A FATOU THEOREM FOR THE SOLUTION OF THE HEAT EQUATION AT THE CORNER POINTS OF A CYLINDER

KIN MING HUI

ABSTRACT. In this paper the author proves existence and uniqueness of the initial-Dirichlet problem for the heat equation in a cylindrical domain $D \times (0, \infty)$ where D is a bounded smooth domain in R^n with zero lateral values. A unique representation of the strong solution is given in terms of measures μ on D and λ on ∂D . We also show that the strong solution $u(x, t)$ of the heat equation in a cylinder converges a.e. $x_0 \in \partial D \times \{0\}$ as (x, t) converges to points on $\partial D \times \{0\}$ along certain nontangential paths.

INTRODUCTION

The existence and uniqueness of the initial-Dirichlet problem for the heat equation in a cylindrical domain $D \times (0, T)$ subject to Dirichlet boundary conditions $u|_{\partial D \times (0, T)} \equiv 0$ where D is a bounded smooth domain in R^n have been studied by a large number of researchers. (See [F, LSU, FGS].)

In this paper, by following the argument of Dahlberg and Kenig [DK2], I prove the existence and uniqueness of the nonnegative strong solution of the initial Dirichlet problem (IDP) for the heat equation in a cylinder $D \times (0, \infty)$, $D \in C^\infty$, with Dirichlet boundary condition $u_{\partial D \times (0, \infty)} \equiv 0$.

In fact I show that corresponding to each nonnegative strong solution $u(x, t)$ of IDP, there exists a pair of measures μ on D and λ on ∂D such that

$$u(x, t) = \int_D G(X, t; Q, 0) d\mu(Q) + \int_{\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) d\lambda(Q)$$

where $G(x, t; Q, s)$ is the Green function for the heat equation and $\partial/\partial N_Q$ is the derivative in the direction of the inward normal at Q .

I also find that the strong solution $u(x, t)$ of the heat equation in a cylinder converges a.e. $x_0 \in \partial D \times \{0\}$ as (x, t) converges to points on $\partial D \times \{0\}$ along certain nontangential path. In fact I prove that

$$\lim_{\substack{(x, t) \in \bar{\Gamma}_\beta(x_0) \\ t \rightarrow 0}} u(x, t) = \frac{\beta}{\sqrt{4\pi}} \cdot \frac{d\lambda}{d\sigma} \quad \text{a.e. } x_0 \in \partial D \times \{0\}$$

Received by the editors June 6, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 35K05, 35K15, 35K20, 35C15, 35D05; Secondary 31B10, 31B25.

Key words and phrases. Heat equation, initial-Dirichlet problem, Fatou theorem at corner points.

©1992 American Mathematical Society
0002-9947/92 \$1.00 + \$.25 per page

where $\bar{\Gamma}_\beta(x_0) = \{(x', x_n, t) \in D \times (0, T) : |x'| \leq Cx_n, x_n = \beta t^{3/2}\}$, $x = (x', x_n)$ is the local coordinate of the point x with respect to the local coordinate system at x_0 with origin at x_0 and with the plane $\{x_n = 0\}$ tangent to ∂D at x_0 , and $\frac{d\lambda}{d\sigma}$ is the Radon-Nikodym derivative of $d\lambda$ with respect to the surface measure $d\sigma$ on ∂D .

While the existence of Fatou type limit on the lateral surface and bottom of a cylinder has been investigated by Fabes, Garofalo, and Salsa [FGS], Fabes and Salsa [FS] in the case of nondegenerate parabolic equation, and by Kemper [K] in the case of heat equation, nothing is known about the behaviour of the solution $u(x, t)$ as (x, t) tends to $\partial D \times \{0\}$. My result is *entirely new*. I have also shown that the index $\frac{3}{2}$ appearing in the definition of $\bar{\Gamma}_\beta(x_0)$ is *essentially sharp*.

By using the argument of M. Grüter and K. O. Widman [GW], we will establish various estimates for the Green's function of the heat equation in $D \times (0, \infty)$ in §1. (The author was informed by Professor Russell Brown that similar estimates were obtained by E. B. Davies [D] using logarithmic Sobolev inequalities.) In subsection 2.1, we will show that any strong solution u of the (IDP) has a trace μ on D and a trace λ on ∂D with $\int_{x \in D} \delta(x) d\mu(x) < \infty$, $\int_{Q \in \partial D} d\lambda(Q) < \infty$, $\delta(x) = \text{dist}(x, \partial D)$ and $\forall \eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$,

$$\lim_{t \rightarrow 0} \int_D u(x, t) \eta(x) dx = \int_D \eta d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N} d\lambda$$

following the same line of proof as [DK2].

In subsection 2.3, we will prove some priori estimates for the strong solutions u of the (IDP) of the heat equation in a cylinder. In subsection 2.4, we will use the methods in [JK], [K] and [FGS] to prove the convergence a.e. on $\partial D \times \{0\}$ of $u(x, t)$ as $(x, t) \rightarrow (Q_0, 0) \in \partial D \times \{0\}$ along the nontangential paths $\Gamma_\beta(Q_0)$ by assuming the everywhere convergence a.e. of the solution $\bar{u}$ of the heat equation at the corner points $\partial D \times \{0\}$ with initial trace $(d\mu, d\lambda) = (0, d\sigma)$ where $d\sigma$ is the surface measure on ∂D along these nontangential paths.

And in subsection 2.5, we will finish the proof by proving the everywhere convergence of such solution $\bar{u}$ along

$$\begin{aligned} \bar{\Gamma}_\beta(Q_0) &= \{(Q, s) = (Q', Q_n, s) \in R^{n-1} \times R^+ \times R^- : \\ &\quad |Q'| \leq MQ_n, Q_n = \beta(-s)^{3/2} \leq \alpha\} \end{aligned}$$

for all corner points $(Q_0, 0) \in \partial D \times \{0\}$ by using layer potential method. Finally in subsection 2.6 we will show that the index $\frac{3}{2}$ on t in the definition of the nontangential cones at corner points are essentially sharp.

1. ESTIMATES FOR THE GREEN FUNCTION

In this section we will establish various estimates on $G(x, t; Q, s)$ the Green kernel of the heat equation in $D \times (0, \infty)$ following basically the line of proofs of [GW] for the estimates on the Green's function for the Laplacian. We will start with a lemma.

Lemma 1.1. *Let u_R be a solution of the heat equation in $D_R \times (0, (2R)^2)$ where $D_R = B(0, 2R) \setminus \overline{B(0, R)}$ with boundary value given by $\phi(x/2R, t/(2R)^2)$ where*

$\phi \in C^\infty(\partial_p(D_1 \times (0, \infty)))$ with

$$\begin{cases} \phi(x, t) \equiv 0 & \text{for } (x, t) \in \partial B(0, 1) \times [0, 1) \cup (B(0, 1) \setminus \overline{B(0, 1/2)}) \times \{0\}, \\ \phi(x, t) \equiv 0 & \text{for } (x, t) \in \partial B(0, 1/2) \times (1/2, 1), \end{cases}$$

and $0 \leq \phi \leq 1$. Then

$$\|\nabla_x u_R(x, t)\|_{L^\infty(D_R \times (0, (2R)^2))} \leq \frac{C}{R} < \infty$$

for some constant $C > 0$ independent of R .

Proof. The lemma follows from the boundary Schauder's estimate. (See [F, p. 65, Theorem 6].)

Corollary 1.2. Let v_R be a solution of the backward heat equation $\partial_s v_R + \Delta v_R = 0$ in $D_R \times (-(2R)^2, 0)$ where $D_R = B(0, 2R) \setminus \overline{B(0, R)}$ with boundary value ϕ where $\phi \in C^\infty$ and

$$\begin{cases} \phi(Q, s) \equiv 1 & \text{for } (Q, s) \in \partial B(0, 2R) \times (-(2R)^2, 0] \\ & \text{or } (Q, s) \in (B(0, 2R) \setminus \overline{B(0, R)}) \times \{0\}, \\ \phi(Q, s) \equiv 0 & \text{for } (Q, s) \in \partial B(0, R) \times (-(2R)^2, -2R^2) \end{cases}$$

and $0 \leq \phi \leq 1$. Then

$$\|\nabla_x v_R(x, t)\|_{L^\infty(D_R \times (-(2R)^2, 0))} \leq \frac{C}{R} < \infty$$

for some constant $C > 0$ independent of R .

Lemma 1.3. For $0 \leq s < t \leq T$, $T > 0$, we have

(i)

$$G(P, t; Q, s) \leq \frac{C}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)}, \quad P, Q \in D,$$

(ii)

$$G(P, t; Q, s) \leq \frac{C\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)}, \quad P, Q \in D,$$

(iii)

$$G(P, t; Q, s) \leq \frac{C\delta(P)\delta(Q)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)}, \quad P, Q \in D,$$

(iv)

$$\left| \frac{\partial G}{\partial N_Q}(P, t; Q, s) \right| \leq \frac{C\delta(P)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)}, \quad P \in D, Q \in \partial D,$$

where $\partial/\partial N_Q$ is the derivative in the direction of the normal to ∂D at the point $Q \in \partial D$, $\delta(P) = \text{dist}(P, \partial D)$, and C is a constant independent of P, Q, t, s .

Proof. (i) is proved in [LSU]. To prove (ii), note that since D is smooth, ∂D satisfies the exterior sphere condition, i.e. there exists a positive constant $h > 0$ such that for each $x \in \partial D$, $\forall 0 < r \leq h$, there exists $x_0 \in D^c$ such that $\overline{B(x_0, r)} \cap \overline{D} = \{x\}$.

Case 1. $\delta(P) \geq h$.

Then

$$1 \leq \frac{\delta(P)}{h} \leq \frac{\delta(P)}{h} \cdot \frac{(t-s)^{1/2}}{(t-s)^{1/2}} \leq C \frac{\delta(P)}{(t-s)^{1/2}}.$$

So by (i)

$$\begin{aligned} G(P, t; Q, s) &\leq \frac{C'}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \cdot \frac{\delta(P)}{(t-s)^{1/2}} \\ &\leq C' \frac{\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)}. \end{aligned}$$

Case 2. $\delta(P) \geq (t-s)^{1/2}/4$.

Then $1/4 \leq \delta(P)/(t-s)^{1/2}$. So by (i)

$$\begin{aligned} G(P, t; Q, s) &\leq \frac{C}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \cdot \frac{4\delta(P)}{(t-s)^{1/2}} \\ &\leq C' \frac{\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)}. \end{aligned}$$

Case 3. $\delta(P) \leq h$ and $\delta(P) \leq (t-s)^{1/2}/4 \leq h$.

We fix $P^* \in \partial D$ with $|P - P^*| = \delta(P)$ and set $R = (t-s)^{1/2}/4$. Since D satisfies the exterior sphere condition, there exists a $P_0 \in D^c$ such that $\overline{B(P_0, R)} \cap \overline{D} = \{P^*\}$. We may also assume without loss of generality that $\overline{PP^*} \subset D$.

Consider the cylinder $Q'_R = D'_R \times (t - (2R)^2, t]$, $D'_R = B(P_0, 2R) \setminus \overline{B(P_0, R)}$. Then $(P, t) \in \overline{Q'_R} \cap D \times (0, \infty)$ and for any $(P', t') \in \overline{Q'_R} \cap \overline{D} \times (0, \infty)$,

$$\begin{aligned} t-s &\geq t'-s \geq t - (2R)^2 - s \\ &\geq t-s - \left(\frac{(t-s)^{1/2}}{2} \right)^2 = \frac{3(t-s)}{4}. \end{aligned}$$

Now we have either

$$(a) \quad |t-s|^{1/2} \leq \frac{|P-Q|}{2}$$

or

$$(b) \quad |t-s|^{1/2} \geq \frac{|P-Q|}{2}.$$

If (a) holds, then $R \leq |P-Q|/8$, so

$$\begin{aligned} |P' - Q| &\geq |P - Q| - |P - P'| \\ &\geq |P - Q| - (|P - P_0| + |P_0 - P'|) \\ &\geq |P - Q| - (2R + 2R) \\ &\geq \frac{|P - Q|}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} G(P', t'; Q, s) &\leq \frac{C}{(t'-s)^{n/2}} e^{-c|P'-Q|^2/(t'-s)} \quad \text{by (i)} \\ &\leq \frac{C'}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)}. \end{aligned}$$

If (b) holds, then $|P - Q|^2/(t - s) \leq 4 \Rightarrow e^{-c|P-Q|^2/(t-s)} \geq e^{-4C'}$. Therefore

$$\begin{aligned} G(P', t'; Q, s) &\leq \frac{C}{(t' - s)^{n/2}} e^{-c|P' - Q|^2/(t' - s)} \quad \text{by (i)} \\ &\leq \frac{C}{(t' - s)^{n/2}} \leq \frac{C''}{(t - s)^{n/2}} e^{-c'|P-Q|^2/(t-s)}. \end{aligned}$$

In both cases, we have

$$G(P', t'; Q, s) \leq C \frac{1}{(t - s)^{n/2}} e^{-c'|P-Q|^2/(t-s)}.$$

If $\bar{u}_R(P', t') = u_R(P' - P_0, t' - (t - (2R)^2))$, $(P', t') \in \overline{Q'_R}$ where u_R is as in Corollary 5.2, then since

$$\begin{cases} G(P', t'; Q, s) = 0 \\ \bar{u}_R(P', t') \geq 0, \quad \forall (P', t') \in \partial D \times [t - (2R)^2, t) \cap \overline{Q'_R} \end{cases}$$

and

$$\begin{cases} G(P', t'; Q, s) \leq C \frac{1}{(t - s)^{n/2}} e^{-c'|P-Q|^2/(t-s)}, \\ \bar{u}_R(P', t') \equiv 1, \quad \forall (P', t') \in \partial_p Q'_R \cap D \times (0, \infty), \end{cases}$$

by applying the maximum principle to the functions

$$G(\cdot, \cdot; Q, s) \quad \text{and} \quad C \frac{1}{(t - s)^{n/2}} e^{-c'|P-Q|^2/(t-s)} \bar{u}_R(\cdot, \cdot)$$

in the region $Q_R \cap D \times (0, \infty)$ we have:

$$G(P, t; Q, s) \leq C \frac{1}{(t - s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \bar{u}_R(P, t)$$

since $(P, t) \in \overline{Q'_R} \cap D \times (0, \infty)$.

$$\leq C \frac{\delta(P)}{(t - s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \cdot \left| \frac{\partial \bar{u}_R}{\partial l_{P^*}}(\bar{P}, t) \right|$$

where $\partial/\partial l_{P^*}$ is the derivative in the direction $\overline{PP^*}$ and $\bar{P}$ is some point on $\overline{PP^*}$.

$$\begin{aligned} &\leq C \frac{\delta(P)}{(t - s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \cdot \frac{C'}{(t - s)^{1/2}} \quad (\text{by Corollary 1.2}) \\ &\leq C' \frac{\delta(P)}{(t - s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)}. \end{aligned}$$

Case 4. $\delta(P) \leq h$ and $\delta(P) \leq h \leq 4h \leq (t - s)^{1/2}$.

We fix $P^* \in \partial D$ as before and set $R = h$. Since now $R = h \leq (t-s)^{1/2}/4$, the same arguments as in Case 3 imply

$$\begin{aligned} G(P, t; Q, s) &\leq C \frac{1}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \bar{u}_R(P, t) \\ &\leq C \frac{\delta(P)}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \\ &\leq C_T \frac{\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)} \quad \text{since } 0 \leq s < t \leq T. \end{aligned}$$

Proof of (iii). The proof of (iii) is analogous to the proof of (ii).

(iv) follows from (iii) by dividing both sides of (iii) by $\delta(Q)$ and then letting $Q \rightarrow Q_0 \in \partial D$ along the normal to ∂D at Q_0 .

Corollary 1.4. *The following is true:*

$$(i) \quad |\nabla_Q G(P, t; Q, s)| \leq C \frac{\delta(P)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)},$$

$$(ii) \quad |\nabla_P G(P, t; Q, s)| \leq C \frac{\delta(Q)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)},$$

$$(iii) \quad |\nabla_Q \nabla_P G(P, t; Q, s)| \leq C \frac{1}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)},$$

for all $0 \leq s < t \leq T$ (for any fixed $T > 0$), $P, Q \in D$, and C is a constant depending only on $T > 0$.

Proof. Case 1. $\delta(Q) \leq (t-s)^{1/2}$.

Apply Schauder interior estimate to the function $G(P, t, \cdot, \cdot)$ in the cylinder $Q'_R = B(Q, R) \times (s, s+R^2)$ with $R = \delta(Q)/2$. We get $\forall i = 1, \dots, n$,

$$\begin{aligned} \left| \frac{\partial G}{\partial Q_i}(P, t; Q, s) \right| \cdot \frac{\delta(Q)}{2} &\leq C \sup_{(Q', s') \in Q'_R} G(P, t; Q', s') \\ &\leq C \sup_{(Q', s') \in Q'_R} \frac{\delta(P)\delta(Q')}{(t-s')^{n+2/2}} e^{-c|P-Q'|^2/(t-s')}. \end{aligned}$$

Now for any $(Q', s') \in Q'_R$,

$$\begin{aligned} s &\leq s' \leq s + R^2 \\ \Rightarrow t-s &\geq t-s' \geq t-s-R^2 \\ &= t-s - \left(\frac{\delta(Q)}{2} \right)^2 \geq t-s - \left(\frac{(t-s)^{1/2}}{2} \right)^2 \\ &\geq \frac{3(t-s)}{4}. \end{aligned}$$

Let Q^* be a point on ∂D such that $|Q - Q^*| = \delta(Q)$. Then

$$\begin{aligned} \delta(Q') &\leq |Q' - Q^*| \leq |Q' - Q| + |Q - Q^*| \\ &\leq \frac{1}{2}\delta(Q) + \delta(Q) = \frac{3\delta(Q)}{2}. \end{aligned}$$

Also we have either

$$(a) \quad (t-s)^{1/2} \leq \frac{|P-Q|}{2},$$

or

$$(b) \quad (t-s)^{1/2} > \frac{|P-Q|}{2}.$$

If (a) holds, then

$$\begin{aligned} |P-Q'| &\geq |P-Q| - |Q-Q'| \geq |P-Q| - R \\ &\geq |P-Q| - \left(\frac{\delta(Q)}{2} \right) \geq |P-Q| - \frac{(t-s)^{1/2}}{2} \\ &\geq |P-Q| - \frac{|P-Q|}{4} = \frac{3|P-Q|}{4}. \end{aligned}$$

So

$$e^{-c|P-Q'|^2/(t-s')} \leq e^{-c'|P-Q|^2/(t-s)}.$$

If (b) holds, then

$$|P-Q|^2/(t-s) \leq 4 \Rightarrow e^{-c'|P-Q|^2/(t-s)} \geq e^{-4C'} \geq e^{-c'|P-Q'|^2/(t-s')}$$

since $e^{-c'|P-Q'|^2/(t-s')} \leq 1$. In both cases we have

$$e^{-c'|P-Q'|^2/(t-s')} \leq C e^{-c|P-Q|^2/(t-s)}.$$

Combining the above inequalities we have

$$\sup_{(Q', s') \in Q'_R} \frac{\delta(P)\delta(Q')}{(t-s')^{n+2/2}} e^{-c|P-Q'|^2/(t-s')} \leq C \frac{\delta(P)\delta(Q)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)}.$$

Hence

$$\begin{aligned} \left| \frac{\partial G}{\partial Q_i}(P, t; Q, s) \right| \cdot \frac{\delta(Q)}{2} &\leq C \frac{\delta(P)\delta(Q)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)} \\ \Rightarrow |\nabla_Q G(P, t; Q, s)| &\leq C \frac{\delta(P)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)}. \end{aligned}$$

Case 2. $\delta(Q) \geq (t-s)^{1/2}$. The proof is the same as in Case 1 except that we apply the Schauder interior estimate to $G(P, t; \cdot, \cdot)$ in $Q'_R = B(Q, R) \times (s, s+R^2)$ with $R = (t-s)^{1/2}/2$, and use (ii) of Lemma 1.3 to control $G(P, t; Q', S')$, $(Q', s') \in Q'_R$.

The proof of (ii) and (iii) is similar to the proof of (i). We omit the details.

2. THE HEAT EQUATION IN A CYLINDER

2.1. The initial Dirichlet problem. In this subsection we will follow the argument in [DK2] to prove that all strong solutions $u(x, t)$ are in one-to-one correspondence with suitable pairs of measures μ on D and λ on ∂D . We will also show that any strong solution u has an explicit representation given by

$$u(x, t) = \int_{Q \in D} G(x, t; Q, 0) d\mu(Q) + \int_{Q \in \partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) d\lambda(Q)$$

where $\partial/\partial N_Q$ is the derivative in the direction of the inward normal N_Q at $Q \in \partial D$.

We will start with some definitions and results of [DK2]. We say that u is a strong solution of the initial Dirichlet problem (IDP) for the equation $\Delta u = \partial_t u$ in $D \times (0, \infty)$ if u is a continuous, nonnegative function in $\bar{D} \times (0, \infty)$, $u \equiv 0$ on $\partial D \times (0, \infty)$ and for all smooth functions η on $\bar{D} \times (0, \infty)$ which vanish on $\partial D \times [\tau_1, \tau_2]$, $\tau_1 > 0$, we have

$$\begin{aligned} \iint_{D \times [\tau_1, \tau_2]} \left[u \Delta \eta + u \frac{\partial \eta}{\partial t} \right] dx dt \\ = \int_D u(x, \tau_2) \eta(x, \tau_2) dx - \int_D u(x, \tau_1) \eta(x, \tau_1) dx. \end{aligned}$$

Lemma 2.1.1 [DK2, Lemma 2]. *Let u_1, u_2 be strong solutions of the (IDP) in $D \times (0, \infty)$. Let $\eta \geq 0$, $\eta \in C_0^\infty(D)$, and let $T < \infty$, $\{\tau_j\} \searrow 0$, $\tau_0 = T$. Then there exist nonnegative measures $\{\lambda_j\}$ on D with $\int_D d\lambda_j \leq \int \eta dx$ and*

$$\int_D [w_1(x, T) - w_2(x, T)] \eta(x) dx = \int_D w_1(x, \tau_j) - w_2(x, \tau_j) d\lambda_j(x)$$

where $w_i(x) = Gu_i(x) = \int_D G(x, y) u_i(y) dy$ (i.e. the Green's potential of u_i), $i = 1, 2$.

Proof. The proof is contained in the proof of Lemma 2.12 of [DK1].

Let u be a strong solution of the (IDP) and let $w = Gu$. Then $\partial w / \partial t = -u \leq 0$ (see (2.25) of [DK1]) and hence $\lim_{t \rightarrow 0} w(x, t)$ exists for each $x \in D$.

Lemma 2.1.2 (Pierre's maximum principle). *Let u_1, u_2 be two strong solutions of the (IDP) in $D \times (0, \infty)$. Suppose that $u_2 \in C(\bar{D} \times [0, \infty))$ and that $w_i = Gu_i$ verify $\lim_{t \rightarrow 0} w_1(x, t) \geq \lim_{t \rightarrow 0} w_2(x, t)$. Then $w_1(x, t) \geq w_2(x, t)$.*

Proof. Same as the proof of Lemma 3 of [DK2].

Lemma 2.1.3. *Let u be a strong solution of the (IDP) in $D \times (0, \infty)$ and $w = Gu$. Then there exists $x_0 \in D$ such that $\lim_{t \rightarrow 0} w(x_0, t) < \infty$.*

Proof. Suppose that $\lim_{t \rightarrow 0} w(x, t) = \infty \quad \forall x \in D$. We claim that this implies that $w(x, t) = \infty \quad \forall x \in D, t > 0$. In order to prove the claim we first note that Lemma 2.1.2 implies that

$$w(x, t) \geq w_f(x, t) = Gu_f \quad \forall f \in C_0^\infty(D)$$

where u_f is the solution of $\Delta u_f = \partial_t u_f$ in $D \times (0, \infty)$ with initial data $u_f(x, 0) = f(x)$, $\forall x \in D$ and boundary data $u_f|_{\partial D \times (0, \infty)} \equiv 0$.

$$\begin{aligned} \Rightarrow w(x, t) &\geq w_{\lambda f}(x, t) = Gu_{\lambda f} = G(\lambda u_f) \\ &= \lambda Gu_f \quad \forall \lambda > 0, f \in C_0^\infty. \end{aligned}$$

Now fix an $f \in C_0^\infty$, $f \geq 0$, and $f(\bar{x}) > 0$ for some $\bar{x} \in D$. Then

$$u_f(x, t) > 0 \quad \forall x \in D, t > 0$$

by the Harnack inequality. So $Gu_f(x, t) > 0$ and thus

$$w(x, t) \geq \lambda Gu_f(x, t) \rightarrow \infty \quad \text{as } \lambda \rightarrow \infty.$$

On the other hand $\forall x \in D, t > 0$,

$$w(x, t) = \int_D G(x, y) u(y, t) dy \leq \|G(x, \cdot)\|_{L^1(D)} \|u(\cdot, t)\|_{L^\infty(D)} < \infty.$$

Contradiction arises. Therefore there must exist an $x_0 \in D$ such that

$$\lim_{t \rightarrow 0} w(x_0, t) < \infty.$$

Lemma 2.1.4. *Let u be a strong solution of the (IDP) in $D \times (0, \infty)$. Then*

$$\sup_{t>0} \int_D u(x, t) \delta(x) dx < \infty \quad \text{and} \quad \int_D h(x) dx < \infty$$

where $h(x) = \lim_{t \rightarrow 0} w(x, t)$.

Proof. The proof follows the argument used in the proof of Lemma 6 in [DK2].

Theorem 2.1.5. *Let u be a strong solution of the (IDP). Then there is a positive Borel measure μ on D with $\int_D \delta(x) d\mu(x) < \infty$ and a positive Borel measure λ on ∂D with $\int_{\partial D} d\lambda < \infty$ such that whenever $\eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$, we have*

$$\lim_{t \rightarrow 0} \int_D u(x, t) \eta(x) dx = \int_D \eta d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N} d\lambda$$

where $\partial/\partial N$ is the derivative in the direction of the inward normal N .

Proof. Let $h(x) = \lim_{t \rightarrow 0} w(x, t)$. By Lemma 6.1.4, $\int_D h(x) dx < \infty$. Hence h is superharmonic in D and we can use the same argument as in the proof of Theorem 7 in [DK2] to finish the proof of Theorem 2.1.5.

Theorem 2.1.6 (Uniqueness). *Suppose u_1 and u_2 are two strong solutions of the (IDP) and that*

$$\lim_{t \rightarrow 0} \int_D u_1(x, t) \eta(x) dx = \lim_{t \rightarrow 0} \int_D u_2(x, t) \eta(x) dx$$

$$\forall \eta \in C^\infty(R^n), \quad \eta|_{\partial D} \equiv 0.$$

Then $u_1 \equiv u_2$.

Proof. The same as the proof of Theorem 8 of [DK2].

We need one more lemma whose proof is not hard.

Lemma 2.1.7. *Let $G(x, t; Q, s)$ be the Green kernel for the heat equation and $\eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$. Then*

$$\int_{x \in D} G(x, t; Q, 0) \eta(x) dx \leq C \delta(Q) < \infty \quad \forall Q \in D, \quad 0 < t \leq 1,$$

for some constant C independent of $Q \in D$ and $0 < t \leq 1$ where $\delta(Q) = \text{dist}(Q, \partial D)$.

Theorem 2.1.8 (Existence). *Given a pair of measures μ on D and λ on ∂D with $\mu \geq 0$, $\lambda \geq 0$, $\int_D \delta(x) d\mu(x) < \infty$, $\int_{\partial D} d\lambda < \infty$, there is a unique strong solution u of the (IDP) such that for $\eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$, we have*

$$\lim_{t \rightarrow 0} \int_D u(x, t) \eta(x) dx = \int_D \eta d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N} d\lambda$$

where $\partial/\partial N$ is the derivative in the direction of the inward normal N . In fact $u(x, t)$ is given explicitly by

$$u(x, t) = \int_{Q \in D} G(x, t; Q, 0) d\mu(Q) + \int_{Q \in \partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) d\lambda(Q)$$

where $G(x, t; Q, s)$ is the Green kernel for the heat equation.

Proof. Let

$$\begin{aligned} u_1(x, t) &= \int_{Q \in D} G(x, t; Q, 0) d\mu(Q), \\ u_2(x, t) &= \int_{Q \in \partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) d\lambda(Q). \end{aligned}$$

By the uniqueness theorem, it suffices to show that both u_1 and u_2 are strong solutions of (IDP) and that $u_1 + u_2$ has initial trace μ on D and λ on ∂D . Clearly both u_1 and u_2 are continuous, nonnegative functions on $D \times (0, \infty)$. Since

$$\begin{aligned} \left| \int_{Q \in D} G(x, t; Q, 0) d\mu(Q) \right| &\leq C \int_D \frac{\delta(Q)}{t^{n+1/2}} e^{-c|x-Q|^2/t} d\mu(Q) \\ &\leq \frac{C}{t^{n+1/2}} \int_D \delta(Q) d\mu(Q) \end{aligned}$$

and

$$\begin{aligned} \int_{\partial D} \left| \frac{\partial G}{\partial N_Q}(x, t; Q, 0) \right| d\lambda(Q) &\leq \int_{\partial D} \frac{C}{t^{n+1/2}} e^{-c|x-Q|^2/t} d\lambda(Q) \\ &\leq \frac{C}{t^{n+1/2}} \int_{\partial D} d\lambda(Q) < \infty, \end{aligned}$$

we have $\forall \psi \in C^\infty(\bar{D} \times (0, \infty))$, $\psi|_{\partial D \times [\tau_1, \tau_2]} \equiv 0$, $\tau_1 > 0$,

$$\begin{aligned} &\iint_{D \times [\tau_1, \tau_2]} \left[u_1 \Delta \psi + u_1 \frac{\partial \psi}{\partial t} \right] dx dt \\ &= \int_D \iint_{D \times [\tau_1, \tau_2]} G(x, t; Q, 0) \left[\Delta \psi + \frac{\partial \psi}{\partial t} \right] dx dt d\mu(Q) \\ &= \int_D \left(\iint_{D \times [\tau_1, \tau_2]} \left[\Delta_x G(x, t; Q, 0) \psi(x, t) \right. \right. \\ &\quad \left. \left. - \frac{\partial G}{\partial t}(x, t; Q, 0) \psi(x, t) \right] dx dt \right) d\mu(Q) \\ &\quad + \int_{Q \in D} \int_D G(x, t; Q, 0) \psi(x, t) dx \Big|_{t=\tau_1}^{t=\tau_2} d\mu(Q) \\ &= \int_D u_1(x, \tau_2) \psi(x, \tau_2) dx - \int_D u_1(x, \tau_1) \psi(x, \tau_1) dx \end{aligned}$$

and

$$\begin{aligned}
& \iint_{D \times [\tau_1, \tau_2]} \left[u_2 \Delta \psi + u_2 \frac{\partial \psi}{\partial t} \right] dx dt \\
&= \int_{\partial D} \iint_{D \times [\tau_1, \tau_2]} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) \left[\Delta \psi + \frac{\partial \psi}{\partial t} \right] dx dt d\lambda(Q) \\
&= \int_{\partial D} \iint_{D \times [\tau_1, \tau_2]} \left[\frac{\partial}{\partial N_Q} \Delta_x G(x, t; Q, 0) \psi(x, t) \right. \\
&\quad \left. - \frac{\partial}{\partial N_Q} \partial_t G(x, t; Q, 0) \psi(x, t) \right] dx dt d\lambda(Q) \\
&\quad + \int_{\partial D} \int_D \frac{\partial G}{\partial N_Q}(x, t; Q, 0) \psi(x, t) dx \Big|_{t=\tau_1}^{t=\tau_2} d\lambda(Q) \\
&= \int_D u_2(x, \tau_2) \psi(x, \tau_2) dx - \int_D u_2(x, \tau_1) \psi(x, \tau_1) dx.
\end{aligned}$$

It remains to show that

$$(i) \quad \lim_{(x, t) \rightarrow (Q_0, t_1)} u_1(x, t) = \lim_{(x, t) \rightarrow (Q_0, t_1)} u_2(x, t) = 0 \quad \forall Q_0 \in \partial D, \quad t_1 > 0,$$

and (ii): u_1 has trace μ on D , 0 on ∂D , and u_2 has trace 0 on D , λ on ∂D .

Proof of (i).

$$\begin{aligned}
u_1(x, t) &= \int_{Q \in D} G(x, t; Q, 0) d\mu(Q) \\
&= \int_{Q \in D} \frac{G(x, t; Q, 0)}{\delta(Q)} \delta(Q) d\mu(Q).
\end{aligned}$$

By Lemma 1.3,

$$G(x, t; Q, 0) \leq C_T \frac{\delta(x) \delta(Q)}{t^{n+2/2}} e^{-c|x-Q|^2/t} \quad \text{for } 0 < t < T$$

for any $T > 0$. The Lebesgue dominated convergence theorem then implies that $u_1(x, t) \rightarrow 0$ as $(x, t) \rightarrow (Q_0, t_1)$, $Q_0 \in \partial D$, $t_1 > 0$. That is,

$$\lim_{(x, t) \rightarrow (Q_0, t_1)} u_1(x, t) = 0 \quad \forall Q_0 \in \partial D, \quad t_1 > 0.$$

Similarly $u_2(x, t) \rightarrow 0$ as $(x, t) \rightarrow (Q_0, t_1)$, $\forall Q_0 \in \partial D$, $t_1 > 0$. Hence u_1 and u_2 are both strong solutions of the (IDP).

Proof of (ii). By Fubini's theorem for any $\eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$,

$$\begin{aligned}
\int_{x \in D} u_1(x, t) \eta(x) dx &= \int_{x \in D} \int_{Q \in D} G(x, t; Q, 0) d\mu(Q) \eta(x) dx \\
&= \int_{Q \in D} \left(\int_{x \in D} G(x, t; Q, 0) \eta(x) dx \right) d\mu(Q) \\
&\rightarrow \int_{Q \in D} \eta d\mu \quad \text{by Lemma 1.7.}
\end{aligned}$$

Also

$$\begin{aligned}
 \int_D u_2(x, t) \eta(x) dx &= \int_{Q \in \partial D} \int_{x \in D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) \eta(x) dx d\lambda(Q) \\
 &= \int_{Q \in \partial D} \frac{\partial}{\partial N_Q} \left(\int_{x \in D} G(x, t; Q, 0) \eta(x) dx \right) d\lambda(Q) \\
 &\rightarrow \int_{Q \in \partial D} \frac{\partial \eta}{\partial N_Q}(Q) d\lambda(Q) \quad \text{as } t \rightarrow 0.
 \end{aligned}$$

2.2. A priori estimate. For any point $S \in \partial D$, there exists a local C^∞ space coordinate system $\phi_S : R^n \rightarrow R$ at S , i.e., there exists $r_0 > 0$ such that

$$\begin{aligned}
 D \cap B(S, r_0) &= \{Q = (Q', Q_n) : Q' \in R^{n-1}, Q_n \in R, Q_n > \phi_S(Q')\} \\
 &\quad \cap B(S, r_0), \\
 \partial D \cap B(S, r_0) &= \{Q = (Q', Q_n) : Q' \in R^{n-1}, Q_n \in R, Q_n = \phi_S(Q')\} \\
 &\quad \cap B(S, r_0)
 \end{aligned}$$

and with the $\{Q_n = 0\}$ plane being tangential to ∂D at S and the origin of the local space coordinate system is at S , i.e., $S = (0, 0)$ in the local space coordinate at S .

Since $\partial D \in C^\infty$, ∂D satisfies the interior cone condition at $S = (0, 0)$, i.e., there exists $M > 0$, $\alpha > 0$, such that

$$\Gamma(S) = \{Q = (Q', Q_n) : |Q'| \leq M Q_n, 0 < Q_n \leq \alpha\} \subset D$$

where (Q', Q_n) is the local space coordinate of Q with respect to the coordinate system (S, ϕ_S) .

We define the nontangential approach to $(S, 0) \in \partial D \times \{0\}$ in $D \times (0, \infty)$ to be the space time cone

$$\begin{aligned}
 \Gamma_\beta(S) &= \{(Q, t) = (Q', Q_n, t) \in R^{n-1} \times R^+ \times R^+ : \\
 &\quad |Q'| \leq M Q_n, Q_n = \beta t^{3/2} \leq \alpha\}
 \end{aligned}$$

where (Q', Q_n) is the local space coordinate of Q with respect to the coordinate system (S, ϕ_S) .

Note that we can choose M and α such that they are independent of ϕ_S and $S \in \partial D$ but depend only on ∂D and r_0 .

Lemma 2.2.1. $\forall \alpha > 0, a > 0$, there exists $C_\alpha = C(a, \alpha) > 0$ such that

$$e^{-a|x-z|^2/t} \leq C_\alpha e^{-a|x|^2/(2t)} \quad \forall x \in R^n, t > 0, |z| \leq \alpha\sqrt{t}.$$

Lemma 2.2.2. Fix an $S \in \partial D$ and let $\phi = \phi_S : R^{n-1} \rightarrow R$ be the C^∞ local coordinate system associated with S as described at the beginning of this section, i.e. there exists $r_0 > 0$ such that

$$\begin{aligned}
 D \cap B(S, r_0) &= \{Q = (Q', Q_n) : Q' \in R^{n-1}, Q_n \in R, Q_n > \phi_S(Q')\} \\
 &\quad \cap B(S, r_0), \\
 \partial D \cap B(S, r_0) &= \{Q = (Q', Q_n) : Q' \in R^{n-1}, Q_n \in R, Q_n = \phi_S(Q')\} \\
 &\quad \cap B(S, r_0).
 \end{aligned}$$

Let ν be a positive Borel measure on D with $\int_D \delta(x) d\nu(x) < \infty$ and $\text{supp } \nu \subset B(S, r_0)$. If $u(x, t) = \int_D G(x, t; Q, 0) d\nu(Q)$, then

$u(x, t) \leq CM(\delta(Q) d\nu(Q))(Q_0) \quad \forall (x, t) \in \Gamma_\beta(Q_0), \quad Q_0 \in B(S, r_0) \cap \partial D$,
where C is independent of $Q_0 \in B(S, r_0) \cap \partial D$ and ν .

$$M(f)(Q_0) = \sup_{\rho > 0} \frac{1}{|B'(Q'_0, \rho)|} \iint_{\substack{|Q' - Q'_0| \leq \rho \\ Q = (Q', Q_n), \quad Q' \in \mathbb{R}^{n-1}}} f(Q) dQ,$$

$$M(\delta(Q) d\nu(Q))(Q_0) = \sup_{\rho > 0} \frac{1}{|B'(Q'_0, \rho)|} \iint_{\substack{|Q' - Q'_0| \leq \rho \\ Q = (Q', Q_n), \quad Q' \in \mathbb{R}^{n-1}}} \delta(Q) d\nu(Q)$$

where $Q_0 = (Q'_0, \phi(Q'_0)) \in \partial D \cap B(S, r_0)$, $Q = (Q', Q_n)$ is the local space coordinate of Q with respect to the local coordinate system (S, ϕ_S) at S as described at the beginning of this section, and $B'(Q'_0, \rho) = \{Q' \in \mathbb{R}^{n-1} : |Q' - Q'_0| \leq \rho\}$.

Proof. By Lemma 1.3,

$$G(x, t; Q, 0) \leq C \frac{\delta(x)\delta(Q)}{t^{n+2/2}} e^{-c|x-Q|^2/t}.$$

So for any $(x, t) \in \Gamma_\beta(Q_0)$, $Q_0 \in B(S, r_0) \cap \partial D$,

$$\begin{aligned} u(x, t) &= \int_{Q \in D} G(x, t; Q, 0) d\nu(Q) \\ &\leq C \frac{\delta(x)}{t^{3/2}} \cdot \frac{1}{t^{n-1/2}} \iint_{Q \in D} e^{-c|x-Q|^2/t} \delta(Q) d\nu(Q) \\ &\leq C' \frac{1}{t^{n-1/2}} \iint_{Q \in D} e^{-c'|Q_0-Q|^2/t} \delta(Q) d\nu(Q) \quad \text{by Lemma 2.2.1} \\ &\leq C' \frac{1}{t^{n-1/2}} \left\{ \iint_{\substack{|Q' - Q'_0| \leq t^{1/2} \\ Q \in D}} e^{-c'|Q'_0 - Q'|^2/t} \delta(Q) d\nu(Q) \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \iint_{\substack{2^k t^{1/2} \leq |Q' - Q'_0| \leq 2^{k+1} t^{1/2} \\ Q \in D}} e^{-c'|Q'_0 - Q'|^2/t} \delta(Q) d\nu(Q) \right\} \\ &\leq C \left(1 + \sum_{k=0}^{\infty} (2^{k+1})^{n-1} e^{-c'2^{2k}} \right) M(\delta(Q) d\nu(Q))(Q_0) \\ &\leq C' M(\delta(Q) d\nu(Q))(Q_0). \end{aligned}$$

Theorem 2.2.3. Let $u_2(x, t) = \int_{\partial D} \partial G / \partial N_Q(x, t; Q, 0) d\lambda(Q)$ where λ is a positive Borel measure on ∂D with $\int_{\partial D} d\lambda < \infty$. Then $u_2(x, t) \leq C\widetilde{M}(d\lambda)(Q_0)$ for all $(x, t) \in \Gamma_\beta(Q_0)$, $Q_0 \in \partial D$, where

$$\widetilde{M}(d\lambda)(Q_0) = \sup_{\rho > 0} \frac{1}{\rho^{n-1}} \int_{\substack{|Q - Q_0| \leq \rho \\ Q \in \partial D}} d\lambda(Q), \quad Q_0 \in \partial D.$$

Proof. By Lemma 1.3,

$$\frac{\partial G}{\partial N_Q}(x, t; Q, 0) \leq C \frac{\delta(x)}{t^{3/2}} \cdot \frac{1}{t^{n-1/2}} e^{-c|x-Q|^2/t}.$$

So

$$\begin{aligned} u_2(x, t) &\leq C \frac{\delta(x)}{t^{3/2}} \cdot \frac{1}{t^{n-1/2}} \int_{Q \in \partial D} e^{-c|x-Q|^2/t} d\lambda(Q) \\ &\leq C \frac{1}{t^{n-1/2}} \int_{Q \in \partial D} e^{-c|x-Q|^2/t} d\lambda(Q) \quad \text{for } (x, t) \in \Gamma_\beta(Q_0) \\ &\leq C' \widetilde{M}(d\lambda)(Q_0) \quad \forall (x, t) \in \Gamma_\beta(Q_0), \quad Q_0 \in \partial D. \end{aligned}$$

2.3. A Fatou theorem for the corner points of a cylinder. In this subsection we will prove the convergence a.e. on $\partial D \times \{0\}$ of the strong solution $u(x, t)$ of the heat equation in a cylinder as $(x, t) \rightarrow (Q_0, 0) \in \partial D \times \{0\}$ along the space time cone $\Gamma_\beta(Q_0)$ as defined in subsection 2.2 by assuming the everywhere convergence of the solution $\bar{u}$ of the heat equation at the corner points $\partial D \times \{0\}$ with initial trace $(d\mu, d\lambda) = (0, d\sigma)$ (where $d\sigma$ is the surface measure on ∂D) along these nontangential paths. We will use the same notation as in subsection 2.2 throughout this subsection.

Lemma 2.3.1. *With the same notation as Lemma 2.2.2, there exists $C > 0$ such that*

$$|\{Q_0 \cap \partial D \cap B(S, r_0) : \lambda \leq M(\delta(Q) d\nu(Q))(Q_0)\}| \leq \frac{C}{\lambda} \int_{B(S, r_0) \cap D} \delta(Q) d\nu(Q).$$

Proof. See [S, Chapter 1].

Theorem 2.3.2. *Let $u_1 = \int_D G(x, t; Q, 0) d\mu(Q)$ with μ being a positive Borel measure on D with $\int_D \delta(Q) d\mu(Q) < \infty$. Then*

$$\lim_{\substack{(x, t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_1(x, t) = 0 \quad \text{a.e. } Q_0 \in \partial D.$$

Proof. Note that in order to prove the theorem, it suffices to show that

$$\lim_{\substack{(x, t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_1(x, t) = 0 \quad \text{a.e. } Q_0 \in \partial D \cap B(S, r_0)$$

for any $S \in \partial D$ and r_0 is as described at the beginning of this subsection.

Hence without loss of generality, we fix an $S \in \partial D$ and assume from now on that $Q_0 \in \partial D \cap B(S, r_0)$. Write

$$E_r = B(S, r_0) \cap \{Q \in D : \delta(Q) \leq r\}.$$

Then for all $(x, t) \in \Gamma_\beta(Q_0)$,

$$\begin{aligned} u_1(x, t) &= \int_{E_r^c \cap D} G(x, t; Q, 0) d\mu(Q) + \int_{E_r} G(x, t; Q, 0) d\mu(Q) \\ &\leq C \int_{Q \in E_r^c \cap D} \frac{\delta(x)\delta(Q)}{t^{n+2/2}} e^{-c|x-Q|^2/t} d\mu(Q) \\ &\quad + CM(\delta(Q)\chi_{E_r} d\mu(Q))(Q_0) \\ &= I_1 + I_2 \end{aligned}$$

by Lemma 1.3 and Lemma 2.2.2 where χ_A is the characteristic function of the set A .

