REYE CONSTRUCTIONS FOR NODAL ENRIQUES SURFACES

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ABSTRACT. A classical Reye congruence X is an Enriques surface of rational equivalence class $(3, 7)$ in the grassmannian $G(1, 3)$ of lines of $\mathbf{P}^3$. X is the locus of lines of $\mathbf{P}^3$ which are included in two quadrics of $W =$ web of quadrics. A generalization to $G(1, t)$ is given (1) for each $t > 2$ there exist Enriques surfaces X of class $(t, 3t - 2)$ in $G(1, t)$, (2) the determinant of the dual of the universal bundle on X is $\mathcal{O}_X(2E + R + K_X)$, with $E =$ isolated elliptic curve, $R^2 = -2$, $E \cdot R = t$, (3) X parameterizes lines of $\mathbf{P}^t$ which are included in a codimension 2 subsystem of W , $W =$ linear system of quadrics of dimension $(\frac{t}{2})$. The paper includes a description of the variety of trisecant lines to a smooth Enriques surface of degree 10 in $\mathbf{P}^5$.

1. INTRODUCTION AND PRELIMINARIES

Let X be an Enriques surface over $\mathbf{C}$,

$$L = \text{Num}(X)$$

the group of its numerical equivalence classes. As it is well known

$$L = \text{Pic}(X)/\text{torsion} \cong \mathbf{Z}^{10}$$

and, as a lattice, it is isomorphic to the orthogonal direct sum $E_8 \oplus H$ (where H is a hyperbolic plane and E_8 is defined as usual, cf. [10, p. 105]). In [10] F. Cossec and I. Dolgachev have studied L in all details with the purpose of describing projective models of X ; among them the so-called *Fano models* are of particular interest. Let us give first their construction; one has [10]:

(1.1) L contains finitely many (modulo isometries) sets of isotropic vectors $\{e_1, \dots, e_{10}\}$ such that

$$e_i \cdot e_j = 1 - \delta_{ij}, \quad \frac{1}{3} \sum e_i \in L;$$

these sets always satisfy the following properties: let $|C|$, $|C'|$ be the two linear systems of numerical class $\frac{1}{3} \sum e_i$, $C - C' = K_X =$ canonical divisor of X , then

$$(1.2) \quad C^2 = 10, \quad p_a(C) = 6, \quad \dim |C| = 5.$$

Assume $|C|$ is irreducible, then

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(1.3) ϕ_C is a morphism of degree 1 and $X_C = \phi_C(X)$ is normal with at most rational double points;

(1.4) each divisor C satisfying (1.2), (1.3) is obtained as in (1.1).

For an isotropic vector $e \in L$ we will say that $2e$ corresponds to an *elliptic pencil* if there exist (exactly) two curves E, E' of class e and such that $|2E| = |2E'|$ is an elliptic pencil. Note that $E - E' \sim K_X$ and $h^0(\mathcal{O}_X(E)) = h^0(\mathcal{O}_X(E')) = 1$; curves E, E' as above are said to be *isolated elliptic curves* on X . Let C be as in (1.2), assume C is very ample then $2e_i$ corresponds to an elliptic pencil $|2E_i| = |E'_i|$. Moreover:

(1.5) E_i, E'_i are contained in X_C as plane cubics; there is no other such curve in X_C .

The same holds for C' . By definition $X_C (X_{C'})$ is a *Fano model*; $C (C')$ a *Fano polarization*. We recall that, on a general X , each Fano polarization is very ample and that C very ample $\Leftrightarrow C'$ very ample. To simplify notations we will write X instead of X_C when no confusion arises.

Now let us describe the contents of this paper: our first purpose was to describe the variety $\text{Tris}(X)$ of trisecant lines to a smooth Fano model X . This is done in §§2 and 3: we show that, excluding one exceptional case, a line L is trisecant to X if and only if L is trisecant to one of the twenty plane cubics in X . The exception is the following: $\text{Tris}(X) = \mathbf{P}^3$ blown up in twenty points; $X = X_{C'}$, where $C' = C + K_X$ is a Reye polarization.

We recall that a *Reye polarization* C' is a special Fano polarization, the special condition on it being

$$X_{C'} \subset G = \text{smooth quadric of } \mathbf{P}^5.$$

If C' is a Reye polarization we will call $X_{C'}$ a *Reye congruence*; the reason is that G is the grassmannian of lines of $\mathbf{P}^3$ and that, traditionally, surfaces in G (i.e. two dimensional families of lines of $\mathbf{P}^3$) were called congruence of lines.

The main modern result on Reye congruences is the following: every Reye congruence X is a nodal Enriques surface and, conversely, on every nodal X admitting a very ample Fano polarization there exists a Reye polarization too [8], [10, vol. II]. We recall that a *nodal Enriques surface* X contains by definition a curve R of arithmetic genus 0 and such that $h^0(\mathcal{O}_R) = 1$. Following [10] we will say that R is an *indecomposable nodal cycle*. By [10, p. 25] a curve R in X is an indecomposable nodal cycle iff it is a fundamental cycle of some rational double point.

