FUNDAMENTAL SOLUTIONS FOR HYPOELLIPTIC DIFFERENTIAL OPERATORS DEPENDING ANALYTICALLY ON A PARAMETER

FRANK MANTLIK

ABSTRACT. Let $P(\lambda, D) = \sum_{|\alpha| \leq m} a_\alpha(\lambda) D^\alpha$ be a differential operator with constant coefficients a_α depending analytically on a parameter λ . Assume that each $P(\lambda, D)$ is hypoelliptic and that the strength of $P(\lambda, D)$ is independent of λ . Under this condition we show that there exists a regular fundamental solution of $P(\lambda, D)$ which also depends analytically on λ .

0. INTRODUCTION

We consider a differential polynomial

$$(0.1) \quad P(\lambda, D) = \sum_{|\alpha| \leq m} a_\alpha(\lambda) D^\alpha, \quad m \in \mathbb{N}, \quad D = -i \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right),$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ denotes a multiindex and $|\alpha| := \sum_{\nu=1}^n |\alpha_\nu|$ the length of α . The coefficients $a_\alpha(\lambda)$ are constant with respect to the variable x but may depend on a parameter $\lambda \in \Lambda$. Assuming that Λ is a complex manifold and the functions a_α are analytic we treat the following problem which has been posed by L. Hörmander [4, II, p. 59]:

- (*) Does there exist an analytic function $f: \Lambda \rightarrow \mathcal{D}'$ (the space of all distributions) such that $f(\lambda)$ is a regular (cf. [4, 10.2.2]) fundamental solution of the differential operator $P(\lambda, D)$ for each $\lambda \in \Lambda$?

F. Trèves [10] showed that if (*) holds and Λ is connected then necessarily $P(\lambda, D)$ is equally strong for each $\lambda \in \Lambda$. Conversely, if this condition is satisfied then for each $\lambda_0 \in \Lambda$ there exists a neighborhood Λ' of λ_0 and an analytic function $f: \Lambda' \rightarrow \mathcal{D}'$ such that $P(\lambda, D)f(\lambda) = \delta$, $\lambda \in \Lambda'$ (δ the Dirac distribution) [11]. The question remained open however whether such a function f could be defined globally in Λ .

In the present article we give a positive answer to problem (*) in the case when the $P(\lambda, D)$ are hypoelliptic (see §4 below). The idea is to use a result of J. Leiterer [6] on “BCAF-sheaves”, which is a strong tool to attack “local-global”-problems for analytic operator functions. For the applicability of this

Received by the editors September 28, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 35B30; Secondary 35E05.

Key words and phrases. Hypoelliptic operators, fundamental solutions, analytic parameter-dependence.

machinery it is necessary to construct suitable Banach spaces of distributions such that the mapping $\lambda \mapsto P(\lambda, D)$ can be considered as an analytic function with values in the bounded, surjective linear operators between these spaces.

The organization of this paper is as follows: In §1 we introduce some distribution spaces $\mathbf{B}_{q,k}^{-\infty}$ and $\mathbf{B}_{q,k}^{-\rho,\eta}$ which are embedded in the well-known spaces $\mathbf{B}_{q,k}^{\text{loc}}$ of Hörmander [4, §10.1]. Our definition of $\mathbf{B}_{q,k}^{-\rho,\eta}$ is motivated by an explicit integral formula for fundamental solutions of $P(\lambda, D)$. In the hypoelliptic case such a representation can be used simultaneously for all λ in a compact subset Λ' of Λ . In order to obtain a global solution of (*) we shall perform a Mittag-Leffler procedure in the Fréchet space $\mathbf{B}_{q,k}^{-\infty}$ which contains $\mathbf{B}_{q,k}^{-\rho,\eta}$ for all values of the parameters ρ, η . The behaviour of a differential operator on $\mathbf{B}_{q,k}^{-\rho,\eta}$ will be clarified in §2. In §3 the definition of parameter-dependent differential operators is formalized in preparation for the statements and proofs of our main results. §4 contains our solution of problem (*) which will be a corollary of a more general theorem.

I would like to thank Professor W. Kaballo for pointing out the problem to me, and for his constructive criticism during the preparation of this manuscript.

1. SOME DISTRIBUTION SPACES

In the sequel, $n \in \mathbb{N}$ denotes a fixed positive integer. We adopt the standard notations for spaces of test functions and distributions (cf. [4]):

$$\begin{aligned} \mathcal{D} &= \mathcal{E}_c^\infty(\mathbb{R}^n), & \mathcal{E}^\infty\text{-functions with compact support;} \\ \mathcal{D}' &= \mathcal{D}'(\mathbb{R}^n), & \text{space of all distributions;} \\ \mathcal{S} &= \mathcal{S}(\mathbb{R}^n), & \text{space of rapidly decreasing } \mathcal{E}^\infty\text{-functions;} \\ \mathcal{S}' &= \mathcal{S}'(\mathbb{R}^n), & \text{space of tempered distributions.} \end{aligned}$$

Recall that each of these spaces carries a natural locally convex vector space topology. The scalar product of two vectors $\zeta, \eta \in \mathbb{C}^n$ will be denoted by $[\zeta, \eta] := \sum_{\nu=1}^n \zeta_\nu \bar{\eta}_\nu$. If $\varphi \in \mathcal{S}$ then the Fourier transform $\hat{\varphi}$ of φ is the function

$$\hat{\varphi}(\zeta) := \int_{\mathbb{R}^n} \exp(-i[\zeta, x]) \varphi(x) dx, \quad \zeta \in \mathbb{R}^n.$$

The Fourier transform $\hat{u}$ of $u \in \mathcal{S}'$ is defined by the formula

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S},$$

where $\langle \cdot, \cdot \rangle$ denotes the distribution pairing. The following definitions are taken from Hörmander [4, §10.1].

1.1. Definition. (a) A function $k: \mathbb{R}^n \rightarrow (0, \infty)$ will be called a *temperate weight function* if there exist constants $C, D > 0$ such that

$$k(\xi + \zeta) \leq (1 + C|\xi|)^D k(\zeta), \quad \xi, \zeta \in \mathbb{R}^n.$$

The set of all such functions will be denoted by $\mathcal{K}$.

(b) If $k \in \mathcal{K}$ and $1 \leq p \leq \infty$ we denote by $\mathbf{B}_{p,k}$ the set of all distributions $u \in \mathcal{S}'$ such that $\hat{u}$ is a function and

$$\|u\|_{p,k} := \left((2\pi)^{-n} \int_{\mathbb{R}^n} |k(\xi) \hat{u}(\xi)|^p d\xi \right)^{1/p} < \infty.$$

In the case $p = \infty$ this expression has to be interpreted as $\text{ess sup}_{\xi \in \mathbb{R}^n} |k(\xi) \hat{u}(\xi)|$.

By [4, Theorem 10.1.7] we have the following embeddings:

$$\mathcal{S} \hookrightarrow \mathbf{B}_{p,k} \hookrightarrow \mathcal{S}',$$

where $\mathfrak{F} \hookrightarrow \mathfrak{G}$ means that the space $\mathfrak{F}$ is a subspace of $\mathfrak{G}$ carrying a stronger topology than that induced by $\mathfrak{G}$. The spaces $\mathbf{B}_{p,k}$ are Banach spaces which, for $1 \leq p < \infty$, contain $\mathcal{D}$ as a dense subset. In this case the dual $(\mathbf{B}_{p,k})'$ of $\mathbf{B}_{p,k}$ is (isometrically) isomorphic to $\mathbf{B}_{p',k'}$, where

$$1/p + 1/p' = 1, \quad k'(\xi) := 1/k(-\xi).$$

In fact, any continuous linear form on $\mathbf{B}_{p,k}$ is given by continuous extension of a form $\mathcal{D} \ni \varphi \mapsto \langle v, \varphi \rangle$ with $v \in \mathbf{B}_{p',k'}$ and the norm of this functional equals $\|v\|_{p',k'}$. Let

$$\mathbf{B}_{p,k}^{\text{loc}} := \{u \in \mathcal{D}' : \psi u \in \mathbf{B}_{p,k}, \psi \in \mathcal{D}\}$$

denote the local space associated with $\mathbf{B}_{p,k}$. This is a Fréchet space with the system of seminorms $u \mapsto \|\psi u\|_{p,k}$, $\psi \in \mathcal{D}$. In what follows we shall also require some modifications of the spaces $\mathbf{B}_{p,k}$.

