ORBITS IN UNIMODULAR HERMITIAN LATTICES

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ABSTRACT. Let L be a unimodular indefinite hermitian lattice over the integers $\mathfrak{o}$ of an algebraic number field, and $N(L, c)$ the number of primitive representations of $c \in \mathfrak{o}$ by L that are inequivalent modulo the action of the integral special unitary group $SU(L)$ on L . The value of $N(L, c)$ is determined from the local representations via a product formula.

1. INTRODUCTION

Let F be an algebraic number field and K a quadratic extension of F . Let V be an indefinite hermitian space over K of finite dimension $n \geq 3$ and $f: V \times V \rightarrow K$ the associated nondegenerate hermitian form on V with respect to the nontrivial automorphism $*$ of K over F where $f(ax, by) = a^*f(x, y)b$. Assume V supports a unimodular lattice L (in the sense of O'Meara [O'M, 82G] for quadratic spaces). Denote by $U(V)$ the unitary group of V and by $U(L)$ the subgroup of isometries in $U(V)$ that leave L invariant. We study the orbits of primitive elements in L under the action of the special unitary group $SU(L) = U(L) \cap SL(V)$. The problem is first solved locally; the global result is then obtained by applying the strong approximation theorem of Shimura [S, 5.12]. The analogous problem for quadratic lattices was considered in [J4]; earlier work on this integral version of Witt's theorem can be found in [J1, J2, N and W]. Kihlfner [K] gives hermitian analogues for the results in [J1].

Let Ω_F be the set of all nontrivial prime spots on F and S the set of all finite prime spots. Denote by $\mathfrak{o} = \mathfrak{o}_F$ the associated Dedekind ring of algebraic integers. Let $\mathfrak{O}_K$ be the integral closure of $\mathfrak{o}_F$ in K . Although L is an $\mathfrak{O}_K$ -module, most of the calculations are done locally at the dyadic primes in S that ramify in K . The localization procedure followed here is essentially that first studied by Shimura [S] (see also [G]). Let $\mathfrak{p}$ be a prime spot of F and $F_{\mathfrak{p}}$ the corresponding local field. Put $K_{\mathfrak{p}} = K \otimes_F F_{\mathfrak{p}}$ and $V_{\mathfrak{p}} = V \otimes_F F_{\mathfrak{p}}$. Making the standard identifications, we have $K \subset K_{\mathfrak{p}}$, $F_{\mathfrak{p}} \subset K_{\mathfrak{p}}$ and $V \subset V_{\mathfrak{p}}$. The hermitian form f on V extends naturally to a hermitian form on $V_{\mathfrak{p}}$. For

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each $\mathfrak{p} \in S$, let $\mathfrak{o}_{\mathfrak{p}}$ be the topological closure of $\mathfrak{o}$ in $F_{\mathfrak{p}}$, and $\mathfrak{O}_{\mathfrak{p}}$ the integral closure of $\mathfrak{o}_{\mathfrak{p}}$ in $K_{\mathfrak{p}}$. Put $L_{\mathfrak{p}} = \mathfrak{O}_{\mathfrak{p}}L \subset V_{\mathfrak{p}}$. Then $L_{\mathfrak{p}}$ is locally unimodular so that $f(L_{\mathfrak{p}}, L_{\mathfrak{p}}) = \mathfrak{O}_{\mathfrak{p}}$ and $L_{\mathfrak{p}}$ has a basis with $\det f(x_i, x_j)$ a unit in $\mathfrak{O}_{\mathfrak{p}}$. Note when $\mathfrak{p}$ splits in K that $K_{\mathfrak{p}} = F_{\mathfrak{p}} \times F_{\mathfrak{p}}$, $\mathfrak{O}_{\mathfrak{p}} = \mathfrak{o}_{\mathfrak{p}} \times \mathfrak{o}_{\mathfrak{p}}$ and the involution $*$ on K becomes $(a, b)^* = (b, a)$ on $K_{\mathfrak{p}}$.

An element $x \in L$ is called *primitive* if $f(x, L) = \mathfrak{O}_K$. We say L represents $c \in \mathfrak{o}$ if there exists a primitive $x \in L$ with $f(x) = f(x, x) = c$ (all our representations are understood to be primitive). Put

$$L(c) = \{x \in L \mid f(x) = c \text{ and } f(x, L) = \mathfrak{O}_K\}$$

and let $N(L, c)$ be the number of orbits in $L(c)$ under the action of $SU(L)$, that is, the number of primitive representation of c modulo the action of $SU(L)$. Similarly, denote by $N(L_{\mathfrak{p}}, c)$ the corresponding number of local primitive representations of $c \in \mathfrak{o}_{\mathfrak{p}}$ modulo the action of $SU(L_{\mathfrak{p}})$. These representation numbers, $N(L, c)$ and $N(L_{\mathfrak{p}}, c)$, will be seen in §§4 and 5 to be finite. Representations of c by x and y are called (locally or globally) equivalent when x and y lie in the same orbit.

Theorem 1.1. *Let L be a unimodular lattice on an indefinite hermitian space V with dimension $n \geq 3$. Let c be a nonzero element in $\mathfrak{o}$ represented by L and assume $V \perp \langle -c \rangle$ has local Witt index at least two for some $\mathfrak{q} \notin S$. Then*

$$N(L, c) = \prod_{\mathfrak{p}} N(L_{\mathfrak{p}}, c)$$

where the product is taken over all dyadic primes $\mathfrak{p} \in S$ that ramify in K .

Note that the Witt index condition on $V \perp \langle -c \rangle$ is always satisfied when there exists a spot $\mathfrak{q} \notin S$ which splits in K . For example, the condition is satisfied if $K \subset \mathbf{R}$, the real field. The values of $N(L, c)$ will be explicitly calculated in some special cases, including $F = \mathbf{Q}$ (see Theorem 6.1), or K a cyclotomic field (see §7). Since the structure of local hermitian lattices is simpler than that of local quadratic lattices, the final results here will be more general than those in [J4].

2. GLOBAL ORBITS

The following result reduces the question of global equivalence of representations of the corresponding local problems.

