ON THE BRAUER GROUP OF TORIC VARIETIES

FRANK R. DEMEYER AND TIMOTHY J. FORD

ABSTRACT. We compute the cohomological Brauer group of a normal toric variety whose singular locus has codimension less than or equal to 2 everywhere.

Associated to each algebraic variety X is the cohomological Brauer group $B'(X) = tors(H^2(X, G_m))$ which is the torsion subgroup of the second étale cohomology group of X with coefficients in the sheaf of units. Except in the easiest cases, calculations of this group are scarce. Toric varieties over an algebraically closed field of characteristic 0 provide a nontrivial class of higher dimensional varieties for which calculations of B'(X) can sometimes be made. These calculations are the purpose of this article.

Each toric variety X is determined by a combinatorical object Δ in real affine space called a fan. Tied into the structure of the fan are arithmetic properties of sublattices of free Z-lattices. Our arguments therefore ultimately reduce questions about B'(X) to calculations with integer matrices.

In §1 we determine the Brauer group B(X) = B'(X) of any nonsingular toric variety X (Theorem 1.1). This group is a direct sum of finitely many copies of finite cyclic groups and copies of \mathbb{Q}/\mathbb{Z} . The algebras generating this group are given explicitly as smash products of cyclic Galois extensions of X. In §2 we consider toric varieties whose singular locus has codimension at most 2 everywhere in X. Let T_N denote the torus identified with an open subset of X, $B'(T_N/X)$ the elements in B'(X) split by T_N , and \tilde{X} a T_N -invariant desingularization of X. In Theorem 2.2 we construct an exact sequence $0 \rightarrow$ $B'(T_N/X) \rightarrow B'(X) \rightarrow B'(\tilde{X}) \rightarrow 0$ which reduces the calculation of B'(X) to the calculation of $B'(T_N/X)$. The hypotheses on X in §2 imply we can assume the associated fan Δ contains cones of dimension at most 2. Corresponding to each cone τ_i of dimension 2 is an irreducible closed subvariety $V_i = \operatorname{orb} \tau_i$ and an affine neighborhood U_{τ_i} of V_i which has a finite cyclic divisor class group $\operatorname{Cl}(U_{\tau_i})$. If Δ has 2-dimensional cones τ_1, \ldots, τ_m , we construct an exact sequence (Theorem 2.3)

$$0 \to \operatorname{Pic}(X) \to \operatorname{Cl}(X) \to \bigoplus_{i=1}^{m} \operatorname{Cl}(U_{\tau_i}) \to \operatorname{B}'(T_N/X) \to 0.$$

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For each prime number p we find a subset τ_1, \ldots, τ_e (after a suitable relabelling) of $\{\tau_1, \ldots, \tau_m\}$ such that $[\bigoplus_{i=1}^e \operatorname{Cl}(U_{\tau_i})]_p \cong [B'(T_N/X)]_p$. We calculate the Brauer group of any toric surface (Corollary 2.9). In this case $B'(T_N/X)$ is nontrivial when there is a cycle of divisors on X whose pairwise consecutive intersections are singular points on X whose local rings all have divisor class groups of order divisible by a common prime p. An analogous statement holds for X of higher dimension. We employ terminology and notation of [12] for toric varieties and [11] for étale cohomology.

1

Following the notation terminology of [12] let r > 0 be an integer and $N = \mathbb{Z}^r$ a free abelian group of rank = r. Let Δ be a finite fan on $N_{\mathbb{R}}$ and $X = T_N \operatorname{emb}(\Delta)$ the associated toric variety containing the r-dimensional torus T_N as an open subset defined over the algebraically closed field k of characteristic 0. Let N' be the subgroup of N generated by $\bigcup_{\sigma \in \Delta} \sigma \cap N$. The basis theorem for finitely generated abelian groups gives a basis n_1, \ldots, n_r of N such that $N' = \mathbb{Z}a_1n_1 \oplus \mathbb{Z}a_2n_2 \oplus \cdots \oplus \mathbb{Z}a_rn_r$ where the a_i are nonnegative integers and $a_i|a_{i+1}$ for $1 \le i \le r-1$. Call $\{a_1, \ldots, a_r\}$ the set of invariant factors of Δ (or $X = T_N \operatorname{emb}(\Delta)$). Let B(X) denote the Brauer group of Azumaya algebras on X and B'(X) the torsion subgroup of $H^2(X, G_m)$ the cohomological Brauer group of X. Our principal result of $\S1$ is

Theorem 1.1. If $X = T_N \operatorname{emb}(\Delta)$ is nonsingular and a_1, \ldots, a_r is the set of invariant factors of X, then

$$\mathbf{B}(X) = \mathbf{B}'(X) \cong \bigoplus_{i=1}^{r-1} \operatorname{Hom}(\mathbb{Z}/a_i, \mathbb{Q}/\mathbb{Z})^{r-i}.$$

We list two special cases of Theorem 1.1.

Corollary 1.2. If Δ contains a cone σ such that dim $\sigma \geq r-1$, then B(X) = (0). *Proof.* Since X is nonsingular, [12, Theorem 1.10] implies there is a basis n_1, \ldots, n_r of N such that $\mathbb{R}_{\geq 0}n_1 + \cdots + \mathbb{R}_{\geq 0}n_{r-1} \subseteq \sigma$. Since each $n_i \in \sigma \cap N$, all of the invariant factors $a_i = 1$ for $1 \leq i \leq r-1$. So B(X) = (0) by Theorem 1.1. \Box

Corollary 1.3. B(X) is finite if and only if $\operatorname{Rank}_{\mathbb{Z}}(N') \ge r - 1$. In this case

$$\mathbf{B}(X) \cong \bigoplus_{i=1}^{r-1} (\mathbb{Z}/a_i)^{r-i} \, .$$

The rest of this section is devoted to a proof of Theorem 1.1. From now on we assume $X = T_N \operatorname{emb}(\Delta)$ is nonsingular. Along the way we will obtain explicit information about the Azumaya algebras on X and show the Brauer group B(X) of Azumaya algebras on X is equal to the cohomological Brauer group $B'(X) = H^2(X, G_m)$.

Let $\Gamma = \{0, \rho_1, \dots, \rho_n\}$ be the fan on $N_{\mathbb{R}}$ consisting of all cones in Δ of dimension ≤ 1 and let $U = T_N \operatorname{emb}(\Gamma)$. The open immersion $U \hookrightarrow X$ induces the isomorphisms of the next lemma.

Lemma 1.4. For each positive integer ν , (a) $H^1(X, \mathbb{Z}/\nu) \cong H^1(U, \mathbb{Z}/\nu)$,

560

- (b) $\mathrm{H}^2(X, \mathbb{Z}/\nu) \cong \mathrm{H}^2(U, \mathbb{Z}/\nu)$,
- (c) $\mathbf{B}'(X) \cong \mathbf{B}'(U)$.

Proof. Let X, U be as above and let Z = X - U. Then part of the long exact sequence of cohomology with supports [11, Proposition 1.25] is

$$\begin{aligned} \mathrm{H}^{1}_{Z}(X,\,\mathbb{Z}/\nu) &\to \mathrm{H}^{1}(X,\,\mathbb{Z}/\nu) \to \mathrm{H}^{1}(U,\,\mathbb{Z}/\nu) \to \mathrm{H}^{2}_{Z}(X,\,\mathbb{Z}/\nu) \\ &\to \mathrm{H}^{2}(X,\,\mathbb{Z}/\nu) \to \mathrm{H}^{2}(U,\,\mathbb{Z}/\nu) \to \mathrm{H}^{2}_{Z}(X,\,\mathbb{Z}/\nu) \end{aligned}$$

since the codimension of Z in X is ≥ 2 , [11, Lemma 9.1, p. 268] implies $H_Z^s(X, \mathbb{Z}/\nu) = (0)$ for s < 4, which proves (a) and (b) in our context. There is an exact sequence [3, Theorem 1.c]

$$0 \rightarrow \mathrm{H}^{2}(X, \mathbf{G}_{m}) \rightarrow \mathrm{H}^{2}(U, \mathbf{G}_{m}) \rightarrow \mathrm{H}^{3}_{Z}(X, \mu)$$

and $H^3_Z(X, \mu) \cong \lim_{\longrightarrow} H^3_Z(X, \mathbb{Z}/\nu) = (0)$, which proves (c). \Box

Notice that $N' = \langle \bigcup_{\sigma \in \Delta} \sigma \cap N \rangle = \langle \bigcup_{i=1}^{n} \rho_i \cap N \rangle$ so as a consequence of Lemma 1.4(c) we can assume that $\Delta = \{0, \rho_1, \dots, \rho_n\}$ and X = U. We write $\rho_k = \mathbb{R}_{\geq 0}\eta_k$ where $\eta_k \in N$ and η_k is primitive (the GDC of the coordinates of η_k is 1). Let n_1, \dots, n_r be a basis for N with $N' = \mathbb{Z}a_1n_1 \oplus \dots \oplus \mathbb{Z}a_rn_r$ and $a_i|a_{i+1}$ for $1 \leq i \leq r$. $(\{a_1, \dots, a_r\}\}$ is the set of invariant factors of Δ .) Let m_1, \dots, m_r be a dual basis for $M = \text{Hom}(N, \mathbb{Z})$. Then $T_N = \text{Spec } k[M]$. An element $\sum a_im_i$ in M is usually identified with the Laurent monomial $x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r}$ and k[M] with $k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$. Let ν be a positive integer and fix a primitive ν th root of unity ζ . Given units α, β in $k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$, the symbol algebra $(\alpha, \beta)_{\nu}$ is the associative k-algebra generated by elements u, v subject to the relations $u^{\nu} = \alpha, v^{\nu} = \beta$; and $uv = \zeta vu$. In what follows, we choose to identify $(x_i, x_j)_{\nu}$ as $(m_i, m_j)_{\nu}$ and work in k(M). By [10, Theorem 6], ${}_{\nu}B(T_N)$ is a free \mathbb{Z}/ν -module with basis given by the set of symbol algebras $\{(m_i, m_j)_{\nu}\}_{1 \leq i < j \leq r}$ for each $\nu \geq 2$. Since T_N is an open subset of X and X is nonsingular, B(X) is a subgroup of $B(T_N)$ by restriction and our object is to identify this subgroup explicitly.