1.2. Definition. Let $\mu \in \mathbb{N}$ and

$$(1.1) \quad \gamma_\mu(x) := \exp(\gamma'_\mu(x) \cdot \sqrt{1 + [x, x]}), \quad x \in \mathbb{R}^n,$$

where $\gamma'_\mu \in \mathcal{E}^\infty(\mathbb{R}^n)$ is defined recursively by

$$\gamma'_0(x) := [x, x], \quad \gamma'_\mu(x) := \log(1 + \gamma'_{\mu-1}(x)).$$

Further, let $1 \leq p \leq \infty$ and $k \in \mathcal{K}$. We define the distribution spaces

$$\mathbf{B}_{p,k}^\mu := \{u = 1/\gamma_\mu \cdot \tilde{u} : \tilde{u} \in \mathbf{B}_{p,k}\}, \quad \mathbf{B}_{p,k}^{-\mu} := \{v = \gamma_\mu \cdot \tilde{v} : \tilde{v} \in \mathbf{B}_{p,k}\}.$$

Obviously these are Banach spaces with the norms $\|u\|_{p,k}^\mu := \|\gamma_\mu \cdot u\|_{p,k}$ resp. $\|v\|_{p,k}^{-\mu} := \|1/\gamma_\mu \cdot v\|_{p,k}$.

Remark. Since $\gamma_\mu, 1/\gamma_\mu \in \mathcal{E}^\infty(\mathbb{R}^n)$ we have $\mathbf{B}_{p,k}^{\pm\mu} \subseteq \mathbf{B}_{p,k}^{\text{loc}}$ by [4, Theorem 10.1.23]. If $p < \infty$ then $\mathcal{D}$ is dense in $\mathbf{B}_{p,k}^\mu$. In this case there corresponds to any continuous linear form l on $\mathbf{B}_{p,k}^\mu$ a unique distribution $\tilde{v} \in \mathbf{B}_{p',k'}$, which satisfies $l(\varphi) = \langle \tilde{v}, \gamma_\mu \varphi \rangle$, $\varphi \in \mathcal{D}$. Clearly, the mapping $l \mapsto v := \gamma_\mu \cdot \tilde{v}$ defines an (isometric) isomorphism of $(\mathbf{B}_{p,k}^\mu)'$ onto $\mathbf{B}_{p',k'}^{-\mu}$.

1.3. Lemma. Let $1 < q \leq \infty$, $k \in \mathcal{K}$ and $0 < \mu < \nu$ (integers). Then we have the embeddings

$$(1.2) \quad \mathbf{B}_{q,k}^{-\nu} \hookrightarrow \mathbf{B}_{q,k}^{-\mu} \hookrightarrow \mathbf{B}_{q,k}^{\text{loc}}.$$

Proof. Let $v = \gamma_\nu \cdot \tilde{v}_1 \in \mathbf{B}_{q,k}^{-\nu}$ with $\tilde{v}_1 \in \mathbf{B}_{q,k}$. Then we have

$$v = \gamma_\mu \cdot (\gamma_\nu/\gamma_\mu) \cdot \tilde{v}_1 =: \gamma_\mu \cdot \tilde{v}_2.$$

Since $\gamma_\nu/\gamma_\mu \in \mathcal{S}$ it follows from [4, Theorem 10.1.15] that $\tilde{v}_2 \in \mathbf{B}_{q,k}$ and $\|\tilde{v}_2\|_{q,k} \leq c\|\tilde{v}_1\|_{q,k}$ where $c < \infty$ only depends on μ, ν and k . Hence $v \in \mathbf{B}_{q,k}^{-\mu}$ and

$$\|v\|_{q,k}^{-\mu} = \|\tilde{v}_2\|_{q,k} \leq c\|\tilde{v}_1\|_{q,k} = c\|v\|_{q,k}^{-\nu},$$

which proves the first embedding in (1.2). The second one can be proved similarly: Fix $\psi \in \mathcal{D}$ and let $v = \gamma_\mu \cdot \tilde{v} \in \mathbf{B}_{q,k}^{-\mu}$, $\tilde{v} \in \mathbf{B}_{q,k}$, be arbitrary. Noting that $\psi \cdot \gamma_\mu \in \mathcal{D} \subseteq \mathcal{S}$ we obtain as above

$$\|\psi v\|_{q,k} = \|\psi \gamma_\mu \tilde{v}\|_{q,k} \leq c' \|\tilde{v}\|_{q,k} = c' \|v\|_{q,k}^{-\mu},$$

with $c' < \infty$ depending only on μ , k , and ψ . Since the topology of $\mathbf{B}_{q,k}^{\text{loc}}$ is given by the seminorms $v \mapsto \|\psi v\|_{q,k}$ we are done. $\square$

1.4. Definition. For any $1 < q \leq \infty$ and $k \in \mathcal{K}$ let

$$\mathbf{B}_{q,k}^{-\infty} := \bigcap_{\mu \in \mathbb{N}} \mathbf{B}_{q,k}^{-\mu}$$

be endowed with the topology which is given by the system of seminorms $\{\|\cdot\|_{q,k}^{-\mu} : \mu \in \mathbb{N}\}$.

Remark. It is not hard to see that $\mathbf{B}_{q,k}^{-\infty}$ is a Fréchet space, and from (1.2) we obtain the embedding

$$(1.3) \quad \mathbf{B}_{q,k}^{-\infty} \hookrightarrow \mathbf{B}_{q,k}^{\text{loc}}.$$

Note that for any $\mu \in \mathbb{N}$ the mapping $v \mapsto 1/\gamma_\mu \cdot v$ embeds $\mathbf{B}_{q,k}^{-\infty}$ continuously into $\mathbf{B}_{q,k}$. The reason why we have introduced the spaces $\mathbf{B}_{q,k}^{-\infty}$ is that they give quite precise information on the growth at infinity of solutions of the equation $P(\lambda, D)f(\lambda) = \delta$ when $P(\lambda, D)$ depends analytically on λ (cf. the remark at the end of §4). In order to solve this equation we have to consider certain subspaces of $\mathbf{B}_{q,k}^{-\infty}$:

1.5. Definition. Let $1 \leq p < \infty$, $k \in \mathcal{K}$, $\rho \geq 0$, and $\eta \in \mathbb{R}^n \setminus \{0\}$. For any $\varphi \in \mathcal{D}$ we put

$$(1.4) \quad \|\varphi\|_{p,k}^{\rho,\eta} := \left((2\pi)^{-n} \int_{|\xi| \leq \rho} \int_{|z|=1} |k(\xi) \hat{\varphi}(\xi + z\eta)|^p \frac{|dz|}{2\pi} d\xi + (2\pi)^{-n} \int_{|\xi| \geq \rho} |k(\xi) \hat{\varphi}(\xi)|^p d\xi \right)^{1/p},$$

where $\xi \in \mathbb{R}^n$, $z \in \mathbb{C}$.