Theorem 2.1. *Let L be a unimodular lattice on a hermitian space V with dimension $n \geq 3$. Let c be a nonzero element in $\mathfrak{o}_F$ and assume $V \perp \langle -c \rangle$ has local Witt index at least two for some $\mathfrak{q} \notin S$. Let $x, y \in L(c)$ be locally equivalent at each $\mathfrak{p} \in S$. Then there exists $\varphi \in SU(L)$ such that $\varphi(x) = y$.*

Proof. By hypothesis, there exist local $\varphi_{\mathfrak{p}} \in SU(L_{\mathfrak{p}})$ such that $\varphi_{\mathfrak{p}}(x) = y$ for each $\mathfrak{p} \in S$. By Witt's theorem, there exists $\theta \in U(V)$ such that $\theta(x) = y$. Let $\det \theta = \eta$. Then the norm $N_{K/F}(\eta) = 1$. Take $r \in V$ with $f(r) \neq 0$ and $f(r, x) = 0$. Since the quasi-symmetry

$$\Psi(r): z \mapsto z - (1 - \eta)f(r, z)f(r)^{-1}r, \quad z \in V,$$

fixes x and $\det \theta \Psi(r)^{-1} = 1$, we may assume $\theta \in SU(V)$. Then $\theta_{\mathfrak{p}}^{-1} \varphi_{\mathfrak{p}}(x) = x$ for each $\mathfrak{p} \in S$. Let $V = Kx \perp W$. Then $\theta_{\mathfrak{p}}^{-1} \varphi_{\mathfrak{p}} \in SU(W_{\mathfrak{p}})$, $\dim W \geq 2$ and

W_q is indefinite for some $q \notin S$. By the strong approximation theorem on $SU(W)$ [S, 5.12] (and modifying the norm notation $\|\cdot\|_p$ from [O'M, 101]), there exists $\psi \in SU(W)$ such that $\|\psi - \theta_p^{-1}\varphi_p\|_p < \varepsilon$ for the finite number of p in S where $\|\theta\|_p \neq 1$, and $\|\psi\|_p = 1$ for all remaining p in S . Extend ψ by the identity to $SU(V)$ and put $\varphi = \theta\psi \in SU(V)$. Then $\varphi(x) = y$ and $\|\varphi\|_p = 1$ for all $p \in S$, provided ε was chosen sufficiently small initially. Hence $\varphi \in SU(L)$. $\square$

The proof of Theorem 1.1 is similar to that for the analogous result in quadratic spaces, namely, Theorem 2.3 in [J4]; it will be given after some local results have been developed.

3. LOCAL ORBITS

The norm and trace mappings from K_p to F_p are denoted by N_p and T_p , respectively. For each $p \in S$, denote by L'_p the sublattice of L_p generated by the $x \in L_p$ with $f(x) \in 2\mathfrak{o}_p$. Trivially, $L'_p = L_p$ for any nondyadic p . Now suppose p splits in K . Then, since the involution on $\mathfrak{O}_p = \mathfrak{o}_p \times \mathfrak{o}_p$ is given by $(a, b)^* = (b, a)$, for any $x \in L_p$ we have $f((1, 0)x) = N_p(1, 0)f(x) = 0$. Hence $(1, 0)x \in L'_p$ and $x = (1, 1)x$ is in L'_p , therefore $L'_p = L_p$. When $p \in S$ is not split in K , let $\mathfrak{P}$ denote the unique maximal ideal in $\mathfrak{O}_p$. Assume $x \in L_p$ is primitive. Then $f(x, L'_p) = \mathfrak{P}^m$ for some $m \geq 0$. Here $m = m_p(x)$ is called the *degree*. Moreover, $m = 0$ when p is nondyadic. By convention we set $m = 0$ when p is split. Call x *characteristic* when $m_p(x) \geq 1$ (p must then be dyadic).

Theorem 3.1. *Assume L_p is unimodular with $n = \text{rank } L_p \geq 3$ and x, y are primitive in L_p . Then there exists $\varphi \in SU(L_p)$ such that $\varphi(x) = y$ if and only if*

- (i) $f(x) = f(y)$,
- (ii) $m_p(x) = m_p(y) = m$,
- (iii) $x - y \in \mathfrak{P}^m L'_p$.

This theorem provides the information needed to calculate $N(L_p, c)$. To prove the theorem we must first determine the structure of L'_p at the dyadic primes and find generators for $SU(L_p)$. Both of those problems have already been investigated in [J3].

When p does not split in K , let $K_p = F_p(\zeta)$ where $\zeta^2 \in \mathfrak{o}_p$ and $\zeta^* = -\zeta$. Fix a prime π in K_p and p in F_p and let $e = \text{ord}_p 2$. In the nondyadic case $\mathfrak{O}_p$ is generated over $\mathfrak{o}_p$ by 1 and ζ provided we choose ζ to be a prime or a unit according as the extension is ramified or not. If p is dyadic, there are the following three possible types of extension of K_p over F_p (see [Jc] or [O'M, 63.2 and 63.3] for more details).

- (i) K_p is an unramified extension of F_p . Then $\zeta^2 = 1 + 4\delta$ with δ a unit in $\mathfrak{o}_p$, and $\mathfrak{O}_p$ consists of the elements $(\alpha + \zeta\beta)/2$ with $\alpha, \beta \in \mathfrak{o}_p$ and $\alpha \equiv \beta \pmod{2}$. Here $2\mathfrak{O}_p = \mathfrak{P}^e$.
- (ii) K_p is a ramified extension of F_p and ζ is a prime in K_p —the ramified prime case. Now we may assume $\pi = \zeta$, $p = \pi\pi^*$ and $\mathfrak{O}_p$ is generated over $\mathfrak{o}_p$ by 1 and π . Here $2\mathfrak{O}_p = \mathfrak{P}^{2e}$.

- (iii) K_p is a ramified extension of F_p and ζ is a unit in K_p —the ramified unit case. We now have $\zeta^2 = 1 - p^{2h+1}\delta$ for some unit δ in $\mathfrak{o}_p$ and some rational integer h with $0 \leq h < e$. Put $\pi = (1 + \zeta)p^{-h}$ so that $\pi\pi^* = p\delta$. Here $\mathfrak{O}_p$ consists of the elements $(\alpha + \zeta\beta)p^{-h}$ with $\alpha, \beta \in \mathfrak{o}_p$ and $\alpha \equiv \beta \pmod{p^h}$; also $2\mathfrak{O}_p = \mathfrak{P}^{2e}$.

In summary, if K_p/F_p is a quadratic extension of fields, $\mathfrak{O}_p$ consists of the elements $(\alpha + \zeta\beta)p^{-h}$ with $\alpha, \beta \in \mathfrak{o}_p$ and $\alpha \equiv \beta \pmod{p^h}$, where we define $h = 0$ in the nondyadic and ramified prime dyadic cases, and $h = e$ in the unramified dyadic case. Note that $T_p(\mathfrak{O}_p) = 2p^{-h}\mathfrak{o}_p$. Let $\mathfrak{U}_p$ denote the group of units in $\mathfrak{O}_p$, and $\mathfrak{u}_p$ the units in $\mathfrak{o}_p$. Then $[u_p : N_p(u_p)] = 2$ when p is ramified dyadic. Also, let f denote the residue class degree of the local field F_p .

Lemma 3.2. *Assume p is ramified dyadic and $c \in \mathfrak{o}_p$. Then there exists $a \in \mathfrak{O}_p$ such that $N_p(a) \equiv c \pmod{2p^{-h}}$.*

Proof. Let $a = \alpha + \pi\beta$ where $\alpha, \beta \in \mathfrak{o}_p$. Then $N_p(a) \equiv \alpha^2 + p\delta\beta^2 \pmod{2p^{-h}}$ (with $\delta = 1$ in the ramified prime case) and $\alpha^2 + p\delta\beta^2 \equiv c \pmod{2p^{-h}}$ can be solved for α and β inductively through successive powers of p . $\square$

The following result was established as Proposition 2.1 in [J3].