Since nodal Enriques surfaces have no general moduli it follows that, for a general Enriques surface, there is no embedding in G as a Fano model. Because of this remark, after the classification of $\text{Tris}(X)$, we came on the following kinds of questions:

Let L be a polarization on X satisfying (1.3), $X_L = \phi_L(X)$, G_r the Plücker embedding of the grassmannian of lines of $\mathbf{P}^r$; when do we have

$$X_L \subset \mathbf{P}^N \cap G_r, \quad r \ll N?$$

($N = \dim |L|$), and what is the special feature of a projective model X_L if X is nodal?

To partially answer them we generalize the construction of Reye congruences to projective models of higher degree $L^2 = 4t - 2 \geq 10$ (§§4 and 5):

Assume the set

$\mathcal{E}_L(t) = \{E \in \text{Div}(X) \mid E = \text{isolated elliptic curve}, E \cdot L = t, h^1(\mathcal{O}_X(L - 2E)) = 0\}$
is not empty, then we show

$$(1.6) \quad X_L \subset G_t \Leftrightarrow L \sim 2E + R + K_X;$$

where $E \in \mathcal{E}_L(t)$, $R =$ indecomposable nodal cycle, $E \cdot R = t$;

$$(1.7) \quad \text{as a surface in } G_t, X \text{ has rational equivalence class } (t, 3t - 2);$$

there exists a linear system W of quadrics of $\mathbf{P}^t$ such that

$$(1.8) \quad \dim(W) = \binom{t}{2} \quad \text{and} \quad X_L = \{l \in G_t \cap \mathbf{P}^N \mid \text{codim}(W_l, W) = 2\}$$

with $W_l = \{Q \in W \mid l \subset Q\}$, $\mathbf{P}^N =$ linear span of X in the Plücker space of G_t .

The case $t = 3$ gives Reye congruences and their classical construction by a web of quadrics of $\mathbf{P}^3$. In view of this we define the projective model appearing in (1.6) as a Reye congruence of index t . Then, in §6, we show that Reye congruences of index t exist for each $t \geq 3$.

Of course an embedding of X in G_t defines a rank 2 vector bundle on X : the restriction to X_L of the universal bundle of G_t . Let $\mathcal{Q}_t$ be the dual of such a bundle; by (1.6), (1.7) $c_1(\mathcal{Q}_t)^2 = 4t - 2$, $c_2(\mathcal{Q}_t) = t$; we can define $\mathcal{Q}_t$ as a *Reye bundle* of index t . This relates our results to those obtained by I. Dolgachev and I. Reider in [11]: there they study the case $t = 3$ and show that $\mathcal{Q}_3$ is stable, (actually the unique stable rank 2 vector bundle on X with Chern class $c_1^2 = 10$, $c_2 = 3$), and extremal i.e. without moduli. Moreover they are interested in the following problem: to find other examples of rank 2 vector bundles $\mathcal{E}$ on an Enriques surface which are stable and extremal.

In this case, computing the dimension of the moduli space, $\mathcal{E}$ must satisfy $c_1(\mathcal{E})^2 = 4t - 2$, $c_2(\mathcal{E}) = t$; therefore our Reye bundle $\mathcal{Q}_t$ seems to be a natural candidate for further examples. During the completion of this paper we learned that stable-extremal rank 2 vector bundles on X were described by Hoil Kim in his Ph.D. thesis [15]. In particular he produces examples $\mathcal{F}_t$ of them with $c_2(\mathcal{F}_t) = t$, any $t \geq 0$. We mention that, applying his results to our situation, it is possible to show that $\mathcal{Q}_t$ is stable-extremal too. Also, we have to mention that the description of $\text{Tris}(X)$ for a Fano model X is independently obtained in [11] as a consequence of the study of global sections of $\mathcal{Q}_3$.

Finally we wish to thank the referee for his help, especially in §6 where our previous degeneration arguments were considerably simplified by his suggestion of using Cremona transformations of $\mathbf{P}^5$ and generic nodal Enriques surfaces.

2. TRISECANTS TO FANO MODELS

Let $V \subset \mathbf{P}^n$ be a smooth projective variety, L a line intersecting V ; in the following we will say that L is a *trisecant line* to V if the scheme $V \cdot L$ is zero dimensional of length ≥ 3 ; (hence, in particular, L is not in V).

Consider a smooth hyperplane section $C = X \cap \mathbf{P}^4$ of a Fano model X ; first we want to study the family $\text{Tris}(C)$ of trisecant lines to C . By Berzolari formulae [12] the expected number of trisecant lines to a smooth, irreducible curve of genus g and degree d in $\mathbf{P}^4$ is $t = \frac{1}{6}(d - 4)(d - 3)(d - 2) - g(d - 4)$,

which gives $t = 20$ in our case. Note that $\mathcal{O}_C(1) \cong \omega_C \otimes \eta$, with $\eta = \mathcal{O}_C(K_X) =$ nontrivial order 2 element of $\text{Pic}(C)$. On the other hand, for a general curve D in $\mathbf{P}^4$ of degree 10 and genus 6, $\mathcal{O}_D(1) \cong \omega_D \otimes \eta$, with $\eta =$ any degree zero line bundle.