From [4, Corollary 1.4] there is an exact sequence

(1)
$$0 \to \mathbf{B}'(X) \to \mathbf{B}(T_N) \xrightarrow{a} \bigoplus_{i=1}^n \mathbf{H}^1(\operatorname{orb} \rho_i, \mathbb{Q}/\mathbb{Z})$$

where orb ρ_i is the T_N -invariant divisor on $X = T_N \operatorname{emb}(\Delta)$ corresponding to the face ρ_i of Δ . Given a symbol algebra $(\alpha, \beta)_{\nu}$ representing a class in $B(T_N)$, the ramification map *a* agrees with the tame symbol (see the discussion following [4, Remark 1.7] and [14, Theorem 8, p. 155]). This means the *k*th coordinate of $a((\alpha, \beta)_{\nu})$, the ramification of $(\alpha, \beta)_{\nu}$ along orb ρ_k , is identified with a cyclic Galois extension of orb ρ_k of degree ν . Over the function field $K(\operatorname{orb} \rho_k)$ this extension is given by adjoining the ν th root of $\alpha^{v_k(\beta)}/\beta^{v_k(\alpha)}$ where v_k is the valuation on K(X) determined by the prime divisor orb ρ_k .

From the remarks above, to determine the ramification of an arbitrary algebra Λ representing an element in $B(T_N)$ along orb ρ_k it suffices to determine $K(\operatorname{orb} \rho_k)$ and $v_k(m_j)$ for each k, j. The following lemma is well known. We include its short proof for completeness and to fix notation.

Lemma 1.5. Let η_k be the primitive vector in $N \cap \rho_k$ and \langle , \rangle the natural inner product from $M \times N \to \mathbb{Z}$.

- (a) $K(\text{orb }\rho_k)$ is the quotient field of $k[\eta_k^{\perp}]$.
- (b) $v_k(m) = \langle m, \eta_k \rangle$.

Proof. Since η_k is primitive there is a primitive $\mu_k \in M$ with $\langle \mu_k, \eta_k \rangle = 1$. Let $\eta_k^{\perp} = \{m \in M | \langle m, \eta_k \rangle = 0\}$. Then $M = \eta_k^{\perp} \oplus \mathbb{Z}\mu_k$ since

$$0 \to \eta_k^\perp \to M \xrightarrow{\varphi} \mathbb{Z} \to 0$$

splits, where $\varphi(m) = \langle m, \eta_k \rangle$. The affine coordinate ring of

$$U_{\rho_k} = T_N \operatorname{emb}\{0, \rho_k\}$$

is $k[\eta_k^{\perp}, \mu_k]$. Localizing X along orb ρ_k is equivalent to localizing U_{ρ_k} along orb ρ_k . The prime ideal corresponding to orb ρ_k is the principal ideal in $k[\eta_k^{\perp}, \mu_k]$ generated by μ_k . Hence μ_k is a local parameter along orb ρ_k . $K(\text{orb } \rho_k)$ is the quotient field of $k[\eta_k^{\perp}, \mu_k]/(\mu_k)$ giving (a). The valuation v_k of any $m \in M$ is the μ_k -coordinate when m is written in terms of the decomposition $M = \eta_k^{\perp} \oplus \mathbb{Z}\mu_k$. Thus $v_k(m) = \langle m, \eta_k \rangle$. \Box

Keeping the notation above, define a homomorphism $\operatorname{ram}_{\operatorname{orb} \rho_k}: {}_{\nu}B(T_N) \to M/\nu M$ by letting $\operatorname{ram}_{\operatorname{orb} \rho_k}(m_i, m_j)_{\nu} = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j + \nu M$ be the assignment on the basis for ${}_{\nu}B(T_N)$, and extending by \mathbb{Z}/ν -linearity.

Lemma 1.6. $(m_i, m_j)_{\nu}$ is unramified along orb ρ_k if and only if

$$\operatorname{ram}_{\operatorname{orb}\rho_{\mu}}(m_i, m_j)_{\nu} = 0.$$

Proof. The ramification of $(m_i, m_j)_{\nu}$ along orb ρ_k corresponds to the cyclic extension of the affine coordinate ring $k[\eta_k^{\perp}]$ of orb ρ_k obtained by adjoining the ν th root of $v_k(m_j)m_i - v_k(m_i)m_j = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j$. (Note $\langle \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j, \eta_k \rangle = 0$ so $\langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j \in \eta_k^{\perp}$.) Thus, $(m_i, m_j)_{\nu}$ is unramified along orb ρ_k if and only if $\langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j \in \nu M$ if and only if ram_{orb $\rho_k}(m_i, m_j)_{\nu} = 0$. \Box}

Let Λ be any Azumaya algebra representing a class in $_{\nu}B(T_N)$. We have seen Λ is equivalent to $\prod_{i < j} (m_i, m_j)_{\nu}^{e_{ij}}$ where $0 \le e_{ij} < \nu$. Moreover the class of Λ determines and is determined by the integers e_{ij} . Associate to the class represented by Λ in $_{\nu}B(T_N)$ the matrix

$$M_{\Lambda} = \begin{bmatrix} 0 & e_{12} & e_{13} & \cdots & e_{1r} \\ -e_{12} & 0 & e_{23} & \cdots & e_{2r} \\ \vdots & & & \vdots \\ \vdots & & & & \vdots \\ -e_{1r} & -e_{2r} & & \cdots & -e_{r-1,r} & 0 \end{bmatrix}$$

Lemma 1.7. (a) The assignment $\Lambda \to M_{\Lambda}$ induces a monomorphism

 $\phi: {}_{\nu}\mathbf{B}(T_N) \to \operatorname{Hom}_{\mathbb{Z}}(N, M/\nu M).$

(b) Λ is unramified along orb ρ_k if and only if $M_{\Lambda} \cdot \eta_k = 0$.

Proof. (a) The matrix M_{Λ} defines the indicated homomorphism $\phi(\Lambda)$ by representing elements in N as column vectors with respect to the basis n_1, \ldots, n_r ; the elements in M as column vectors with respect to the dual basis m_1, \ldots, m_r and following left multiplication by M_{Λ} by reduction modulo νM . Since multiplication of symbols corresponds to addition of exponents modulo ν , it is clear that ϕ is a homomorphism. If $\phi(\Lambda) = M_{\Lambda} = 0$, then each $e_{ij} = 0$, so $\Lambda = 0$ in $B(T_N)$. Thus ϕ is a monomorphism.

(b) Write $\eta_k = \sum_{i=1}^r \eta_{ki} n_i$ and let $\Lambda = \prod_{i < j} (m_i, m_j)_{\nu}^{e_{ij}}$. Then

$$\operatorname{ram}_{\operatorname{orb}\rho_k}(m_i, m_j)_{\nu} = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j = \eta_{kj} m_i - \eta_{ki} m_j.$$

Hence

$$\operatorname{ram}_{\operatorname{orb}\rho_{k}}(\Lambda) = \operatorname{ram}_{\operatorname{orb}\rho_{k}}\left(\prod_{i < j}(m_{i}, m_{j})_{\nu}^{e_{ij}}\right)$$
$$= \sum_{i < j}e_{ij}\eta_{kj}m_{i} - e_{ij}m_{j} + \nu M$$
$$= \sum_{i=1}^{r}\sum_{j=i+1}^{r}e_{ij}\eta_{kj}m_{i} - \sum_{i=1}^{r}\sum_{j=i+1}^{r}e_{ij}\eta_{ki}m_{j} + \nu M$$
$$= \sum_{i=1}^{r}\sum_{j=i+1}^{r}e_{ij}\eta_{kj}m_{i} - \sum_{j=1}^{r}\sum_{i=j+1}^{r}e_{ji}\eta_{kj}m_{i} + \nu M$$
$$= M_{\Lambda} \cdot \begin{bmatrix}\eta_{k1}\\\vdots\\\eta_{kr}\end{bmatrix}.$$

As we observed in the proof of Lemma 1.6, $\operatorname{ram}_{\operatorname{orb} \rho_k}(\Lambda) = m + \nu M$ for some $m \in \eta_k^{\perp}$. The ramification of Λ along orb ρ_k is the cyclic extension of $k[\eta_k^{\perp}]$ obtained by adjoining the ν th root of m and this extension is split (Λ is unramified along orb ρ_k) if and only if $m \in \nu \eta_k^{\perp}$. Since η_k^{\perp} is a direct summand of M, Λ is unramified along orb ρ_k if and only if $m \in \nu M$ if and only if

$$M_{\Lambda} \cdot \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kr} \end{bmatrix} = 0. \quad \Box$$

Theorem 1.8. Let $X = T_N \operatorname{emb}(\Delta)$ be a nonsingular toric variety and a_1, \ldots, a_r the set of invariant factors of X. Then B'(X) is the subgroup of $B(T_N)$ represented by algebra classes $\prod_{i < j} (m_i, m_j)_{\nu_i}^{e_{ij}}$ where $\nu_i | a_i, 1 \le i \le r$.