By the Lebesgue dominated convergence theorem, I_1 vanishes as $(x, t) \in \Gamma_\beta(Q_0) \rightarrow (Q_0, 0)$. Hence for any $\lambda > 0$,

$$\begin{aligned} & \left| \left\{ Q_0 \in \partial D \cap B(S, r_0) : \lambda \leq \overline{\lim}_{\substack{(x,t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_1(x, t) \right\} \right| \\ & \leq |\{Q_0 \in \partial D \cap B(S, r_0) : \lambda \leq CM(\delta(Q)\chi_{E_r} d\mu(Q))(Q_0)\}| \\ & \leq \frac{C}{\lambda} \int_{Q \in B(S, r_0) \cap D} \delta(Q) \chi_{E_r} d\mu(Q) \\ & \leq \frac{C}{\lambda} \int_{\substack{Q \in B(S, r_0) \cap D \\ \delta(Q) \leq r}} \delta(Q) d\mu(Q) \rightarrow 0 \quad \text{as } r \rightarrow 0 \end{aligned}$$

since $\int_{B(S, r_0) \cap D} \delta(Q) d\mu(Q) \leq \int_D \delta(Q) d\mu(Q) < \infty$. Hence

$$\left| \left\{ Q_0 \in \partial D \cap B(S, r_0) : \lambda \leq \overline{\lim}_{\substack{(x,t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_1(x, t) \right\} \right| = 0 \quad \forall \lambda > 0.$$

Therefore

$$\lim_{\substack{(x,t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_1(x, t) = 0 \quad \text{a.e. } Q_0 \in \partial D.$$

Lemma 2.3.3. Let λ be a positive Borel measure on ∂D with $\int_{\partial D} d\lambda < \infty$ and let

$$u_2(x, t) = \int_{\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) d\lambda(Q).$$

Then for almost every $Q_0 \in \partial D$ we have

$$\lim_{\substack{(x,t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} \left| u_2(x, t) - f(Q_0) \int_{\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) d\sigma(Q) \right| = 0$$

where $d\sigma$ is the surface measure on ∂D and $f = d\lambda/d\sigma$ is the Radon-Nikodym derivative of $d\lambda$ with respect to $d\sigma$. (Note that $\int_{\partial D} f d\sigma \leq \int_{\partial D} d\lambda < \infty$ and $f \geq 0$ a.e. $[d\sigma]$.)

Proof. Write $d\lambda = f d\sigma + d\nu_s$ where $d\nu_s \perp d\sigma$. We will assume from now on that $(x, t) \in \Gamma_\beta(Q_0)$ in the proof. Now since $d\nu_s \perp d\sigma$,

$$\lim_{r \rightarrow 0} \frac{1}{r^{n-1}} \int_{\substack{|Q - Q_0| \leq r \\ Q \in \partial D}} |d\nu_s|(Q) = 0 \quad \text{a.e. } Q_0 \in \partial D.$$

Also a.e. $Q_0 \in \partial D$ are Lebesgue points of f . Let $Q_0 \in \partial D$ be a Lebesgue point of f satisfying

$$\lim_{r \rightarrow 0} \frac{1}{r^{n-1}} \int_{\substack{|Q - Q_0| \leq r \\ Q \in \partial D}} |d\nu_s|(Q) = 0 \quad \text{and} \quad 0 \leq f(Q_0) < \infty.$$

Then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\frac{1}{r^{n-1}} \int_{\substack{|Q-Q_0| \leq r \\ Q \in \partial D}} |d\nu_s|(Q) \leq \varepsilon$$

and

$$\frac{1}{r^{n-1}} \int_{\substack{|Q-Q_0| \leq r \\ Q \in \partial D}} |f(Q) - f(Q_0)| d\sigma(Q) \leq \varepsilon \quad \forall 0 < r \leq \delta.$$

Choose $k_0 \in \mathbb{Z}^+$, $k_0 = k_0(t)$, such that $2^{k_0} t^{1/2} \leq \delta < 2^{k_0+1} t^{1/2}$. Then

$$\begin{aligned} & \left| u_2(x, t) - f(Q_0) \int_{\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) d\sigma(Q) \right| \\ & \leq \int_{\substack{|Q-Q_0| \leq t^{1/2} \\ Q \in \partial D}} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) (|f(Q) - f(Q_0)| d\sigma(Q) + |d\nu_s|(Q)) \\ & \quad + \sum_{k=0}^{k_0} \int_{2^k t^{1/2} \leq |Q-Q_0| \leq 2^{k+1} t^{1/2}} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) \\ & \quad \cdot (|f(Q) - f(Q_0)| d\sigma(Q) + |d\nu_s|(Q)) \\ & \quad + \int_{\substack{|Q-Q_0| \geq \delta \\ Q \in \partial D}} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) ((f(Q) - f(Q_0)) d\sigma(Q) + d\nu_s(Q)). \end{aligned}$$

Since $|x - Q| = |x - Q_0 + Q_0 - Q|$ and $(x, t) \in \Gamma_\beta(Q_0) \Rightarrow \delta(x) \leq |x - Q_0| \leq C t^{3/2} \leq C \sqrt{t}$, by Lemma 2.2.1 and Lemma 1.3 the last term above is dominated by

$$\begin{aligned} & C \frac{\delta(x)}{t^{n+2/2}} \int_{\substack{|Q-Q_0| \geq \delta \\ Q \in \partial D}} e^{-c|x-Q|^2/t} (d\lambda(Q) + |f(Q_0)| d\sigma(Q)) \\ & \leq \frac{C'}{t^{n-1/2}} \int_{\substack{|Q-Q_0| \geq \delta \\ Q \in \partial D}} e^{-c|Q_0-Q|^2/t} (d\lambda(Q) + |f(Q_0)| d\sigma(Q)) \\ & \leq \frac{C}{t^{n-1/2}} e^{-c'\delta^2/t} \left| \int_{\partial D} d\lambda(Q) + |f(Q_0)| d\sigma(Q) \right| \rightarrow 0 \quad \text{as } t \rightarrow 0. \end{aligned}$$

While the first two terms are dominated by

$$\begin{aligned}
& C \frac{\delta(x)}{t^{3/2}} \cdot \frac{1}{t^{n-1/2}} \int_{\substack{|Q-Q_0| \leq t^{1/2} \\ Q \in \partial D}} e^{-c|x-Q|^2/t} (|f(Q) - f(Q_0)| d\sigma(Q) + |d\nu_s|(Q)) \\
& + C \frac{\delta(x)}{t^{3/2}} \cdot \sum_{k=0}^{k_0} \frac{(2^{k+1})^{n-1}}{(2^{k+1}t^{1/2})^{n-1}} \cdot \int_{\substack{2^k t^{1/2} \leq |Q-Q_0| \leq 2^{k+1} t^{1/2} \\ Q \in \partial D}} e^{-c|x-Q|^2/t} \\
& \quad \cdot (|f(Q) - f(Q_0)| d\sigma(Q) + |d\nu_s|(Q)) \\
& \leq \frac{C}{t^{n-1/2}} \int_{\substack{|Q-Q_0| \leq t^{1/2} \\ Q \in \partial D}} (|f(Q) - f(Q_0)| d\sigma(Q) + |d\nu_s|(Q)) \\
& + C \sum_{k=0}^{k_0} \frac{(2^{k+1})^{n-1}}{(2^{k+1}t^{1/2})^{n-1}} \cdot \int_{\substack{|Q-Q_0| \leq 2^{k+1} t^{1/2} \\ Q \in \partial D}} e^{-c|x-Q|^2/t} \\
& \quad \cdot (|f(Q) - f(Q_0)| d\sigma(Q) + |d\nu_s|(Q)) \\
& \leq C \left(1 + \sum_{k=0}^{\infty} 2^{(k+1)(n-1)} e^{-c'2^{2k}} \right) \cdot \varepsilon \leq C' \varepsilon.
\end{aligned}$$

Therefore

$$\lim_{\substack{(x,t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} \left| u_2(x,t) - f(Q_0) \int_{\partial D} \frac{\partial G}{\partial N_Q}(x,t;Q,0) d\sigma(Q) \right| \leq C' \varepsilon$$

for all $\varepsilon > 0$.

$$\Rightarrow \lim_{\substack{(x,t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} \left| u_2(x,t) - f(Q_0) \int_{\partial D} \frac{\partial G}{\partial N_Q}(x,t;Q,0) d\sigma(Q) \right| = 0$$

and the proof is completed.

Theorem 2.3.4. Let $\bar{u}(x,t) = \int_{\partial D} \partial G / \partial N_Q(x,t;Q,0) d\sigma(Q)$ with $d\sigma$ as the surface measure on ∂D and $\Gamma_\beta(Q_0)$ as in subsection 2.2. Then

$$\lim_{\substack{(x,t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} \bar{u}(x,t) = \frac{\beta}{\sqrt{4\pi}} \quad \forall Q_0 \in \partial D.$$

Proof. Same as the proof of Theorem 2.4.1 of subsection 2.4.

Theorem 2.3.5. Let $u(x,t)$ be a strong solution of the (IDP) in $D \times (0, \infty)$ with initial trace μ on D and λ on ∂D . (See Theorem 2.1.8 for its existence.) Then

$$\lim_{\substack{(x,t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u(x,t) = \frac{\beta}{\sqrt{4\pi}} \cdot \frac{d\lambda}{d\sigma}(Q_0) \quad \text{a.e. } Q_0 \in \partial D$$

where $d\lambda/d\sigma$ is the Radon-Nikodym derivative of $d\lambda$ with respect to the surface measure $d\sigma$ on ∂D and $\Gamma_\beta(Q_0)$ is as defined in subsection 2.2.

Proof. By Theorems 2.1.5, 2.1.6, 2.1.8, $u(x, t) = u_1(x, t) + u_2(x, t)$ where

$$u_1(x, t) = \int_{Q \in D} G(x, t; Q, 0) d\mu(Q),$$

$$u_2(x, t) = \int_{Q \in \partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) d\lambda(Q).$$

By Theorem 2.3.2,

$$\lim_{\substack{(x, t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_1(x, t) = 0 \quad \text{a.e. } Q_0 \in \partial D.$$

By Theorem 2.3.3 and Theorem 2.3.4,

$$\lim_{\substack{(x, t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_2(x, t) = \frac{\beta}{\sqrt{4\pi}} \cdot \frac{d\lambda}{d\sigma}(Q_0) \quad \text{a.e. } Q_0 \in \partial D.$$

Therefore

$$\begin{aligned} \lim_{\substack{(x, t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u(x, t) &= \lim_{\substack{(x, t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_1(x, t) + \lim_{\substack{(x, t) \in \Gamma_\beta(Q_0) \\ t \rightarrow 0}} u_2(x, t) \\ &= \frac{\beta}{\sqrt{4\pi}} \cdot \frac{d\lambda}{d\sigma}(Q_0) \end{aligned}$$

for a.e. $Q_0 \in \partial D$.

2.4. A Fatou theorem for the solution with initial trace $(0, d\sigma)$. We will start this section with a definition. We let

$$\begin{aligned} \bar{\Gamma}_\beta(Q_0) &= \{(Q, s) = (Q', Q_n, s) \in R^{n-1} \times R^+ \times R^- : |Q'| \leq M Q_n, \\ &\quad Q_n = \beta(-s)^{3/2} \leq \alpha\} \end{aligned}$$

where $Q_0 \in \partial D$ and $Q = (Q', Q_n)$ is the local space coordinate of $Q \in D$ with respect to the coordinate system $\phi = \phi_{Q_0}$ at Q_0 as defined in subsection 2.2, α is the constant defined in the beginning of subsection 2.2, and M is the Lipschitz constant for ∂D . The main result of this section is the following theorem.

Theorem 2.4.1. *Let $v(Q, s) = \int_{\partial D} \partial G / \partial N_x(x, 0; Q, s) d\sigma(x)$, $s < 0$, with $d\sigma$ being the surface measure on ∂D . Then*

$$\lim_{\substack{(Q, s) \in \bar{\Gamma}_\beta(Q_0) \\ s \rightarrow 0}} v(Q, s) = \frac{\beta}{\sqrt{4\pi}} \quad \forall Q_0 \in \partial D.$$

Before proving the theorem we would like to assume without loss of generality that $Q_0 = 0$ and ϕ is the local coordinate system at $Q_0 = 0$ with $\nabla \phi(0) = 0$, $\phi \in C_0^\infty(R^{n-1})$.

We first note that if D' is the image of D under the transformation $(x', x_n) \rightarrow (x', x'_n)$ with $x'_n = x_n - \phi(x')$, $x = (x', x_n) \in D$ being the local coordinate of x with respect to $(0, \phi)$, and if $\tilde{u}((x', x'_n), t) = u((x', x'_n + \phi(x')), t)$, $(x', x'_n) \in D'$, then

$$\begin{aligned} \Delta u - \partial_t u &= 0 \quad \text{in } D \times (-\infty, \infty) \\ \Leftrightarrow \operatorname{div}(a_{ij}(x') \nabla \tilde{u}) - \partial_t \tilde{u} &= 0 \quad \text{in } D' \times (-\infty, \infty) \end{aligned}$$

where $a_{ij}(x') = \delta_{ij}$, $a_{1j}(x') = a_{j1} = -\partial\phi/\partial x_j$, $i, j = 1, \dots, n-1$, $a_{nn} = 1 + |\nabla_{x'}\phi(x')|^2$. It is not hard to see that $A(x') = (a_{ij})$ is a uniformly elliptic matrix with the eigenvalues bounded above and below by positive constants independent of x' , $\forall(x', x_n) \in D'$, and depends on ∂D only.

If $G(x, t; y, \tau) = G((x', x_n), t; (y', y_n), \tau)$ is the Green's function for the heat equation $\Delta - \partial_t = 0$ in $D \times (-\infty, \infty)$, then

$$\tilde{G}((x', x'_n), t; (y', y'_n), \tau) = G((x', x'_n + \phi(x')), t; (y', y'_n + \phi(y')), \tau)$$

is the Green's function for $\operatorname{div}(a_{ij}\nabla) - \partial_t = 0$ in $D' \times (-\infty, \infty)$.

Idea of proof of Theorem 2.4.1. We first note that

$$\int_{x \in \partial D} \frac{\partial G}{\partial N_x}(x, 0; y, \tau) d\sigma(x) = \left(\int_{\substack{x \in \partial D \\ |x'| > \varepsilon}} + \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \right) \frac{\partial G}{\partial N_x}(x, 0; y, \tau) d\sigma(x).$$

The first term vanishes as $(y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow 0$ by the Lebesgue dominated convergence theorem since its integrand

$$\begin{aligned} &\leq C \frac{\delta(y)}{|\tau|^{n+2/2}} \exp\left(-C \frac{|x-y|^2}{|\tau|}\right) \quad \text{by Lemma 1.3} \\ &\leq \frac{C}{|\tau|^{n-1/2}} \exp\left(-C' \frac{\varepsilon^2}{|\tau|}\right) \quad \text{for } (y, \tau) \in \bar{\Gamma}_\beta(0), |y| \leq \frac{\varepsilon}{2}, \\ &\rightarrow 0 \quad \text{as } \tau \rightarrow 0. \end{aligned}$$

While the second term

$$\begin{aligned} &= \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \frac{\partial G}{\partial N_x}(x, 0; y, \tau) d\sigma(x) \\ &= \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \nabla_x G(x, 0; y, \tau) \cdot (-\nabla_{x'}\phi(x'), 1) dx' \\ &= - \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \nabla_{x'} G(x, 0; y, \tau) \cdot \nabla_{x'}\phi(x') dx' + \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \frac{\partial G}{\partial x_n}(x, 0; y, \tau) dx'. \end{aligned}$$

The term

$$\begin{aligned} &\left| \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \nabla_{x'} G \cdot \nabla_{x'}\phi dx' \right| \\ &\leq C \sup_{|x'| \leq \varepsilon} \{|\nabla_{x'}^2\phi(x')| |x'|\} \cdot \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \frac{\delta(y)}{|\tau|^{n+2/2}} \exp\left(-C \frac{|x-y|^2}{|\tau|}\right) dx' \\ &\leq C \sup_{|x'| \leq \varepsilon} |\nabla_{x'}^2\phi(x')| \cdot \varepsilon \cdot \int_{|x'| \leq \varepsilon} \frac{1}{|\tau|^{n-1/2}} \exp\left(-C' \frac{|x'|^2}{|\tau|}\right) dx' \\ &\leq C' \varepsilon. \end{aligned}$$

Therefore it remains to consider

$$\int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \frac{\partial G}{\partial x_n}(x, 0; y, \tau) dx' = \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \frac{\partial \tilde{G}}{\partial x_n}(x, 0; y, \tau) dx'$$

and show that it converges to $\beta/\sqrt{4\pi}$ as $(y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow 0$. To show this we note that $\tilde{G}(x, t; y, s) = G_y(x, t; y, s) + E(x, t; y, s)$ where

$$E(x, t; y, s) = \int_s^t d\lambda \int_{D'} G_\eta(x, t; \eta, \lambda) \Phi(\eta, \lambda; y, s) d\eta.$$

(See Lemma 2.4.6 below.) $G_y(x, t; z, s)$ is the Green's function for

$$L^y u = \sum a_{ij}(y') \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = 0, \quad y = (y', y'_n),$$

from which $\partial \tilde{G}/\partial x_n = \partial G_y/\partial x_n + \partial E/\partial x_n$. Since $x \in \partial D'$, $|x'| \leq \varepsilon \leq r_0 \Rightarrow x'_n = 0$, $d\sigma|_{\partial D' \cap \{|x| \leq \varepsilon\}} = dx'$. If

$$\int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} |\nabla_x E| d\sigma(x) \rightarrow 0 \quad \text{as } (y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow 0,$$

then

$$\lim_{\substack{(y, \tau) \in \bar{\Gamma}_\beta(0) \\ \tau \rightarrow 0}} \int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} \frac{\partial \tilde{G}}{\partial x_n} dx' = \lim_{\substack{(y, \tau) \in \bar{\Gamma}_\beta(0) \\ \tau \rightarrow 0}} \int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} \frac{\partial G_y}{\partial x_n} dx'.$$

On the other hand, layer potential method implies that the function

$$\Rightarrow w_y(x, 0; y, \tau) = \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)$$

satisfies

$$\begin{aligned} w_y(x, 0; y, \tau) &= 2 \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, 0; y, \tau) \\ &\quad - 2 \int_\tau^0 d\lambda \int_{\eta \in \partial D'} H_y(x, 0; \eta, \lambda) w_y(\eta, \lambda; y, \tau) d\sigma(\eta) \end{aligned}$$

where $Z_y(x, t; \eta, \tau)$ is the fundamental solution for $L^y u = 0$ and

$$H_y(x, t; \eta, \lambda) = \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, t; \eta, \lambda),$$

$$\begin{aligned} (n_i(x))_{i=1}^n &= N_x = \text{unit inward normal at } x \in \partial D' \\ &= (0, \dots, 0, 1) \quad \text{if } |x| \leq \varepsilon, \quad x \in \partial D'. \end{aligned}$$

So

$$\begin{aligned} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) &= \sum_j a_{nj}(y) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) \\ &= \sum_{1 \leq j \leq n-1} -\frac{\partial \phi}{\partial y_j}(y') \cdot \frac{\partial G_y}{\partial x_j} + \frac{\partial G_y}{\partial x_n}. \end{aligned}$$

Therefore

$$\left| \int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} \sum_{1 \leq j \leq n-1} -\frac{\partial \phi}{\partial y_j}(y') \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) dx' \right| \\ \leq C \sup_{|x'| \leq \varepsilon} |\nabla \phi(x')| \cdot \int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} \frac{\delta(y)}{|\tau|^{n+2/2}} \exp \left(-C \frac{|x-y|^2}{|\tau|} \right) dx' \leq C' \varepsilon$$

as before. Hence

$$\begin{aligned} \int_{x \in \partial D} \frac{\partial G}{\partial N_x} d\sigma(x) &\approx \int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} \frac{\partial G}{\partial x_n} dx = \int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} \frac{\partial \tilde{G}}{\partial x'_n} dx' \\ &\approx \int_{|x'| \leq \varepsilon} \frac{\partial G_y}{\partial x'_n} dx' + \int_{|x'| \leq \varepsilon} \frac{\partial E}{\partial x'_n} dx' \\ &\approx \int_{|x'| \leq \varepsilon} \frac{\partial G_y}{\partial x'_n} dx' \approx \int_{|x'| \leq \varepsilon} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) dx' \\ &= 2 \int_{|x'| \leq \varepsilon} \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, 0; y, \tau) dx' \\ &\quad - 2 \int_{|x'| \leq \varepsilon} \int_{\tau}^0 d\lambda \int_{\eta \in \partial D'} H_y(x, 0; \eta, \lambda) w_y(\eta, \lambda; y, \tau) d\sigma(\eta) dx' \\ &\rightarrow \frac{\beta}{\sqrt{4\pi}} \end{aligned}$$

if the first term above $\rightarrow \beta/\sqrt{4\pi}$ and the second term above $\rightarrow 0$ as $(y, \tau) (\in \bar{\Gamma}_\beta(0)) \rightarrow 0$.

By the above discussion, we see that in order to prove Theorem 2.4.1, it is natural to first investigate properties of $\tilde{G}$. We would let $G_y(x, t; \eta, \tau)$ be the Green's function for the equation

$$L^y u = \sum a_{ij}(y) \frac{\partial^2 u}{\partial x_i \partial x_j}(x, t) - \frac{\partial u}{\partial t}(x, t) = 0 \quad \text{in } \partial D' \times (-\infty, \infty)$$

and

$$\begin{aligned} Z_y(x, t; \eta, \lambda) \\ = \frac{1}{(\det a^{ij}(y))^{1/2}} \cdot \frac{1}{(4\pi(t-\tau))^{n/2}} \exp \left\{ -\frac{\sum a^{ij}(y)(x_i - \eta_i)(x_j - \eta_j)}{4(t-\tau)} \right\} \end{aligned}$$

be its fundamental solution throughout this section where $(a_{ij}(x))$ is as defined on page 624 and $(a^{ij}) = (a_{ij})^{-1}$. We will also assume without loss of generality that $\partial D'$ is smooth throughout this section. Since the adjoint equation for L^y is

$$L^{y*} v = \sum a_{ij}(y) \frac{\partial^2 v}{\partial z_i \partial z_j}(z, \tau) + \frac{\partial v}{\partial t}(z, \tau) = 0,$$

maximum principle holds for solutions of L^y and L^{y*} . Therefore Lemma 1.3 and Lemma 1.4 remains valid for G_y . For the sake of completeness, we state them here again.

Lemma 2.4.2. For $-T \leq \tau < t \leq T$, $T > 0$, we have

(i)

$$G_y(x, t; z, \tau) \leq \frac{C}{(t - \tau)^{n/2}} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right), \quad x, z \in D',$$

(ii)

$$G_y(x, t; z, \tau) \leq C \frac{\delta(z)}{(t - \tau)^{n+1/2}} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right), \quad x, z \in D',$$

(iii)

$$G_y(x, t; z, \tau) \leq C \frac{\delta(x)\delta(z)}{(t - \tau)^{n+2/2}} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right), \quad x, z \in D',$$

(iv)

$$\left| \frac{\partial G_y}{\partial N_x}(x, t; z, \tau) \right| \leq C \frac{\delta(z)}{(t - \tau)^{n+2/2}} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right),$$

$$x \in \partial D', \quad z \in D',$$

where $\partial/\partial N_x$ is the derivative in the direction of the normal to $\partial D'$ at the point $x \in \partial D'$, $\delta(x) = \text{dist}(x, \partial D')$, and C is a constant independent of x, z, t, τ .

Corollary 2.4.3. The following is true:

(i) $|\nabla_z G_y(x, t; z, \tau)| \leq C \frac{\delta(x)}{(t - \tau)^{n+2/2}} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right),$

(ii) $|\nabla_x G_y(x, t; z, \tau)| \leq C \frac{\delta(z)}{(t - \tau)^{n+2/2}} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right)$

for all $-T \leq \tau < t \leq T$ (for any fixed $T > 0$), $x, z \in \overline{D'}$, and C is a constant independent of y and depending only on $T > 0$.

The following lemma comes from [LSU, Theorem 16.3].

Lemma 2.4.4. The Green's function $G_y(x, t; z, \tau)$ for

$$L^y = \sum a_{ij}(y) \partial_{x_i}^2 \partial_{x_j} - \partial_t = 0 \quad \text{in } D' \times (-\infty, \infty)$$

satisfies the following inequalities:

(i) $|\partial_t^r \partial_x^s G_y(x, t; z, \tau)| \leq \frac{C}{(t - \tau)^{n+2r+s/2}} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right),$

(ii) $|\partial_t^r \partial_x^s G_y(x, t; z, \tau) - \partial_t^r \partial_x^s G_y(x, t'; z, \tau)|$

$$\leq C(t - t')^{3/2-r-s/2}(t' - \tau)^{-(n+3)/2} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right),$$

where $2r + s = 1, 2$ and $\tau < t' < t$, and

(iii) $|\partial_t^r \partial_x^s G_y(x, t; z, \tau) - \partial_t^r \partial_x^s G_y(\bar{x}, t; z, \tau)|$

$$\leq C \frac{|x - \bar{x}|}{(t - \tau)^{n+3/2}} \exp \left(-C \frac{|x'' - z|^2}{t - \tau} \right)$$

where $2r + s = 2$ and $\bar{x}$ is that one of the points x, x' which is closest to y , C is a positive constant independent of y .

The following lemma comes from [F, Chapter 1, Lemma 3].

Lemma 2.4.5. If $-\infty < \alpha < n/2 + 1$, $-\infty < \beta < n/2 + 1$, then

$$\begin{aligned} & \int_{\sigma}^t \int_{R^n} (t-\lambda)^{-\alpha} \exp\left(-\frac{h|x-\eta|^2}{4(t-\lambda)}\right) (\lambda-\sigma)^{-\beta} \exp\left(-\frac{h|\eta-y|^2}{4(\lambda-\sigma)}\right) d\eta d\lambda \\ &= \left(\frac{4\pi}{h}\right)^{n/2} B\left(\frac{n}{2}-\alpha+1, \frac{n}{2}-\beta+1\right) (t-\sigma)^{n/2+1-\alpha-\beta} \exp\left(-\frac{h|x-y|^2}{4(t-\sigma)}\right) \end{aligned}$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, Γ is the gamma function.

Lemma 2.4.6. If $\tilde{G}(x, t; y, \tau)$ is the Green's function for $Lu = \operatorname{div}(a_{ij}(x')\nabla u) - \partial_t u = 0$ in $D' \times (-\infty, \infty)$, $(x', x'_n) \in D'$. Then the following integral identity holds:

$$\begin{aligned} \tilde{G}(x, t; y, \tau) &= G_y(x, t; y, \tau) + E(x, t; y, \tau), \\ E(x, t; y, \tau) &= \int_{\tau}^t d\lambda \int_{D'} G_{\eta}(x, t; \eta, \lambda) \Phi(\eta, \lambda; y, \tau) d\eta \end{aligned}$$

with Φ being determined from the integral identity

$$\begin{aligned} \Phi(x, t; y, \tau) &= LG_y(x, t; y, \tau) \\ &+ \int_{\tau}^t d\lambda \int_{D'} LG_{\eta}(x, t; \eta, \lambda) \Phi(\eta, \lambda; y, \tau) d\eta. \end{aligned}$$

Proof. The proof is exactly the same as the construction of the fundamental solution of $Lu = 0$ given in [F, Chapter 1] except that we use Lemma 2.4.4 to control G_y and replace Theorem 1 of Chapter 1 of [F] by the following lemma. We omit the details of its proof.

Lemma 2.4.7. Let $f(y, \tau)$ be a continuous bounded function in $D' \times (T_0, T_1)$. Then $J(x, t, \tau) = \int_{D'} G_y(x, t; y, \tau) f(y, \tau) dy$ is a continuous function in (x, t, τ) , $x \in \overline{D'}$, $T_0 \leq \tau < t \leq T_1$ and $\lim_{\tau \rightarrow t} J(x, t, \tau) = f(x, t)$.

Corollary 2.4.8. Let v_R be the solution of the adjoint equation

$$L^{y*} v_R = \sum a_{ij}(y') \frac{\partial^2 v_R}{\partial z_i \partial z_j}(z, \tau) + \frac{\partial v_R}{\partial \tau}(z, \tau) = 0 \quad \text{in } D_R \times (-2R^2, 0)$$

of L^y where $D_R = B(0, 2R) \setminus \overline{B(0, R)}$ with boundary value $\phi(z/R, \tau/R^2)$, $\phi \in C^\infty$, $0 \leq \phi \leq 1$, and

$$\begin{cases} \phi(z, \tau) \equiv 1 & \text{for } (z, \tau) \in \partial B(0, 2) \times (-4, 0] \cup (B(0, 2) \setminus \overline{B(0, 1)}) \times \{0\}, \\ \phi(z, \tau) \equiv 0 & \text{for } (z, \tau) \in \partial B(0, 1) \times (-2, -4). \end{cases}$$

Then

$$\|\nabla_z v_R\|_{L^\infty(D_R \times (-4R^2, 0))} \leq C/R < \infty$$

for some constant C independent of $y \in D'$ and $R > 0$.

Proof. The lemma follows from the boundary Schauder's estimate. (See [F, Theorem 6, p. 65].)

Lemma 2.4.9. With the same notation as in Lemma 2.4.6, then the function

$$\begin{aligned} F(x, t; z, \tau) &= LG_y(x, t; z, \tau) \\ &= \operatorname{div}(a_{ij}(x')\nabla_x G_y(x, t; z, \tau)) - \partial_t G_y(x, t; z, \tau) \end{aligned}$$

satisfies the adjoint equation $L^{y*}F = 0$ where

$$L^{y*} = \sum a_{ij}(y') \frac{\partial^2}{\partial z_i \partial z_j} + \frac{\partial}{\partial \tau}.$$

$$F(x, t; z, \tau)|_{\substack{z \in \partial D' \\ t > \tau}} \equiv 0$$

and

$$(i) \quad |F(x, t; z, \tau)| \leq C \left\{ \frac{|x - y|}{(t - \tau)^{n+2/2}} + \frac{1}{(t - \tau)^{n+1/2}} \right\} \cdot \exp \left\{ -C \frac{|x - z|^2}{t - \tau} \right\},$$

$$(ii) \quad |F(x, t; z, \tau)| \leq C_\alpha \left\{ \frac{|x - y| \delta(z)^{1-\alpha}}{(t - \tau)^{(n+3-\alpha)/2}} + \frac{\delta(z)^{1-\alpha}}{(t - \tau)^{(n+2-\alpha)/2}} \right\} \cdot \exp \left\{ -C \frac{|x - z|^2}{t - \tau} \right\}$$

for some constants $C_\alpha, C > 0$, C_α, C both independent of $y \in \partial D'$, $-1 < \tau < t < 1$, $x, z \in \partial D'$, $0 \leq \alpha < 1$.

Proof. We first note that

$$L^{y*}F(x, t; z, \tau) = L^{y*}LG_y(x, t; z, \tau) = 0$$

and

$$\begin{aligned} G_y(x, t; z, \tau)|_{\substack{z \in \partial D' \\ t > \tau}} &= 0 \\ \Rightarrow \partial_x^r \partial_t^s G_y(x, t; z, \tau)|_{\substack{z \in \partial D' \\ t > \tau}} &= 0 \quad \forall r, s = 0, 1, 2, \dots \\ \Rightarrow F(x, t; z, \tau) &= LG_y(x, t; z, \tau)|_{\substack{z \in \partial D' \\ t > \tau}} = 0. \end{aligned}$$

Also

$$\begin{aligned} |F(x, t; z, \tau)| &= |LG_y(x, t; z, \tau)| \\ &= \left| \sum a_{ij}(x') \frac{\partial^2 G_y}{\partial x_i \partial x_j}(x, t; z, \tau) - \Delta_{y'} \phi(y') \frac{\partial G_y}{\partial x_n}(x, t; z, \tau) - \frac{\partial G_y}{\partial t}(x, t; z, \tau) \right| \\ &= \left| \sum (a_{ij}(x') - a_{ij}(y')) \frac{\partial^2 G_y}{\partial x_i \partial x_j}(x, t; z, \tau) - \Delta_{y'} \phi(y') \frac{\partial G_y}{\partial x_n}(x, t; z, \tau) \right| \\ &\leq C \frac{|x - y|}{(t - \tau)^{n+2/2}} \exp \left(-C \frac{|x - z|^2}{t - \tau} \right) \\ &\quad + C \frac{1}{(t - \tau)^{n+1/2}} \cdot \exp \left(-C \frac{|x - z|^2}{t - \tau} \right). \end{aligned}$$

Proof of (ii). The proof of (ii) of Lemma 2.4.9 will be similar to the proof of (ii) of Lemma 1.4. We first note that since $\partial D'$ is smooth, $\partial D'$ satisfies the exterior sphere condition, i.e. there exists a positive constant $h > 0$ such that for each $x \in \partial D'$, $\forall 0 < r \leq h$, there exists $x_0 \in D'^c$ such that $\overline{B(x_0, r)} \cap \overline{D'} = \{x\}$.

Case 1. $\delta(z) \geq h$, then

$$1 \leq \frac{\delta(z)^{1-\alpha}}{h^{1-\alpha}} \leq \frac{\delta(z)^{1-\alpha}}{h^{1-\alpha}} \cdot \frac{(t - \tau)^{1-\alpha/2}}{(t - \tau)^{1-\alpha/2}} \leq C_\alpha \frac{\delta(z)^{1-\alpha}}{(t - \tau)^{1-\alpha/2}}$$

since $|t|, |\tau| \leq 1$. So by (i),

$$\begin{aligned} & |F(x, t; z, \tau)| \\ & \leq C_\alpha \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp \left(-C \frac{|x-z|^2}{t-\tau} \right) \frac{\delta(z)^{1-\alpha}}{(t-\tau)^{1-\alpha/2}} \\ & \leq C'_\alpha \left\{ \frac{|x-y|\delta(z)^{1-\alpha}}{(t-\tau)^{n+3-\alpha/2}} + \frac{\delta(z)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}} \right\} \exp \left(-C \frac{|x-z|^2}{t-\tau} \right). \end{aligned}$$

Case 2. $\delta(z) \geq (t-\tau)^{1/2}/4$. The proof is similar to the proof of Case 1.

Case 3. $\delta(z) \leq h$ and $\delta(z) \leq (t-\tau)^{1/2}/4 \leq h$.

We fix $z^* \in \partial D'$ with $|z - z^*| = \delta(z)$ and set $R = (t-\tau)^{1/2}/4$. Since $\partial D'$ satisfies the exterior sphere condition, there exists $z_0 \in \partial D'^c$ such that $\overline{B(z_0, R)} \cap \overline{D'} = \{z^*\}$. We may also assume $\overline{zz^*} \subset D'$. Consider the cylinder $\overline{Q_R} = \overline{D_R} \times (\tau, \tau + (2R)^2)$, $D_R = B(z_0, 2R) \setminus \overline{B(z_0, R)}$. Then, we have $(z, \tau) \in \overline{Q_R} \cap \overline{D'} \times (-1, 1)$. Similar to the proof of Case 3 of Lemma 1.4, for any $(\bar{z}, \bar{\tau}) \in \overline{Q_R} \cap \overline{D'} \times [-1, 1]$

$$|F(x, t; \bar{z}, \bar{\tau})| \leq C \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp \left(-C' \frac{|x-z|^2}{t-\tau} \right).$$

If $\bar{v}_R(\bar{z}, \bar{\tau}) = v_R(\bar{z} - z_0, \bar{\tau} - (\tau + 4R^2))$, $(\bar{z}, \bar{\tau}) \in \overline{Q_R}$, where v_R is as in Corollary 2.4.8, then since

$$\begin{cases} F(x, t; \bar{z}, \bar{\tau}) = 0, \\ \bar{v}_R(\bar{z}, \bar{\tau}) \geq 0 \quad \forall (\bar{z}, \bar{\tau}) \in \overline{Q_R} \cap \partial D' \times (-\infty, \infty) \end{cases}$$

and

$$\begin{cases} |F(x, t; \bar{z}, \bar{\tau})| \leq C \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp \left(-C \frac{|x-z|^2}{t-\tau} \right), \\ \bar{v}_R(\bar{z}, \bar{\tau}) \equiv 1 \end{cases}$$

for all

$$\begin{aligned} (\bar{z}, \bar{\tau}) & \in \{\partial B(z_0, 2R) \times (\tau, \tau + (2R)^2)\} \\ & \cup (B(z_0, 2R) \setminus B(z_0, R)) \times \{\tau + (2R)^2\} \cap \overline{D} \times (-\infty, \infty), \end{aligned}$$

by the maximum principle applied to the functions $F(x, t, \cdot, \cdot)$ and

$$C \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp \left(-C \frac{|x-z|^2}{t-\tau} \right) \bar{v}_R(\cdot, \cdot)$$

in the region $Q_R \cap D \times (-\infty, \infty)$, we have

$$\begin{aligned} |F(x, t; z, \tau)| & \leq C \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp \left(-C \frac{|x-z|^2}{t-\tau} \right) \bar{v}_R(z, \tau) \\ & \leq C \frac{\delta(z)}{(t-\tau)^{1/2}} \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp \left(-C \frac{|x-z|^2}{t-\tau} \right) \\ & = C_\alpha \left\{ \frac{|x-y|\delta(z)^{1-\alpha}}{(t-\tau)^{n+3-\alpha/2}} + \frac{\delta(z)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}} \right\} \exp \left(-C \frac{|x-z|^2}{t-\tau} \right) \end{aligned}$$

(since $\delta(z) \leq (t-\tau)^{1/2}/4 \Rightarrow (\delta(z)/(t-\tau)^{1/2})^\alpha \leq 4^{-\alpha}$).

Case 4. $\delta(z) \leq h$ and $\delta(z) \leq h \leq 4h \leq (t - \tau)^{1/2}$.

Proof of Case 4 is similar to Case 3. We omit the details.

Lemma 2.4.10. *With the same notation as 2.4.6, the function $\Phi(x, t; y, \tau)$ satisfies the inequality*

$$|\Phi(x, t; y, \tau)| \leq C_\alpha \frac{\delta(y)^{1-\alpha}}{(t - \tau)^{n+2-\alpha/2}} \exp\left(-C \frac{|x - y|^2}{t - \tau}\right), \quad 0 < \alpha < 1,$$

where C_α is a constant independent of $x, y \in D'$ and $t, \tau, -1 < \tau < t < 1$.

Proof. Since Φ satisfies the integral identity,

$$\begin{aligned} \Phi(x, t; y, \tau) &= LG_y(x, t; y, \tau) \\ &\quad + \int_\tau^t d\lambda \int_{\partial D'} LG_\eta(x, t; \eta, \lambda) \Phi(\eta, \lambda; y, \tau) d\eta. \end{aligned}$$

Therefore $\Phi(x, t; y, \tau) = \sum_{\nu=1}^\infty (LG_y)_\nu(x, t; y, \tau)$ where $(LG_y)_1(x, t; y, \tau) = LG_y(x, t; y, \tau)$ and

$$(LG_y)_{\nu+1}(x, t; y, \tau) = \int_\tau^t d\lambda \int_{\partial D'} LG_\eta(x, t; \eta, \lambda) (LG_y)_\nu(\eta, \lambda; y, \tau) d\eta$$

for all $\nu = 1, 2, \dots$. By Lemma 2.4.9,

$$\begin{aligned} |LG_y(x, t; y, \tau)| & (= |F(x, t; y, \tau)|) \\ &\leq C_\alpha \left\{ \frac{|x - y| \delta(y)^{1-\alpha}}{(t - \tau)^{n+3-\alpha/2}} + \frac{\delta(y)^{1-\alpha}}{(t - \tau)^{n+2-\alpha/2}} \right\} \exp\left(-C \frac{|x - y|^2}{t - \tau}\right) \\ &\leq C'_\alpha \frac{\delta(y)^{1-\alpha}}{(t - \tau)^{n+2-\alpha/2}} \exp\left(-C' \frac{|x - y|^2}{t - \tau}\right). \end{aligned}$$

Also from the proof of Lemma 2.4.9,

$$\begin{aligned} |LG_\eta(x, t; \eta, \lambda)| &\leq C \frac{|x - \eta|}{(t - \lambda)^{n+2/2}} \exp\left(-C \frac{|x - \eta|^2}{t - \lambda}\right) \\ &\quad + C \frac{1}{(t - \lambda)^{n+1/2}} \exp\left(-C \frac{|x - \eta|^2}{t - \lambda}\right) \\ &\leq C' \frac{1}{(t - \lambda)^{n+1/2}} \exp\left(-C \frac{|x - \eta|^2}{t - \lambda}\right). \end{aligned}$$

Therefore

$$\begin{aligned} |(LG_y)_2(x, t; y, \tau)| &\leq C_\alpha \int_\tau^t d\lambda \int_{D'} \frac{1}{(t - \lambda)^{n+1/2}} \exp\left(-C \frac{|x - \eta|^2}{t - \lambda}\right) \\ &\quad \cdot \frac{\delta(y)^{1-\alpha}}{(\lambda - \tau)^{n+2-\alpha/2}} \exp\left(-C \frac{|\eta - y|^2}{\lambda - \tau}\right) d\eta \\ &= C_\alpha \frac{\delta(y)^{1-\alpha}}{(t - \tau)^{n+1-\alpha/2}} \exp\left(-C \frac{|x - y|^2}{t - \tau}\right) \frac{\Gamma(1/2)\Gamma(\alpha/2)}{\Gamma(\alpha + 1/2)} \end{aligned}$$

by Lemma 2.4.5. And in general

$$\begin{aligned} |(LG_y)_{\nu+1}(x, t; y, \tau)| \\ \leq C_\alpha \frac{\Gamma(1/2)^\nu \Gamma(\alpha/2)}{\Gamma(\alpha + \nu/2)} \frac{\delta(y)^{1-\alpha}}{(t - \tau)^{n+2-\nu-\alpha/2}} \exp\left(-C \frac{|x - y|^2}{t - \tau}\right) \end{aligned}$$

for all $\nu = 0, 1, 2, \dots$. Therefore $\sum_{\nu=0}^{\infty} (LG_y)_{\nu+1}(x, t; y, \tau)$ is absolutely convergent and

$$\begin{aligned} |\Phi(x, t; y, \tau)| &\leq \sum_{\nu=0}^{\infty} |(LG_y)_{\nu+1}(x, t; y, \tau)| \\ &\leq C_{\alpha} \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}} \exp\left(-C \frac{|x-y|^2}{t-\tau}\right). \end{aligned}$$

Lemma 2.4.11. *With the same notation as Lemma 2.4.6, then the function*

$$\begin{aligned} E(x, t; y, \tau) &= \tilde{G}(x, t; y, \tau) - G_y(x, t; y, \tau) \\ &= \int_{\tau}^t d\lambda \int_{D'} G_y(x, t; \eta, \lambda) \Phi(\eta, \lambda; y, \tau) d\eta \end{aligned}$$

satisfies the following inequality

$$|\nabla_x E(x, t; y, \tau)| \leq C_{\alpha} \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+1-\alpha/2}} \exp\left(-C \frac{|x-y|^2}{t-\tau}\right), \quad 0 < \alpha < 1,$$

where C_{α} is a constant independent of $x, y \in D'$ and $t, \tau, -1 < \tau < t < 1$.

Proof. By Lemma 2.4.4,

$$|\nabla_x G_{\eta}(x, t; y, \tau)| \leq \frac{C}{(t-\tau)^{n+1/2}} \exp\left(-C \frac{|x-y|^2}{t-\tau}\right).$$

So

$$\begin{aligned} |\nabla_x E(x, t; y, \tau)| &\leq \int_{\tau}^t d\lambda \int_{D'} |\nabla_x G_y(x, t; \eta, \lambda)| |\Phi(\eta, \lambda; y, \tau)| d\eta \\ &\leq C'_{\alpha} \int_{\tau}^t d\lambda \int_{D'} \frac{1}{(t-\lambda)^{n+1/2}} \exp\left(-C \frac{|x-\eta|^2}{t-\lambda}\right) \\ &\quad \cdot \frac{\delta(y)^{1-\alpha}}{(\lambda-\tau)^{n+2-\alpha/2}} \exp\left(-C \frac{|\eta-y|^2}{\lambda-\tau}\right) d\eta \\ &\leq C'_{\alpha} \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+1-\alpha/2}} \exp\left(-C \frac{|x-y|^2}{t-\tau}\right) \end{aligned}$$

by Lemma 2.4.5 and Lemma 2.4.10.

Lemma 2.4.12. *With the same notation as Lemma 2.4.6,*

$$\lim_{\substack{(y, \tau) \in \bar{\Gamma}_{\beta}(0) \\ \tau \rightarrow 0}} \int_{\partial D'} |\nabla_x E(x, 0; y, \tau)| d\sigma(x) = 0.$$

Proof. Choose $0 < \alpha < 1/2$ in Lemma 2.4.11. Then for $(y, \tau) \in \bar{\Gamma}_{\beta}(0)$,

$$\begin{aligned} &\int_{\partial D'} |\nabla_x E(x, 0; y, \tau)| d\sigma(x) \\ &\leq C_{\alpha} \frac{\delta(y)^{1-\alpha}}{(-\tau)^{n+1-\alpha/2}} \int_{\partial D'} \exp\left(-C \frac{|x-y|^2}{-\tau}\right) d\sigma(x) \\ &\leq C'_{\alpha} \frac{(-\tau)^{3(1-\alpha)/2}}{(-\tau)^{n+1-\alpha/2}} \int_{\partial D'} \exp\left(-C' \frac{|x|^2}{-\tau}\right) d\sigma(x) \\ &\leq C_{\alpha} (-\tau)^{(1-2\alpha)/2} \rightarrow 0 \quad \text{as } \tau \rightarrow 0. \end{aligned}$$

Lemma 2.4.13. *The function*

$$\sum_{i,j} a_{ij}(y') n_i(x) \frac{\partial G_y}{\partial x_j}(x, t; y, \tau), \quad x \in \partial D', \quad y = (y', y'_n) \in D',$$

where $-1 < \tau < t < 1$ and $(n_i(x))_{i=1}^n = N_x$ is the inward normal to $\partial D'$ at x , satisfies the following integral equation:

$$\begin{aligned} \sum_{i,j} a_{ij}(y') n_i(x) \frac{\partial G_y}{\partial x_j}(x, t; y, \tau) \\ = 2H_y(x, t; y, \tau) - 2 \int_{\tau}^t d\lambda \int_{\eta \in \partial D'} H_y(x, t; \eta, \lambda) w_y(\eta, \lambda; y, \tau) d\sigma(\eta) \end{aligned}$$

where

$$\begin{aligned} w_y(\eta, \lambda; y, \tau) &= \sum a_{ij}(y') n_i(\eta) \frac{\partial G_y}{\partial \eta_j}(\eta, \lambda; y, \tau), \\ H_y(x, t; \eta, \lambda) &= \sum a_{ij}(y') n_i(x) \frac{\partial Z_y}{\partial x_j}(x, t; \eta, \lambda), \\ Z_y(x, t; \eta, \lambda) &= \frac{1}{(\det(a^{ij}(y)))^{1/2}} \\ &\quad \cdot \frac{1}{(4\pi(t-\lambda))^{n/2}} \exp \left\{ -\frac{\sum a^{ij}(y)(x_i - \eta_i)(x_j - \eta_j)}{4(t-\tau)} \right\}, \\ (a^{ij}(y)) &= A^{-1} = (a_{ij}(y))^{-1}. \end{aligned}$$

Also

$$\int_{\tau}^t \int_{\partial D'} |w_y(\eta, \lambda; y, \tau)| d\sigma(\eta) \leq C < \infty, \quad C \text{ independent of } y \in D'.$$

Proof. We first observe that $Z_y(x, t; \eta, \lambda)$ is the fundamental solution of

$$L^y u = \sum a_{ij}(y) \frac{\partial^2 u}{\partial x_i \partial x_j}(x, t) - \frac{\partial u}{\partial t}(x, t) = 0.$$

By the theory of layer potentials [LSU, p. 409] and [PO] the Green's function $G_y(x, t; z, \tau)$ for the equation $L^y = 0$ in $D' \times (-\infty, \infty)$ is then given by

$$G_y(x, t; z, \tau) = Z_y(x, t; z, \tau) - g_y(x, t; z, \tau)$$

where

$$g_y(x, t; z, \tau) = \int_{\tau}^t d\lambda \int_{\partial D'} Z_y(x, t; \eta, \lambda) w_y(\eta, \lambda; z, \tau) d\sigma(\eta)$$

for all $x \in R^n$, $z \in D$, $\tau < t$, with the density w_y being determined from the integral equation

$$\begin{aligned} w_y(x, t; z, \tau) &= 2H_y(x, t; z, \tau) \\ &\quad - 2 \int_{\tau}^t d\lambda \int_{\partial D'} H_y(x, t; \eta, \lambda) w_y(\eta, \lambda; z, \tau) d\sigma(\eta). \end{aligned}$$

Now for $x \in \partial D'$,

$$\begin{aligned} \lim_{\substack{\bar{x} \in \overline{D'}^c \rightarrow x \\ \bar{x} - x \perp N_x}} \sum a_{ij}(y) n_i(\bar{x}) \frac{\partial g_y}{\partial x_j}(\bar{x}, t; z, \tau) &= H_y(x, t; z, \tau) \\ &= W_y(x, t; z, \tau) + \frac{1}{2} w_y(x, t; z, \tau) \end{aligned}$$

where

$$W_y(x, t; z, \tau) = \int_{\tau}^t d\lambda \int_{\partial D'} H_y(x, t; \eta, \lambda) w_y(\eta, \lambda; z, \tau) d\sigma(\eta)$$

and

$$\begin{aligned} \lim_{\substack{\bar{x} \in \overline{D'} \rightarrow x \\ \bar{x} - x \perp N_x}} \sum a_{ij}(y) n_i(\bar{x}) \frac{\partial g_y}{\partial x_j}(\bar{x}, t; z, \tau) &= W_y(x, t; z, \tau) - \frac{1}{2} w_y(x, t; z, \tau) \\ &= H_y(x, t; z, \tau) - w_y(x, t; z, \tau). \end{aligned}$$

Therefore $\forall x \in \partial D'$,

$$\begin{aligned} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, t; z, \tau) &= \lim_{\substack{\bar{x} \in \overline{D'} \rightarrow x \\ \bar{x} - x \perp N_x}} \sum a_{ij}(y) n_i(\bar{x}) \frac{\partial G_y}{\partial x_j}(\bar{x}, t; z, \tau) \\ &= \lim_{\substack{\bar{x} \in \overline{D'} \rightarrow x \\ \bar{x} - x \perp N_x}} \left\{ \sum a_{ij}(y) n_i(\bar{x}) \left(\frac{\partial Z_y}{\partial x_j}(\bar{x}, t; z, \tau) - \frac{\partial g_y}{\partial x_j}(\bar{x}, t; z, \tau) \right) \right\} \\ &= \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, t; z, \tau) - (H_y(x, t; z, \tau) - w_y(x, t; z, \tau)) \\ &= w_y(x, t; z, \tau). \end{aligned}$$

Also by the result of [LSU, p. 411], $\int_{\tau}^t d\lambda \int_{\partial D'} |w_y(\eta, \lambda; y, \tau)| d\sigma(\eta) \leq C < \infty$ for some constant C independent of y .

Lemma 2.4.14. *Use the same notation as Lemma 2.4.6. For any $0 < 2\varepsilon \leq r_0$ (r_0 as in the beginning of subsection 2.2),*

$$2 \int_{\substack{|x| \leq \varepsilon \\ x \in \partial D'}} H_y(x, 0; y, \tau) d\sigma(x) \rightarrow \frac{\beta}{\sqrt{4\pi}} \quad \text{as } (y, \tau) \in \overline{\Gamma_{\beta}}(0) \rightarrow (0, 0).$$

Proof. For $x \in \partial D'$, $|x| \leq \varepsilon$, $(y, \tau) \in \overline{\Gamma_{\beta}}(0)$, $|y| \leq \varepsilon$,

$$\begin{aligned} \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, 0; y, \tau) &= \sum_{i,j,k} a_{ij}(y) n_i(x) \left(-\frac{a^{jk}(y)(x_k - y_k)}{2(-\tau)} \right) Z_y(x, 0; y, \tau) \\ &= \frac{-\langle x - y, N_x \rangle}{2(-\tau)} Z_y(x, 0; y, \tau) \\ &= \frac{y'_n}{2(-\tau)} Z_y(x, 0; y, \tau). \end{aligned}$$

Therefore

$$\begin{aligned}
& 2 \int_{\substack{|x| \leq \varepsilon \\ x \in \partial D'}} H_y(x, 0; y, \tau) d\sigma(x) \\
&= \frac{y_n - \phi(y')}{\sqrt{4\pi}(-\tau)^{3/2}} \cdot \frac{1}{(\det a^{ij}(y))^{1/2}(4\pi(-\tau))^{n-1/2}} \\
&\quad \cdot \int_{|x'| \leq \varepsilon} \exp\left(-\frac{\sum a^{ij}(y)(x_i - y_i)(x_j - y_j)}{4(-\tau)}\right) dx' \\
&= \frac{y_n - \phi(y')}{\sqrt{4\pi}(-\tau)^{3/2}} \left(\frac{1}{(\det a^{ij}(y))^{1/2}} - 1 \right) \\
&\quad \cdot \frac{1}{(4\pi(-\tau))^{n-1/2}} \int_{|x'| \leq \varepsilon} \exp\left(-\frac{\sum a^{ij}(y)(x_i - y_i)(x_j - y_j)}{4(-\tau)}\right) dx' \\
&\quad + \frac{y_n - \phi(y')}{\sqrt{4\pi}(-\tau)^{3/2}} \cdot \frac{1}{(4\pi(-\tau))^{n-1/2}} \\
&\quad \cdot \int_{|x'| \leq \varepsilon} \exp\left(-\frac{\sum a^{ij}(y)(x_i - y_i)(x_j - y_j)}{4(-\tau)}\right) dx' \\
&= I_1 + I_2.
\end{aligned}$$

$$\begin{aligned}
|I_1| &\leq \frac{|y_n| + |\phi(y')|}{\sqrt{4\pi}(-\tau)^{3/2}} \left| \frac{1}{(\det a^{ij}(y))^{1/2}} - 1 \right| \\
&\quad \cdot \frac{1}{(4\pi(-\tau))^{n-1/2}} \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \exp\left(-C \frac{|x - y|^2}{(-\tau)}\right) dx' \\
&\leq C \left| \frac{1}{(\det(a^{ij}(y')))^{1/2}} - 1 \right| \\
&\quad \cdot \frac{1}{(4\pi(-\tau))^{n-1/2}} \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \exp\left(-C' \frac{|x'|^2}{(-\tau)}\right) dx' \\
&\leq C' \left| \frac{1}{(\det(a^{ij}(y')))^{1/2}} - 1 \right| \rightarrow 0 \quad \text{as } (y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow (0, 0)
\end{aligned}$$

since $(a^{ij}(y))^{-1} \rightarrow I = \text{identity matrix}$ as $y \rightarrow 0$.

For I_2 , since $|y'| \leq C(-\tau)^{3/2}$, we have

$$\frac{y_n - \phi(y')}{(-\tau)^{3/2}} = \frac{y_n}{(-\tau)^{3/2}} + \frac{O(y'^2)}{(-\tau)^{3/2}} = \beta + O(\tau^{3/2}) \rightarrow \beta \quad \text{as } \tau \rightarrow 0$$

and

$$\begin{aligned}
& \frac{1}{(4\pi(-\tau))^{n-1/2}} \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \exp\left(-\frac{\sum a^{ij}(y)(x_i - y_i)(x_j - y_j)}{4(-\tau)}\right) dx' \\
& \rightarrow \frac{1}{(4\pi)^{n-1/2}} \int_{\bar{x}' \in \mathbb{R}^{n-1}} \exp(-|\bar{x}'|^2/4) d\bar{x}' = 1 \quad \text{as } \tau \rightarrow 0.
\end{aligned}$$

Therefore $I_2 \rightarrow \beta/\sqrt{4\pi}$ as $(y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow (0, 0)$. So

$$2 \int_{\substack{|x| \leq \varepsilon \\ x \in \partial D'}} H_y(x, 0; y, \tau) d\sigma(x) = I_1 + I_2 \rightarrow \frac{\beta}{\sqrt{4\pi}} \quad \text{as } (y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow (0, 0).$$

Lemma 2.4.15. *With the same notation as Lemma 2.4.13, we have*

$$\begin{aligned} & \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) dx' \\ &= \int_{\substack{|x| \leq \varepsilon \\ x' \in \partial D'}} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) d\sigma(x) \\ &\rightarrow \frac{\beta}{\sqrt{4\pi}} \quad \text{as } (y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow (0, 0). \end{aligned}$$

Proof. By Lemma 2.4.13,

$$\begin{aligned} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) &= 2H_y(x, 0; y, \tau) \\ &\quad - 2 \int_{\tau}^0 d\lambda \int_{\eta \in \partial D'} H_y(x, 0; \eta, \lambda) w_y(\eta, \lambda; y, \tau) d\sigma(\eta). \end{aligned}$$

Also $x'_n = 0$ for $x \in \partial D'$, $|x| \leq \varepsilon$, therefore

$$\begin{aligned} & \int_{\substack{|x| \leq \varepsilon \\ x' \in \partial D'}} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) d\sigma(x) \\ &= \int_{\substack{|x| \leq \varepsilon \\ x' \in \partial D'}} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) dx' \\ &= \int_{\substack{|x| \leq \varepsilon \\ x' \in \partial D'}} (-2) \int_{\tau}^0 d\lambda \left\{ \int_{\substack{|\eta| \leq 2\varepsilon \\ \eta \in \partial D'}} + \int_{\substack{|\eta| > 2\varepsilon \\ \eta \in \partial D'}} \right\} \\ &\quad \cdot H_y(x, 0; \eta, \lambda) w_y(\eta, \lambda; y, \tau) d\sigma(\eta) dx' \\ &\quad + 2 \int_{\substack{|x| \leq \varepsilon \\ x' \in \partial D'}} H_y(x, 0; y, \tau) dx' \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Now for $x, \eta \in \partial D'$, $|x|, |\eta| \leq 2\varepsilon$, we have $x'_n = \eta'_n = 0$ and $N_x = (0, \dots, 0, 1)$. So

$$\begin{aligned} H_y(x, 0; \eta, \lambda) &= \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, 0; \eta, \lambda) \\ &= \frac{-\langle N_x, x - \eta \rangle}{2(-\lambda)} Z_y(x, 0; \eta, \lambda) \\ &= 0. \end{aligned}$$

Therefore $J_1 = 0$. For $|x| \leq \varepsilon$, $|\eta| \geq 2\varepsilon$, $x, \eta \in \partial D'$, $(y, \tau) \in \bar{\Gamma}_\beta(0)$, $-1 < \tau < \lambda < 0$, since

$$\begin{aligned} |H_y(x, 0; \eta, \lambda)| &= \left| \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, 0; \eta, \lambda) \right| \\ &= \left| \frac{-\langle N_x, x - \eta \rangle}{2(-\lambda)} \right| |Z_y(x, 0; \eta, \lambda)| \\ &\leq C \frac{1}{(-\lambda)^{n+1/2}} \exp \left(-C \frac{\varepsilon^2}{(-\lambda)} \right). \end{aligned}$$

Therefore $|J_2|$ is

$$\begin{aligned} &\leq C \int_{|x'| \leq \varepsilon} \int_{\tau}^0 d\lambda \int_{\substack{\eta \in \partial D' \\ |\eta| \geq 2\varepsilon}} \frac{1}{(-\lambda)^{n+1/2}} \exp \left(-C \frac{\varepsilon^2}{(-\lambda)} \right) |w(\eta, \lambda, y, \tau)| d\sigma(\eta) dx' \\ &\leq C \varepsilon^{n-1} \sup_{\tau \leq \lambda \leq 0} \frac{C}{|\lambda|^{n-1/2}} \exp \left(-C \frac{\varepsilon^2}{|\lambda|} \right) \rightarrow 0 \quad \text{as } (y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow (0, 0). \end{aligned}$$

Also by Lemma 2.4.14, $J_3 \rightarrow \beta/\sqrt{4\pi}$ as $(y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow (0, 0)$. Therefore

$$\lim_{\substack{(y, \tau) \in \bar{\Gamma}_\beta(0) \\ \tau \rightarrow 0}} \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \sum a_{ij}(y') n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) dx' = \frac{\beta}{\sqrt{4\pi}}.$$

Lemma 2.4.16. *With the same notation as Lemma 2.4.6 and Lemma 2.4.13, we have*

$$\lim_{\substack{(y, \tau) \in \bar{\Gamma}_\beta(0) \\ \tau \rightarrow 0}} \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \frac{\partial \tilde{G}}{\partial x'_n}(x, 0; y, \tau) dx' = \frac{\beta}{\sqrt{4\pi}}.$$

Proof. By Lemma 2.4.6,

$$\begin{aligned} &\int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \frac{\partial \tilde{G}}{\partial x'_n}(x, 0; y, \tau) dx' \\ &= \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \frac{\partial G_y}{\partial x'_n}(x, 0; y, \tau) dx' \\ &\quad + \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \frac{\partial E}{\partial x'_n}(x, 0; y, \tau) dx' \\ &= I_1 + I_2. \end{aligned}$$

By Lemma 2.4.12, $I_2 \rightarrow 0$ as $(y, \tau) \in (\bar{\Gamma}_\beta(0)) \rightarrow (0, 0)$. Also

$$\begin{aligned} \left| I_1 - \frac{\beta}{\sqrt{4\pi}} \right| &\leq \left| I_1 - \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \sum a_{ij}(y') n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) dx' \right| \\ &\quad + \left| \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \sum a_{ij}(y') n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) dx' - \frac{\beta}{\sqrt{4\pi}} \right| \\ &= E_1 + E_2. \end{aligned}$$

By Lemma 2.4.15, $E_2 \rightarrow 0$ as $(y, \tau) \in (\bar{\Gamma}_\beta(0)) \rightarrow (0, 0)$. While

$$\begin{aligned} E_1 &= \left| I_1 - \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \sum a_{nj}(y') \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) dx' \right| \quad (n_i(x))_{i=1}^n = (0, \dots, 0, 1) \\ &\leq C \left\{ \sum_{1 \leq j \leq n-1} |a_{nj}(y')| + |a_{nn}(y') - 1| \right\} \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \max_j \left| \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) \right| dx' \\ &\leq C \left\{ \sum_{1 \leq j \leq n-1} \left| \frac{\partial \phi}{\partial y_j}(y') \right| + |\nabla_{y'} \phi(y')|^2 \right\} \\ &\quad \cdot \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \frac{\delta(y)}{|\tau|^{n+2/2}} \exp \left(-C \frac{|x - y|^2}{|\tau|} \right) dx' \\ &\leq C' \left\{ \sum_{1 \leq j \leq n-1} \left| \frac{\partial \phi}{\partial y_j}(y') \right| + |\nabla_{y'} \phi(y')|^2 \right\} \\ &\quad \cdot \frac{1}{|\tau|^{n-1/2}} \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \exp \left(-C' \frac{|x'|^2}{|\tau|} \right) dx' \\ &\leq C' \left\{ \sum_{1 \leq j \leq n-1} \left| \frac{\partial \phi}{\partial y_j}(y') \right| + |\nabla_{y'} \phi(y')|^2 \right\} \rightarrow 0 \end{aligned}$$

as $(y, \tau) \in (\bar{\Gamma}_\beta(0)) \rightarrow (0, 0)$ since $\nabla \phi(0) = 0$. Hence $|I_1 - \beta/\sqrt{4\pi}| \rightarrow 0$ as $(y, \tau) \in (\bar{\Gamma}_\beta(0)) \rightarrow (0, 0)$. So

$$\lim_{\substack{(y, \tau) \in \bar{\Gamma}_\beta(0) \\ \tau \rightarrow 0}} \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \frac{\partial \tilde{G}}{\partial x'_n}(x, 0; y, \tau) dx' = \frac{\beta}{\sqrt{4\pi}}.$$

We are now ready for the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1.

$$\begin{aligned}
& \left| \int_{x \in \partial D} \frac{\partial G}{\partial N_x}(x, 0; y, \tau) d\sigma(x) - \frac{\beta}{\sqrt{4\pi}} \right| \\
& \leq \int_{\substack{x \in \partial D \\ |x'| \geq \varepsilon}} \left| \frac{\partial G}{\partial N_x}(x, 0; y, \tau) \right| d\sigma(x) \\
& \quad + \left| \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} \nabla_x G(x, 0; y, \tau) \cdot (-\nabla \phi(x'), 1) dx' - \frac{\beta}{\sqrt{4\pi}} \right| \\
& \leq \int_{\substack{x \in \partial D \\ |x'| \geq \varepsilon}} \left| \frac{\partial G}{\partial N_x}(x, 0; y, \tau) \right| d\sigma(x) \\
& \quad + \sup_{|x'| \leq \varepsilon} |\nabla \phi(x')| \cdot \int_{\substack{x \in \partial D \\ |x'| \leq \varepsilon}} |\nabla_x G(x, 0; y, \tau)| dx' \\
& \quad + \left| \int_{\substack{|x'| \leq \varepsilon \\ x' \in \mathbb{R}^{n-1}}} \frac{\partial \tilde{G}}{\partial x'_n}(x, 0; y, \tau) dx' - \frac{\beta}{\sqrt{4\pi}} \right| \\
& = I'_1 + I'_2 + I'_3
\end{aligned}$$

since

$$\frac{\partial G}{\partial x_n}(x, 0; y, \tau) = \frac{\partial \tilde{G}}{\partial x'_n}(x, 0; y, \tau) \quad \text{and} \quad \left| \frac{\partial(x', x_n)}{\partial(x', x'_n)} \right| = 1$$

with $x'_n = x_n - \phi(x')$. Now

$$\begin{aligned}
\int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} |\nabla_x G(x, 0; y, \tau)| dx' & \leq \int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} \frac{\delta(y)}{|\tau|^{n+2/2}} \exp\left(-C \frac{|x-y|^2}{|\tau|}\right) dx' \\
& \leq \int_{\substack{x \in \partial D' \\ |x'| \leq \varepsilon}} \frac{1}{|\tau|^{n-1/2}} \exp\left(-C \frac{|x'|^2}{|\tau|}\right) dx' \leq C'
\end{aligned}$$

for $(y, \tau) \in \bar{\Gamma}_\beta(0)$.

Therefore $|I'_2| \leq C'' \sup_{|x'| \leq \varepsilon} |\nabla^2 \phi(x')| |x'| \leq C''' \varepsilon$.

By Lemma 2.4.16, $I'_3 \rightarrow 0$ as $(y, \tau) \in \bar{\Gamma}_\beta(0) \rightarrow 0$. Also $I'_1 \rightarrow 0$ as $\tau \rightarrow 0$. Therefore

$$\begin{aligned}
& \lim_{\substack{(y, \tau) \in \bar{\Gamma}_\beta(0) \\ \tau \rightarrow 0}} \left| \int_{x \in \partial D} \frac{\partial G}{\partial N_x}(x, 0; y, \tau) d\sigma(x) - \frac{\beta}{\sqrt{4\pi}} \right| \leq C\varepsilon \quad \forall \varepsilon > 0 \\
& \Rightarrow \lim_{\substack{(y, \tau) \in \bar{\Gamma}_\beta(0) \\ \tau \rightarrow 0}} \int_{x \in \partial D} \frac{\partial G}{\partial N_x}(x, 0; y, \tau) d\sigma(x) = \frac{\beta}{\sqrt{4\pi}}.
\end{aligned}$$

2.5. An example. In this section we will give an example of a solution of the heat equation in R^2 to show that the index $3/2$ on t in the definition of the space-time cone for the corner points of a cylinder is essentially sharp.

By the result of P. Hartman and A. Wintner [HW] and Hattermer [H], if

$$\theta(x, t) = \sum_{k=-\infty}^{\infty} (4\pi t)^{-1/2} \exp\left(-\frac{(x+2k)^2}{4t}\right),$$

then the Green function for the heat equation in the region $D_1 \times (0, \infty)$, where $D_1 = [0, 1]^2$, is given by

$$G((x_1, x_2), t; (y_1, y_2), \tau) = \prod_{i=1}^2 \{\theta(x_i - y_i, t - \tau) - \theta(x_i + y_i, t - \tau)\}$$

where $0 \leq s < t < \infty$. Therefore

$$\left. \frac{\partial G}{\partial N_{(y_1, y_2)}} \right|_{y_1=0} = -2\theta_{x_1}(x_1, t - \tau) \cdot \{\theta(x_2 - y_2, t - \tau) - \theta(x_2 + y_2, t - \tau)\}$$

where $\partial/\partial N_{(y_1, y_2)}$ is the derivative in the direction of the inward normal $N_{(y_1, y_2)}$ to ∂D_1 at $(y_1, y_2) \in \partial D_1$. By the representation theorem in subsection 2.2,

$$u(x, t) = -2\theta_{x_1}(x_1, t - \tau) \cdot \int_0^1 \{\theta(x_2 - y_2, t - \tau) - \theta(x_2 + y_2, t - \tau)\} dy_2$$

with $x = (x_1, x_2) \in D_1$, $t > 0$, is the strong solution of the (IDP) in $D_1 \times (0, \infty)$ with initial trace 0 on D_1 and trace $d\lambda$ on ∂D_1 that equals dx_2 on $\{0\} \times [0, 1]$ and equals 0 on $\partial D_1 \setminus \{0\} \times [0, 1]$.

Fix a point $(x_1^0, x_2^0) = (0, x_2^0) \in \partial D$, $0 < x_2^0 < 1$, and let $(x, t) \in D_1 \times (0, \infty)$. Then

$$\int_0^1 \{\theta(x_2 - y_2, t - \tau) - \theta(x_2 + y_2, t - \tau)\} dy_2$$

is the bounded solution of the heat equation in $[0, 1] \times (0, \infty)$ with initial value 1 on $(0, 1) \times \{0\}$ and boundary value 0 on the lateral sides of $[0, 1] \times (0, \infty)$. Therefore

$$\int_0^1 \{\theta(x_2 - y_2, t - \tau) - \theta(x_2 + y_2, t - \tau)\} dy_2 \rightarrow 1 \quad \text{if } |x_2 - x_2^0| \leq Ct^{3/2} \rightarrow 0.$$

On the other hand,

$$\begin{aligned} -2\theta_{x_1}(x_1, t - \tau) &= 2 \sum_{k=-\infty}^{\infty} (4\pi t)^{-1/2} \cdot \frac{2(x_1 + 2k)}{4t} \exp\left(-\frac{(x_1 + 2k)^2}{4t}\right) \\ &= \frac{x_1}{\sqrt{4\pi t^{3/2}}} \exp\left(-\frac{x_1^2}{4t}\right) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(x_1 + 2k)}{\sqrt{4\pi t^{3/2}}} \exp\left(-\frac{(x_1 + 2k)^2}{4t}\right). \end{aligned}$$

If $x_1 = \beta t^a$, then the second term always goes to 0 as $t \rightarrow 0$ while the first term will converge to 0 if $a > 3/2$, ∞ if $a < 3/2$, $\beta/\sqrt{4\pi}$ if $a = 3/2$. Hence

$$\begin{aligned} u(x, t) &\rightarrow 0 \quad \text{as } t \rightarrow 0 \text{ if } a > 3/2 \text{ and } x_1 = \beta t^a, \quad |x_2 - x_2^0| \leq Ct^{3/2}, \\ &\rightarrow \infty \quad \text{as } t \rightarrow 0 \text{ if } a < 3/2 \text{ and } x_1 = \beta t^a, \quad |x_2 - x_2^0| \leq Ct^{3/2}, \\ &\rightarrow \frac{\beta}{\sqrt{4\pi}} \quad \text{as } t \rightarrow 0 \text{ if } a = 3/2 \text{ and } x_1 = \beta t^a, \quad |x_2 - x_2^0| \leq Ct^{3/2}. \end{aligned}$$

Therefore the index $3/2$ on t in the definition of the space-time cone is essentially sharp.

ACKNOWLEDGMENT

This paper is part of the thesis [HU] I wrote under the direction of Professor Carlos E. Kenig, my thesis advisor. I wish to thank him for suggesting this problem to me and for his constant encouragement and advice during the preparation of the paper. I would also like to thank Professor Robert Fefferman and Professor Russell M. Brown for many helpful discussions of the problem.

REFERENCES

- [D] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Univ. Press, New York, 1989.
- [DK1] B. E. J. Dahlberg and C. E. Kenig, *Nonnegative solutions of the generalized porous medium equation*, Rev. Mat. Iberoamericana **2** (1986), 267–305.
- [DK2] ———, *Nonnegative solutions of the initial Dirichlet problem for generalized porous medium equation in cylinders*, J. Amer. Math. Soc. **1** (1988), 401–412.
- [F] A. Friedman, *Partial differential equations of parabolic type*, Krieger, Malabar, Florida, 1983.
- [FGS] E. B. Fabes, N. Garofalo, and S. Salsa, *A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations*, Illinois J. Math. **30** (1986), 536–565.
- [FS] E. B. Fabes and S. Salsa, *Estimate of caloric measure and the initial Dirichlet problem for the heat equation in Lipschitz cylinders*, Trans. Amer. Math. Soc. **279** (1983), 635–650.
- [GW] Michael Grüter and K. O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), 303–342.
- [H] J. R. Hattemer, *Boundary behaviour of temperatures. I*, Studia Math. **25** (1964), 111–155.
- [HU] Kin Ming Hui, Thesis (in preparation).
- [HW] Philip Hartman and A. Wintner, *On the solutions of the equation of heat conduction*, Amer. J. Math. **72** (1950), 367–395.
- [JK] D. S. Jerison and C. E. Kenig, *Boundary value problems on Lipschitz domains*, Studies in Partial Differential Equations, MAA Studies in Math., 1982.
- [K] J. T. Kemper, *Temperatures in several variables: kernel functions, representations, and parabolic boundary values*, Trans. Amer. Math. Soc. **167** (1972), 243–262.
- [LSU] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, Transl. Math. Mono., Vol. 23, Amer. Math. Soc., Providence, R.I., 1968.
- [Po] W. Pogorzelski, *Étude d'une fonction de Green et du problème aux limites pour l'équation parabolique normale*, Ann. Polon. Math. **4** (1958), 288–307.
- [S] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1971.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

Current address: Institute of Mathematics, Academia Sinica, Nankang, Taipei, Taiwan, R.O.C.

E-mail address: makmhui@twinas886.bitnet