So C is a special element of its Hilbert scheme and, at least for this reason, we need to analyze more in detail $\text{Tris}(C)$. For this consider any curve C of genus 6 embedded in $\mathbf{P}^4$ by a very ample linear system $\omega_C \otimes \eta$ with $\eta =$ nontrivial order 2 element of $\text{Pic}(C)$; let

$$(2.1) \quad \begin{aligned} C(3) &= 3\text{-symmetric product of } C, \\ a: C(3) &\rightarrow \text{Pic}^0(C) \text{ the Abel map,} \\ W &= a(C(3)), \\ W_\eta &= \text{translation of } W \text{ by } \eta; \text{ then} \end{aligned}$$

(2.2) **Proposition.** *Let $d \in C(3)$. There exists a trisecant line containing d if and only if $h^0(\eta(d)) = 1$.*

Proof. The condition d is contained in a trisecant line is equivalent to $h^0(\omega_C \otimes \eta(-d)) = 3$. Since η is isomorphic to its dual $h^0(\omega_C \otimes \eta(-d)) = h^1(\eta(d))$, (Serre duality). Finally, $h^1(\eta(d)) = 3 \Leftrightarrow h^0(\eta(d)) = 1$.

(2.3) **Corollary.** *d is contained in a trisecant line if and only if $a(d) \in W \cap W_\eta$.*

Computing $W \cdot W_\eta$ we obtain by Poincaré formulae $W \cdot W_\eta = (\Theta/3!)^2 = 20$, ($\Theta =$ theta divisor in $\text{Pic}^0(C)$). At this point we need a transversality condition for W and W_η in $\text{Pic}^0(C)$:

(2.4) **Proposition.** *W and W_η are transversal at $a(d)$ if and only if $h^0(\eta(2d)) = 1$.*

Proof. Since $h^0(\eta(d)) = 1$ there exists a unique $d' \in C(3)$ such that $\eta \cong \mathcal{O}_C(d' - d)$. By (2.2), (2.3) $a(d') \in W \cap W_\eta$ and $h^0(\eta(d')) = h^0(\mathcal{O}_C(d)) = 1$. Moreover, since $\omega_C \otimes \eta$ is very ample, $\text{Supp}(d) \cap \text{Supp}(d') = \emptyset$ [10, 0.6]. Writing the derivative of Abel map as in [1, 4.1] one obtains the standard identifications

$$(2.5) \quad H^0(\omega_C) \cong T_{\text{Pic}^0(C), a(d)}, \quad H^0(\omega_C(-d)) \cong T_{W, a(d)}$$

since the derivative of the translation by η is the identity, one has

$$(2.6) \quad T_{W_\eta, a(d)} = T_{W, a(d')} = H^0(\omega_C(-d')).$$

Since $\text{Supp}(d) \cap \text{Supp}(d') = \emptyset$ we have $H^0(\omega_C(-d - d')) = H^0(\omega_C(-d)) \cap H^0(\omega_C(-d'))$. Therefore, by Riemann-Roch and Serre duality,

$$h^0(\omega_C(-d - d')) = 0 \Leftrightarrow h^1(\eta(2d)) = 0 \Leftrightarrow h^0(\eta(2d)) = 1 \Leftrightarrow W, W_\eta$$

are transversal at $a(d)$ (and also at $a(d')$).

Let us apply Proposition (2.4) to a smooth hyperplane section $C = \mathbf{P}^4 \cap X$ of a smooth Fano model. Of course the 10 pairs E_i, E'_i ($i = 1 \cdots 10$) of plane cubic curves define 10 pairs L_i, L'_i of trisecant lines to C and the divisors

$$d_i = E_i \cdot C = L_i \cdot C, \quad d'_i = E'_i \cdot C = L'_i \cdot C$$

on C . Note that $\mathcal{O}_C(d'_i - d_i) \cong \mathcal{O}_C(K_X) \cong \eta$, where $\omega_C \otimes \eta = \mathcal{O}_C(1)$. Therefore we have

$$\eta(2d_i) \cong \mathcal{O}_C(E_i + E'_i) \cong \eta(2d'_i)$$

and the exact sequence

$$(2.8) \quad 0 \rightarrow \mathcal{O}_X(-C + E_i + E'_i) \rightarrow \mathcal{O}_X(E_i + E'_i) \rightarrow \eta(2d_i) \rightarrow 0.$$

Since $h^1(\mathcal{O}_X(E_i + E'_i)) = 0$ this gives the exact sequence

$$(2.9) \quad 0 \rightarrow H^0(\mathcal{O}_X(E_i + E'_i)) \rightarrow H^0(\eta(2d_i)) \rightarrow H^1(\mathcal{O}_X(-C + E_i + E'_i)) \rightarrow 0$$

with $\dim H^0(\mathcal{O}_X(E_i + E'_i)) = 1$. Therefore we have shown

(2.10) **Proposition.** *W and W_η are transversal at $a(d_i)$, $a(d'_i)$ if and only if $H^1(\mathcal{O}_X(-C + E_i + E'_i)) = 0$.*

The main point is now the following known result:

(2.11) **Proposition.** *Let X be a smooth Fano model polarized by C , then $h^1(\mathcal{O}_X(-C + E_i + E'_i)) = 0 \quad \forall i = 1 \cdots 10$ unless $C + K_X$ is a Reye polarization.*

Proof. Cf. [8] and the next section.

(2.12) **Theorem.** *Let X be a smooth Fano model polarized by C , π_i, π'_i be the planes containing E_i, E'_i ; π_i^*, π'^*_i their dual planes. Assume $C + K_X$ is not a Reye polarization then*

$$\text{Tris}(X) = \bigcup_{i=1}^{10} (\pi_i^* \cup \pi'^*_i).$$

Proof. Let $L \in \text{Tris}(X)$, by definition L is not in X so that there exists a $\mathbf{P}^4$ transversal to X and containing L . Let $C = X \cap \mathbf{P}^4$, l a degree 3 divisor contained in the 0-cycle $L \cdot C$. Then $a(l) \in W \cap W_\eta \subset \text{Pic}^0(C)$. By (2.4), (2.11) the scheme $W \cdot W_\eta$ contains the points $a(d_i)$, $a(d'_i)$, ($i = 1 \cdots 10$), as isolated components. Let Z be possibly some different excess intersection component of $W \cdot W_\eta$. Let us show directly that $Z = 0$: consider in $C(3) \times \text{Pic}^0(C)$ the incidence correspondence $I = \{(d, \xi) / h^0(\xi(d)) \geq 1\}$. The projection $p_1: I \rightarrow C(3)$ has fibre isomorphic to W , ($\forall d \in C(3)$). Hence I is irreducible, $\dim(I) = 6$. The projection $p_2: I \rightarrow \text{Pic}^0(C)$ has fibre $W \cdot W_\xi$ over ξ ($W_\xi = W$ translated by ξ); moreover p_2 is surjective and $\deg(p_2) = W^2 = 20$. Applying Stein factorization to p_i it follows that the number of connected components of $p_2^{-1}(\xi) = W \cap W_\xi$ cannot exceed 20. Hence $W \cdot W_\eta = \sum(a(d_i) + a(d'_i))$, $l = d_i$ or d'_i for some i , $L \in \pi_i^*$ or π'^*_i .

(2.13) **Remark.** Take a smooth Fano model X satisfying the assumption of Theorem 2.12, then a general point p of X is not contained in a trisecant line to X . Hence projecting X from p one obtains a smooth Enriques surface $S \subset \mathbf{P}^4$ of degree 9 and sectional genus 6 which has been blown up in one point. The existence of such an S was previously conjectured. A proof has been recently given in [13] by using a different method.

3. TRISECANTS TO ADJOINTS TO REYE MODELS

To complete the results of the previous section we need to describe $\text{Tris}(X_C)$ in the case $C + K_X =$ very ample Reye polarization. With this purpose we

give an explicit projective construction for X_C which essentially comes from the results of [8, 10]. Nevertheless we present it as a special case of the more general situation to be discussed in §4. Let

$$(3.1) \quad V = \text{vector space}, \quad \dim(V) = t + 1, \quad \mathbf{P}' = \mathbf{P}(V).$$

Consider the Segre product

$$(3.2) \quad \mathbf{P}' \times \mathbf{P}' \subset \mathbf{P}^{\otimes} = \mathbf{P}(V \otimes V)$$

(i.e. the projectivized set of indecomposable vectors of $V \otimes V$) and the involution

$$(3.3) \quad I_t: \mathbf{P}' \times \mathbf{P}' \rightarrow \mathbf{P}' \times \mathbf{P}'$$

induced by the linear map $v_1 \otimes v_2 \rightarrow v_2 \otimes v_1$. Let Q^+ , Q^- be its eigenspaces, they are generated respectively by vectors $(v_1 \otimes v_2 + v_2 \otimes v_1)$ and $(v_1 \otimes v_2 - v_2 \otimes v_1)$. Hence, as usual, we identify Q^+ , Q^- to $\text{Sym}^2 V$, $\wedge^2 V$ by the isomorphisms $(v_1 \otimes v_2 + v_2 \otimes v_1) \rightarrow v_1 v_2$ and $(v_1 \otimes v_2 - v_2 \otimes v_1) \rightarrow v_1 \wedge v_2$. Correspondingly we have in $\mathbf{P}^{\otimes}$ the projectivized eigenspaces

$$(3.4) \quad \mathbf{P}^+ = \mathbf{P}(\text{Sym}^2 V), \quad \mathbf{P}^- = \mathbf{P}\left(\wedge^2 V\right)$$

which are the set of fixed points of I_t . We are interested in the linear projections

$$(3.5) \quad p_+: \mathbf{P}^{\otimes} \rightarrow \mathbf{P}^+, \quad p_-: \mathbf{P}^{\otimes} \rightarrow \mathbf{P}^-$$

respectively of centers $\mathbf{P}^-$, $\mathbf{P}^+$; restricting these projections to $\mathbf{P}' \times \mathbf{P}'$ we get

$$(3.6) \quad \mathbf{P}^- \xleftarrow{p_-} \mathbf{P}' \times \mathbf{P}' \xrightarrow{p_+} \mathbf{P}^+$$

with $p_+(v_1 \otimes v_2) = v_1 v_2$, $p_-(v_1 \otimes v_2) = v_1 \wedge v_2$. Therefore

$$(3.7) \quad p_-(\mathbf{P}' \times \mathbf{P}') = G_t, \quad p_+(\mathbf{P}' \times \mathbf{P}') = \Sigma_t$$

with G_t = Plücker embedding of the grassmannian of lines of $\mathbf{P}^t$, Σ_t = 2-symmetric product of $\mathbf{P}^t$. Since p_+ is the quotient map $\mathbf{P}' \times \mathbf{P}' \rightarrow \mathbf{P}' \times \mathbf{P}' / \langle I_t \rangle$ it turns out that $\text{Sing}(\Sigma_t) = p_+(\Delta)$, where Δ is the diagonal of $\mathbf{P}' \times \mathbf{P}'$; it is known that each point of $\text{Sing}(\Sigma_t)$ occurs with multiplicity $2t - 2$. Finally $\deg \Sigma_t = \frac{1}{2} \binom{2t}{t}$.

Consider the case $t = 3$, then $\mathbf{P}^- = \mathbf{P}^5$, $\mathbf{P}^+ = \mathbf{P}^9$, $\mathbf{P}^{\otimes} = \mathbf{P}^{15}$. Let $\Sigma = \Sigma_3$, $G = G_3$; fix any 5-dimensional projective space

$$(3.8) \quad \Lambda \subset \mathbf{P}^+$$

intersecting properly Σ along the reduced, irreducible surface of degree 10

$$(3.9) \quad X = \Lambda \cdot \Sigma.$$

Assume X is normal with at most rational double points, then

(3.10) **Proposition.** *X is a Fano model. Let C be its polarization, then $C + K_X$ is a Reye polarization.*

Proof. Since X has no point of multiplicity 4, $\text{Sing}(\Sigma) \cap X = \emptyset$. Let $\tilde{X} = p_+^{-1}(X)$, then $p_+: \tilde{X} \rightarrow X$ is an étale double covering and $i = I_t / \tilde{X}$ its associated fixed points free involution; moreover $\tilde{X}$ is normal with at most rational

double points. Note that $\tilde{X}$ is a complete intersection in $\mathbf{P}^3 \times \mathbf{P}^3$ of four hyperplane sections; by Bertini's theorem this implies that $\tilde{X}$ is connected and, since it is normal, irreducible. Finally, by adjunction formula, $\tilde{X}$ is a $K3$ -surface. Since i is fixed-point-free and $X = \tilde{X}/\langle i \rangle$, X is an Enriques surface and, by (1.4), a Fano model.

Now consider the other projection $p_-: \tilde{X} \rightarrow \mathbf{P}^-$; $p_-(\tilde{X})$ is a Fano model of X which is contained in G , hence a Reye congruence. Let C' be its polarization: clearly $C \approx C'$; on the other hand the pull back of $\mathcal{O}_X(C' - C)$ to $\tilde{X}$ is trivial. Since, as an étale double covering of X , $\tilde{X}$ is defined by $\mathcal{O}_X(K_X)$, it follows $C - C' \sim K_X$.

(3.11) **Lemma.** *Let X_C be a smooth Fano model;*

$$(u, v) = (h^1(\mathcal{O}_X(E_i + E'_i - C)), h^1(\mathcal{O}_X(C' - 2E_i))).$$

Then (u, v) must be one of the pairs $(0, 0)$, $(1, 0)$, $(0, 1)$.

Proof. Let for instance $u > 0$; applying Riemann Roch it follows that $C' - E_i - E'_i \sim R$ with R effective, $p_a(R) = 0$. Consider the exact sequence

$$0 \rightarrow H^0(\mathcal{O}_X(-R)) \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_R) \rightarrow H^1(\mathcal{O}_X(-R)) \rightarrow 0,$$

by Serre duality $v = h^1(\mathcal{O}_X(-R))$. Since R is a curve of degree 4 in the smooth model X_C it is easy to check that $v > 0 \Leftrightarrow R$ is not connected. Assume $v > 0$ then $R = A + B$ with A, B effective, $A \cap B = \emptyset$. Since $E_i \cdot (A + B) = 3$ we have $E_i \cdot A$ or $E_i \cdot B \leq 1$. Hence the linear system $|E_i + A + B| = |C' - E'_i|$ is not numerically 2-connected. On the other hand, $|C' - E'_i|$ is base-point-free [14] hence numerically 2-connected [10, 4.3.4, 4.4.1], contradiction. Therefore it must be $v = 0$. Since R is connected and $p_a(R) = 0$, $h^0(\mathcal{O}_X(R)) = 1$. Hence $h^0(\mathcal{O}_X(R)) = h^0(\mathcal{O}_X(C' - E_i - E'_i)) = h^1(\mathcal{O}_X(E_i + E'_i - C)) = u = 1$. A completely similar argument works if we assume $v > 0$.

(3.12) **Lemma.** *Assume $C' = C + K_X$ is a very ample Reye polarization, then $h^1(\mathcal{O}_X(C' - 2E_i)) = 0$, $i = 1 \dots 10$.*

Proof. Let π_i, π'_i be the supporting planes of the cubic curves E_i, E'_i in the smooth Reye model $X_{C'}$; it is very well known that $\pi_i \cap \pi'_i \neq \emptyset$. Hence $h^0(\mathcal{O}_X(C' - E_i - E'_i)) = h^1(\mathcal{O}_X(E_i + E'_i - C)) > 0$ and, by Lemma 3.11, $h^1(\mathcal{O}_X(C' - 2E_i)) = 0$.

(3.13) **Proposition.** *Let X be a smooth Fano model, then the following conditions are equivalent:*

- (1) $h^1(\mathcal{O}_X(E_i + E'_i - C)) = 1$ for some $i = 1 \dots 10$,
- (2) $C \sim 2E + R$, $E =$ isolated elliptic curve, $R =$ indecomposable nodal cycle, $E \cdot R = 3$,
- (3) $C' = C + K_X$ is a Reye polarization,
- (4) $h^1(\mathcal{O}_X(E_i + E'_i - C)) = 1$ for each $i = 1 \dots 10$,
- (5) $X_C = \Lambda \cdot \Sigma$ as in (3.9).

Proof. Consider as in (4.11) $\mathcal{E}_C = \{E \in \text{Div}(X)/E = \text{isolated elliptic curve}, E \cdot C = 3, h^1(\mathcal{O}_X(C' - 2E)) = 0\}$. By Lemmas (3.11), (3.12) the cubic curves E_i, E'_i belong to $\mathcal{E}_C(3)$. Then the equivalence of (1), (2), (3), (4) is just the case $t = 3$ of Theorem (4.12). Finally (3) $\Rightarrow$ (5) by Lemma 5.6 and (5) $\Rightarrow$ (3) by Proposition 3.10.

(3.14) **Proposition.** *Let X be a smooth Fano model polarized by C . Assume $C + K_X$ is Reye then*

- (1) *$\text{Tris}(X)$ is the blowing up of $\mathbf{P}^3$ in 20 points;*
- (2) *the union of trisecant lines to X is a determinantal quartic hypersurface $\Delta = \{\det(d_{ij}) = 0\}$, where (d_{ij}) is a 4×4 symmetric matrix of linear forms.*

Proof. By (3.11), $C + K_X = \text{Reye polarization} \Leftrightarrow X = \Lambda \cdot \Sigma \subset \mathbf{P}(\text{Sym}^2 V)$ as in (3.9). $\mathbf{P}(\text{Sym}^2 V)$ is the parameter space for quadrics Q in $\mathbf{P}(V^*) = \mathbf{P}^3$ and Σ parametrizes rank 2 quadrics. Hence Λ is a 5-dimensional linear system of quadrics of $\mathbf{P}^{3*}$ and

$$X = \Lambda \cdot \Sigma = \{Q \in \Lambda / \text{rank}(Q) = 2\}.$$

Let $p \in \mathbf{P}^{3*}$, $\Lambda_p = \{Q \in \Lambda / p \in \text{Sing}(Q), \dim(\Lambda_p) = d\}$. If $d = 1$ then, counting properly multiplicities, Λ_p contains three rank 2 quadrics and it is a trisecant line to X . In the same way $d = 2 \Rightarrow \Lambda_p \cdot X = \text{plane cubic}$: we know that there are exactly 20 points o_i ($i = 1 \cdots 20$) such that $d = 2$. If $d = 3$ then Λ contains a rank 1 quadric, which is a point of multiplicity 4 for $\Lambda \cdot \Sigma = X$: against our assumptions. Let P be a 3-secant line to X , Δ the quartic hypersurface parametrizing singular quadrics of Λ . Since $\text{Sing}(\Delta) = \Lambda \cdot \Sigma = X$ it follows that $P \subset \Delta$. Since, by definition P is not in X , all the members of P have rank 3 but for three of them having rank 2. Looking at the projective classification of pencil of quadrics one can deduce that, in our case, P has a (unique) singular base point p . This yields a morphism $f: \text{Tris}(X) \rightarrow \mathbf{P}^3$ sending P to p . f blows up $\mathbf{P}^3$ in the points o_i . The union of trisecant lines to X is the quartic hypersurface considered above. This completes the proof: for brevity we omitted some easy details.

4. ENRIQUES SURFACES IN GRASSMANNIANS

In this section we construct projective models X of Enriques surfaces which are contained in the intersection of G_t with a $(2t - 1)$ -space: $X \subset \mathbf{P}^{2t-1} \cap G_t$. The degree of X in $\mathbf{P}^{2t-1}$ is $4t - 2$, the rational equivalence class of X in G_t is $(t, 3t - 2)$. These models are the natural generalization of classical Reye congruences.

Let $\mathcal{H}_t$, $t \geq 3$, be the Hilbert scheme of normal Enriques surfaces of degree $4t - 2$, $\mathcal{R}_t$ the Hilbert scheme of normal Enriques surfaces in G_t having rational equivalence class $(t, 3t - 2)$, $f: \mathcal{R}_t \rightarrow \mathcal{H}_t$ the obvious morphism between these two Hilbert schemes. As a consequence of our generalization we would like to conjecture the following:

- (4.1) $f(\mathcal{R}_t)$ is a closed codimension one subset of $\mathcal{H}_t$ and its points correspond to all nodal Enriques surfaces.

For Fano models ($t = 3$) this is generically true, [8, 10]. Let L be an irreducible polarization on X such that

$$(4.2) \quad L^2 = 4t - 2, \quad \phi_L \text{ is a degree 1 morphism,} \quad X_L \text{ is normal.}$$

(4.3) **Definition.** We say that the pair (X, L) fits in a Reye diagram of index t if the following conditions are satisfied

(1) there exists a commutative diagram

$$\begin{array}{ccccc}
 \mathbf{P}^- & \xrightarrow{p_-} & \mathbf{P}^\otimes & \xrightarrow{p_+} & \mathbf{P}^+ \\
 \cup & & \cup & & \cup \\
 G_t & \xrightarrow{p_-} & \mathbf{P}^t \times \mathbf{P}^t & \xrightarrow{p_+} & \Sigma_t \\
 \cup & & \cup & & \cup \\
 X_{L'} & \xrightarrow{p_-/\tilde{X}} & \tilde{X} & \xrightarrow{p_+/\tilde{X}} & X_L
 \end{array}
 \quad (4.4)$$

where $\tilde{X}$ is the $K3$ cover of X' , $L' = K_X + L$ and for the top arrow we use the same notations of (3.1)–(3.6).

(2) The projections of $\tilde{X}$ on the two factors of $\mathbf{P}^t \times \mathbf{P}^t$ are not contained in a hyperplane.

(4.5) **Proposition.** *Assume $L = 2E + R$, with $E =$ isolated elliptic curve, $R =$ indecomposable nodal cycle, $E \cdot L = t$. Then the pair (X, L) fits in a Reye diagram of index t .*

Proof. $L^2 = 4t - 2$; let $\pi: \tilde{X} \rightarrow X$ be the $K3$ étale double cover of X , $i: \tilde{X} \rightarrow \tilde{X}$ the induced involution, $\tilde{L} = \pi^*L$. Since R is connected π^*R splits

$$\pi^*R = R_1 + R_2,$$

$\pi/R_j: R_j \rightarrow R =$ isomorphism ($j = 1, 2$). This yields two polarizations of degree $2t - 2$ on $\tilde{X}$:

$$L_1 = R_1 + \tilde{E}, \quad L_2 = R_2 + \tilde{E}$$

with $|\tilde{E}| = |\pi^*E| =$ elliptic pencil. Since $h^1(\mathcal{O}_{\tilde{X}}(R_j)) = 0$ the sequence

$$(4.6) \quad 0 \rightarrow H^0(\mathcal{O}_{\tilde{X}}(R_j)) \rightarrow H^0(\mathcal{O}_{\tilde{X}}(L_j)) \rightarrow H^0(\mathcal{O}_{\tilde{E}}(L_j)) \rightarrow 0$$

is exact and $\dim |L_j| = t$. Let $\phi_j: \tilde{X} \rightarrow \mathbf{P}^t$ be the map defined by $|L_j|$, $X_j = \phi_j(\tilde{X})$: by (4.6) and the very ampleness of $\mathcal{O}_{\tilde{E}}(L_j)$, $\phi_j/\tilde{E}$ is an embedding. This easily implies $\deg(\phi_j) = 1$. Then, applying standard properties of linear systems on a $K3$ surface, it follows that ϕ_j is a morphism and X_j is normal with at most rational double points. Since $|L_1 + L_2| = |\pi^*L|$ the map $\phi_{\tilde{L}}$ has the same properties; since π is étale and $\pi^*L \sim \tilde{L} \sim \pi^*L'$ the same is still true for $|L|$, $|L'|$ on X . Consider

$$\tilde{X} \xrightarrow{\Psi} \mathbf{P}^t \times \mathbf{P}^t \xrightarrow{\sigma} \mathbf{P}^\otimes$$

($\sigma =$ Segre embedding as in (3.2); $\Psi = (\phi_1 \times \phi_2)$). $\sigma \cdot \Psi$ is the map defined by the vector space $\text{Im}(\mu) \subset H^0(\mathcal{O}_{\tilde{X}}(\tilde{L}))$ where

$$\mu: H^0(\mathcal{O}_{\tilde{X}}(L_1)) \otimes H^0(\mathcal{O}_{\tilde{X}}(L_2)) \rightarrow H^0(\mathcal{O}_{\tilde{X}}(\tilde{L}))$$

is the multiplication map. First we want to show that μ is surjective. Observe that $|\tilde{L} - \tilde{E}|$ has no fixed components (it contains $R_1 + |L_2|$ and $|L_1| + R_2$) and positive self-intersection. Therefore $h^1(\mathcal{O}_{\tilde{X}}(\tilde{L} - \tilde{E})) = 0$ and the sequence

$$(4.7) \quad 0 \rightarrow H^0(\mathcal{O}_{\tilde{X}}(\tilde{L} - \tilde{E})) \rightarrow H^0(\mathcal{O}_{\tilde{X}}(\tilde{L})) \rightarrow H^0(\mathcal{O}_{\tilde{E}}(\tilde{L})) \rightarrow 0$$

is exact. Let $T = H^0(\mathcal{O}_{\tilde{X}}(L_1)) \otimes H^0(\mathcal{O}_{\tilde{X}}(L_2))$, $T_{\tilde{E}} = H^0(\mathcal{O}_{\tilde{E}}(L_1)) \otimes H^0(\mathcal{O}_{\tilde{E}}(L_2))$; using (4.6) we get a homomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & T & \longrightarrow & T_{\tilde{E}} \longrightarrow 0 \\ & & \mu_1 \downarrow & & \mu \downarrow & & \mu_2 \downarrow \\ 0 & \longrightarrow & H^0(\mathcal{O}_{\tilde{X}}(\tilde{L} - \tilde{E})) & \longrightarrow & H^0(\mathcal{O}_{\tilde{X}}(\tilde{L})) & \longrightarrow & H^0(\mathcal{O}_{\tilde{E}}(\tilde{L})) \longrightarrow 0 \end{array}$$

where μ_1, μ_2 are the induced multiplication maps. The surjectivity of μ_2 for two very ample line bundles $\mathcal{O}_{\tilde{E}}(L_j)$ on an elliptic curve is standard. To finish we must show that μ_1 is surjective. Observe that $\dim H^0(\mathcal{O}_{\tilde{X}}(\tilde{L} - \tilde{E})) = 2t$, $\dim K = 2t + 1$. Fix $h_j \in H^0(\mathcal{O}_{\tilde{X}}(L_j))$, $h_j \neq 0$ and vanishing on $\tilde{E}$. Then K is spanned by the vector spaces

$$K_1 = \langle h_1 \rangle \otimes H^0(\mathcal{O}_{\tilde{X}}(L_2)), \quad K_2 = H^0(\mathcal{O}_{\tilde{X}}(L_1)) \otimes \langle h_2 \rangle.$$

Therefore: $w \in \text{Ker}(\mu_1) \Rightarrow (w = h_1 \otimes s_2 + s_1 \otimes h_2 \text{ and } \text{div}(h_1 s_2) = \text{div}(h_2 s_1)) \Rightarrow \tilde{E} + \text{div}(s_2) = \tilde{E} + \text{div}(s_1)$. Since R_1, R_2 are disjoint and we can choose $\tilde{E}$ irreducible, it follows $\text{div}(s_j) = E_j + R_j$; ($E_j \in |\tilde{E}|$). Hence $\text{Ker}(\mu_1)$ is contained in $H^0(\mathcal{O}_{\tilde{X}}(L_1 - R_1)) \otimes H^0(\mathcal{O}_{\tilde{X}}(L_2 - R_2)) \cong H^0(\mathcal{O}_{\tilde{X}}(\tilde{E})) \otimes H^0(\mathcal{O}_{\tilde{X}}(\tilde{E}))$. The multiplication map on the latter vector space has 1-dimensional kernel. Hence $\dim \text{Ker}(\mu_1) = 1$ and μ, μ_1 are surjective. Now it is quite easy to reconstruct diagram (4.4) for the pair (X, L) : since μ is surjective $\text{Ker} \mu$ has codimension $h^0(\mathcal{O}_{\tilde{X}}(\tilde{L})) = 4t$ in $V^* \otimes V^* = H^0(\mathcal{O}_{\tilde{X}}(L_1)) \otimes H^0(\mathcal{O}_{\tilde{X}}(L_2))$.

Moreover $\tilde{X}_{\tilde{L}}$ spans $\mathbf{P}(K)$, with $K = \text{Ker}(\mu)^\perp = \{u \in V \otimes V / h(u) = 0, h \in \text{Ker}(\mu)\}$. Since $i^* L_1 \sim L_2$ we can assume that i is induced by I_t as in (3.3) and that $\mathbf{P}(K)$ is an invariant space of I_t . Let $j: K^* \rightarrow K^*$ be the involution induced by I_t on K^* then, under the restriction isomorphism $K^* \cong H^0(\mathcal{O}_{\tilde{X}}(\tilde{L}))$, the eigenspaces of j are $\pi^* H^0(\mathcal{O}_X(L))$, $\pi^* H^0(\mathcal{O}_X(L'))$. Therefore, with the same notations of (3.4), we have $p_+(\tilde{X}) = X_L$ or $X_{L'}$, ($p_-(\tilde{X}) = X_L$ or $X_{L'}$). This implies that the condition (1) of (4.3) is satisfied up to showing $\pi^* H^0(\mathcal{O}_X(L)) = +1$ eigenspace of i : we omit for brevity the proof of this last fact. The surjectivity of μ implies that condition (2) of (4.3) is also satisfied.

(4.8) **Corollary.** $h^1(\mathcal{O}_X(L' - 2E)) = 0$.

Proof. Observe that $h^1(\mathcal{O}_X(L' - 2E)) = h^1(\mathcal{O}_X(-R))$ by Serre duality and that the condition $h^1(\mathcal{O}_X(-R)) = 0$ is equivalent to $h^0(\mathcal{O}_R) = 1$.

Let $CH^*(G_t)$ be the Chow ring of G_t ; we recall that $CH^2(G_t)$ is isomorphic to $\mathbf{Z}^2$ and generated by

$$\begin{aligned} \sigma_{11} &= \text{class of } \{l \in G_t / l \subset H\}, & (H \text{ a given hyperplane}), \\ \sigma_{20} &= \{l \in G_t / l \cap M \neq \emptyset\}, & (M \text{ a given codimension 3 subspace}). \end{aligned}$$

Therefore, by the intersection pairing, $CH^2(G_t) \cong CH^{2t-4}(G_t)$ and the rational equivalence class of a surface S in G_t is $(\sigma_{11} \cdot S, \sigma_{20} \cdot S)$.

(4.9) **Corollary.** The rational equivalence class of $X_{L'}$ is $(t, 3t - 2)$.

Proof. