note here that both F_1 and F_2 are in $\mathcal{L}$ (to see that $F_1 \in \mathcal{L}$ just observe that for $T, S \in \mathcal{A}$, $TNSP = NTNSP$; so that $TF_1 = F_1TF_1$. The proof for F_2 is similar). Also $F_1 + F_2 \leq N$.

For the projection $N \in \mathcal{L}$ above we associated projections $E, E_1, E_2, Q, P, F_1, F_2$ etc. This can be done for every projection $L \in \mathcal{L}$ and then we shall write $E(L), E_1(L), E_2(L)$, etc.

Lemma 2.6. Fix $N \in \mathcal{L}$ as above and let F_1, F_2 be the projections defined above. Then, for every projection $M \in \mathcal{L}$,

- (1) $I - \varphi(MF_1) \in \mathcal{L}_1$,
- (2) $\varphi(MF_2) \in \mathcal{L}_1$.

Proof. We shall prove (1). The proof of (2) is similar. Note that from Lemma 2.4 we know:

- (i) $E_2\mathcal{A}_1E(I - \varphi(F_1)) = \{0\}$,
- (ii) $(I - \varphi(F_2))E\mathcal{A}_1(I - E_1) = \{0\}$, and from Lemma 2.5,
- (iii) $F_1F_2 = 0$.

Now (ii) and (iii) imply that $\varphi(MF_1)\mathcal{A}_1(I - E_1) = \{0\}$. To prove (1) we have to show that

$$\varphi(MF_1)\mathcal{A}_1(I - \varphi(MF_1)) = \{0\}.$$

Write $G = MF_1$. Then $G \in \mathcal{L}$ and $E(G)$, $E_1(G)$, etc. are well defined. Applying (ii)–(iii) to G , in place of N , we get

$$(i') \quad E_2(G)\mathcal{A}_1 E(G)(I - \varphi(F_1(G))) = \{0\},$$

$$(ii') \quad (I - \varphi(F_2(G)))E(G)\mathcal{A}_1(I - E_1(G)) = \{0\},$$

$$(iii') \quad F_1(G)F_2(G) = 0.$$

As $E(G) = \varphi(G) \leq \varphi(N) = E \leq I - E_2$ and $E(G) = E_1(G) - E_2(G)$, it is easy to check that $E(G) = E_2 \vee E_1(G) - E_2 \vee E_2(G)$; hence, replacing $E_i(G)$, $i = 1, 2$, by $E_i(G) \vee E_2$, we can assume that $E_i(G) \geq E_2$. (Recall that the choice of $E_1(G)$ and $E_2(G)$ was arbitrary.)

We now get, from (i'), $E_2\mathcal{A}_1 E(G)(I - \varphi(F_1(G))) = \{0\}$, i.e.

$$(1 - \varphi(F_1(G)))E(G)\mathcal{A}_1^* E_2 = \{0\}.$$

But $E(G) = \varphi(G) = \varphi(F_1)\varphi(M)$. Hence $(1 - \varphi(F_1(G)))\varphi(M)\varphi(F_1)\mathcal{A}_1^* E_2 = \{0\}$. As $\varphi(F_1)$ is the projection onto $[E\mathcal{A}_1^* E_2(H)]$ and $E \geq \varphi(F_1)$, it is the projection onto $[\varphi(F_1)\mathcal{A}_1^* E_2(H)]$. We therefore have

$$0 = (I - \varphi(F_1(G)))\varphi(M)\varphi(F_1) = (I - \varphi(F_1(G)))\varphi(G).$$

It follows that $G(I - F_1(G)) = 0$. But $F_1(G) \leq G$; hence $G = F_1(G)$. As $F_2(G) \leq G - F_1(G)$, $F_2(G) = 0$ and, using (ii') we get

$$(*) \quad E(G)\mathcal{A}_1(I - E_1(G)) = \{0\}.$$

Also $E(G)\mathcal{A}_1 E_2(G) = \{0\}$ as $E_2(G) \in \mathcal{L}_1$ and $E_2(G)E(G) = 0$. Combining this with (*) we have

$$E(G)\mathcal{A}_1(I - E(G)) = \{0\}.$$

Hence $\varphi(MF_1) = \varphi(G) = E(G) \in \mathcal{L}_1^\perp$. $\square$

We shall now write $P_0 = \bigvee \{P \in \mathcal{L} \cap \mathcal{L}^\perp; \mathcal{A}P \subseteq \mathcal{L}'\}$ and note that $\mathcal{A}P_0 \subseteq \mathcal{L}'$.

Lemma 2.7. For N and M in $\mathcal{L}$, $F_1(N)F_2(M) \leq P_0$.

Proof. Write $L = F_1(N)F_2(M) \in \mathcal{L}$. As $L \leq F_1(N)$ and $L \leq F_2(M)$, Lemma 2.6 implies that $\varphi(L) \in \mathcal{L}_1 \cap \mathcal{L}_1^\perp$. Note that $\mathcal{L} \cap \mathcal{L}^\perp$ is the set of all projections in $\mathcal{A}'$. Thus $\varphi(\mathcal{L} \cap \mathcal{L}^\perp) = \mathcal{L}_1 \cap \mathcal{L}_1^\perp$ and thus $L \in \mathcal{L} \cap \mathcal{L}^\perp$. In fact, for every $N_1 \in \mathcal{L}$ $N_1 L \leq L$ and the same argument shows that $N_1 L \in \mathcal{L} \cap \mathcal{L}^\perp$. Hence $\mathcal{L}L \subseteq \mathcal{L} \cap \mathcal{L}^\perp$ from which it easily follows that $\mathcal{A}L \subseteq \mathcal{L}'$. Hence $L \leq P_0$. $\square$

We now write $F_+ = \bigvee \{F_2(M): M \in \mathcal{L}\}$ and $F_- = \bigvee \{F_1(M): M \in \mathcal{L}\}$. If $N \in \mathcal{L}$ and $N \leq F_+$ then $N = \bigvee \{NF_2(M): M \in \mathcal{L}\}$ and $\varphi(N) = \bigvee \{\varphi(NF_2(M)): M \in \mathcal{L}\} \in \mathcal{L}_1$ (by Lemma 2.6). Similarly whenever $N \in \mathcal{L}$ and $N \leq F_-$, $\varphi(N) \in \mathcal{L}_1^\perp$. Note also that $F_+F_- \leq P_0$ (by Lemma 2.7). Suppose that $P_0 = 0$ so that $F_+F_- = 0$.

Write $A = F_+ + F_- \in \mathcal{L}$ and note that $\varphi(I - A) = (I - \varphi(F_+)) - \varphi(F_-)$ and $I - \varphi(F_+)$, $\varphi(F_-) \in \mathcal{L}_1^\perp$.

We can thus apply Lemma 2.5 and Lemma 2.6 with $\mathcal{A}^*$, $\mathcal{A}_1^*$ replacing $\mathcal{A}$ and $\mathcal{A}_1$, $\mathcal{L}^\perp$ and $\mathcal{L}_1^\perp$ replacing $\mathcal{L}$ and $\mathcal{L}_1$, φ^* (defined by $\varphi^*(T^*) = \varphi(T)^*$) replacing φ and $I - A$, F_+ and F_- replacing N , Q and P respectively to get the following.

Lemma 2.8. Suppose $P_0 = 0$ and let A_+ and A_- be the projections onto $[(I - A)\mathcal{A} * F_+(H)]$ and $[(I - A)\mathcal{A} * F_-(H)]$ respectively. Then

- (1) $A_+A_- = 0$.
- (2) For every projection $M \in \mathcal{L}$, $\varphi((I - M)A_-) \in \mathcal{L}_1$. $\square$

Lemma 2.9. $(I - A)\mathcal{A} \subseteq \mathcal{L}'$.

Proof. Fix $N \in \mathcal{L}$. Recall that $F_1(N)$ is the projection onto $[NAP(H)]$, $F_2(N)$ is the projection onto $[N\mathcal{A}Q(H)]$, $Q + P = I - N$ and $F_1(N) + F_2(N) \leq A$. Hence

$$(I - A)N\mathcal{A}(I - N) = \{0\}.$$

also $(I - N)AN = \{0\}$ (as $N \in \mathcal{L}$). We therefore have

$$(I - N)(I - A)\mathcal{A}N = \{0\} = N(I - A)\mathcal{A}(I - N).$$

Hence $(I - A)\mathcal{A} \subseteq \mathcal{L}'$. $\square$

Finally, write $E_+ = F_+ + A_+$ and $E_- = F_- + A_-$.