The theorem of Paley-Wiener-Schwartz [4, §7.3] ensures that $\|\varphi\|_{p,k}^{\rho,\eta}$ is finite for each $\varphi \in \mathcal{D}$. Obviously, $\|\cdot\|_{p,k}^{\rho,\eta}$ is a norm on $\mathcal{D}$. We consider the space

$$\mathbf{B}_{p',k'}^{-\rho,\eta} := \{v \in \mathbf{B}_{p',k'}^{\text{loc}} : \|v\|_{p',k'}^{-\rho,\eta} := \sup\{|\langle v, \varphi \rangle| : \varphi \in \mathcal{D}, \|\varphi\|_{p,k}^{\rho,\eta} \leq 1\} < \infty\}$$

endowed with the norm $\|\cdot\|_{p',k'}^{-\rho,\eta}$. Note that $\|\cdot\|_{p,k}^{0,\eta} = \|\cdot\|_{p,k}$ and thus $\mathbf{B}_{p',k'}^{-0,\eta} = \mathbf{B}_{p',k'}$ (isometrically).

Definition (1.4) is motivated by the fact that for any hypoelliptic polynomial P and with suitable $\rho > 0$, $\eta \in \mathbb{R}^n$, the formula

$$\langle f_P, \varphi \rangle := (2\pi)^{-n} \left(\int_{|\xi| \leq \rho} \int_{|z|=1} \frac{\hat{\varphi}(-\xi - z\eta)}{P(\xi + z\eta)} \frac{dz}{2\pi i z} d\xi + \int_{|\xi| \geq \rho} \frac{\hat{\varphi}(-\xi)}{P(\xi)} d\xi \right)$$

defines a fundamental solution f_P of $P(D)$. The idea of using such a representation goes back to L. Hörmander. It can be shown that $f_P \in \mathbf{B}_{\infty,P}^{-\rho,\eta}$, where

$\tilde{P} \in \mathcal{K}$ is defined as in §2 below.

1.6. Lemma. *Let $1 \leq p < \infty$, $k \in \mathcal{K}$, $0 \leq \rho_1 \leq \rho_2$, and $\eta_1, \eta_2 \in \mathbb{R}^n \setminus \{0\}$ such that $\eta_2 = t\eta_1$ with some $t \geq 1$. Then the estimate*

$$(1.5) \quad \|\varphi\|_{p,k}^{\rho_1, \eta_1} \leq \|\varphi\|_{p,k}^{\rho_2, \eta_2}, \quad \varphi \in \mathcal{D},$$

holds, and

$$(1.6) \quad \mathbf{B}_{p',k'}^{-\rho_1, \eta_1} \hookrightarrow \mathbf{B}_{p',k'}^{-\rho_2, \eta_2}.$$

Proof. With the notation

$$w_\tau(\xi) := \int_{|z|=1} |\hat{\varphi}(\xi + z\tau\eta_1)|^p \frac{|dz|}{2\pi}, \quad \varphi \in \mathcal{D},$$

we have $|\hat{\varphi}(\xi)|^p = w_0(\xi) \leq w_1(\xi) \leq w_t(\xi)$, $\xi \in \mathbb{R}^n$, (cf. [8, E.II, §3.3]). Using this, the verification of (1.5) is immediate. The embedding (1.6) follows directly from (1.5) and the definition of $\mathbf{B}_{p',k'}^{-\rho, \eta}$. $\square$

1.7. Lemma. *Let $1 \leq p < \infty$, $k \in \mathcal{K}$, $\mu \in \mathbb{N}$, $\rho \geq 0$, and $\eta \in \mathbb{R}^n \setminus \{0\}$. Then there exists a constant $c < \infty$ such that*

$$(1.7) \quad \|\varphi\|_{p,k}^{\rho, \eta} \leq c \|\varphi\|_{p,k}^{\mu}, \quad \varphi \in \mathcal{D}.$$

Proof. First we note that for any $\varphi \in \mathcal{D}$

$$\begin{aligned} (\|\varphi\|_{p,k}^{\rho, \eta})^p &\leq (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{|z|=1} |k(\xi) \hat{\varphi}(\xi + z\eta)|^p \frac{|dz|}{2\pi} d\xi \\ (1.8) \quad &= (2\pi)^{-n} \int_{|z|=1} \int_{\mathbb{R}^n} |k(\xi) \mathcal{F}(\exp(-i[z\eta, \cdot])\varphi)(\xi)|^p d\xi \frac{|dz|}{2\pi} \\ &= \int_{|z|=1} (\|\exp(-i[z\eta, \cdot])\varphi\|_{p,k})^p \frac{|dz|}{2\pi} \end{aligned}$$

(cf. the proof of Lemma 1.6). Here we have used the notation $\mathcal{F}(v) := \hat{v}$. Now consider the functions $\chi_z(x) := \exp(-i[z\eta, x])/\gamma_\mu(x)$, $|z| = 1$. It is an easy exercise to check that $\{\chi_z : |z| = 1\}$ is a bounded subset of $\mathcal{S}$. With the weight function $M_k \in \mathcal{K}$,

$$M_k(\xi) := \sup_{\zeta \in \mathbb{R}^n} k(\xi + \zeta)/k(\zeta), \quad \xi \in \mathbb{R}^n,$$

(cf. [4, §10.1]) we have $\mathcal{S} \hookrightarrow \mathbf{B}_{1, M_k}$ [4, Theorem 10.1.7], hence $\sup_{|z|=1} \|\chi_z\|_{1, M_k} =: c < \infty$. It follows from [4, Theorem 10.1.15] that

$$\sup_{|z|=1} \|\chi_z \psi\|_{p,k} \leq \sup_{|z|=1} \|\chi_z\|_{1, M_k} \|\psi\|_{p,k} = c \|\psi\|_{p,k}, \quad \psi \in \mathcal{D}.$$

From (1.8) we thus obtain

$$\begin{aligned} \|\varphi\|_{p,k}^{\rho, \eta} &\leq \left(\int_{|z|=1} (\|\chi_z \gamma_\mu \varphi\|_{p,k})^p \frac{|dz|}{2\pi} \right)^{1/p} \\ &\leq c \cdot \|\gamma_\mu \varphi\|_{p,k} = c \cdot \|\varphi\|_{p,k}^{\mu}. \quad \square \end{aligned}$$

1.8. Corollary. *Under the assumptions of Lemma 1.7 the mapping $v \mapsto \langle v, \cdot \rangle$ identifies $\mathbf{B}_{p',k'}^{-\rho, \eta}$ isometrically with the dual of the normed space $(\mathcal{D}, \|\cdot\|_{p,k}^{\rho, \eta})$. In particular $\mathbf{B}_{p',k'}^{-\rho, \eta}$ is complete. Furthermore we have*

$$(1.9) \quad \mathbf{B}_{p',k'}^{-\rho, \eta} \hookrightarrow \mathbf{B}_{p',k'}^{-\mu}.$$

Proof. Clearly, $v \mapsto \langle v, \cdot \rangle$ defines an isometric embedding of $\mathbf{B}_{p',k'}^{-\rho,\eta}$ into $(\mathcal{D}, \|\cdot\|_{p,k}^{\rho,\eta})'$. We have to show that it is onto. So let l be a continuous linear form on $(\mathcal{D}, \|\cdot\|_{p,k}^{\rho,\eta})$. By Lemma 1.7 we have for any $\varphi \in \mathcal{D}$

$$(1.10) \quad |l(\varphi)| \leq \|l\| \cdot \|\varphi\|_{p,k}^{\rho,\eta} \leq c\|l\| \cdot \|\varphi\|_{p,k}^{\mu}.$$