Proof. The exact sequence (1) and Lemma 1.7 imply $_{\nu}B'(X)$ consists of those algebra classes Λ in $_{\nu}B(T_N)$ such that $M_{\Lambda} \cdot \eta_k = 0$ for the primitive vector η_k on each 1-dimensional cone ρ_k in Δ ($1 \le k \le n$). If N' is the subgroup of N generated by $\bigcup_{\sigma \in \Delta} \sigma \cap N$ then N' is generated by $\{\eta_k\}_{k=1}^n$ so Λ represents a class in $_{\nu}B'(X)$ if and only if M_{Λ} vanishes on N'. For each $\nu > 0$ we have

the commutative diagram with exact rows and columns



Taking the direct limit over all $\nu \ge 2$ gives a monomorphism

 $\mathbf{B}'(X) \to \operatorname{Hom}_{\mathbb{Z}}(N/N', M \otimes \mathbb{Q}/\mathbb{Z}).$

Let n_1, \ldots, n_r be a basis for N such that $N' = \mathbb{Z}a_1n_1 \oplus \cdots \oplus \mathbb{Z}a_rn_r$ and $a_i|a_{i+1}$ for $1 \le i \le r-1$. That is a_1, \ldots, a_r is the set of invariant factors for X. Then $\operatorname{Hom}_{\mathbb{Z}}(N/N', M \otimes \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(\bigoplus(\mathbb{Z}n_i/\mathbb{Z}a_in_i), M \otimes \mathbb{Q}/\mathbb{Z})$. This means B'(X) is contained in the subgroup of $B(T_N)$ of algebra classes $\prod_{i< j} (m_i, m_j)_{\nu_i}^{e_{ij}}$ where $0 < \nu_i$ and $\nu_i|a_i, 1 \le i \le r$. Conversely, if $\nu_i|a_i$ and $\nu_i \ge 1$ then the matrix M_{Λ} for $(m_i, m_j)_{\nu_i}$ has a +1 in the *ij*th entry and a -1 in the *ji*th entry. A typical element in N' is $x = \lambda_1 a_1 n_1 + \cdots + \lambda_r a_r n_r$ and $M_{\Lambda} \cdot x = \lambda_j a_j m_i - \lambda_i a_i m_j \in a_i M$. Thus $(m_i, m_j)_{\nu_i}$ represents an element in B'(X). So $B'(X) = \{\prod_{i < j} (m_i, m_j)_{\nu_i}^{e_{ij}} | 0 < \nu_i$ and $\nu_i | a_i \}$. \Box

Now it follows that $B'(X) \cong \bigoplus_{i=1}^{r-1} \operatorname{Hom}(\mathbb{Z}/a_i, \mathbb{Q}/\mathbb{Z})^{r-i}$. To complete the proof of Theorem 1.1 it suffices to show B(X) = B'(X). It suffices to find an Azumaya algebra Λ on X such that $K(X) \otimes \Lambda$ is equivalent to $(m_i, m_j)_{\nu_i}$ for each $\nu_i | a_i$.

Lemma 1.9. Let X be as in Theorem 1.8 and let $N' = \langle \bigcup_{\sigma \in \Delta} \sigma \cap N \rangle$. Let $\nu \ge 2$ and let $M_{\nu} = \{m \in M \mid \langle m, n' \rangle \equiv 0 \pmod{\nu} \text{ for all } n' \in N' \}$. If (ν, a_i) is the greatest common divisor of ν and a_i , then

$$\mathrm{H}^{1}(X, \mathbb{Z}/\nu) \cong M_{\nu}/\nu M \cong \bigoplus_{i=1}^{\prime} \mathbb{Z}/(\nu, a_{i}).$$

Proof. Restriction induces an embedding $H^1(X, \mathbb{Z}/\nu) \to H^1(T_N, \mathbb{Z}/\nu)$. The correspondence which assigns to each element $m \in M$ the cyclic extension of T_N obtained by adjoining the ν th root of m induces an isomorphism $M/\nu M \cong H^1(T_N, \mathbb{Z}/\nu)$. An element $m + \nu M$ corresponds to an element of $H^1(X, \mathbb{Z}/\nu)$ if and only if $K(X)(m^{1/\nu})$ is unramified along orb ρ_k for $1 \le k \le n$ if and only if the restriction of m to orb ρ_k is a unit in the coordinate ring $k[\eta_k^{\perp}, \mu_k]$ of orb ρ_k if and only if $v_k(m) \equiv 0 \pmod{\nu}$ if and only if $\langle m, \eta_k \rangle \equiv 0 \pmod{\nu}$ where unexplained notation is as in Lemma 1.5. Thus

$$H^{1}(X, \mathbb{Z}/\nu) = \{m \in M \mid \langle m, \eta_{k} \rangle \equiv 0 \pmod{\nu} \ (1 \le k \le n)\} + \nu M$$
$$= \{m \in M \mid \langle m, n' \rangle \equiv 0 \pmod{\nu} \text{ for all } n' \in N'\} + \nu M$$
$$= M_{\nu}/\nu M.$$

But $N' = \mathbb{Z}a_1n_1 \oplus \cdots \oplus \mathbb{Z}a_rn_r$ so it is easy to check that

$$M_{\nu}/\nu M \cong \bigoplus_{i=1}^{\prime} \mathbb{Z}\left(\frac{\nu}{(a_i,\nu)}m_i\right)/\mathbb{Z}(\nu m_i) \cong \bigoplus_{i=1}^{\prime} \mathbb{Z}/(a_i,\nu). \quad \Box$$

Lemma 1.10. If X is as in Theorem 1.8, then B(X) = B'(X).

Proof. It suffices to show each $(m_i, m_j)_{\nu_i}$ is in the image of the cup product map $H^1(X, \mathbb{Z}/\nu_i) \times H^1(X, \mu_{\nu_i}) \to B(X)$ when $\nu_i | a_i$, since cup products correspond to taking smash products of cyclic Galois extensions and thus are Azumaya algebras (e.g. [6]).

If $a_j = 0$, both $m_i, m_j \in M_{\nu_i}$ since $\nu_i/(a_i, \nu_i) = \nu_i/\nu_i = 1 = \nu_i/(0, \nu_i)$. If $a_j \neq 0$, then $(m_i, m_j)_{\nu_i} \sim (m_i, m_j)_{a_j}^{a_j/\nu_i} \sim ((a_j/\nu_i)m_i, m_j)_{a_j}$. But $a_j/(a_i, a_j) = a_j/a_i$ which divides a_j/ν_i since $\nu_i|a_i$. Thus $(a_j/\nu_i)m_i$ and m_j are both in M_{a_j} and $(m_i, m_j)_{\nu_i}$ is equivalent to an algebra in the image of the cup product map $H^1(X, \mathbb{Z}/a_j) \times H^1(X, \mu_{a_j}) \rightarrow a_j B(X)$. So B(X) = B'(X). \Box

As a result of observations made so far, we can show the following proposition.

Proposition 1.11. Let $\prod_{i < j} (m_i, m_j)_{\nu}^{e_{ij}}$ represent a class in $B(T_N)$ of order ν . Let M_{Λ} be the matrix transformation in $Hom(N, M/\nu M)$ defined in Lemma 1.7 and let t be the rank of kernel (M_{Λ}) . Then there exists a direct summand P of M with rank(P) = r - t and an Azumaya algebra L over k[P] with $\Lambda \cong k[M] \otimes_{k[P]} L$. No direct summand of M of smaller rank has this property. Proof. Find a basis n_1, \ldots, n_r of N such that $ker(M_{\Lambda}) = \mathbb{Z}b_1n_1 \oplus \mathbb{Z}b_2n_2 \oplus \cdots \oplus \mathbb{Z}b_rn_r$ and $b_i|b_{i+1}$ for $1 \le i \le r-1$. Since $ker(M_{\Lambda})$ has rank t, $b_i \ne 0$ and $b_{t+i} = 0$ for $i \ge 1$.

Let P be a direct summand of M and assume Λ is obtained by extending an algebra over k[P]. Let m'_1, \ldots, m'_s be a basis for P and extend this basis to a basis for M. We can assume $\Lambda = \prod_{i < j \leq s} (m'_i, m'_j)_{\nu}^{e'_{ij}}$. If n'_1, \ldots, n'_r is the dual basis to m'_1, \ldots, m'_r , then the matrix of the transformation M_{Λ} with respect to this new basis pair has a kernel which contains a direct summand of N of rank r-s. Therefore $t \geq r-s$ so $s \geq r-t$. Now let m_1, \ldots, m_r be a dual basis for M with respect to n_1, \ldots, n_r . The matrix M_{Λ} with respect to this new basis is

$$M_{\Lambda} = \begin{bmatrix} 0 & 0 & \\ 0 & e_{t+1,t} & \cdots & e_{t+1,r} \\ 0 & -e_{t+1,t} & 0 & \cdots & \\ \vdots & & \vdots & \\ & -e_{t+1,r} & & \cdots & 0 \end{bmatrix}$$

So Λ is defined on the torus $k[m_{t+1}, \ldots, m_r, -m_{t+1}, \ldots, -m_r]$ and we can take $P = \langle m_{t+1}, \ldots, m_r \rangle$. The rank of P is r - t. \Box

2

In this section we continue to let Δ be a finite fan on $N_{\mathbb{R}}$ and $X = T_N \operatorname{emb}(\Delta)$ the associated toric variety containing the *r*-dimensional torus T_N as an open subset. Assume Δ consists of cones of dimension ≤ 2 and let $\Delta(2) = \{\tau_1, \ldots, \tau_m\}$. Let $U_i = U_{\tau_i}, V_i = V(\tau_i) = \operatorname{orb}(\tau_i)$ and let $V = V_1 \cup \cdots \cup V_m$. Then $X - V = T_N \operatorname{emb}(\Delta - \Delta(2))$ is nonsingular. In this situation our first lemma gives information about the étale cohomology groups of the affine open subsets U_i of X. **Lemma 2.1.** (a) For each i and each $p \ge 0$, we have a short exact sequence

$$0 \to \mathrm{H}^p(U_i, \, \mathbf{G}_m) \to \mathrm{H}^p(U_i - V_i, \, \mathbf{G}_m) \to \mathrm{H}^{p+1}_{V_i}(U_i, \, \mathbf{G}_m) \to 0.$$

(b)
$$H^p(U_i, G_m) \cong H^p(T_{r-2}, G_m)$$
 where T_{r-2} is a torus of dimension $r-2$.

Proof. First we check that $H^p(U_i, G_m)$ is torsion for $p \ge 2$. For notational simplicity we suppress the subscript *i* from τ_i , U_i , and V_i . Now τ is a two-dimensional cone in $N_{\mathbb{R}}$. Let $\overline{\tau}$ be τ viewed as a two-dimensional cone in $\mathbb{R}\tau$. Then $U = U_{\overline{\tau}} \times T_{r-2}$ where T_{r-2} is an (r-2)-dimensional torus. Let R be the affine coordinate ring of $U_{\overline{\tau}}$, and R^h the henselization of R at the maximal ideal m corresponding to the closed point orb $\overline{\tau}$. Let $R[X, X^{-1}]$ denote the affine coordinate ring of U and let $U^h = \operatorname{Spec} R^h[X, X^{-1}]$. Let $V^h = V \times U^h$. Then V^h is the closed set corresponding to $I = mR^h[X, X^{-1}]$. Let $V^h = V \times U^h$. Then V^h is the closed set corresponding to $I = mR^h[X, X^{-1}]$. It is the *m*-adic completion of R. By [13, p. 127], we see that $(R^h[X, X^{-1}], I)$ is a Hensel pair. By [5, p. 35] $\operatorname{Cl}(R^h[X, X^{-1}])$ embeds into $\operatorname{Cl}(\widehat{R}[X, X^{-1}])$. Since the singularity of U is given by a finite cyclic group action [12, p. 30], it is well known that $\operatorname{Cl}(\widehat{R}[X, X^{-1}]) = \operatorname{Cl}(\widehat{R})$ is also finite cyclic [2, Satz 2.11]. Thus $\operatorname{Cl}(U^h)$ is finite. The long exact sequences for the pairs $V \subseteq U$ and $V^h \subseteq U^h$ give the commutative diagram

with exact rows. By excision $H_V^p(U, G_m) \cong H_{V^h}^p(U^h, G_m)$ [11, p. 92]. By [15] $H^p(U^h, G_m) \cong H^p(V^h, G_m) = H^p((\text{orb}\,\overline{\tau}) \times T_{r-2}, G_m)$ which is torsion for $p \ge 2$ since T_{r-2} is smooth [7, p. 71]. Again by [7, p. 71] $H^p(U^h - V^h, G_m)$ and $H^p(U - V, G_m)$ are torsion for $p \ge 2$. But $H^1(U^h - V^h, G_m) = \text{Pic}(U^h - V^h) = Cl(U^h - V^h) = Cl(U^h)$ is torsion. It now follows that $H^p(U, G_m)$ is torsion for $p \ge 2$.

The natural map $U \times A^1 \to U$ and Kummer theory induce the commutative diagram

for all $p \ge 2$ and $n \ge 2$. By [11, p. 240] β is an isomorphism for $p \ge 2$. Since Pic $U = 0 = \text{Pic}(U \times A^1)$, α is an isomorphism for p = 2. Therefore, γ is an isomorphism for p = 2 and all $n \ge 2$. Taking the inductive limit over all n, we have $H^2(U \times A^1, G_m) \cong H^2(U, G_m)$. By induction on p we see that $H^p(U \times A^1, G_m) \cong H^p(U, G_m)$ for all $p \ge 2$.

We can give the coordinate ring $k[\mathscr{S}_{\tau}]$ of U a grading by the nonnegative integers such that the degree = 0 subring is the coordinate ring of T_{r-2} . Since $H^{p}(U \times A^{1}, G_{m}) = H^{p}(U, G_{m})$, [8, Theorem 1.1] implies $H^{p}(U, G_{m}) \cong$ $H^{p}(T_{r-2}, G_{m})$, which proves (b). We have a commutative diagram

$$\begin{array}{cccc} \mathrm{H}^{p}(T_{r-2},\,\mathrm{G}_{m}) &\cong & \mathrm{H}^{p}(U,\,\mathrm{G}_{m}) & \stackrel{\alpha}{\longrightarrow} & \mathrm{H}^{p}(U-V,\,\mathrm{G}_{m}) \\ & & & & \downarrow^{\gamma} & & & \\ & & & & & H^{p}(T_{r},\,\mathrm{G}_{m}) \end{array}$$

where the maps β , γ , δ are induced from restriction. Since β is injective, α is injective and Lemma 2.1 now follows. \Box

Theorem 2.2. Let Δ be a fan which consists of cones of dimension ≤ 2 . Let Δ' be a nonsingular fan obtained from Δ by subdividing the two-dimensional faces of Δ and let $\tilde{X} = T_N \operatorname{emb}(\Delta')$. Then the sequence $0 \to B'(T_N/X) \to B'(X) \to B'(\tilde{X}) \to 0$ (with natural maps) is exact.

Proof. Let $\pi: \tilde{X} \to X$ be the desingularization resulting from the subdivision Δ' of Δ [12, Corollary 1.18] and let $\tilde{U}_i = \pi^{-1}(U_i)$. From the long exact sequence of cohomology with supports, and the observation that V is a disjoint union of closed sets V_i (see pp. 92–93 of [11]) we have a commutative diagram with exact rows

$$0 \to \mathbf{B}'(T_N/X) \to \mathbf{B}'(X) \to \mathbf{B}'(X-V) \longrightarrow \bigoplus_{i=1}^m \mathbf{H}^3_{V_i}(U_i, \mathbf{G}_m)$$
$$\downarrow^{\alpha} \qquad \qquad \downarrow^{=} \qquad \qquad \qquad \downarrow^{\beta}$$
$$0 \longrightarrow \mathbf{B}'(\widetilde{X}) \to \mathbf{B}'(\widetilde{X} - \pi^{-1}(V)) \longrightarrow \bigoplus_{i=1}^m \mathbf{H}^3_{\pi^{-1}(V_i)}(\widetilde{U}_i, \mathbf{G}_m)$$

The second row is exact since \tilde{X} is nonsingular. First check β is injective. For each *i*, Lemma 2.1 yields the commutative diagram with exact rows

$$0 \longrightarrow \mathbf{B}'(U_i) \longrightarrow \mathbf{B}'(U_i - V_i) \longrightarrow \mathbf{H}^3_{V_i}(U_i, \mathbf{G}_m) \longrightarrow 0$$

$$\downarrow^{\alpha_i} \qquad \qquad \downarrow^{=} \qquad \qquad \downarrow^{\beta_i}$$

$$0 \longrightarrow \mathbf{B}'(\widetilde{U}_i) \longrightarrow \mathbf{B}'(\widetilde{U}_i - \pi^{-1}(V_i)) \longrightarrow \mathbf{H}^3_{\pi^{-1}(V_i)}(\widetilde{U}_i, \mathbf{G}_m)$$

Here $B'(U_i) = B'(U_{\overline{\tau}_i} \times T_{r-2}) = B(T_{r-2})$ by Lemma 2.1 and $B'(U_i - V_i) = H^2(U_i - V_i, G_m)$ since $U_i - V_i$ is nonsingular. If $\Delta'(\tau_i)$ is the fan whose cones are the cones of Δ' contained in τ_i , then $\Delta'(\tau_i)$ is a nonsingular fan whose one dimensional faces lie in a plane. The invariants for $\widetilde{U}_i = T_N \operatorname{emb}(\Delta'(\tau_i))$ are $\{1, 1, 0, \ldots, 0\}$ and Theorem 1.1 implies $B'(\widetilde{U}_i) = B'(\widetilde{U}_{\overline{\tau}_i} \times T_{r-2}) = B(T_{r-2})$ so α_i is an isomorphism. Since $\ker \beta_i = \operatorname{coker} \alpha_i$, β_i is injective so β is injective. But $\ker \beta = \operatorname{coker} \alpha$, so α is an epimorphism and the theorem follows. \Box

As a result of Theorem 2.2 and our analysis of the Brauer groups of nonsingular toric varieties in §1, we are left with the study of $B'(T_N/X)$.

Theorem 2.3. Let Δ be a fan which consists of cones of dimension ≤ 2 . Let $\Delta(2) = \{\tau_1, \ldots, \tau_m\}$. Let $X = T_N \operatorname{emb}(\Delta)$ and let $U_i = U_{\tau_i}$ be the open subsets

of X associated to the τ_i . Then there is an exact sequence

$$0 \to \operatorname{Pic}(X) \to \operatorname{Cl}(X) \to \bigoplus_{i=1}^{m} \operatorname{Cl}(U_i) \to \operatorname{B}'(T_N/X) \to 0.$$

Proof. Let $V_i = V(\tau_i) = \operatorname{orb}(\tau_i)$ and let $V = V_1 \cup \cdots \cup V_m$. From the long exact sequence of cohomology with supports in the closed set V we have (since V is the disjoint union of the closed sets V_i)

(1)

$$\cdots \to \mathrm{H}^{1}(X, \mathbf{G}_{m}) \to \mathrm{H}^{1}(X - V, \mathbf{G}_{m}) \to \bigoplus_{i=1}^{m} \mathrm{H}^{2}_{V_{i}}(X, \mathbf{G}_{m})$$

$$\to \mathrm{H}^{2}(X, \mathbf{G}_{m}) \to \mathrm{H}^{2}(X - V, \mathbf{G}_{m}) \to \bigoplus_{i=1}^{m} \mathrm{H}^{3}_{V_{i}}(X, \mathbf{G}_{m}) \to \cdots$$

Since V has codimension 2 in X, and X-V is nonsingular, $H^1(X-V, G_m) = Pic(X-V) = Cl(X-V) = Cl(X)$. Since U_i is an open neighborhood of V_i , $H^p_{V_i}(X, G_m) = H^p_{V_i}(U_i, G_m)$ for all $p \ge 0$ [11, p. 93]. From Lemma 2.1 with p = 1 we get the exact sequences

$$0 \to \operatorname{Pic} U_i \to \operatorname{Cl} U_i \to \operatorname{H}^2_{V_i}(U_i, \mathbf{G}_m) \to 0 \qquad (1 \le i \le m) \,.$$

Lemma 2.1(b) gives $\operatorname{Pic} U_i = \operatorname{Pic} T_{r-2} = 0$ so $\operatorname{Cl}(U_i) = \operatorname{H}^2_{V_i}(U_i, G_m)$. Since τ_i is simplicial, $\operatorname{Cl}(U_i) = \operatorname{Pic}(U_i - V_i)$ is torsion [12, Proposition 2.1]. Since X - V is nonsingular, $\operatorname{B}'(X - V) = \operatorname{H}^2(X - V, G_m)$ and $\operatorname{B}'(X - V) \to \operatorname{B}(T_N)$ is injective. But $\operatorname{Pic} X \to \operatorname{Cl}(X)$ is injective [5]. With these identifications (1) reduces to the sequence of the theorem. \Box

Corollary 2.4. In the context of Theorem 2.3, if $\operatorname{rank}_{\mathbb{Z}}(N) = r \leq 3$ and $m \geq 1$, then

$$0 \to \operatorname{Pic}(X) \to \operatorname{Cl}(X) \to \bigoplus_{i=1}^{m} \operatorname{Cl}(U_i) \to \operatorname{B}'(X) \to 0$$

is exact.

Proof. We need to check $B'(X) = B'(T_N/X)$. $H^2(U_i, G_m) = H^2(T_{r-2}, G_m) = 0$ for $r \le 3$ (Lemma 2.1 and [10]). From Lemma 2.1(a) we have (since $U_i - V_i$ is nonsingular) $B'(U_i - V_i) = H^2(U_i - V_i, G_m) = H^3_{V_i}(U_i, G_m) = H^3_{V_i}(X, G_m)$ so (1) becomes

$$0 \to \operatorname{Pic} X \to \operatorname{Cl}(X) \to \bigoplus_{i=1}^{m} \operatorname{Cl}(U_i) \to \operatorname{B}'(X) \to \operatorname{B}'(X-V) \to \bigoplus_{i=1}^{m} \operatorname{B}'(U_i-V_i).$$

Since X - V is nonsingular, restriction induces a monomorphism $B'(X - V) \rightarrow B'(U_i - V_i)$ for each *i* and the corollary follows. \Box

The object of the rest of this section is to give an algorithm for finding for each prime p a subset τ_1, \ldots, τ_e of the two-dimensional faces of Δ such that $[\bigoplus_{i=1}^e \operatorname{Cl}(U_{\tau_i})]_p \cong [B'(T_N/X)]_p$. (For G a finite abelian group, G_p is the Sylow *p*-subgroup.) In particular, the exact sequence of Theorem 2.3 is splitexact. To the fan Δ we associated a bipartite graph Γ . The vertex set of Γ is $\Delta(1)\cup\Delta(2) = \{\rho_1, \ldots, \rho_n\}\cup\{\tau_1, \ldots, \tau_m\}$ and there is an edge in Γ connecting ρ_j and τ_j if and only if ρ_j is a face of τ_i . If Y is the T_N -invariant divisor $X - T_N$ on X, then Γ is the graph associated to Y in the sense of [4]. A cycle Z in Γ (i.e., Z is homeomorphic to the unit circle) determines a finite set τ_1, \ldots, τ_l of two-dimensional cones and ρ_1, \ldots, ρ_l of one-dimensional faces of Δ configured as follows:



If Δ_Z is the subfan of Δ consisting of the cones $\{0, \rho_1, \ldots, \rho_l, \tau_1, \ldots, \tau_l\}$, we will show that the cohomological Brauer group of $T_N \operatorname{emb}(\Delta_Z)$ is cyclic of order the greatest common divisor of $\{|\operatorname{Cl} U_{\tau_i}|\}_{i=1}^l$. Of course, there may be many such cycles in Γ and the last step in the analysis is to choose for each prime p a list of cycles $\{Z_i\}_{i=1}^e$ and for each Z_i a face τ_i such that $[\bigoplus_{i=1}^e \operatorname{Cl}(U_{\tau_i})]_p \cong [\mathbf{B}'(T_N/X)]_p$.

We adopt the following notation: for each two-dimensional cone τ_i in Δ $(1 \le i \le m)$ let ρ_{i1} and ρ_{i2} be the one-dimensional faces of τ_i so $\tau_i = \rho_{i1} + \rho_{i2}$. We have observed Cl(X) = Cl(X - V) = Pic(X - V) and $Cl(U_i) = Cl(U_i - V_i) = Pic(U_i - V_i)$. Now we want to present Pic(X - V) and $Pic(U_i - V_i)$ in terms of support functions on the fan $\Delta - \{\tau_1, \ldots, \tau_m\}$. If we let ρ_1, \ldots, ρ_n be the one-dimensional cones in Δ , then we can identify the support functions on $\Delta - \{\tau_1, \ldots, \tau_m\}$ with the direct sum of copies of \mathbb{Z} indexed by the ρ_i . If $\Delta_i = \{0, \rho_{i1}, \rho_{i2}\}$, then $U_i - V_i = T_N \operatorname{emb}(\Delta_i)$. It follows from [12, Corollary 2.5] that the sequences

(2)
$$M \to \bigoplus_{i=1}^{n} \mathbb{Z}\rho_{i} \to \operatorname{Cl}(X) \to 0,$$
$$M \xrightarrow{\beta} \mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2} \to \operatorname{Cl}(U_{i}) \to 0$$

are exact. Combining these sequences with the exact sequence of Theorem 2.3 gives a commutative diagram with exact rows and columns

It is routine to check that $\operatorname{im} \alpha + \operatorname{im} \beta = \ker \delta \gamma$. As a result we have a fundamental exact sequence which we exploit for the remainder of this section: (in this sequence $\psi = \delta \gamma$),

(3)
$$\left(\bigoplus_{j=1}^{n} \mathbb{Z}\rho_{j}\right) \oplus \left(\bigoplus_{i=1}^{m} M\right) \xrightarrow{\alpha+\beta} \bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2}) \xrightarrow{\psi} \mathbf{B}'(T_{N}/X) \to 0.$$

Let Γ be the graph associated to Δ . Observe that Γ has 2m edges since each τ_i has exactly two one-dimensional faces ρ_{i1} and ρ_{i2} . The free abelian group $\bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$ is called the edge space of Γ . If we write Γ as a union of its connected components Γ_i we get a corresponding decomposition of Δ into subfans Δ_i with $\Delta_i \cap \Delta_j = \{0\}$ whenever $i \neq j$. The decomposition of Δ gives an open cover of X where the elements in the open over are $T_N \operatorname{emb}(\Delta_i) = X_i$ and $X_i \cap X_j = T_N$ whenever $i \neq j$. With this notation we can prove

Proposition 2.5. The natural map $B'(X) \to \bigoplus_i B'(X_i)$ induces an isomorphism $B'(T_N/X) \cong \bigoplus_i B'(T_N/X_i)$.