Lemma 3.3. *Let F_p be a dyadic local field with p not split in K . Then $L'_p = \{r \in L_p \mid p^h f(r) \in 2\mathfrak{o}_p\}$. In particular, $L_p = L'_p$ when K_p is an unramified extension of F_p .*

Corollary 3.4. $0 \leq m_p(x) \leq e - h$ for all primitive $x \in L_p$.

Proof. Let $f(x, z) = 1$ where $z \in L_p$. In the nonsplit dyadic cases take $a \in \mathfrak{P}^{e-h}$. Then $az \in L'_p$ and consequently $m_p(x) \leq e - h$. $\square$

Since L_p is a unimodular $\mathfrak{O}_p$ -lattice with rank at least three, it is split by a hyperbolic plane (if p splits in K this can be easily verified, otherwise see [Jc, 7.1, 8.1a, 10.3]). Hence $L_p = H_p \perp M_p$ where $H_p = \mathfrak{O}_p u + \mathfrak{O}_p v$ is a hyperbolic plane with $f(u) = f(v) = 0$ and $f(u, v) = 1$. This choice of u and v will be fixed throughout the local discussion.

Lemma 3.5. *Let F_p be a dyadic local field with p ramified in K . Then $L_p = H_p \perp J_p \perp B_p$ where J_p is a sum of hyperbolic planes and rank $B_p \leq 2$. If n is odd, $B_p = \mathfrak{O}_p w$, with $f(w)$ a unit, and*

$$L'_p = H_p \perp J_p \perp \mathfrak{P}^{e-h}w.$$

If n is even, then $B_p = \mathfrak{O}_p z + \mathfrak{O}_p w$ with $f(z, w) = 1$ and $z \in L'_p$, and

$$L'_p = H_p \perp J_p \perp (\mathfrak{O}_p z + \mathfrak{P}^k w)$$

where $k = \max\{0, e - h - \text{ord}_p f(w)\}$.

Proof. The splitting of L_p follows from [Jc, 10.3]. If $B_p = \mathfrak{O}_p z + \mathfrak{O}_p w$ with $f(z, w) = 1$ we can arrange that $z \in L'_p$ as follows. Replace z by $z' = z + aw$ with $a \in \mathfrak{O}_p$ chosen such that $N_p(a) \equiv f(z)f(w)^{-1} \pmod{2p^{-h}}$ by Lemma 3.2. Then $p^h f(z') \in 2\mathfrak{o}_p$. Multiply z' by a unit to recover $f(z', w) = 1$. The structure of L'_p follows from Lemma 3.3. $\square$

We now define the standard isometries needed in the unitary group $U(L_p)$. Put $\mu = (1, 0)$ when p is split. Otherwise, fix $2\mu = 1$ except when F_p is dyadic and K_p is either an unramified or a ramified unit extension of F_p ; in these exceptional cases fix $2\mu = 1 + \zeta \in p^h \mathfrak{O}_p$. Then $T_p(\mu) = 1$. For $s \in M'_p = M_p \cap L'_p$, define the *Eichler transformation* $E(u, s)$ by

$$E(u, s)(x) = x - f(u, x)s + f(s, x)u - \mu f(s)f(u, x)u, \quad x \in L_p.$$

Then $E(u, s) \in SU(L_p)$ (note that $\mu f(s) \in \mathfrak{O}_p$ by Lemma 3.3). Let $\mathcal{E}$ denote the subgroup of $SU(L_p)$ generated by the Eichler transformations $E(u, s)$ and $E(v, s)$ with $s \in M'_p$.

Let λ in $\mathfrak{O}_p$ have $T_p(\lambda) = 0$. The *transvection* $T_\lambda(u)$ is defined by

$$T_\lambda(u)(x) = x + \lambda f(u, x)u, \quad x \in L_p.$$

Then $T_\lambda(u)$ and $T_\lambda(v)$ belong to $SU(L_p)$.

Let $\nu \neq 0$ in $\mathfrak{O}_p$ satisfy $T_p(\nu) = N_p(\nu)$. For $r \in L_p$ with $\nu f(r)^{-1}$ in $\mathfrak{O}_p$, define the *quasi-symmetry* $\Psi_\nu(r)$ by

$$\Psi_\nu(r)(x) = x - \nu f(r, x)f(r)^{-1}r, \quad x \in L_p.$$

Then $\Psi_\nu(r) \in U(L_p)$ and $\det \Psi_\nu(r) = 1 - \nu$.

Lemma 3.6. $U(H_p)$ is generated by quasi-symmetries and transvections.

Proof. We will reduce $\varphi \in U(H_p)$ to the identity using quasi-symmetries and transvections. Let $\varphi(u) = au + bv$. Since $\Psi_2(u - v)$ interchanges u and v , assume $a \in \mathfrak{u}_p$. Put $\lambda = -ba^{-1}$. Then $T_p(\lambda) = 0$, since $f(au + bv) = 0$, and $T_\lambda(v)\varphi(u) = au$. If $T_p(a) \neq 0$, put $r = au - v$ and $\nu a = T_p(a) = -f(r)$ so that $T_p(\nu) = N_p(\nu)$. Then $\Psi_\nu(r)(au) = au - r = v$ and we may assume $\varphi(u) = u$ after applying $\Psi_2(u - v)$. If, however, $T_p(a) = 0$, then $T_{a^*}(v)(au) = au + N_p(a)v$ with $N_p(a) \in \mathfrak{u}_p$. After interchanging u and v , it can again be assumed that $\varphi(u) = u$ since $T_p(N_p(a)) \neq 0$. Now $\varphi(v) = cu + v$ with $T_p(c) = 0$ and the proof can be completed with the transvection $T_{-c}(u)$.

Proof of Theorem 3.1 (necessity). Condition (i) is clearly necessary and, except in the ramified dyadic situations, (ii) and (iii) are vacuous since $L_p = L'_p$. Assume, therefore, p is ramified dyadic and there exists $\varphi \in SU(L_p)$ with $\varphi(x) = y$. Condition (ii) is necessary since $\varphi(L'_p) = L'_p$. It remains to show that (iii) is satisfied. Let $L_p = H_p \perp J_p \perp B_p$ as in Lemma 3.5. By Theorem 4.2 in [J3], $U(L_p)$ is generated by $\mathcal{E}$, $U(H_p)$ and at most one symmetry in $U(B_p)$. Observe first that $\theta(x) \equiv x \pmod{\mathfrak{P}^m L'_p}$ when θ is an Eichler transformation $E(u, s)$ or $E(v, s)$ with $s \in M'_p$, or when θ is an element in $U(H_p)$ (since $f(\theta(x) - x, H_p) \subset \mathfrak{P}^m$). Moreover, if $\Psi_\lambda(t) \in U(H_p)$, then $f(t) \in 2p^{-h}\mathfrak{o}_p$ and, consequently, $\lambda \in 2p^{-h}\mathfrak{O}_p$. It remains to study the effect of a quasi-symmetry Ψ_ν from $U(B_p)$ in φ . Since $\det \varphi = 1$ and $\det \Psi_\nu = 1 - \nu$, and since $U(H_p)$ is generated by transvections and quasi-symmetries, there will also have to be quasi-symmetry from $U(H_p)$ in φ ; consequently $\nu \in 2p^{-h}\mathfrak{O}_p$. When $B_p = \mathfrak{O}_p w$, $f(w)$ is a unit, $\mathfrak{P}^{e-h}w \subset L'_p$ and $\Psi_\nu(w)(x) \equiv x \pmod{\mathfrak{P}^m L'_p}$ since $m \leq e - h$. When $\text{rank } B_p = 2$, take B_p as in Lemma 3.5. If $k = 0$ there is nothing to prove since $L_p = L'_p$. Finally assume $k \geq 1$. The quasi-symmetry needed is $\Psi_\nu(r)$ with $r = w - f(w)z$ (see [J3, p. 477]). Then $\Psi_\nu(r)(x) \equiv x \pmod{\mathfrak{P}^m L'_p}$ since $\mathfrak{P}^k r \subset L'_p$.

Proof of Theorem 3.1 (sufficiency). Assume primitive x, y in L_p satisfy the three conditions given in Theorem 3.1. Let $x = au + bv + r$ with $a, b \in \mathfrak{P}^m$ and $r \in M_p$. We prove first there exists $\varphi \in SU(L_p)$ such that $\varphi(x) = \pi^m u + b'v + r'$. Note that when the quasi-symmetry $\Psi_\nu(t)$ lies in $U(H_p)$, with $t \in H_p$, then $\Psi_\nu(t)\Psi_\nu(w)^{-1} \in SU(L_p)$, where w is as in Lemma 3.5 (since $\text{ord}_p f(w) \leq \text{ord}_p f(t)$). Assume, therefore, $\text{ord}_p a \leq \text{ord}_p b$ (otherwise use $\Psi_2(u-v)$ to interchange u and v). If $\text{ord}_\pi a > m$ there exists $t_1 \in M'_p$ such that $\text{ord}_\pi f(t_1, r) = m$. The coefficient of u in $E(u, t_1)(x)$ now has order m and we may assume $a = \pi^m \varepsilon$ with $\varepsilon \in \mathfrak{U}_p$. Apply the isometry $u \mapsto \varepsilon^{-1}u$, $v \mapsto \varepsilon^*v$ to x ; although this isometry needed not be in $SU(L_p)$, by Lemma 3.6 it can be multiplied by a quasi-symmetry $\Psi_\nu(w)$ so that the product is in $SU(L_p)$. We may now assume $a = \pi^m$. Likewise, assume $y = au + cv + s$ with $s \in M_p$. By condition (iii), $x - y = (b' - c)v + (r' - s) \in \mathfrak{P}^m L'_p$. Hence $t' = a^{-1}(r' - s) \in M'_p$ and $E(v, t')(x) = au + c'v + s$. Put $\lambda = a^{-1}(c - c')$. Then $T_p(\lambda) = 0$, since $f(x) = f(y)$, and $T_\lambda(v)E(v, t')(x) = y$, completing the proof.

Corollary 3.7. Assume $\text{rank } L_p = n \geq 3$ and p is not ramified dyadic. Then $N(L_p, c) = 1$ for any $c \in \mathfrak{o}_p$.

Proof. First $N(L_p, c) \leq 1$ by Theorem 3.1 since $L_p = L'_p$. Also, given $c \in \mathfrak{o}_p$, there now exists $a \in \mathfrak{D}_p$ with $T_p(a) = c$. Put $x = u + av$. Then $f(x) = c$ and $N(L_p, c) \geq 1$.

Proof of Theorem 1.1. Partition $L(c)$ into orbits under $SU(L)$ and let $O(L, c)$ denote the collection of these orbits. Then $N(L, c) = |O(L, c)|$ and there exists a natural mapping

$$\Gamma: O(L, c) \rightarrow \prod_{\mathfrak{p}} O(L_{\mathfrak{p}}, c)$$

into the Cartesian product of the corresponding local orbits $O(L_p, c)$. By Corollary 3.7, the product is essentially over the dyadic primes $p \in S$ which ramify in K . The map is injective by Theorem 2.1 so it remains to show that Γ is surjective. We are given primitive $x_p \in L_p$ with $f(x_p) = c$ for each dyadic $p \in S$ that ramifies. Since $n \geq 3$ there exist similar $x_p \in L_p$ for all remaining $p \in S$. Also, by hypothesis, there exists primitive $r \in L$ with $f(r) = c$. By Witt's theorem, as in the proof of Theorem 2.1, there exist $\theta_p \in SU(V_p)$ such that $\theta_p(x_p) = r$ for each dyadic $p \in S$ that ramifies, and by Theorem 3.1 corresponding θ_p in $SU(L_p)$ for the remaining $p \in S$. There now exists $\psi \in SU(V)$, by strong approximation [S, 5.12], such that $\|\psi - \theta_p^{-1}\|_p$ is small for all ramifying dyadic primes, while $\|\psi\|_p = 1$ at the remaining $p \in S$. Put $y = \psi(r)$ so that $f(y) = c$. Then y is close to x_p , and hence primitive in L_p , for all ramifying dyadic p . For the remaining $p \in S$, $\psi \in SU(L_p)$ and hence $y \in L$ is primitive. Moreover, $\psi\theta_p \in SU(L_p)$ and $\psi\theta_p(x_p) = y$ for all ramifying dyadic primes. Hence the orbit of y in $O(L, c)$ is mapped by Γ onto the product of the orbits of x_p , for the ramifying dyadic primes.

Remark. Theorem 1.1 is still true if the assumption " L represents c " is replaced by " V_q represents c for all $q \notin S$." Then there exists $r \in V$ with $f(r) = c$ by the Hasse Minkowski Theorem. Since $r \in L_p$ for almost all $p \in S$, the above proof is easily modified by including the finite number of exceptions in the approximations $\|\psi - \theta_p^{-1}\|_p$ small.

4. LOCAL REPRESENTATIONS: n ODD

We now compute $N(L_p, c)$ when $n = \text{rank } L_p$ is odd and p is ramified dyadic. As in Lemma 3.5,

$$L_p = H_p \perp J_p \perp \mathfrak{O}_p w \quad \text{and} \quad L'_p = H_p \perp J_p \perp \mathfrak{P}^{e-h} w.$$

The discriminant $dL_p = (-1)^{(n-1)/2} f(w) N_p(\mathfrak{U}_p)$ is an invariant of L_p . The rank and the discriminant determine L_p when n is odd by [Jc, 10.4]. Thus for each odd rank n , there exist two classes of unimodular lattices. Define $\delta(L_p) = 1$ when $dL_p = (-1)^{(n-1)/2} N_p(\mathfrak{U}_p)$, and $\delta(L_p) = -1$ otherwise. When $\delta(L_p) = 1$ we can assume $f(w) = 1$.

Lemma 4.1. *Assume $n \geq 3$ odd with p ramified dyadic. Then*

- (i) $N(L_p, c) \geq 1$ for all $c \in \mathfrak{o}_p$,
- (ii) $N(L_p, c) = 1$ for all $c \in p\mathfrak{o}_p$.

Proof. Let $c \in \mathfrak{o}_p$ and $x = u + bv + aw$ with $a \in \mathfrak{O}_p$ chosen by Lemma 3.2 such that $N_p(a)f(w) \equiv c \pmod{2p^{-h}}$. Take $b \in \mathfrak{O}_p$ such that $f(x) = T_p(b) + N_p(a)f(w) = c$. Thus $N(L_p, c) \geq 1$. Now assume characteristic $y \in L_p$ represents $c \in p\mathfrak{o}_p$. Then $f(y, L'_p) \subset \mathfrak{P}$ and $y \equiv aw \pmod{\mathfrak{P}L_p}$ for some unit $a \in \mathfrak{U}_p$. But then $c = f(y) \equiv N_p(a)f(w) \pmod{\mathfrak{P}}$ and c must be a unit. It follows that $N(L_p, c) \leq 1$, since all noncharacteristic representations are equivalent by Theorem 3.1.

Lemma 4.2. *Assume $n \geq 3$ is odd with p ramified prime. Then*

$$N(L_p, c) = 1 + e \quad \text{for all units } c \in \mathfrak{u}_p.$$

Proof. Let $x = \zeta^m(u + bv) + aw$ with $0 \leq m \leq e$, $b \in \mathfrak{o}_p$ and $a \in \mathfrak{U}_p$. Then $f(x, L'_p) = \mathfrak{P}^m$ so that $m_p(x) = m$. Also, $f(x) = 2p^m b + N_p(a)f(w) = c$ provided the congruence $N_p(a)f(w) \equiv c \pmod{2p^m}$ can be solved for $a = \alpha + \zeta\beta \in \mathfrak{O}_p$. Since $m \leq e$, the congruence

$$N_p(a)f(w) = (\alpha^2 - p\beta^2)f(w) \equiv c \pmod{2p^m}$$

can be solved for $\alpha, \beta \in \mathfrak{o}_p$ inductively through successive powers of p . Hence there exist $1 + e$ inequivalent representations of c , one for each value of m . Moreover, if y is a second representation of c with $m_p(y) = m$, then $y = \zeta^m r + a'w$ for some $r \in H_p \perp J_p$ and $a' \in \mathfrak{U}_p$. Take x as above. Since $f(r) \in T_p(\mathfrak{O}_p) = 2\mathfrak{o}_p$, it follows that $N_p(a^{-1}a') \equiv 1 \pmod{2p^m}$ and, consequently, $a \equiv a' \pmod{\mathfrak{P}^{m+e}}$. Thus $x - y \in \mathfrak{P}^m L'_p$ and the two representations are equivalent by Theorem 3.1. Hence $N(L_p, c) = 1 + e$.

Lemma 4.3. *Assume $n \geq 3$ is odd and p is ramified unit. Then there exists a characteristic representation of $c \in \mathfrak{u}_p$ with degree m , $1 \leq m \leq e - h$, if and only if there exists $a \in \mathfrak{U}_p$ with $N_p(a) \equiv cf(w)^{-1} \pmod{2p^{m-h}}$.*

Proof. Assume $x = \pi^m r + aw$ represent c , where $r \in H_p \perp J_p$ and $a \in \mathfrak{U}_p$. Then $c = f(x) \equiv N_p(a)f(w) \pmod{2p^{m-h}}$. Conversely, if $a \in \mathfrak{U}_p$ satisfies this congruence, put $x' = \pi^m(u + bv) + aw$ with $b \in \mathfrak{O}_p$ chosen such that $f(x') = c$.

Corollary 4.4. *Assume $f = e - h = 1$ with p ramified unit. Then L_p characteristically represents c if and only if $c \equiv f(w) \pmod{p^2}$.*

Proof. Let $a = \alpha + \pi\beta \in \mathfrak{U}_p$ so that $N_p(a) = \alpha^2 + 2p^{-h}\alpha\beta + p\delta\beta^2$. Since $|\mathfrak{o}_p/p| = 2$, it follows that $\alpha \equiv \delta \equiv 1 \pmod{p}$ and, by the lemma there is a characteristic representation if and only if $c \equiv f(w) \pmod{p^2}$.

Lemma 4.5. Assume $n \geq 3$ is odd, $h = e - 1$ and $\mathfrak{p}$ is ramified unit. Then $N(L_{\mathfrak{p}}, c) = 1$ or 3 .

Proof. By Theorem 3.1 and Lemma 4.3, $N(L_{\mathfrak{p}}, c) = 1$ unless there is an $a = \alpha + \pi\beta \in \mathfrak{U}_{\mathfrak{p}}$ with $N_{\mathfrak{p}}(a) = \alpha^2 + 2p^{-h}\alpha\beta + p\delta\beta^2 \equiv cf(w)^{-1} \pmod{p^2}$. Moreover, when such an $a \in \mathfrak{U}_{\mathfrak{p}}$ exists, there is a characteristic representation of $c = f(x)$ with $x = \pi(u + bv) + aw$. Let $a' = a + \pi\gamma$ where $\gamma \in \mathfrak{u}_{\mathfrak{p}}$ and $\gamma\delta \equiv 2p^{-e}\alpha \pmod{p}$. Then $N_{\mathfrak{p}}(a') \equiv N_{\mathfrak{p}}(a) \pmod{p^2}$ and there exists $b' \in \mathfrak{O}_{\mathfrak{p}}$ such that $c = f(x')$ with $x' = \pi(u + b'v) + a'w$. Therefore $x - x' \notin \mathfrak{P}L'_{\mathfrak{p}}$ and $N(L_{\mathfrak{p}}, c) \geq 3$. Now let $y = \pi r + dw$ be any characteristic representation of c . Then $N_{\mathfrak{p}}(da^{-1}) \equiv 1 \pmod{p^2}$ and it follows that $d \equiv a, a' \pmod{p}$. Hence $y - x \in \mathfrak{P}L'_{\mathfrak{p}}$ or $y - x' \in \mathfrak{P}L'_{\mathfrak{p}}$ so that $N(L_{\mathfrak{p}}, c) \leq 3$ by Theorem 3.1.

5. LOCAL REPRESENTATIONS: n EVEN

Assume $n = \text{rank } L_{\mathfrak{p}} \geq 4$ is even and $\mathfrak{p}$ is ramified dyadic. Then, as in Lemma 3.5,

$$L_{\mathfrak{p}} = H_{\mathfrak{p}} \perp J_{\mathfrak{p}} \perp (\mathfrak{O}_{\mathfrak{p}}z + \mathfrak{O}_{\mathfrak{p}}w) \quad \text{and} \quad L'_{\mathfrak{p}} = H_{\mathfrak{p}} \perp J_{\mathfrak{p}} \perp (\mathfrak{O}_{\mathfrak{p}}z + \mathfrak{P}^k w)$$

where $k = \max\{0, e - h - \text{ord}_{\mathfrak{p}} f(w)\}$. There are no characteristic representations when $k = 0$ since then $L_{\mathfrak{p}} = L'_{\mathfrak{p}}$, that is, $L_{\mathfrak{p}}$ is an even lattice. The discriminant $dL_{\mathfrak{p}} = (-1)^{(n-2)/2}(f(z)f(w) - 1)N_{\mathfrak{p}}(\mathfrak{U}_{\mathfrak{p}})$ is an invariant of $L_{\mathfrak{p}}$. Define $\delta(L_{\mathfrak{p}}) = 0$ when $dL_{\mathfrak{p}} = (-1)^{n/2}N_{\mathfrak{p}}(\mathfrak{U}_{\mathfrak{p}})$; in particular, we can assume that $f(z) = 0$ in this case. Otherwise, define $\delta(L_{\mathfrak{p}}) = 2$. The invariants n , $dL_{\mathfrak{p}}$ (or $\delta(L_{\mathfrak{p}})$) and $\text{ord}_{\mathfrak{p}} f(w)$ uniquely determine $L_{\mathfrak{p}}$ by [Jc, 10.4].

Lemma 5.1. Assume $n \geq 4$ is even and $c \in \mathfrak{o}_{\mathfrak{p}}$ with $\mathfrak{p}$ ramified dyadic. Then

- (i) $N(L_{\mathfrak{p}}, c) \leq 1$ when c is a unit,
- (ii) $N(L_{\mathfrak{p}}, c) = 0$ for all $c \in \mathfrak{o}_{\mathfrak{p}}$ with $\text{ord}_{\mathfrak{p}} c < e - h - k$,
- (iii) $N(L_{\mathfrak{p}}, c) \geq 1$ for all $c \in \mathfrak{o}_{\mathfrak{p}}$ with $\text{ord}_{\mathfrak{p}} c \geq e - h - k$,
- (iv) $N(L_{\mathfrak{p}}, c) = 1$ when $k = 0$ for all $c \in \mathfrak{o}_{\mathfrak{p}}$ with $\text{ord}_{\mathfrak{p}} c \geq e - h$.

Proof. Assume primitive $x \in L_{\mathfrak{p}}$ represents $c \in \mathfrak{o}_{\mathfrak{p}}$. If x is characteristic, then $f(x, L'_{\mathfrak{p}}) \subset \mathfrak{P}$ and $x \equiv az \pmod{\mathfrak{P}L_{\mathfrak{p}}}$ for some unit $a \in \mathfrak{U}_{\mathfrak{p}}$. Then $c = f(x) \equiv N_{\mathfrak{p}}(a)f(z) \equiv 0 \pmod{\mathfrak{P}}$, since $f(z) \in 2p^{-h}\mathfrak{o}_{\mathfrak{p}}$ by Lemma 3.3. This proves (i) since there is at most one noncharacteristic representation (up to equivalence). Since $\text{ord}_{\mathfrak{p}} f(w) \geq e - h - k$, the lattice $L_{\mathfrak{p}}$ cannot represent any element c with $\text{ord}_{\mathfrak{p}} c < e - h - k$; this proves (ii). As in Lemma 4.1, there is a noncharacteristic representation of each $c \in \mathfrak{o}_{\mathfrak{p}}$ with $\text{ord}_{\mathfrak{p}} c \geq e - h - k$. When $k = 0$ this is the only representation (up to equivalence).

Lemma 5.2. Assume $n \geq 4$ is even, $e = 1$, $k > 0$ and $\mathfrak{p}$ ramified prime. Then $N(L_{\mathfrak{p}}, c) = 2^f$ for $c \in 2\mathfrak{o}_{\mathfrak{p}}$.

Proof. Since $k > 0$ and $e = 1$ it follows that $k = 1$ and $f(w)$ is a unit. Also $f(z) \in 2\mathfrak{o}_{\mathfrak{p}}$ by Lemma 3.3. Put $x = \zeta(u + bv) + dz + a\zeta w$ with $a, b \in \mathfrak{o}_{\mathfrak{p}}$ and $d \in \mathfrak{u}_{\mathfrak{p}}$. Then $f(x) = 2pb + d^2f(z) + pa^2f(w)$. For fixed d , choose $a \in \mathfrak{o}_{\mathfrak{p}}$ such that $pa^2f(w) \equiv c - d^2f(z) \pmod{4}$, and then b such that $f(x) = c$. Varying d , this gives $2^f - 1$ inequivalent characteristic representations of c . Hence $N(L_{\mathfrak{p}}, c) \geq 2^f$, since there also exists a noncharacteristic representation of c .

from Lemma 5.1. Let $y = \zeta r + d'z + a'\zeta w$ be a characteristic representation of c , where r lies in $H_{\mathfrak{p}} \perp J_{\mathfrak{p}}$. Then $d \equiv d' \pmod{\mathfrak{P}}$ for one of the x 's above and $y - x \equiv (a' - a)\zeta w \pmod{\mathfrak{P}L'_{\mathfrak{p}}}$. Since $f(y) = c = f(x)$, it follows that $N_{\mathfrak{p}}(a') \equiv a^2 \pmod{2}$ and hence $a' \equiv a \pmod{\mathfrak{P}}$. Thus x and y are equivalent representations, and $N(L_{\mathfrak{p}}, c) \leq 2^f$.

Lemma 5.3. *Assume $n \geq 4$ is even, with $\mathfrak{p}$ ramified unit, $h = e - 1$, $k > 0$ and $\delta(L_{\mathfrak{p}}) = 0$. Then*

- (i) $N(L_{\mathfrak{p}}, c) = 2^{f+1} - 1$ when $\text{ord}_{\mathfrak{p}} c \geq 2$,
- (ii) $N(L_{\mathfrak{p}}, c) = 2^f - 1$ when $\text{ord}_{\mathfrak{p}} c = 1$.

Proof. Since $k > 0$ and $h = e - 1$, it follows that $k = 1$ and $f(w)$ is a unit. Since $dL_{\mathfrak{p}} = (-1)^{n/2}N_{\mathfrak{p}}(\mathfrak{U}_{\mathfrak{p}})$ and $\text{ord}_{\mathfrak{p}} f(w)$ determine $L_{\mathfrak{p}}$, we may assume $f(z) = 0$ and $f(w) = 1$. Put $x = \pi(u + bv) + dz + a\pi w$ with $a \in \mathfrak{o}_{\mathfrak{p}}$, $b \in \mathfrak{O}_{\mathfrak{p}}$ and $d \in \mathfrak{u}_{\mathfrak{p}}$. Then $f(x) = p\delta T_{\mathfrak{p}}(b) + ad2p^{-h} + p\delta a^2$.

Assume first $\text{ord}_{\mathfrak{p}} c \geq 2$. Fix $d \in \mathfrak{u}_{\mathfrak{p}}$ and put $a = 0$ or $a = -2p^{-e}d\delta^{-1} \in \mathfrak{u}_{\mathfrak{p}}$, and choose $b \in \mathfrak{O}_{\mathfrak{p}}$ such that $f(x) = c$. Since there are $2^f - 1$ choices for $d \pmod{p}$, this gives $2(2^f - 1)$ inequivalent characteristic representations of c , and hence $N(L_{\mathfrak{p}}, c) \geq 2^{f+1} - 1$. Conversely, if $y = \pi r + d'z + a'\pi w$ represents c , then $d' \equiv d \pmod{\mathfrak{P}}$ for one of the x 's constructed above. It follows that $a' \equiv 0 \pmod{\mathfrak{P}}$ or $a' \equiv 2p^{-e}d\delta^{-1} \pmod{\mathfrak{P}}$, and hence y is equivalent to one of these x 's. Thus $N(L_{\mathfrak{p}}, c) = 2^{f+1} - 1$.

Now assume $p^{-1}c$ is a unit. Put $b = 0$ and choose $a \in \mathfrak{u}_{\mathfrak{p}}$ such that $a^2\delta \not\equiv p^{-1}c \pmod{P}$. Then choose $d \in \mathfrak{u}_{\mathfrak{p}}$ so that $f(x) = c$. Since there are $2^f - 2$ choices for $a \pmod{p}$, this gives $2^f - 2$ inequivalent characteristic representations of c . Hence $N(L_{\mathfrak{p}}, c) \geq 2^f - 1$. Conversely, any characteristic representation of c is equivalent to one of those just constructed, so that $N(L_{\mathfrak{p}}, c) = 2^f - 1$.

Lemma 5.4. *Assume $n \geq 4$ is even, with $\mathfrak{p}$ ramified unit, $h = e - 1$, $k > 0$ and $\delta(L_{\mathfrak{p}}) = 2$. Then*

- (i) $N(L_{\mathfrak{p}}, c) = 1$ when $\text{ord}_{\mathfrak{p}} c \geq 2$,
- (ii) $N(L_{\mathfrak{p}}, c) = 2^f + 1$ when $\text{ord}_{\mathfrak{p}} c = 1$.

Proof. Since $\delta(L_{\mathfrak{p}}) = 2$ and $k = 1$, $V_{\mathfrak{p}}$ cannot be hyperbolic so that $f(w)$ is a unit and the binary lattice $\mathfrak{O}_{\mathfrak{p}}z + \mathfrak{O}_{\mathfrak{p}}w$ must be anisotropic. Hence

$$\text{ord}_{\mathfrak{p}} f(dz + \pi aw) = 1$$

for all $d \in \mathfrak{u}_{\mathfrak{p}}$ and $a \in \mathfrak{O}_{\mathfrak{p}}$. Therefore, $L_{\mathfrak{p}}$ cannot characteristically represent any $c \in p^2\mathfrak{o}_{\mathfrak{p}}$, proving (i). Now assume $\text{ord}_{\mathfrak{p}} c = 1$. Fix $a \in \mathfrak{o}_{\mathfrak{p}}$. Since $f(z + \pi aw) = p\varepsilon$ for some unit ε , there exist $d \in \mathfrak{u}_{\mathfrak{p}}$ and $b \in \mathfrak{O}_{\mathfrak{p}}$ such that $x = \pi(u + bv) + d(z + \pi aw)$ represents c . Hence there exist 2^f inequivalent characteristic representations of c and $N(L_{\mathfrak{p}}, c) \geq 2^f + 1$. Conversely, since $a \pmod{\mathfrak{P}}$ uniquely determines $d \pmod{\mathfrak{P}}$, any characteristic representation of c is equivalent to one constructed above. Hence $N(L_{\mathfrak{p}}, c) = 2^f + 1$.

6. QUADRATIC EXTENSIONS OF $\mathbf{Q}$

Let $F = \mathbf{Q}$, $K = \mathbf{Q}(\sqrt{m})$, with m a square free integer, and $\mathfrak{o}_F = \mathbf{Z}$. Let p be a rational prime. Then p splits in K if either $p = 2$ and $m \equiv 1 \pmod{8}$,

or p is odd and $\left(\frac{m}{p}\right) = 1$. Otherwise, for $p = 2$, we have an unramified extension if $m \equiv 5 \pmod{8}$, a ramified unit extension with $h = 0$ if $m \equiv 3 \pmod{4}$, and a ramified prime extension if m is even. Since $-1 \notin N_p(\mathcal{U}_p)$ at the dyadic prime when $m \equiv 3 \pmod{4}$, we may assume $f(w) = \delta(L_p)$ in L_p when n is odd, and $f(z) = \delta(L_p)$ and $f(w) = 1$ when n is even and L_p is not even, in this situation.

Theorem 6.1. *Let L be a unimodular lattice on an indefinite hermitian space V of dimension $n \geq 3$ over $K = \mathbf{Q}(\sqrt{m})$. Let $c \neq 0$ be a rational integer represented by L , and assume the local Witt index of $V \perp \langle -c \rangle$ at the real prime is at least two when $m < 0$. Then*

- (i) $N(L, c) = 3$ when $m \equiv 3 \pmod{4}$, $c \equiv \delta(L_p) \pmod{4}$ and L_p is not even,
- (ii) $N(L, c) = 2$ when $m \equiv 2 \pmod{4}$, $n \equiv c \pmod{2}$ and L_p is not even,
- (iii) $N(L, c) = 1$ otherwise.

Proof. This follows from Theorem 1.1 and the local results in the two previous sections since $e = f = 1$ for any ramifying dyadic prime.

Remark. This theorem also provides information about the orbits in $L(c)$ under the action of the group $U(L)$. Clearly, when $N(L, c) = 1$ there is also only one orbit under the action of $U(L)$. However, when $N(L, c) > 1$ there will also be at least two orbits under $U(L)$, for no locally characteristic vector in $L(c)$ can be mapped by $U(L)$ into a locally noncharacteristic vector.

Example. Assume $m \equiv 3 \pmod{4}$ and L is a free $\mathcal{O}_K$ -module with orthogonal basis $x_1, \dots, x_n$, where $f(x_i) = -1$ for $1 \leq i \leq r$ and $f(x_i) = 1$ for $r+1 \leq i \leq n$. Then $dL_2 = (-1)^r \in N_2(\mathcal{U}_2)$ if and only if r is even (and then $\delta(L_2) = 0$ or 1). Assume $1 \leq r < n$ where $m < 0$ so that L is indefinite. Let $c \in \mathbf{Z}$ be nonzero. When $m < 0$, assume $r \geq 2$ if $c < 0$, and $r \leq n-2$ if $c > 0$, so that the index condition on $V \perp \langle -c \rangle$ is satisfied. Then $N(L, c) = 3$ if either n is even and $c \equiv 2r \pmod{4}$, or n is odd and $c \equiv 2r+1 \pmod{4}$; otherwise $N(L, c) = 1$.

7. CYCLOTOMICS FIELDS

We now consider hermitian forms over the cyclotomic field $K = \mathbf{Q}(\omega)$, with ω a primitive m th root of unity, and $F = \mathbf{Q}(\omega) \cap \mathbf{R} = \mathbf{Q}(\omega + \omega^*)$ the maximal real subfield of K . The involution $*$ on K is complex conjugation. Let $\mathfrak{o} = \mathfrak{o}_F$ be the ring of algebraic integers in F .

Theorem 7.1. *Let L be a unimodular lattice on an hermitian space V of dimension $n \geq 3$ over $K = \mathbf{Q}(\omega)$ where ω is a primitive m th root of unity and $m \geq 3$ is odd. Assume L represents $c \in \mathfrak{o}_F$, $c \neq 0$, and the local Witt index of $V \perp \langle -c \rangle$ is at least two for some archimedean prime spot. Then $N(L, c) = 1$.*

Proof. When $m \geq 3$ is odd, the prime 2 is unramified in K and, consequently, $N(L, c) \leq 1$ for any nonzero $c \in \mathfrak{o}_F$, by Theorem 1.1. $\square$

When $m = 2^k \geq 4$, it follows that $\iota \in K$ where $\iota^2 = -1$, and hence $K = F(\iota)$. Moreover, 2 totally ramifies in K . When $\mathfrak{p} \in S$ is the unique

dyadic prime spot, K_p/F_p is a dyadic ramified unit extension with $e = \text{ord}_2 p = \frac{1}{2}\phi(2^k) = 2^{k-2}$. Since $\iota^2 = 1 - p^e \varepsilon$ with ε a unit in $\mathfrak{o}_p$, it follows that $h = 0$ when $e = 1$, and $e > h \geq \frac{1}{2}(e - 1)$, in general. Therefore, $e = 2$ and $h = 1$ when $m = 8$.

Lemma 7.2. *Let $K = \mathbf{Q}(\omega)$ where ω is a primitive m th root of unity with $m = 2^k \geq 4$. If $\mathfrak{p}$ is the unique prime spot over 2, then K_p/F_p is a ramified unit extension with $e = m/4$ and $h = e - 1$.*

Proof. It remains to show $h = e - 1$ when $m \geq 16$. Put $p = \omega + \omega^* \in \mathfrak{o}_p$. Since e is a 2-power, $\binom{e}{e/2} \equiv 2 \pmod{4}$, and $\binom{e}{i} \equiv 0 \pmod{4}$ for $1 \leq i < e/2$. Hence $\omega^e = \iota$ and $p^e = (\omega + \omega^*)^e \equiv 2 \pmod{4}$, so that p is prime in F_p . Put

$$a = 1 + p^{e/2} + p^{3e/4} + p^{7e/8} + \cdots + p^{(e-1)e/e}.$$

Then $a \in \mathfrak{o}_p$ and $(a\iota)^2 \equiv 1 - p^{2e-1} \pmod{4}$. Since $K_p = F_p(a\iota)$ it follows that $h = e - 1$. $\square$

Now return to the general cyclotomic field $K = \mathbf{Q}(\omega)$ where $m = 2^k m'$ with $k \geq 2$ and m' odd. Let $\mathfrak{p}$ be a dyadic prime spot in S . Denote by g the number of dyadic spots in S . Each $\mathfrak{p}$ will now ramify in K . As before, let $e = \text{ord}_p 2$ be the ramification index in F_p and hence also in F . Then $e = 2^{k-2}$, since 2 is unramified in $\mathbf{Q}(\omega')$ where ω' is a primitive m' th root of unity. Let f denote the residue class degree of p in F (and hence also in $K = F(\iota)$). Then $f \geq 1$ is minimal such that $2^f \equiv 1 \pmod{m'}$ (with $f = 1$ when $m' = 1$) and $2efg = \phi(m)$. Hence $fg = \phi(m')$. Define $g' = g'(L)$ ($\leq g$) to be the number of dyadic primes $\mathfrak{p}$ where $\delta(L_p) = 1$ when n is odd, and where $\delta(L_p) = 0$ and L_p is not an even lattice when n is even. Define g'' ($\leq g - g'$) to be the number of dyadic primes where $\delta(L_p) = 2$ and L_p is not even.

Theorem 7.3. *Let L be a unimodular lattice on an hermitian space V of dimension $n \geq 3$ over $K = \mathbf{Q}(\omega)$, where ω is a primitive m th root of unity. Assume L represents the nonzero integer $c \in \mathbf{Z}$ and that the local Witt index of $V \perp \langle -c \rangle$ is at least two at some archimedean prime spot. Then, when $m \equiv 0 \pmod{8}$,*

- (i) $N(L, c) = (2^{f+1} - 1)^{g'}$ when $n \equiv c \equiv 0 \pmod{2}$,
- (ii) $N(L, c) = 3^{g'}$ when $n \equiv c \equiv 1 \pmod{2}$,
- (iii) $N(L, c) = 1$ otherwise;

and when $m \equiv 4 \pmod{8}$,

- (iv) $N(L, c) = (2^{f+1} - 1)^{g'}$ when $2n \equiv c \equiv 0 \pmod{4}$,
- (v) $N(L, c) = (2^f - 1)^{g'}(2^f + 1)^{g''}$ when $n \equiv 0 \pmod{2}$ and $c \equiv 2 \pmod{4}$,
- (vi) $N(L, c) = 3^{g'}$ when $n \equiv c \equiv 1 \pmod{2}$, provided f is even when $c \equiv 3 \pmod{4}$,
- (vii) $N(L, c) = 3^{g-g'}$ where n and f are odd, and $c \equiv 3 \pmod{4}$,
- (viii) $N(L, c) = 1$ otherwise.

Proof. Apply Theorem 1.1 and the local results in §§4 and 5. The condition $m \equiv 0 \pmod{8}$ ensures that locally $e \geq 2$ with $e - h = 1$ for any dyadic prime, so that $c \equiv 0, 1 \pmod{p^2}$. However, $e = 1$ and $p = 2$ when $m \equiv 4 \pmod{8}$. Parts (i), (iv) and (v) then follow from Lemmas 5.3 and 5.4.

Now assume n is odd. As in Lemma 7.2, take $p^e \equiv 2 \pmod{4}$ and $\zeta^2 \equiv 1 - p^{2e-1} \pmod{4}$. Consider first $\delta(L_p) = -1$. Choose $b \in \mathfrak{u}_p$ such that $X^2 + X \equiv b \pmod{p}$ has no roots; in particular, take $b = 1$ when f is odd. Put $f(w) = 1 + pb \notin N_p(\mathfrak{u}_p)$. When $e \geq 2$, the congruence in Lemma 4.3 has no solutions for c odd, and hence there are no characteristic representations of c . When $e = 1$, so that $p = 2$ and $\pi = 1 + \iota$, the congruence has a solution if and only if $c \equiv 3 \pmod{4}$ and f is odd (that is when $b = 1$). Finally let $\delta(L_p) = 1 = f(w)$. If c is odd, the congruence in Lemma 4.3 can always be solved when $e \geq 2$, or when $c \equiv 1 \pmod{4}$. However, if $e = 1$ and $c \equiv 3 \pmod{4}$, the congruence can only be solved when f is even, since it reduces to solving $X^2 + X \equiv 1 \pmod{2}$. The remaining parts of the theorem now follow from Lemma 4.5.

Remarks. The value of $N(L, c)$ can be computed when $c \notin \mathbb{Z}$, but it now also depends on the varying values of $\text{ord}_p c$ at the dyadic primes. The comment following Theorem 6.1 also applies here.

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