Since $\mathbf{B}_{p',k'}^{-\mu}$ is the dual space of $\mathbf{B}_{p,k}^{\mu}$ there exists a distribution $v \in \mathbf{B}_{p',k'}^{-\mu} \subseteq \mathbf{B}_{p',k'}^{\text{loc}}$ such that $l(\varphi) = \langle v, \varphi \rangle$, $\varphi \in \mathcal{D}$. By (1.10) we have $v \in \mathbf{B}_{p',k'}^{-\rho,\eta}$ which proves the first assertion. Next we show the embedding (1.9): If $v \in \mathbf{B}_{p',k'}^{-\rho,\eta}$ it follows from Lemma 1.7 that

$$|\langle v/\gamma_{\mu}, \varphi \rangle| \leq \|v\|_{p',k'}^{-\rho,\eta} \|\varphi/\gamma_{\mu}\|_{p,k}^{\rho,\eta} \leq c\|v\|_{p',k'}^{-\rho,\eta} \|\varphi\|_{p,k}, \quad \varphi \in \mathcal{D}.$$

Since $\mathcal{D}$ is dense in $\mathbf{B}_{p,k}$ this implies that $v/\gamma_{\mu} \in \mathbf{B}_{p',k'}$ and $\|v\|_{p',k'}^{-\mu} = \|v/\gamma_{\mu}\|_{p',k'} \leq c\|v\|_{p',k'}^{-\rho,\eta}$. $\square$

We conclude this section by observing that (1.9) yields

$$(1.11) \quad \mathbf{B}_{q,k}^{-\rho,\eta} \hookrightarrow \mathbf{B}_{q,k}^{-\infty}$$

for any $1 < q \leq \infty$, $k \in \mathcal{K}$, $\rho \geq 0$, and $\eta \in \mathbb{R}^n \setminus \{0\}$.

2. DIFFERENTIAL OPERATORS IN $\mathbf{B}_{q,k}^{-\rho,\eta}$

In this section we investigate how a differential operator with constant coefficients acts in the spaces $\mathbf{B}_{q,k}^{-\rho,\eta}$. If $P(x) = \sum_{|\alpha| \leq m} a_{\alpha} x^{\alpha}$ is a polynomial in $x \in \mathbb{R}^n$ we consider the differential expression $P(D) := \sum_{|\alpha| \leq m} a_{\alpha} D^{\alpha}$ where $D := -i\partial$, $\partial := (\partial/\partial x_1, \dots, \partial/\partial x_n)$. With a polynomial $P(x)$ we also associate the function

$$\tilde{P}(\xi) := \left(\sum_{\beta} |P^{(\beta)}(\xi)|^2 \right)^{1/2}, \quad \xi \in \mathbb{R}^n,$$

where $P^{(\beta)} := \partial^{\beta} P$ and the sum extends over all multi-indices β . Note that $\tilde{P} \in \mathcal{K}$ for any polynomial P (cf. [4, §10.1]). Now let $P, Q \in \mathbb{C}[x_1, \dots, x_n]$ be any two polynomials. We say that Q is *weaker* than P if $\sup_{\xi \in \mathbb{R}^n} \tilde{Q}(\xi)/\tilde{P}(\xi) < \infty$. In this case we write $Q < P$. P and Q are said to be *equally strong* (notation: $P \approx Q$) if $Q < P$ and $P < Q$. Obviously, $\approx$ is an equivalence relation on $\mathbb{C}[x_1, \dots, x_n]$. For the properties of $<$ and $\approx$ see [4, §10.1].

2.1. Proposition. *Let $P, P_0 \in \mathbb{C}[x_1, \dots, x_n] \setminus \{0\}$ with $P < P_0$. Let $1 < q \leq \infty$, $k \in \mathcal{K}$, $\rho > 0$, and $\eta \in \mathbb{R}^n \setminus \{0\}$. Then the operator $P(D)$ maps $\mathbf{B}_{q,kP_0}^{-\rho,\eta}$ continuously into $\mathbf{B}_{q,k}^{-\rho,\eta}$.*

Proof. With $p := q'$ and $k_0 := k\tilde{P}_0$ we have for any $\varphi \in \mathcal{D}$:

$$\begin{aligned}
 (2\pi)^n (\|P(-D)\varphi\|_{p, k'_0}^{\rho, \eta})^p &= \int_{|\xi| \leq \rho} \int_{|z|=1} |k'_0(\xi) P(-\xi - z\eta) \hat{\varphi}(\xi + z\eta)|^p \frac{|dz|}{2\pi} d\xi \\
 &\quad + \int_{|\xi| \geq \rho} |k'_0(\xi) P(-\xi) \hat{\varphi}(\xi)|^p d\xi \\
 (2.1) \quad &\leq c \cdot \int_{|\xi| \leq \rho} \int_{|z|=1} |k'_0(\xi) \tilde{P}(-\xi) \hat{\varphi}(\xi + z\eta)|^p \frac{|dz|}{2\pi} d\xi \\
 &\quad + \int_{|\xi| \geq \rho} |k'_0(\xi) \tilde{P}(-\xi) \hat{\varphi}(\xi)|^p d\xi \\
 &\leq (2\pi)^n (c' \|\varphi\|_{p, k'}^{\rho, \eta})^p.
 \end{aligned}$$

Here we have used the fact that $k'_0(\xi) \tilde{P}(-\xi) = k'(\xi) \tilde{P}(-\xi) / \tilde{P}_0(-\xi) \leq c_0 k'(\xi)$ and

$$\begin{aligned}
 |P(-\xi - \zeta)| &= \left| \sum_{\beta} \frac{1}{\beta!} P^{(\beta)}(-\xi) (-\zeta)^\beta \right| \\
 &\leq c_1 (1 + |\zeta|)^m \tilde{P}(-\xi) \leq c_2 (1 + |\zeta|)^m \tilde{P}_0(-\xi),
 \end{aligned}$$

if m is the degree of P . Hence

$$(2.2) \quad \|P(-D)\varphi\|_{q', (k\tilde{P}_0)'}^{\rho, \eta} \leq c' \|\varphi\|_{q', k'}^{\rho, \eta}, \quad \varphi \in \mathcal{D}.$$

Now, if $v \in \mathbf{B}_{q, k\tilde{P}_0}^{-\rho, \eta} \subseteq \mathbf{B}_{q, k\tilde{P}_0}^{\text{loc}}$ it follows from [4, Theorem 10.1.22] that $P(D)v \in \mathbf{B}_{q, k}^{\text{loc}}$. Furthermore, (2.2) implies that

$$\begin{aligned}
 |\langle P(D)v, \varphi \rangle| &= |\langle v, P(-D)\varphi \rangle| \leq \|v\|_{q, k\tilde{P}_0}^{-\rho, \eta} \|P(-D)\varphi\|_{q', (k\tilde{P}_0)'}^{\rho, \eta} \\
 &\leq c' \|v\|_{q, k\tilde{P}_0}^{-\rho, \eta} \|\varphi\|_{q', k'}^{\rho, \eta}
 \end{aligned}$$

for any $\varphi \in \mathcal{D}$. In particular this means that $P(D)v \in \mathbf{B}_{q, k}^{-\rho, \eta}$ and $\|P(D)v\|_{q, k}^{-\rho, \eta} \leq c' \|v\|_{q, k\tilde{P}_0}^{-\rho, \eta}$. $\square$

2.2. Proposition. Let $P, P_0 \in \mathbb{C}[x_1, \dots, x_n] \setminus \{0\}$ with $P \approx P_0$. Let $1 < q \leq \infty$ and $k \in \mathcal{K}$. Assume that $\rho > 0$, $\eta \in \mathbb{R}^n \setminus \{0\}$ are chosen such that with some $c > 0$

$$\begin{aligned}
 \tilde{P}_0(-\xi) &\leq c \cdot |P(-\xi)| \quad \text{if } \xi \in \mathbb{R}^n, |\xi| \geq \rho; \\
 P(-\xi - z\eta) &\neq 0 \quad \text{if } \xi \in \mathbb{R}^n, |\xi| \leq \rho, z \in \mathbb{C}, |z| = 1.
 \end{aligned}$$

Then the operator $P(D): \mathbf{B}_{q, k\tilde{P}_0}^{-\rho, \eta} \rightarrow \mathbf{B}_{q, k}^{-\rho, \eta}$ is surjective.