Proof. Assume $\Delta = \Delta_1 \cup \Delta_2$ where Δ_1 and Δ_2 are fans with $\Delta_1 \cap \Delta_2 = \{0\}$. It is sufficient to prove $B'(T_N/X) \cong B'(T_N/X_1) \oplus B'(T_N/X_2)$ where $X_i = T_N \operatorname{emb}(\Delta_i)$ (i = 1, 2). Let $\Delta_1(1) = \{\rho_1, \ldots, \rho_{n_1}\}$ and $\Delta_1(2) = \{\tau_1, \ldots, \tau_{m_1}\}$ and $\Delta_2(1) = \{r_1, \ldots, r_{n_2}\}$ and $\Delta_2(2) = \{t_1, \ldots, t_{m_2}\}$. Also let $\tau_i = \rho_{i1} + \rho_{i2}$ and $t_j = r_{j1} + r_{j2}$ where $\rho_{ik} \in \Delta_1(1)$ and $r_{jl} \in \Delta_2(1)$. With respect to this decomposition the exact sequence (3) decomposes as

$$\begin{bmatrix} \prod_{j=1}^{n_1} \mathbb{Z}\rho_j \oplus \bigoplus_{i=1}^{m_1} M \end{bmatrix} \oplus \begin{bmatrix} \prod_{j=1}^{n_2} \mathbb{Z}r_j \oplus \bigoplus_{i=1}^{m_2} M \end{bmatrix} \xrightarrow{(\alpha_1 + \beta_1) \oplus (\alpha_2 + \beta_2)} \\ \begin{bmatrix} \bigoplus_{i=1}^{m_1} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2}) \end{bmatrix} \oplus \begin{bmatrix} \prod_{i=1}^{m_2} (\mathbb{Z}r_{i1} \oplus \mathbb{Z}r_{i2}) \end{bmatrix} \to \mathbf{B}'(T_N/X) \to \mathbf{0}.$$

But $\operatorname{coker}(\alpha_1 + \beta_1) \oplus (\alpha_2 + \beta_2) = \mathbf{B}'(T_N/X_1) \oplus \mathbf{B}'(T_N/X_2)$ by (3) so $\mathbf{B}'(T_N/X) = \mathbf{B}'(T_N/X_1) \oplus \mathbf{B}'(T_N/X_2)$. \Box

Notice in Proposition 2.5 that if X_i corresponds to a connected component of Γ containing no two-dimensional faces τ_i as vertices, then $X_i = T_N \operatorname{emb}\{0, \rho\}$ for some one-dimensional cone ρ in Δ . In this case X_i is nonsingular and $B'(T_N/X_i) = 0$. Thus, as a result of Proposition 2.5 we can assume Γ is connected and at least one vertex of Γ is a two-dimensional cone in Δ .

We now determine a matrix representation for the map $\alpha + \beta$ in (3). Let $\tau = \rho_1 + \rho_2 \in \Delta(2)$ and consider the map

$$M \xrightarrow{\beta} \mathbb{Z}\rho_1 \oplus \mathbb{Z}\rho_2 \to \operatorname{Cl}(U_\tau) \to 0$$

as in (2). Pick a basis n_1, \ldots, n_r for N and a dual basis m_1, \ldots, m_r for M. Let η_i be a primitive element in N with $\rho_i = \mathbb{R}_{\geq 0}\eta_i$. The matrix of β with respect to the basis pair $\{m_1, \ldots, m_r\}, \{\rho_1, \rho_2\}$ is the $2 \times r$ matrix whose i, jth entry is $\langle m_j, \eta_i \rangle$. But $\langle m_j, \eta_i \rangle$ is the *j*th coordinate of η_i so we can write this matrix as $\binom{\eta_1}{\eta_2}$ where we think of η_i as a row vector. Therefore the map β in (3)

$$\bigoplus_{i=1}^m M \xrightarrow{\beta} \bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$$

has a matrix representation which is a direct sum of $2 \times r$ matrices $\binom{\eta_{i1}}{\eta_{i2}}$ where η_{i1} and η_{i2} are the primitive generators of ρ_{i1} and ρ_{i2} expressed with respect to the basis $\{n_1, \ldots, n_r\}$. To determine the matrix for the homomorphism

$$\bigoplus_{j=1}^{n} \mathbb{Z}\rho_{j} \xrightarrow{\alpha} \bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$$

given in (3) we observe the *j*th column of this matrix is $\alpha(\rho_j)$. Thus the *j*th column has a 1 in the row determined by ρ_{ik} if $\rho_j = \rho_{ik}$. Otherwise this entry is 0. The matrix of the homomorphism $\alpha + \beta$ of (3) is then

$$Q = \left[\alpha(\rho_1) \cdots \alpha(\rho_n) \middle| \begin{array}{c} \binom{\eta_{11}}{\eta_{12}} & & \\ & \ddots & \\ & & & \binom{\eta_{m1}}{\eta_{m2}} \end{array} \right]$$

Note Q is an integral matrix with 2m rows and n + rm columns, and we can identify $im(\alpha+\beta)$ with the column space of Q. Since $B'(T_N/X) \cong coker(\alpha+\beta)$ from (3), calculating $B'(T_N/X)$ is reduced to determining the column space of Q. Our first observation is a straightforward calculation:

(4)
$$Q \cdot \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \\ -I_r \\ \vdots \\ -I_r \end{bmatrix} = (0)$$

where I_r is the $r \times r$ identity matrix. Thus the last r columns of Q containing $\binom{\eta_{m1}}{\eta_{m2}}$ are linear combinations of the preceding columns. We now assume Γ is connected, and let T be a spanning tree for Γ . We observe that in Γ each vertex τ_i is joined by edges $\tau_{i-\rho_{i1}}$, $\tau_{i-\rho_{i2}}$ to vertices ρ_{i1} , ρ_{i2} so there are 2m edges in Γ . Since Γ is connected, there are n + m - 1 edges in T [1].



Thus, if c_1, \ldots, c_e denote the edges of Γ that are not in T, then e = m - n + 1and for each *i* at least one of $\tau_{i-}\rho_{i1}$, $\tau_{i-}\rho_{i2}$ is in T. By reindexing we can assume $c_1 = \tau_{1-}\rho_{12}, \ldots, c_e = \tau_{e-}\rho_{e2}$. We identify c_j with the basis vector ρ_{j2} in the edge space $\bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$. For $1 \le i \le m$ let n_{i1} be a primitive vector in N with $\mathbb{R}_{\ge 0}n_{i1} = \rho_{i1}$ and choose n_{i2} in N with $\{n_{i1}, n_{i2}\}$ a basis for $\mathbb{R}\tau_i \cap N$. We can extend $\{n_{i1}, n_{i2}\}$ to a basis $\{n_{i1}, n_{i2}, \ldots, n_{ir}\}$ for N. With respect to this basis we can write $\eta_{i1} = n_{i1}$, $\eta_{i2} = a_i n_{i1} + b_i n_{i2}$ where the η_{ij} are as in Q. With respect to these basis choices for N and corresponding dual basis choices for M, and after deleting columns consisting of zeros, the matrix Q for $\alpha + \beta$ becomes

$$Q = \begin{array}{ccccc} & & \rho_1 & \rho_2 & \cdots & \rho_n & & \alpha_1 \beta_1 & & \alpha_i \beta_i & & \alpha_m \beta_m \\ \hline \tau_1 - \rho_{12} & & & \\ \tau_i - \rho_{i2} & & \\ \tau_m - \rho_{m1} & & \\ \tau_m - \rho_{m2} & & & \\ \end{array} \begin{pmatrix} & & & & \\ \alpha(\rho_1) & \alpha(\rho_2) \cdots \alpha(\rho_n) & & & \\ & & & \\ & & & \\$$

where the first *n* columns span im α and the last 2m columns span im β . We checked in (4) above that the last 2 columns labeled α_m and β_m are linear combinations of the preceding n + 2(m-1) columns. It follows from (2) that $|b_i| = |\operatorname{Cl}(U_{\tau_i})| = |\operatorname{Cl} U_i|$. The columns β_1, \ldots, β_e are b_1c_1, \ldots, b_ec_e .

Theorem 2.6. Let Δ be a fan on $N_{\mathbb{R}}$ and let $X = T_N \operatorname{emb}(\Delta)$. Assume all the cones in Δ have dimension ≤ 2 . Assume the two-dimensional faces τ_1, \ldots, τ_m and one-dimensional faces ρ_1, \ldots, ρ_n of Δ can be ordered so that $\tau_i \cap \tau_{i+1} = \rho_i$ $(1 \leq i \leq m-1)$ and $\tau_m \cap \tau_1 = \rho_m$. Let b_i be the order of $\operatorname{Cl}(U_{\tau_i})$. Then $\mathbf{B}'(T_N/X)$ is cyclic of order $\operatorname{gcd}\{b_1, \ldots, b_m\}$.

Proof. Using Proposition 2.5 and the hypotheses, we can assume that the graph Γ is connected and consists of one cycle as shown:



We take the spanning tree T for Γ to be the graph obtained from Γ by deleting the edge $c_1 = \tau_{1-}\rho_m$. Let C be the matrix whose only column is c_1 and form the augmented matrix [Q|C]:

Here c_1 corresponds to the edge $\tau_{1-}\rho_m$ and $\bigoplus_{i=1}^m (\mathbb{Z}\rho_i \oplus \mathbb{Z}\rho_{i-1})$ is the edge space of Γ where $\rho_0 = \rho_m$. We observed that $B'(T_N/X)$ is the quotient of the edge space by the column space of Q. Let [B|C] be the matrix whose columns are the columns labeled $\rho_1, \ldots, \rho_m, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, c_1$. We check that the columns of [B|C] form a basis for the edge space by using column operations to reduce to a permutation matrix. Use column c_1 to eliminate a_1 from column α_1 and the 1 in the entry with row index $\tau_{1-}\rho_m$ and column index ρ_m . Then use the 1 in the new column α_1 to eliminate the 1 in the entry with row index $\tau_{1-}\rho_{1}$ and column index ρ_{1} . Use the new column ρ_{1} to eliminate the a_{2} in column α_2 . Continue inductively, eliminating a_3, \ldots, a_{m-2} from columns indexed $\alpha_3, \ldots, \alpha_{m-2}$ and the ones in the entries with row index $\tau_{i-}\rho_i$ and column index ρ_i , for $2 \le i \le m-2$. At the last step use the remaining 1 in column ρ_{m-2} to eliminate a_{m-1} in column α_{m-1} . Use 1 in the new column α_{m-1} to eliminate the 1 in the entry in row $\tau_{m-1}-\rho_{m-1}$, column ρ_{m-1} . The result is a matrix whose Z-rank is 2m, which shows that c_1 generates the quotient of the edge space $\bigoplus_{i=1}^{m} (\mathbb{Z}\rho_i \oplus \mathbb{Z}\rho_{i-1})$ $(\rho_0 = \rho_m)$ by the column space of Q. Recall that the last two columns of Q are a linear combination of the preceeding ones. Thus to calculate this quotient we simply project each of the columns β_1, \ldots, β_m on $\mathbb{Z}c_1$. These projections follow the recursive pattern:

Column vector	Projection on $\mathbb{Z}c_1$
$\beta_1 = b_1 c_1$	b_1c_1
$\beta_2 = b_2(\rho_1 - (\alpha_1 - a_1c_1))$	$b_2 a_1 c_1$
$\beta_3 = b_3(\rho_2 - (\alpha_2 - a_2(\rho_1 - (\alpha_1 - a_1c_1))))$	$b_3 a_2 a_1 c_1$
:	
β_m	$b_m a_{m-1} \cdots a_2 a_1 c_1$

The subgroup generated by the projections of the columns β_i on $\mathbb{Z}c_1$ is the subgroup generated by dc_1 where $d = \gcd\{b_1, b_2a_1, \ldots, b_ma_{m-1}\cdots a_2a_1\}$. Since $\gcd(a_i, b_i) = 1$ for $1 \le i \le m$, we see $d = \gcd\{b_1, \ldots, b_m\}$. But $|b_i|$ is the order of $\operatorname{Cl}(U_{\tau_i})$, so the theorem follows. \Box

To extend Theorem 2.6 it is necessary to introduce some additional notation. Suppose the graph Γ we have associated to the fan Δ is connected and let T be a spanning tree for Γ . Since each vertex labeled by a two-dimensional face τ_i is connected by exactly two edges to vertices ρ_{i1} and ρ_{i2} corresponding to the one-dimensional faces of τ_i in Δ , each τ_i is a vertex in T. If $\Delta(2) = \{\tau_1, \ldots, \tau_m\}$, designate τ_m as the root node for T. Let C be the matrix whose columns are c_1, \ldots, c_e and let [Q|C] be the augmented matrix similar to that used in the proof of Theorem 2.6. Then τ_1, \ldots, τ_e are leaf nodes of T and $c_i = \tau_{i-}\rho_{i2}$ for $1 \le i \le e$. For e < i < m relabel ρ_{i1} and ρ_{i2} if necessary so the edge $\tau_{i-}\rho_{i1}$ is closer to the root node τ_m than the edge $\tau_{i-}\rho_{i2}$. In our previous analysis this amounts to permuting the basis of the edge space $\bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1}\oplus\mathbb{Z}\rho_{i2})$. This does not affect the columns labeled $\alpha_1, \beta_1, \ldots, \alpha_e, \beta_e$ in Q. Let [B|C] be the matrix obtained from [Q|C] by deleting from Q the columns labeled $\beta_1, \ldots, \beta_{m-1}, \beta_m, \alpha_m$. We note that the column space of B depends on the choice of τ_m .

Lemma 2.7. The columns of [B|C] form a basis for $\bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$, the edge space, for any choice of root node τ_m .

Proof. If Γ is a tree, then $T = \Gamma$ and e = 0. In this case we need to show that the columns of B span the edge space. Each leaf node of Γ must be ρ_{i2} for some i since each τ_i is incident to two edges in $\Gamma = T$ and ρ_{i1} is closer to the root node τ_m than ρ_{i2} . We call the pair (τ_i, ρ_{i2}) a leaf node pair. If i = m, then it is possible for ρ_{i1} to be a leaf node. This is the only exception and will be treated in the basis step for our induction below. Assume (τ_i, ρ_{i2}) is a leaf node pair and $i \neq m$. In Δ , ρ_{i2} is a face of exactly one two-dimensional cone τ_i . Thus the column indexed by ρ_{i2} in B has exactly one nonzero entry which is a 1 in the row indexed $\tau_{i-}\rho_{i2}$ as indicated below.

$$\tau_i - \rho_{il} \begin{bmatrix} \rho_{il} & \rho_{i2} & \alpha_i \\ 1 & 0 & | 1 \\ 0 & 1 & | a_i \end{bmatrix}$$

Use the column indexed ρ_{i2} to eliminate a_i in the column α_i by an elementary column operation, then use the new column α_i to eliminate the 1 in the $\tau_{i-}\rho_{i1}$ entry of column ρ_{i1} . After these two steps are performed, we say we have pruned the leaf node pair from the tree $\Gamma = T$. The two columns indexed ρ_{i2} and α_i are now elementary basis vectors in our basis for the edge space and appear in no further column operations. After the columns indexed ρ_{i2} and α_i are deleted, the remaining matrix is the matrix we would associate to the fan Δ' obtained from Δ be deleting the cones ρ_{i2} and τ_i . Apply this leaf pruning algorithm iteratively to reduce to the case where Γ is the tree $\rho_{m1-}\tau_{m-}\rho_{m2}$. The matrix B for this tree is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus our algorithm reduces the original matrix B, using elementary column operations, to a permutation matrix.

If Γ is not a tree, let $\tau_{1-}\rho_{12}, \ldots, \tau_{e-}\rho_{e2}$ be the edges of Γ which are not in T. Fix $i, 1 \le i \le e$. Since $\tau_{i-}\rho_{i1}$ is in T, it follows that τ_i is a leaf node of T. We can use the column indexed c_i and elementary column operations to eliminate the entry a_i from the column indexed α_i and the entry 1 in the column indexed ρ_{i2} and row indexed $\tau_{i-}\rho_{i2}$ in the matrix [B|C]. Use the new column α_i to eliminate the 1 in the $\tau_{i-}\rho_{i1}$ entry of column ρ_{i1} . Repeat this step for $i = 1, \ldots, e$. Observe that the $2e \times 2e$ submatrix of [B|C] whose columns are indexed α_i $(1 \le i \le e), c_i$ $(1 \le i \le e)$ and whose rows are indexed $\tau_{i-}\rho_{ij}$ $(1 \le i \le e \text{ and } 1 \le j \le 2)$ has rank 2e. If we delete this submatrix from [B|C] (which corresponds to deleting τ_1, \ldots, τ_e from the fan Δ), the resulting matrix is the one we would associate to the fan $\Delta' = \Delta - \{\tau_1, \ldots, \tau_e\}$. The graph of Δ' is a tree so the result follows from the first part of the proof. \Box

Corollary 2.8. Let Δ be a fan on \mathbb{R}^r whose cones all have dimension ≤ 2 . If the graph Γ associated to Δ is a disjoint union of trees and $X = T_N \operatorname{emb}(\Delta)$, then $\mathbf{B}'(T_N/X) = 0$.

Proof. By Proposition 2.5 we can assume Γ is connected, so the hypotheses imply Γ is a tree. By Lemma 2.7 the columns of B span the edge space of Γ . But the column space of B is contained in $im(\alpha + \beta)$ in (3) $B'(T_N/X) = 0$. \Box

Corollary 2.9. Let Δ be a fan on \mathbb{R}^2 and $X = T_N \operatorname{emb}(\Delta)$ the associated toric surface.

- (a) If $\Delta = \{0\}$, then $B(X) \cong \mathbb{Q}/\mathbb{Z}$.
- (b) If $\Delta \neq \{0\}$ and $|\Delta| \neq \mathbb{R}^2$ (i.e., X is not complete), then B(X) = 0.
- (c) If $|\Delta| = \mathbb{R}^2$ (i.e., X is complete), $\Delta(1) = \{\rho_1, \ldots, \rho_n\}$ and $N' = \langle \rho_1 \cap N, \ldots, \rho_n \cap N \rangle$, then $B(X) \cong N/N'$.

Proof. Every toric surface is a normal projective surface [12] so by [9, Corollary 9] B(X) = B'(X). If $\Delta = \{0\}$, then $X = T_N$ is nonsingular and since r = 2, $B(X) = \mathbb{Q}/\mathbb{Z}$ [10]. If $\Delta \neq \{0\}$ and $|\Delta| \neq \mathbb{R}^2$, then the graph Γ associated to Δ is a disjoint union of trees. By Proposition 2.5 we assume Γ is a tree and contains at least one two-dimensional cone τ . By Corollary 2.8 and Corollary 2.4, B(X) = 0. If $|\Delta| = \mathbb{R}^2$, then Γ is a cycle. Corollary 2.4 implies B(X) = $B(T_N/X)$. If $\Delta(2) = \{\tau_1, \ldots, \tau_m\}$, then Theorem 2.6 implies B(X) is cyclic of order $gcd\{|Cl(U_{\tau_i})|\}_{i=1}^m$. Let ρ_1, \ldots, ρ_m be the one-dimensional cones in Δ with $\tau_i = \rho_i + \rho_{i+1}$ $(1 \le i \le m)$ where $\rho_{m+1} = \rho_1$ and let $\rho_i = \mathbb{R}_{\ge 0}\eta_i$ for primitive vectors η_1, \ldots, η_m in N. Choose a basis n_1, n_2 for N with $n_1 = \eta_1$ and write $\eta_1 = a_i n_1 + b_i n_2$. Then $N/N' = N/\langle \eta_1, \ldots, \eta_m \rangle$ is cyclic of order $gcd\{b_1, \ldots, b_m\}$. On the other hand for $(1 \le i \le m - 1)$, $Cl(U_{\tau_i})$ is cyclic of order $|det[\frac{a_i a_{i+1}}{b_i b_{i+1}}]|$ and $|Cl(U_{\tau_m})| = |det[\frac{a_m 1}{b_m 0}]|$ by (2). Since $gcd(a_i, b_i) = 1$ for each i, an easy calculation shows $gcd\{|Cl(U_{\tau_i})|\}_{i=1}^m = gcd\{b_1, \ldots, b_m\}$. \Box

To determine the *p*-subgroups of $B'(T_N/X)$ for each prime number *p*, we introduce some additional notation and terminology. Let Γ be a finite edgeweighted graph such that to each edge E is associated the positive integer weight(E). Let v_p be the p-adic valuation on \mathbb{Z} and set the p-weight of $E = \text{weight}_{p}(E) = v_{p}(\text{weight}(E))$. If Γ_{1} is a subgraph of Γ , let weight_p(Γ_{1}) = $\sum \text{weight}_p(E)$ where the summation is over all edges E in Γ_1 . A p-maximal spanning tree for Γ is a spanning tree T for Γ such that weight_p(T) is maximal among the *p*-weights of all spanning trees. It is clear that every connected graph has a p-maximal spanning tree. Let T be a p-maximal spanning tree for Γ and let c denote an edge of Γ which is not an edge of T. Since T is a spanning tree, the subgraph Γ_1 of Γ obtained by adding the edge c to T contains a cycle Z which is unique since there is a unique path between any two vertices of the tree T. Suppose there is some edge E in Z with weight_n(E) < weight_n(c). Then we could obtain a spanning tree of larger *p*-weight by deleting the edge E from Γ_1 . This means that if T is a p-maximal spanning tree for Γ , c is an edge of Γ not in T and Z is the unique cycle in the graph $T \cup \{c\}$, then c is an edge of minimal p-weight in Z.

Let Γ be the (connected) graph we have associated to the fan Δ whose cones all have dimension ≤ 2 . Assign weights to the edges $\tau_{i-}\rho_{ij}$ of Γ by setting weight $(\tau_{i-}\rho_{ij}) = b_i = |\operatorname{Cl}(U_{\tau_i})|$ (recall $\operatorname{Cl}(U_{\tau_i}) = \mathbb{Z}/b_i$ is cyclic from (2)). Let T be a *p*-maximal spanning tree for Γ . We have labeled the edges of Γ not in T as $\tau_{1-}\rho_{12}, \tau_2 - \rho_{22}, \ldots, \tau_{e-}\rho_{e2}$. We call the set of 2-dimensional cones $\{\tau_1, \ldots, \tau_e\}$ in Δ a *p*-minimal set of cones in Δ . If Γ is not connected, then we can decompose Δ as a union of fans Δ_i with $\Delta_i \cap \Delta_j = \{0\}$ when $i \neq j$ and the graphs Γ_i associated to Δ_i are connected. We define a *p*-minimal set of cones in Δ to be the union of *p*-minimal sets of cones in each Δ_i .

Theorem 2.10. Let Δ be a fan on $N_{\mathbb{R}}$ and assume every cone in Δ has dimension ≤ 2 . Let $\{\tau_1, \ldots, \tau_e\}$ be a p-minimal set of cones in Δ and let $|\operatorname{Cl}(U_{\tau_i})| = b_i$. If $X = T_N \operatorname{emb}(\Delta)$, then $\operatorname{B}'(T_N/X)_p \cong [\bigoplus_{i=1}^e \mathbb{Z}/b_i]_p$. This isomorphism is induced by the epimorphism ψ of (3).

Proof. By Proposition 2.5 and the discussion preceeding the theorem, we can assume the graph Γ associated to the fan Δ is connected. Let T be a p-maximal spanning tree for Γ . Continuing the analysis that was begun in

the proof of Theorem 2.6 we consider the matrix [Q|C] defined there. If $\psi : \bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2}) \to B'(T_N/X)$ is the epimorphism given in (3) and $\widehat{\mathbb{Z}}_p$ is the *p*-adic integers, we have an epimorphism

$$\psi_p: \bigoplus_{i=1}^m (\widehat{\mathbb{Z}}_p \rho_{i1} \oplus \widehat{\mathbb{Z}}_p \rho_{i2}) \to \mathbf{B}'(T_N/X)_p.$$

It follows from Lemma 2.7 that $\{c_1, \ldots, c_e\}$ generates $\operatorname{coker}(\alpha + \beta)$ so $\{\psi_p(c_1), \ldots, \psi_p(c_e)\}$ generates $B'(T_N/X)_p$. We check

$$0 = \langle \psi_p(c_j) \rangle \cap \langle \psi_p(c_1), \ldots, \psi_p(c_{j-1}), \psi_p(c_{j+1}), \ldots, \psi_p(c_e) \rangle$$

and $\psi_p(c_j)$ has order $p^{v_p(b_j)}$ for $1 \le j \le e$ by identifying these elements in $B'(T_N/X)$ with their corresponding preimages $c_j + image(\alpha + \beta) \in coker(\alpha + \beta)$ in (3) and then checking the corresponding statements in $coker(\alpha + \beta)$.

Fix j and let π be a permutation of $\{1, \ldots, m\}$ with π chosen so $\pi(1) = j$ where the edge $c_j = \tau_{j-}\rho_s$ and the cycle in $T \cup \{c_j\}$ is

$$\tau_j = \tau_{\pi(1)-} \rho_{1-} \tau_{\pi(2)-} \rho_{2-} \cdots \tau_{\pi(s)-} \rho_{s-} \tau_{\pi(1)}.$$

Choose the vertex $\tau_{\pi(s)}$ as the root node for T. By Lemma 2.7 we know the columns of [B|C] form a basis for $\bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$. The column space of B is a submodule of image $(\alpha + \beta)$. Project the submodule $\widehat{\mathbb{Z}}_{p}\beta_{1} + \cdots + \widehat{\mathbb{Z}}_{p}\beta_{m}$ of image $(\alpha + \beta)$ onto a $\widehat{\mathbb{Z}}_{p}$ -submodule of the column space of C over $\widehat{\mathbb{Z}}_{p}$. Then coker $(\alpha + \beta)_{p}$ is the quotient module.

If $s+1 \leq i \leq m$ we check the projection of $\beta_{\pi(i)}$ on $\mathbb{Z}_p c_j$ is 0. The selection of $\tau_{\pi(s)}$ as the root node for T gives a partial order on the vertices of T. Let T_i be the subtree of T with root node $\rho_{\pi(i)2}$. This means the vertices v in T_i are those for which the unique path from v to $\tau_{\pi(s)}$ contains $\rho_{\pi(i)2}$. In the expression for $\beta_{\pi(i)}$ as a linear combination of the columns of [B|C], the columns of C that appear are those c_k which when considered as edges of Γ are incident to some vertex in T_i (see the proof of Theorem 2.6). But neither τ_j nor ρ_s are in T_i since $i \geq s+1$ so the projection of $\beta_{\pi(i)}$ on c_j has coefficient = 0.

If $1 \le i \le s$ let the projection of $\beta_{\pi(i)}$ on $\bigoplus_{k=1}^{e} \widehat{\mathbb{Z}}_{p}c_{k}$ be $\sum_{k=1}^{e} d_{ki}c_{k}$. We say in the proof of Theorem 2.6 that $b_{\pi(i)}|d_{ji}$ (where $b_{\pi(i)} = |\operatorname{Cl}(U_{\pi(i)})|$). The projections of $\beta_{\pi(1)}, \ldots, \beta_{\pi(m)}$ on $\bigoplus_{k=1}^{e} \widehat{\mathbb{Z}}_{p}c_{k}$ are the columns of the $e \times m$ matrix (d_{ki}) . We have observed above that $d_{j(s+1)} = \cdots = d_{jm} = 0$. The definitions of β_{j} and c_{j} imply $\beta_{j} = b_{j}c_{j}$ so $d_{k1} = 0$ if $k \ne j$. Also $b_{\pi(i)}|d_{ji}$ for $2 \le i \le s$ and we chose j with $v_{p}(b_{j}) = \min\{v_{p}(b_{\pi(1)}), \ldots, v_{p}(b_{\pi(s)})\}$. Thus after elementary column operations over $\widehat{\mathbb{Z}}_{p}$ the column space of (d_{ki}) is equal to the column space of

$$\begin{bmatrix} 0 & d_{12} & \cdots & d_{1m} \\ \vdots & \vdots & & \vdots \\ 0 & d_{(j-1)2} & & & \\ b_j & 0 & \cdots & 0 \\ 0 & d_{(j+1)2} & & & \\ \vdots & \vdots & & \vdots \\ 0 & d_{e2} & \cdots & d_{em} \end{bmatrix}$$

Therefore c_1, \ldots, c_e represents a basis for $\operatorname{coker}(\alpha + \beta)_p$, $\mathbf{B}'(T_N/X)_p = \langle \psi_p(c_1) \rangle \oplus \cdots \oplus \langle \psi_p(c_e) \rangle$, and $\langle \psi_p(c_j) \rangle$ is cyclic of order $p^{v_p(b_j)}$. \Box

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DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, COLORADO 80523

E-mail address: demeyer@euclid.math.colostate.edu

DEPARTMENT OF MATHEMATICS, FLORIDA ATLANTIC UNIVERSITY, BOCA RATON, FLORIDA 33431 *E-mail address*: ford@acc.fau.edu