Lemma 2.10. Suppose $P_0 = 0$. Then

- (1) E_+ and E_- are in $\mathcal{L} \cap \mathcal{L}^\perp$.
- (2) $E_+ + E_- = I$.
- (3) For every $M \in \mathcal{L}$, $\varphi(ME_+) \in \mathcal{L}_1$ and $\varphi(ME_-) \in \mathcal{L}_1^\perp$.

Proof. First we show that $E_+ \in \mathcal{L}$. As $F_+ \in \mathcal{L}$ it suffices to show that $(I - E_+)\mathcal{A}A_+ = 0$. As $I - E_+ = F_- + (I - A_+)(I - A)$, we shall show

- (i) $F_-\mathcal{A}A_+ = \{0\}$; and
- (ii) $(I - A_+)(I - A)\mathcal{A}A_+ = \{0\}$.

For (i) note that $A_+A_- = 0$ and A_- is the projection on $[(I - A)\mathcal{A} * F_-(H)]$; hence $A_+(I - A)\mathcal{A} * F_- = \{0\}$ and, thus, $F_-\mathcal{A}(I - A)A_+ = 0$. As $A_+ = (I - A)A_+$, this proves (i). (ii) follows immediately from the fact that $(I - A)\mathcal{A} \subseteq \mathcal{L}'$ (Lemma 2.9). Hence $E_+ \in \mathcal{L}$. The proof that $E_- \in \mathcal{L}$ is almost identical and is omitted.

Now write $A_0 = I - E_+ - E_- \leq I - F_+ - F_- = I - A$. As $(I - A)\mathcal{A} \subseteq \mathcal{L}'$ (Lemma 2.9), also $A_0\mathcal{A} \subseteq \mathcal{L}'$. Note also that, as $A_0A_+ = A_0A_- = 0$ we have, $A_0\mathcal{A} * F_+ = A_0\mathcal{A} * F_- = \{0\}$; hence $A_0\mathcal{A} * A = \{0\}$ and, thus, $A\mathcal{A}A_0 = \{0\}$. As $(I - A)\mathcal{A} \subseteq \mathcal{L}'$, we have $(I - A)\mathcal{A}A_0 \subseteq \mathcal{L}'$. Therefore $\mathcal{A}A_0 = A\mathcal{A}A_0 + (I - A)\mathcal{A}A_0 \subseteq \mathcal{L}'$. Hence we have $\mathcal{A}A_0 \subseteq \mathcal{L}'$ and $A_0\mathcal{A} \subseteq \mathcal{L}'$ and this implies that $A_0 \in \mathcal{A}'$ (as $(I - A_0)\mathcal{A}A_0 \subseteq (I - A_0)\mathcal{L}'A_0 = \{0\}$ and, similarly $A_0\mathcal{A}(I - A_0) = \{0\}$); i.e. $A_0 \leq P_0 = 0$.

This shows that $E_+ + E_- = I$ and, as $E_+, E_- \in \mathcal{L}$ we have $E_+, E_- \in \mathcal{L} \cap \mathcal{L}^\perp$. This proves parts (1) and (2). For (3) we fix $M \in \mathcal{L}$. We want to show that $\varphi(ME_+) \in \mathcal{L}_1$. As $E_+ \in \mathcal{L} \cap \mathcal{L}^\perp \subseteq \mathcal{A}'$ we can replace $\mathcal{L}$ by $\mathcal{L}E_+$ and $\mathcal{A}$ by $\mathcal{A}E_+$; i.e. we assume $E_+ = I$. Then $F_+ + A_+ = I$.

Note first that by Lemma 2.8(2) we know that $\varphi((I - M)A_+) \in \mathcal{L}_1^\perp$. Hence we have

$$(*) \quad \varphi((I - M)A_+)\mathcal{A}_1\varphi(MA_+) = \varphi((I - M)A_+)\mathcal{A}_1\varphi((I - M)A_+)\varphi(MA_+) = \{0\}.$$

For every $T \in \mathcal{A}$, $F_+(I - M)TMA_+ = 0$ (as $M \in \mathcal{L}$) and $MA_+T(I - M)F_+ = 0$ (as $F_+ \in \mathcal{L}$ and $A_+F_+ = 0$). Hence

$$\begin{aligned} 0 &= \varphi(F_+(I - M)TMA_+ + MA_+T(I - M)F_+) \\ &= \varphi(F_+(I - M))\varphi(T)(MA_+) + \varphi(MA_+)\varphi(T)\varphi((I - M)F_+). \end{aligned}$$

As $\varphi(F_+) \in \mathcal{L}$ and $\varphi(A_+)\varphi(F_+) = 0$, $\varphi(MA_+)\varphi(T)\varphi((I - M)F_+) = 0$. Hence $\varphi(F_+(I - M))\mathcal{A}_1\varphi(MA_+) = \{0\}$.