Proof. Since $\inf\{|P(-\xi - z\eta)| / \tilde{P}_0(-\xi) : |\xi| \leq \rho, |z| = 1\} =: c_1 > 0$ a similar calculation as in (2.1) yields

$$(2.3) \quad \|P(-D)\varphi\|_{q', (k\tilde{P}_0)'}^{\rho, \eta} \geq c_2 \|\varphi\|_{q', k'}^{\rho, \eta}, \quad \varphi \in \mathcal{D},$$

with $c_2 := \min\{c_1, c^{-1}\}$. Now let $w \in \mathbf{B}_{q, k}^{-\rho, \eta}$ be given. Then by (2.3) the mapping $P(-D)\varphi \mapsto \langle w, \varphi \rangle$ is a well-defined continuous linear form on the

subspace $P(-D)\mathcal{D}$ of $(\mathcal{D}, \|\cdot\|_{q', (kP_0)'}^{\rho, \eta})$. By Corollary 1.8 and the Hahn-Banach theorem there exists a distribution $v \in \mathbf{B}_{q, kP_0}^{-\rho, \eta}$ such that

$$\langle P(D)v, \varphi \rangle = \langle v, P(-D)\varphi \rangle = \langle w, \varphi \rangle, \quad \varphi \in \mathcal{D},$$

i.e., $P(D)v = w$. $\square$

We shall see in §4 that the assumptions of Proposition 2.2 can be satisfied if P is hypoelliptic.

3. HYPOELLIPTIC FAMILIES OF DIFFERENTIAL OPERATORS

For any fixed polynomial $P_0 \in \mathbb{C}[x_1, \dots, x_n]$ we consider the sets

$$\mathbb{W}(P_0) := \{P \in \mathbb{C}[x_1, \dots, x_n] : P < P_0\},$$

$$\mathbb{E}(P_0) := \{P \in \mathbb{C}[x_1, \dots, x_n] : P \approx P_0\}.$$

It is clear that $\mathbb{W}(P_0)$ only contains polynomials of degree less than or equal to the degree of P_0 . Hence $\mathbb{W}(P_0)$ is a finite-dimensional complex vector space and $\mathbb{E}(P_0) \subseteq \mathbb{W}(P_0)$. The following result is due to Hörmander.

3.1. Lemma. *The set $\mathbb{E}(P_0)$ is a domain of holomorphy in $\mathbb{W}(P_0)$.*

Proof. According to Hörmander [4, Theorem 10.4.7] there exists a compact subset $\mathcal{L}_0$ of $\mathbb{W}(P_0)' \setminus \{0\}$ such that

$$\mathbb{E}(P_0) = \{P \in \mathbb{W}(P_0) : l(P) \neq 0 \text{ for each } l \in \mathcal{L}_0\}.$$

The compactness of $\mathcal{L}_0$ at once implies that $\mathbb{E}(P_0)$ is open. Of course, $\mathbb{E}(P_0) \neq \emptyset$ since $P_0 \in \mathbb{E}(P_0)$. For any $l \in \mathcal{L}_0$ set

$$\mathbb{E}_l := \{P \in \mathbb{W}(P_0) : l(P) \neq 0\}.$$

This is a domain of holomorphy in $\mathbb{W}(P_0)$ since the function $P \mapsto 1/l(P)$ is analytic in $\mathbb{E}_l$ and singular everywhere on the boundary of $\mathbb{E}_l$. Because of $\mathbb{E}(P_0) = \bigcap_{l \in \mathcal{L}_0} \mathbb{E}_l = \text{interior}(\mathbb{E}(P_0))$ we conclude from [3, 2.5.7] that $\mathbb{E}(P_0)$ is also a domain of holomorphy. $\square$

Remark. In general, the set $\mathbb{E}(P_0)$ is not connected as can be seen from the following example: Consider the polynomials $P_0(x) := x + iy$, $P_1(x) := x - iy$ in two variables. We have $P_0 \approx P_1$. However it is not possible to find a path from P_0 to P_1 in $\mathbb{W}(P_0)$ without leaving the set $\mathbb{E}(P_0)$. Hence P_0 and P_1 belong to different components of $\mathbb{E}(P_0)$.

Now consider a polynomial $P(\lambda, x) = \sum_{|\alpha| \leq m} a_\alpha(\lambda)x^\alpha$ where the coefficients a_α (constant with respect to x) are functions of a parameter $\lambda \in \Lambda$.

3.2. Definition. We say that the family $\{P(\lambda, \cdot) : \lambda \in \Lambda\}$ is an $\mathbb{E}(P_0)$ -family, P_0 a fixed polynomial, if $P(\lambda, \cdot) \approx P_0$ for each $\lambda \in \Lambda$. It is called a *continuous* resp. *analytic* (etc.) $\mathbb{E}(P_0)$ -family if the functions a_α are continuous resp. analytic in Λ . Note that by Lemma 3.1 the family $\{P : P \in \mathbb{E}(P_0)\}$ itself is an analytic $\mathbb{E}(P_0)$ -family with parameter manifold $\Lambda = \mathbb{E}(P_0)$ which is a domain of holomorphy in $\mathbb{W}(P_0)$.

3.3. Remark. Let $\{P(\lambda, \cdot) : \lambda \in \Lambda\}$ be an $\mathbb{E}(P_0)$ -family. With any fixed basis $\{P_0, P_1, \dots, P_{r-1}\}$ of $\mathbb{W}(P_0)$ we can write

$$(3.1) \quad P(\lambda, x) = \sum_{s=0}^{r-1} b_s(\lambda)P_s(x)$$

where the b_s are complex-valued functions of λ . Since the family $\{P_0, P_1, \dots, P_{r-1}\}$ is linearly independent the functions b_s are continuous, analytic (etc.) iff each a_α has this property.

3.4. Definition. A polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ is called *hypoelliptic* if one of the following equivalent conditions [4, Theorem 11.1.3] is satisfied.

- (i) If $d_P(\xi)$ is the distance of $\xi \in \mathbb{R}^n$ to the set $\{\zeta \in \mathbb{C}^n: P(\zeta) = 0\}$ then $d_P(\xi) \rightarrow \infty$ when $\xi \rightarrow \infty$ in $\mathbb{R}^n$.
- (ii) $P^{(\alpha)}(\xi)/P(\xi) \rightarrow 0$ when $\xi \rightarrow \infty$ in $\mathbb{R}^n$ and $\alpha \neq 0$.
- (iii) There exist constants $c, C > 0$ such that

$$|P^{(\alpha)}(\xi)|/|P(\xi)| \leq C|\xi|^{-c|\alpha|}$$

if $\xi \in \mathbb{R}^n$ and $|\xi|$ is sufficiently large.

An $\mathbb{E}(P_0)$ -family will be called hypoelliptic if P_0 is hypoelliptic. In this case each polynomial $P \in \mathbb{E}(P_0)$ is hypoelliptic, too [4, Theorem 11.1.9]. On the other hand, if P_0 and P are both elliptic and have the same degree then $P \in \mathbb{E}(P_0)$ [4, Theorem 10.4.9].

4. THE MAIN RESULTS

For any complex manifold Λ and a locally convex vector space $\mathcal{E}$ we denote by $\mathcal{H}(\Lambda, \mathcal{E})$ the set of all analytic functions on Λ with values in $\mathcal{E}$ (cf. [5, §16.7]). This space will be endowed with the topology of uniform convergence on compact subsets of Λ .

Let $P_0 \neq 0$ be a fixed hypoelliptic polynomial, Λ a complex manifold and $\{P(\lambda, \cdot): \lambda \in \Lambda\}$ an analytic hypoelliptic $\mathbb{E}(P_0)$ -family.

4.1. Theorem. Let $1 < q \leq \infty$ and $k \in \mathcal{H}$. Assume that Λ is a Stein manifold. Then for any $g \in \mathcal{H}(\Lambda, \mathbf{B}_{q,k})$ there exists $f \in \mathcal{H}(\Lambda, \mathbf{B}_{q,k}^{\text{loc}} \widetilde{P_0})$ such that

- (1) $P(\lambda, D)f(\lambda) = g(\lambda)$, $\lambda \in \Lambda$;
- (2) for each $\mu \in \mathbb{N}$ the function $\lambda \mapsto f(\lambda)/\gamma_\mu$ is analytic with values in $\mathbf{B}_{q,k} \widetilde{P_0}$ where γ_μ is given by (1.1).

Condition 4.1(2) means precisely that $f \in \mathcal{H}(\Lambda, \mathbf{B}_{q,k}^{-\infty} \widetilde{P_0})$. If g is a constant then Λ may be an arbitrary complex manifold:

4.2. Corollary. Let $1 < q \leq \infty$ and $k \in \mathcal{H}$. Let Λ be an arbitrary complex manifold. Then for any $g_0 \in \mathbf{B}_{q,k}$ there exists $f \in \mathcal{H}(\Lambda, \mathbf{B}_{q,k}^{\text{loc}} \widetilde{P_0})$ such that $P(\lambda, D)f(\lambda) \equiv g_0$. Additionally, 4.1(2) is satisfied.

Proof. By Lemma 3.1 we may consider the family $\{P: P \in \mathbb{E}(P_0)\}$ as an analytic $\mathbb{E}(P_0)$ -family over itself and $\Lambda = \mathbb{E}(P_0)$ is a domain of holomorphy in $\mathbb{W}(P_0)$, hence Stein. From Theorem 4.1 we obtain a function $\tilde{f} \in \mathcal{H}(\mathbb{E}(P_0), \mathbf{B}_{q,k}^{-\infty} \widetilde{P_0})$ such that $P(D)\tilde{f}(P) = g_0$, $P \in \mathbb{E}(P_0)$. Since the mapping $\lambda \mapsto p(\lambda) := P(\lambda, \cdot)$ is analytic with values in $\mathbb{E}(P_0)$ we have $f := \tilde{f} \circ p \in \mathcal{H}(\Lambda, \mathbf{B}_{q,k}^{-\infty} \widetilde{P_0})$ and $P(\lambda, D)f(\lambda) \equiv g_0$. $\square$

Since the Dirac distribution δ belongs to $\mathbf{B}_{\infty,1}$ (1 denotes the weight function which is identically 1) we immediately obtain a solution of problem (*) posed in the introduction:

4.3. Corollary. *Let $q = \infty$, $k \equiv 1$ and $g(\lambda) \equiv \delta$. Let Λ be an arbitrary complex manifold. Then there exists $f \in \mathcal{H}(\Lambda, \mathbf{B}_{\infty, \tilde{P}_0}^{\text{loc}})$ such that $P(\lambda, D)f(\lambda) \equiv \delta$. Additionally, 4.1(2) is satisfied.*

4.4. Remark. If Λ is an open subset of $\mathbb{R}^d$ then the analogues of Theorem 4.1 and its corollaries hold with “analytic” replaced by “real analytic.”

Proof. Let $\Lambda_1 \subseteq \mathbb{C}^d$ be a complex neighborhood of Λ such that the coefficients b_s in the representation (3.1) of $P(\lambda, x)$ and the function g are analytic on Λ_1 . Since $\mathbb{E}(P_0)$ is an open subset of $\mathbb{W}(P_0)$ we can choose Λ_1 in such a way that $\{P(\lambda, \cdot): \lambda \in \Lambda_1\}$ is still an $\mathbb{E}(P_0)$ -family. By a result of Grauert [1, §3] there exists a complex Stein neighborhood Λ_2 of Λ such that $\Lambda \subseteq \Lambda_2 \subseteq \Lambda_1$. Now it suffices to apply Theorem 4.1 for the parameter manifold Λ_2 and take the restriction of the solution f to Λ . $\square$

If $\mathfrak{F}, \mathfrak{G}$ are Banach spaces we denote by $\mathcal{L}(\mathfrak{F}, \mathfrak{G})$ the space of all bounded linear operators from $\mathfrak{F}$ to $\mathfrak{G}$ equipped with the operator norm topology. The following result is due to J. Leiterer [6, Theorems 2.3(iv) and 5.1, and Corollary 5.4]. It will play an essential role in the proof of Theorem 4.1.

4.5. Theorem. *Let $\mathfrak{F}, \mathfrak{G}$ be Banach spaces and Λ a complex Stein manifold. Let $\mathfrak{T} \in \mathcal{H}(\Lambda, \mathcal{L}(\mathfrak{F}, \mathfrak{G}))$ such that $\mathfrak{T}(\lambda)\mathfrak{F} = \mathfrak{G}$ for each $\lambda \in \Lambda$. Then*

- (a) *There exists for each function $g \in \mathcal{H}(\Lambda, \mathfrak{G})$ a function $f \in \mathcal{H}(\Lambda, \mathfrak{F})$ such that $\mathfrak{T}(\lambda)f(\lambda) = g(\lambda)$, $\lambda \in \Lambda$.*
- (b) *For any open subset Λ' of Λ let $\mathcal{N}(\Lambda') := \{f \in \mathcal{H}(\Lambda', \mathfrak{F}): \mathfrak{T}(\lambda)f(\lambda) \equiv 0\}$. If Λ' is holomorphically convex then the set $\mathcal{N}(\Lambda)|_{\Lambda'}$ of restrictions to Λ' of functions in $\mathcal{N}(\Lambda)$ is dense in $\mathcal{N}(\Lambda')$.*

It is our aim to apply this theorem in the case where the $\mathfrak{T}(\lambda)$ are differential operators generated by the formal expressions $P(\lambda, D)$. The following lemma is quoted from F. Trèves [9, p. 23].

4.6. Lemma. *Let Λ be a compact topological space and $\{Q(\lambda, \cdot): \lambda \in \Lambda\}$ a continuous hypoelliptic $\mathbb{E}(Q_0)$ -family, $Q_0 \neq 0$. Then there exists a constant $c > 0$ such that*

$$(4.1) \quad c^{-1}(1 + |Q_0(\xi)|) \leq 1 + |Q(\lambda, \xi)| \leq c(1 + |Q_0(\xi)|), \quad \lambda \in \Lambda, \quad \xi \in \mathbb{R}^n.$$

4.7. Corollary. *Same assumptions as in Lemma 4.6. Then there exist constants $\rho, c > 0$ such that*

$$(4.2) \quad c^{-1}\tilde{Q}_0(\xi) \leq |Q(\lambda, \xi)| \leq c\tilde{Q}_0(\xi), \quad \lambda \in \Lambda, \quad \xi \in \mathbb{R}^n, \quad |\xi| \geq \rho.$$

Proof. Choose $c_1 > 0$ such that (4.1) holds with $c = c_1$. Since by 3.4(ii) we have $|Q_0(\xi)| \rightarrow \infty$ ($\xi \rightarrow \infty$ in $\mathbb{R}^n$) one can find $\rho > 0$ with the property that $|Q_0(\xi)| \geq 2c_1^{-1}$, $|\xi| \geq \rho$; hence

$$1 + |Q_0(\xi)| - c_1 \geq \frac{1}{2}(1 + |Q_0(\xi)|), \quad |\xi| \geq \rho.$$

Now (4.1) yields

$$|Q(\lambda, \xi)| \geq c_1^{-1}(1 + |Q_0(\xi)|) - 1 \geq (2c_1)^{-1}(1 + |Q_0(\xi)|), \quad \lambda \in \Lambda, \quad |\xi| \geq \rho.$$

By 3.4(iii) we have $\tilde{Q}_0(\xi) \leq c_2(1 + |Q_0(\xi)|)$, $\xi \in \mathbb{R}^n$, with some constant $c_2 > 0$. Hence, $\tilde{Q}_0(\xi) \leq 2c_1c_2|Q(\lambda, \xi)|$, $\lambda \in \Lambda$, $|\xi| \geq \rho$. The second inequality in (4.2) follows from the second inequality in (4.1). $\square$

4.8. **Lemma.** *Same assumptions as in Lemma 4.6. Further, let $\eta \in \mathbb{R}^n \setminus \{0\}$ and $\rho > 0$ be fixed. Then there exists $t > 0$ such that*

$$(4.3) \quad Q(\lambda, \xi + z\eta) \neq 0, \quad \lambda \in \Lambda, \quad \xi \in \mathbb{R}^n, \quad |\xi| \leq \rho, \quad z \in \mathbb{C}, \quad |z| \geq t.$$

Proof. Consider the polynomial $q_{\lambda, \xi}(z) := Q(\lambda, \xi + z\eta)$, $z \in \mathbb{C}$. Since $|q_{\lambda, \xi}(z)| \rightarrow \infty$ ($z \rightarrow \infty$ in $\mathbb{R}$) by 3.4(ii), $q_{\lambda, \xi}$ is nonconstant. (4.1) implies that the degree of $q_{\lambda, \xi}$ is independent of $\lambda \in \Lambda$. Since for fixed $\xi \in \mathbb{R}^n$ the polynomial $Q(\lambda, \xi + \cdot)$ is equally strong as $Q(\lambda, \cdot)$, the same argument shows that the degree of $q_{\lambda, \xi}$ is independent of $\xi \in \mathbb{R}^n$ too. Since $\Lambda \times \{\xi \in \mathbb{R}^n : |\xi| \leq \rho\}$ is compact there exists $t > 0$ such that the zeros z of $q_{\lambda, \xi}$ satisfy $|z| < t$ for $\lambda \in \Lambda$, $|\xi| \leq \rho$. $\square$

Proof of Theorem 4.1. First we choose an exhausting sequence of open submanifolds Λ_ν of Λ such that each Λ_ν is holomorphically convex, $\overline{\Lambda_\nu}$ is compact and $\overline{\Lambda_\nu} \subseteq \Lambda_{\nu+1}$, $\nu \in \mathbb{N}$. Note that $\{Q(\lambda, \cdot) : \lambda \in \Lambda\}$ is an analytic hypoelliptic $\mathbb{E}(Q_0)$ -family where we have set $Q(\lambda, x) := P(\lambda, -x)$, $Q_0(x) := P_0(-x)$. Hence by Corollary 4.7 there exists an increasing sequence $0 < \rho_1 \leq \rho_2 \leq \dots$ such that with some constants $c_\nu > 0$,

$$(4.4) \quad c_\nu^{-1} \tilde{Q}_0(\xi) \leq |Q(\lambda, \xi)| \leq c_\nu \tilde{Q}_0(\xi), \quad \lambda \in \overline{\Lambda_\nu}, \quad \xi \in \mathbb{R}^n, \quad |\xi| \geq \rho_\nu.$$

Fix any vector $\eta \in \mathbb{R}^n$, $|\eta| = 1$. From Lemma 4.8 we get the existence of a sequence $0 < t_1 \leq t_2 \leq \dots$ such that

$$(4.5) \quad Q(\lambda, \xi + z t_\nu \eta) \neq 0, \quad \lambda \in \overline{\Lambda_\nu}, \quad \xi \in \mathbb{R}^n, \quad |\xi| \leq \rho_\nu, \quad z \in \mathbb{C}, \quad |z| = 1.$$

We set $\eta_\nu := t_\nu \eta$ and define the spaces

$$\mathfrak{F}_\nu := \mathbf{B}_{q, k \tilde{P}_0}^{-\rho_\nu, \eta_\nu}, \quad \mathfrak{G}_\nu := \mathbf{B}_{q, k}^{-\rho_\nu, \eta_\nu}, \quad \nu \in \mathbb{N}.$$

By (1.3), (1.6), and (1.11) we have

$$(4.6) \quad \mathfrak{F}_\nu \hookrightarrow \mathfrak{F}_{\nu+1} \hookrightarrow \mathfrak{F} := \mathbf{B}_{q, k \tilde{P}_0}^{-\infty} \hookrightarrow \mathbf{B}_{q, k \tilde{P}_0}^{\text{loc}},$$

$$(4.7) \quad \mathbf{B}_{q, k} \hookrightarrow \mathfrak{G}_\nu \hookrightarrow \mathfrak{G}_{\nu+1} \hookrightarrow \mathbf{B}_{q, k}^{\text{loc}}$$

(recall that $\mathbf{B}_{q, k} = \mathbf{B}_{q, k}^{-0, \eta}$). Now let $\{P_0, P_1, \dots, P_{r-1}\}$ be a basis of $\mathbb{W}(P_0)$. According to Remark 3.3 we can write $P(\lambda, x) = \sum_{s=0}^{r-1} b_s(\lambda) P_s(x)$ with $b_s \in \mathcal{H}(\Lambda, \mathbb{C})$, $s = 0, \dots, r-1$. By Proposition 2.1 each $P_s(D)$ induces an operator $\mathfrak{T}_{s, \nu} \in \mathcal{L}(\mathfrak{F}_\nu, \mathfrak{G}_\nu)$ such that $\mathfrak{T}_{s, \nu} u = P_s(D)u$, $u \in \mathfrak{F}_\nu$, in the distribution sense. Hence the mapping $\lambda \mapsto \mathfrak{T}_\nu(\lambda) := \sum_{s=0}^{r-1} b_s(\lambda) \mathfrak{T}_{s, \nu} = P(\lambda, D)|_{\mathfrak{F}_\nu}$ is analytic with values in $\mathcal{L}(\mathfrak{F}_\nu, \mathfrak{G}_\nu)$. It is clear that

$$(4.8) \quad \mathfrak{T}_{\nu+1}(\lambda)|_{\mathfrak{F}_\nu} = \mathfrak{T}_\nu(\lambda), \quad \lambda \in \Lambda_\nu.$$

From (4.4), (4.5), and Proposition 2.2 we conclude that $\mathfrak{T}_\nu(\lambda)\mathfrak{F}_\nu = \mathfrak{G}_\nu$ for each $\lambda \in \Lambda_\nu$. By (4.7) we have $\mathfrak{g} \in \mathcal{H}(\Lambda, \mathfrak{G}_\nu)$. It follows from part (a) of Theorem 4.5 that there exists for each ν a function $\tilde{f}_\nu \in \mathcal{H}(\Lambda_\nu, \mathfrak{F}_\nu)$ such that

$$\mathfrak{T}_\nu(\lambda)\tilde{f}_\nu(\lambda) = \mathfrak{g}(\lambda), \quad \lambda \in \Lambda_\nu.$$

Now let $\sigma_1 \leq \sigma_2 \leq \dots$ be an increasing fundamental sequence of seminorms of the Fréchet space $\mathfrak{F}$. Set $f_1 := \tilde{f}_1$ and assume that we have already constructed functions $f_\nu \in \mathcal{H}(\Lambda_\nu, \mathfrak{F}_\nu)$, $\nu = 1, \dots, \mu$. Consider then $\mathfrak{d}_{\mu+1}(\lambda) :=$

$\tilde{f}_{\mu+1}(\lambda) - f_{\mu}(\lambda)$, $\lambda \in \Lambda_{\mu}$. Obviously, $\mathfrak{d}_{\mu+1} \in \mathcal{H}(\Lambda_{\mu}, \mathfrak{F}_{\mu+1})$ and we may assume inductively that $\mathfrak{T}_{\mu+1}(\lambda)\mathfrak{d}_{\mu+1}(\lambda) = 0$, $\lambda \in \Lambda_{\mu}$. By part (b) of Theorem 4.5 there exists for arbitrary $\varepsilon_{\mu+1} > 0$ a function $c_{\mu+1} \in \mathcal{H}(\Lambda_{\mu+1}, \mathfrak{F}_{\mu+1})$ with the properties

$$\begin{aligned}\mathfrak{T}_{\mu+1}(\lambda)c_{\mu+1}(\lambda) &= 0, & \lambda \in \Lambda_{\mu+1}, \\ \sup_{\lambda \in \Lambda_{\mu-1}} \|\mathfrak{d}_{\mu+1}(\lambda) - c_{\mu+1}(\lambda)\|_{\mathfrak{F}_{\mu+1}} &\leq \varepsilon_{\mu+1},\end{aligned}$$

where for convenience we put $\Lambda_0 := \emptyset$. Since $\mathfrak{F}_{\mu+1} \hookrightarrow \mathfrak{F}$ we can choose $\varepsilon_{\mu+1}$ so small that $\sup_{\lambda \in \Lambda_{\mu-1}} \sigma_{\mu}(\mathfrak{d}_{\mu+1}(\lambda) - c_{\mu+1}(\lambda)) \leq 2^{-\mu}$. With this choice of $c_{\mu+1}$ we set $f_{\mu+1}(\lambda) := \tilde{f}_{\mu+1}(\lambda) - c_{\mu+1}(\lambda)$, $\lambda \in \Lambda_{\mu+1}$. In this way one obtains a sequence of functions $f_{\nu} \in \mathcal{H}(\Lambda_{\nu}, \mathfrak{F}_{\nu}) \subseteq \mathcal{H}(\Lambda_{\nu}, \mathfrak{F})$ with the properties

$$(4.9) \quad \mathfrak{T}_{\nu}(\lambda)f_{\nu}(\lambda) = P(\lambda, D)f_{\nu}(\lambda) = g(\lambda), \quad \lambda \in \Lambda_{\nu},$$

$$(4.10) \quad \sup_{\lambda \in \Lambda_{\nu-1}} \sigma_{\nu}(f_{\nu+1}(\lambda) - f_{\nu}(\lambda)) \leq 2^{-\nu}.$$

Hence the limit $f(\lambda) := \lim_{\nu \rightarrow \infty} f_{\nu}(\lambda)$ exists in $\mathfrak{F}$ for each $\lambda \in \Lambda$, and $f \in \mathcal{H}(\Lambda, \mathfrak{F}) \subseteq \mathcal{H}(\Lambda, \mathbf{B}_{q, k\tilde{P}_0}^{\text{loc}})$. It follows directly from the definition of $\mathfrak{F}$ that for each $\mu \in \mathbb{N}$ the function $\lambda \mapsto f(\lambda)/\gamma_{\mu}$ is analytic with values in $\mathbf{B}_{q, k\tilde{P}_0}^{\text{loc}}$. Since $P(\lambda, D): \mathbf{B}_{q, k\tilde{P}_0}^{\text{loc}} \rightarrow \mathbf{B}_{q, k}^{\text{loc}}$ is continuous [4, Theorem 10.1.22] we conclude from (4.6), (4.7), (4.9), and (4.10) that $P(\lambda, D)f(\lambda) \equiv g(\lambda)$. The proof is complete. $\square$

Remark. It is known that each differential operator with constant coefficients possesses a tempered fundamental solution [2, 7]. Thus we might have tried to search for a solution $f: \Lambda \rightarrow \mathcal{S}'$ of the equation $P(\lambda, D)f(\lambda) \equiv \delta$. However, let us consider the ordinary differential operator $P(\lambda, D) = \frac{d}{dx} - \lambda$ where $\lambda \in \mathbb{C} = \Lambda$ is a parameter. Denoting by $H(x)$ the Heaviside function, each fundamental solution of $P(\lambda, D)$ can be written in the form $f_{\lambda}(x) = (H(x) + c_{\lambda})\exp(\lambda x)$, c_{λ} a constant. If now it is required that $f_{\lambda} \in \mathcal{S}'$ then we have to take $c_{\lambda} = -1$ for $\text{Re}(\lambda) > 0$, $c_{\lambda} = 0$ for $\text{Re}(\lambda) < 0$. This example shows that in general it is not even possible to find a continuous function $f: \Lambda \rightarrow \mathcal{S}'$ satisfying $P(\lambda, D)f(\lambda) \equiv \delta$. At the same time we see that the solution space $\mathfrak{F} = \mathbf{B}_{q, k\tilde{P}_0}^{-\infty}$ which occurs in the proof of Theorem 4.1 cannot be chosen much smaller, for the space $\mathfrak{F}$ has to contain, for any $a > 0$, distributions in $\mathbf{B}_{q, k\tilde{P}_0}^{\text{loc}}$ which “grow” at infinity like $\exp(a\sqrt{1 + [x, x]})$. However, F. Trèves [11] has shown that in the situation of Corollary 4.3 it is possible locally (in Λ) to choose $f(\lambda)$ with arbitrarily small exponential growth.

While this paper was in print the author succeeded in eliminating the assumption of hypoellipticity from Theorem 4.1 and its corollaries. A more general result will appear in Ann. Inst. Fourier (Grenoble).

REFERENCES

1. H. Grauert, *On Levi's problem and the embedding of real-analytic manifolds*, Ann. of Math. **68** (1958), 460–472.
2. L. Hörmander, *On the division of distributions by polynomials*, Ark. Mat. **3** (1958), 555–568.

3. —, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N.J., 1966.
4. —, *The analysis of linear partial differential operators*. I, II, Grundlehren Math. Wiss., Vols. 256, 257, Springer, 1983.
5. H. Jarchow, *Locally convex spaces*, Math. Leitfäden, Teubner, Stuttgart, 1981.
6. J. Leiterer, *Banach coherent analytic Fréchet sheaves*, Math. Nachr. **85** (1978), 91–109.
7. S. Lojasiewicz, *Sur le problème de division*, Studia Math. **18** (1959), 87–136.
8. I. I. Priwalow, *Randeigenschaften analytischer Funktionen*; Hochschulbücher für Math. **25** (1956).
9. F. Trèves, *Opérateurs différentiels hypoelliptiques*, Ann. Inst. Fourier (Grenoble) **9** (1959), 1–73.
10. —, *Un théorème sur les équations aux dérivées partielles à coefficients constants dépendant de paramètres*, Bull. Soc. Math. France **90** (1962), 473–486.
11. —, *Fundamental solutions of linear partial differential equations with constant coefficients depending on parameters*, Amer. J. Math. **84** (1962), 561–577.

FACHBEREICH MATHEMATIK, UNIVERSITÄT DORTMUND, POSTFACH 50 05 00, D-4600 DORTMUND 50, GERMANY