As $I - M = (A_+ + F_+)(I - M) = A_+(I - M) + F_+(I - M)$, combining this with (*) we get

$$\varphi(I - M)\mathcal{A}_1\varphi(MA_+) = \{0\}.$$

As $\varphi(MF_+) \in \mathcal{L}_1$ (see the discussion following Lemma 2.7),

$$\varphi(I - M)\mathcal{A}_1\varphi(M) = \varphi(I - M)\mathcal{A}_1(\varphi(MA_+) + \varphi(MF_+)) = \{0\}.$$

Thus $\varphi(MA_+) \in \mathcal{L}_1$.

The proof that $\varphi(MA_-) \in \mathcal{L}^\perp$ is similar and is omitted. $\square$

Proposition 2.11. *Let $\varphi: \mathcal{A} \rightarrow \mathcal{A}_1$ be a Jordan partial *-isomorphism of $\mathcal{A}$ onto $\mathcal{A}_1$. Then there are projections E_+ , E_- and P_0 in $\mathcal{L} \cap \mathcal{L}^\perp$ such that $E_+ + E_- + P_0 = I$ and*

- (1) $AP_0 = \mathcal{L}'P_0$ (hence $\varphi(\mathcal{L}P_0) \subseteq \mathcal{L}_1 \cap \mathcal{L}_1^\perp$);
- (2) for every $M \in \mathcal{L}$, $\varphi(ME_+) \in \mathcal{L}_1$ and, in fact, $\varphi(\mathcal{L}E_+) = \mathcal{L}_1\varphi(E_+)$; and
- (3) for every $M \in \mathcal{L}$, $\varphi(ME_-) \in \mathcal{L}_1^\perp$ and, in fact $\varphi(\mathcal{L}E_-) = \mathcal{L}_1^\perp\varphi(E_-)$.

Proof. Let $P_0 = \bigvee \{P \in \mathcal{L} \cap \mathcal{L}^\perp : \mathcal{A}P \subseteq \mathcal{L}'P\}$ (as above) and replace $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{L}$, $\mathcal{L}_1$ by $\mathcal{A}P_0^\perp$, $\mathcal{A}_1\varphi(P_0^\perp)$, $\mathcal{L}P_0^\perp$, $\mathcal{L}_1\varphi(P_0^\perp)$ respectively; i.e. assume $P_0 = 0$. Then let E_+ and E_- be the projections constructed above. By Lemma 2.10 we have $E_+ + E_- = I$. Part (1) now follows from the definition of P_0 . From Lemma 2.10 we know that $\varphi(\mathcal{L}E_+) \subseteq \mathcal{L}_1\varphi(E_+)$ and $\varphi(\mathcal{L}E_-) \subseteq \mathcal{L}_1^\perp\varphi(E_-)$. Now write $P'_0 = \bigvee \{P \in \mathcal{L}'_1 \cap \mathcal{L}_1^\perp : \mathcal{A}_1P \subseteq \mathcal{L}'_1P\}$. It is clear that $\varphi(P_0) = P'_0$ and since we assume that $P_0 = 0$, also $P'_0 = 0$. If E'_+ and E'_- are the projections constructed for φ^{-1} then $E'_+ + E'_- = I$. If $\varphi(\mathcal{L}E_+) \neq \mathcal{L}_1\varphi(E_+)$ then there exists an element $G_0 \in \mathcal{L}_1$ such that $G_0 \leq \varphi(E_+)$ and $G_0 \notin \varphi(\mathcal{L}E_+)$. By Lemma 2.10, $\varphi^{-1}(G_0E'_+) \in \mathcal{L}$ and $\varphi^{-1}(G_0E'_-) \in \mathcal{L}^\perp$. Now, if $G_0E'_- = 0$, then $\varphi^{-1}(G_0) = \varphi^{-1}(G_0E'_+) \in \mathcal{L}$ and also $\varphi^{-1}(G_0) \leq E_+$; hence $\varphi^{-1}(G_0) \in \mathcal{L}E_+$, and we arrive at a contradiction. Therefore $G_0E'_- \neq 0$ and we write $G = G_0E'_-$. Then $G \leq \varphi(E_+)$ and $\varphi^{-1}(G) \in \mathcal{L}^\perp$. Thus $I - \varphi^{-1}(G) \in \mathcal{L}$ and, by Lemma 2.10, $\varphi(E_+) - G = \varphi((I - \varphi^{-1}(G))E_+) \in \mathcal{L}_1$. Since $E_+ \in \mathcal{L} \cap \mathcal{L}^\perp$ (Lemma 2.10), $\varphi(E_+) \in \mathcal{L}_1 \cap \mathcal{L}_1^\perp$. Combining these facts we get $G = \varphi(E_+)(I - (\varphi(E_+) - G)) \in \mathcal{L}_1^\perp$; hence $G \in \mathcal{L}_1 \cap \mathcal{L}_1^\perp$. Thus $\varphi^{-1}(G) \in \mathcal{L} \cap \mathcal{L}^\perp$ and, since $G_0 = G_0E'_+ + G_0E'_- = G_0E'_+ + G$, we have $\varphi^{-1}(G_0) = \varphi^{-1}(G_0E'_+) + \varphi^{-1}(G) \in \mathcal{L}$. Since $\varphi^{-1}(G_0) \leq E_+$, $\varphi^{-1}(G_0) \in \mathcal{L}E_+$. This contradicts the choice of G_0 and proves that $\varphi(\mathcal{L}E_+) = \mathcal{L}_1\varphi(E_+)$. The fact that $\varphi(\mathcal{L}E_-) = \mathcal{L}_1^\perp\varphi(E_-)$ is proved in a similar way. $\square$

Let E_+ , E_- and P_0 be as in Proposition 2.11. We first assume that $E_+ = I$ (i.e. $E_- = P_0 = 0$). Then $\varphi(\mathcal{L}) = \mathcal{L}_1$. The following argument was used in [MT2].

Since $\mathcal{L}''$ is a commutative von Neumann algebra there is a maximal abelian selfadjoint algebra (i.e. a masa) $\mathcal{R}$ such that $\mathcal{L}'' \subseteq \mathcal{R} \subseteq \mathcal{L}' = \mathcal{A} \cap \mathcal{A}^*$. The restriction of φ to $\mathcal{R}$ is a Jordan *-isomorphism and therefore its image, $\varphi(\mathcal{R})$ is an abelian selfadjoint algebra, written $\mathcal{R}_1$. As φ^{-1} , restricted to $\mathcal{L}'_1$, is Jordan *-isomorphism of the von Neumann algebra $\mathcal{L}'_1$ onto $\mathcal{L}'$ it preserves commutativity. Thus if $T \in \mathcal{R}'_1 \subseteq \mathcal{L}'_1$ then $\varphi^{-1}(T) \in \mathcal{R}' = \mathcal{R}$; hence

$T \in \mathcal{R}_1$. This shows that $\mathcal{R}_1 = \varphi(\mathcal{R})$ is a masa. Now the restriction of φ to $\mathcal{R}$ is a Jordan $*$ -isomorphism of masa's and, thus, is a $*$ -isomorphism. By [KR, Theorem 9.3.1] we know that φ , restricted to $\mathcal{R}$, can be implemented by some unitary operator V ; i.e. $\varphi(T) = V^*TV$, $T \in \mathcal{R}$.

Since $\varphi(\mathcal{L}) = \mathcal{L}_1$, $V^*\mathcal{L}V = \mathcal{L}_1$ and $V^*\mathcal{A}V = \mathcal{A}_1$. If we now write $\psi(T) = V\varphi(T)V^*$ then ψ is a Jordan partial $*$ -isomorphism from $\mathcal{A}$ onto itself leaving $\mathcal{R}$ (and, in particular, $\mathcal{L}''$) elementwise fixed.

Lemma 2.12. *Suppose $E_+ = I$ and ψ is as above. Then for every interval $E_2 - E_1$ (with $E_1, E_2 \in \mathcal{L}$ and $E_1 \leq E_2$) and $T \in \text{Alg } \mathcal{L}$ we have*

$$\psi(T(E_2 - E_1)) = \psi(T)(E_2 - E_1) \quad \text{and} \quad \psi((E_2 - E_1)T) = (E_2 - E_1)\psi(T).$$

Proof. We have $TE_2(I - E_1) = (E_2 - E_1)T(E_2 - E_1) + E_1T(E_2 - E_1)$ (as $(I - E_2)T(E_2 - E_1) = 0$). Also $(E_2 - E_1)TE_1 = 0$ (as $E_1 \in \mathcal{L}$) and thus

$$\begin{aligned} \psi(T(E_2 - E_1)) &= \psi((E_2 - E_1)T(E_2 - E_1)) + \psi(E_1T(E_2 - E_1) + (E_2 - E_1)TE_1) \\ &= (E_2 - E_1)\psi(T)(E_2 - E_1) + E_1\psi(T)(E_2 - E_1) + (E_2 - E_1)\psi(T)E_1 \\ &= (E_2 - E_1)\psi(T)(E_2 - E_1) + E_1\psi(T)(E_2 - E_1) \\ &= \psi(T)(E_2 - E_1). \end{aligned}$$

The other statement is proved in a similar way. $\square$

We shall also need the following lemma.

Lemma 2.13. *For every $N \in \mathcal{L}$ we write $G_1(N)$ for the projection onto $[N\mathcal{A}(I - N)(H)]$ and $G_2(N)$ for the projection onto $[(I - N)\mathcal{A}^*N(H)]$. Let $G_1 = \bigvee \{G_1(N) : N \in \mathcal{L}\}$ and $G_2 = \bigvee \{G_2(N) : N \in \mathcal{L}\}$. Then $G_1 \vee G_2 \geq I - P_0$. Hence if $P_0 = 0$ then $G_1 \vee G_2 = I$.*

Proof. Write $F = I - G_1 \vee G_2$. As $FG_1 = FG_2 = 0$ we have $FN\mathcal{A}(I - N) = 0$ and $F(I - N)\mathcal{A}^*N = 0$ for every $N \in \mathcal{L}$. Hence $N(F\mathcal{A})(I - N) = N(\mathcal{A}F)(I - N) = 0$ for every $N \in \mathcal{L}$. It follows that both $F\mathcal{A}$ and $\mathcal{A}F$ are contained in $\mathcal{A}^*$. As both are also contained in $\mathcal{A}$, we have $F\mathcal{A} \subseteq \mathcal{L}'$ and $\mathcal{A}F \subseteq \mathcal{L}'$ which implies that $F \leq P_0$. $\square$

Proposition 2.14. *Assume that $E_+ = I$ and ψ is as above then ψ is multiplicative.*

Proof. From Lemma 2.13 it follows that we can find sequences $\{N_i\}_{i=1}^\infty$ and $\{M_i\}_{i=1}^\infty$ of projections in $\mathcal{L}$ such that $(\bigvee_i G_1(N_i)) \vee (\bigvee_i G_2(M_i)) = I$. We can, therefore, find a sequence of projections $\{E_i\}$ in $\mathcal{L}''$ such that:

- (i) each E_i is an interval in $\mathcal{L}$. (Note that $G_1(N) \in \mathcal{L}$ and $G_2(N) \in \mathcal{L}^\perp$ for every $N \in \mathcal{L}$).
- (ii) For $i \neq j$, $E_iE_j = 0$;
- (iii) $\sum_i E_i = I$; and
- (iv) For each i there is some $N \in \mathcal{L}$ such that either $E_i \leq G_1(N)$ or $E_i \leq G_2(N)$.

Now fix $T, S \in \mathcal{A}$. For $i \neq j$ we have

$$\begin{aligned} \psi(E_iTSE_j) &= \psi(E_iTSE_j + SE_jE_iT) = \psi(E_iT)\psi(SE_j) + \psi(SE_j)\psi(E_iT) \\ &= E_i\psi(T)\psi(S)E_j + \psi(S)E_jE_i\psi(T) = E_i\psi(T)\psi(S)E_j. \end{aligned}$$

(We use Lemma 2.12 to see that $\psi(E_i T) = E_i \psi(T)$ and $\psi(SE_j) = \psi(S)E_j$.)
Hence

$$(*) \quad \psi(E_i T S E_j) = E_i \psi(T) \psi(S) E_j, \quad i \neq j.$$

Suppose E_i is such that $E_i \leq G_1(N)$ for some $N \in \mathcal{L}$. For every $R \in \mathcal{A}$ we have

$$\psi(E_i T E_i S E_i N R (I - N)) = \psi(E_i T E_i S E_i N R (I - N)) + \psi(N R (I - N) E_i S E_i T E_i)$$

(as $E_i \leq G_1(N) \leq N$). Since ψ is a Jordan isomorphism,

$$\begin{aligned} \psi(E_i T E_i S E_i N R (I - N)) &= \psi(E_i T E_i) \psi(E_i S E_i) \psi(N R (I - N)) \\ &+ \psi(N R (I - N)) \psi(E_i S E_i) \psi(E_i T E_i) = E_i \psi(T) E_i \psi(S) E_i N \psi(R) (I - N) \\ &+ N \psi(R) (I - N) E_i \psi(S) E_i \psi(T) E_i = E_i \psi(T) E_i \psi(S) E_i N \psi(R) (I - N). \end{aligned}$$

We get

$$\psi(E_i T E_i S E_i N R (I - N)) = E_i \psi(T) E_i \psi(S) E_i N \psi(R) (I - N)$$

and, replacing S by $T E_i S$ and T by I we get

$$\begin{aligned} \psi(E_i T E_i S E_i N R (I - N)) &= E_i \psi(T E_i S) E_i N \psi(R) (I - N) \\ &= \psi(E_i T E_i S E_i) N \psi(R) (I - N) \end{aligned}$$

and from these two equations we conclude:

$$E_i \psi(T) E_i \psi(S) E_i N \psi(R) (I - N) = \psi(E_i T E_i S E_i) N \psi(R) (I - N).$$

Since this holds for all $R \in \mathcal{A}$ and since $E_i = E_i G_1(N)$ we get (recall that $G_1(N)$ is the projection onto $[N \mathcal{A} (I - N)(H)]$),

$$(**) \quad \psi(E_i T E_i S E_i) = E_i \psi(T) E_i \psi(S) E_i.$$

Using the fact that ψ is a Jordan homomorphism again we get

$$\begin{aligned} \psi(E_i T S E_i + E_i S E_i T E_i) &= E_i \psi(E_i T S E_i + S E_i E_i T) E_i \\ &= E_i (\psi(E_i T) \psi(S E_i) + \psi(S E_i) \psi(E_i T)) E_i \\ &= E_i \psi(T) \psi(S) E_i + E_i \psi(S) E_i \psi(T) E_i \end{aligned}$$

and, using (**) with T and S reserved, we have

$$\psi(E_i T S E_i + E_i S E_i T E_i) = E_i \psi(T) \psi(S) E_i + \psi(E_i S E_i T E_i).$$

Hence

$$(***) \quad \psi(E_i T S E_i) = E_i \psi(T) \psi(S) E_i.$$

This was shown to hold for E_i with the property that $E_i \leq G_1(N)$. We wish to show it also for E_i satisfying $E_i \leq G_2(N)$ for some $N \in \mathcal{L}$. For that simply replace $\mathcal{A}$ by $\mathcal{A}^*$, ψ by ψ^* (where $\psi^*(T) = \psi(T)^*$, $T \in \mathcal{A}^*$), T by S^* and S by T^* and note that $G_2(N)$ is now $G_1(I - N)$ (with $\mathcal{A}^*$ in place of $\mathcal{A}$). We now get,

$$\psi^*(E_i S^* T^* E_i) = E_i \psi^*(S^*) \psi^*(T^*) E_i$$

and, thus, by taking adjoints, $\psi(E_i T S E_i) = E_i \psi(T) \psi(S) E_i$; so that (***) holds for every i .

Now we write

$$\psi(TS) = \sum_{ij} E_i \psi(TS) E_j = \sum_i E_i \psi(TS) E_i + \sum_{i \neq j} E_i \psi(TS) E_j$$

and, using $(*)$, $(***)$, we conclude

$$\psi(TS) = \sum_i E_i \psi(T) \psi(S) E_i + \sum_{i \neq j} E_i \psi(T) \psi(S) E_j = \psi(T) \psi(S). \quad \square$$

We are now ready to prove the main theorem.

Theorem 2.15. *Let $\mathcal{L}$ and $\mathcal{L}_1$ be commutative lattices of projections acting on a separable Hilbert space H . Let $\theta: \text{Alg } \mathcal{L} \rightarrow \text{Alg } \mathcal{L}_1$ be a partial triple product isomorphism onto $\text{Alg } \mathcal{L}_1$. (In particular this holds for a linear surjective isometry.) Let $U = \theta(I)$ and $\varphi: \text{Alg } \mathcal{L} \rightarrow \text{Alg } \mathcal{L}_1$ be defined by $\varphi(T) = U^* \theta(T)$, $T \in \text{Alg } \mathcal{L}_1$. Then there is a projection $E \in \mathcal{L} \cap \mathcal{L}^\perp \subseteq (\text{Alg } \mathcal{L})'$ such that :*

- (1) φ , restricted to $(\text{Alg } \mathcal{L})E$, is an isomorphism of $(\text{Alg } \mathcal{L})E$ onto $(\text{Alg } \mathcal{L}_1)\varphi(E)$ mapping $\mathcal{L}E$ onto $\mathcal{L}_1\varphi(E)$.
- (2) φ , restricted to $(\text{Alg } \mathcal{L})(I - E)$, is an anti-isomorphism of $(\text{Alg } \mathcal{L})(I - E)$ onto $(\text{Alg } \mathcal{L}_1)\varphi(I - E)$ mapping $\mathcal{L}(I - E)$ onto $\mathcal{L}_1^\perp \varphi(I - E)$.

Proof. Recall (Proposition 2.11) that there are projections E_+ , E_- and P_0 , in $\mathcal{L} \cap \mathcal{L}^\perp$, such that $I = E_+ + E_- + P_0$ and $\varphi(\mathcal{L}E_+) = \mathcal{L}_1\varphi(E_+)$, $\varphi(\mathcal{L}E_-) = \mathcal{L}_1^\perp \varphi(E_-)$ and $\mathcal{A}P_0 \subseteq \mathcal{L}'P_0$ (where $\mathcal{A} = \text{Alg } \mathcal{L}$). Since we can decompose $\mathcal{A}$ as the direct sum of $\mathcal{A}E_+$, $\mathcal{A}E_-$ and $\mathcal{A}P_0$, it suffices to prove the result for the restrictions of φ to each of these algebras.

For $\mathcal{A}E_+$ we can assume $E_+ = I$ and use Proposition 2.15 (with ψ as in the discussion preceding Lemma 2.12) to conclude that ψ is multiplicative and, thus, so is φ (as $\varphi(T) = V^* \psi(T)V$ and V is unitary).

For $\mathcal{A}E_-$ we can assume that $E_- = I$, i.e. $\varphi(\mathcal{L}) \subseteq \mathcal{L}_1^\perp$. Fix an involution J of H , i.e. an isometric conjugative linear mapping J of H onto H such that $J^2 = I$. Then it is easy to check that the map $T \mapsto JT^*J$ is a $*$ -anti-isomorphism of $B(H)$ onto itself. Define $\psi_0(T) = \varphi(JT^*J)$ for $T \in (\text{alg}(J\mathcal{L}J))^* = \text{Alg}(J\mathcal{L}J)^\perp$. This defines a Jordan partial $*$ -isomorphism from $\text{Alg}(J\mathcal{L}J)^\perp$ onto $\text{Alg } \mathcal{L}_1$ that maps I into I and $(J\mathcal{L}J)^\perp$ into $\mathcal{L}_1$. Hence, by the part just proved above, φ_0 is an isomorphism and it follows that φ is an anti-isomorphism.

For $\mathcal{A}P_0 = \mathcal{L}'P_0$ the result follows from [K, Theorem 10]. $\square$

Corollary 2.16. *Every partial triple product isomorphism of CSL algebras is continuous.*

Proof. This follows from Theorem 2.15 and Theorem 3.1 of [GM]. (The argument used in the proof of Theorem 2.15 can be used to show that [GM, Theorem 3.1] holds for anti-isomorphisms.) $\square$

Corollary 2.17. *Let $\mathcal{L}$ and $\mathcal{L}_1$ be commutative subspace lattices and suppose that there is an isometry from $\text{Alg } \mathcal{L}$ onto $\text{Alg } \mathcal{L}_1$. If $\mathcal{L}$ is completely distributive, then so is $\mathcal{L}_1$ and the results of [MT2] can be applied.*

Proof. This follows from Theorem 2.15 and [GM, Corollary 2.2]. $\square$

Remark. Combining Theorem 2.15 (for an isometry θ) with Theorem 2.1) of [DP] provides an alternative proof of the main result of [MT2] (see Theorem 1.2).

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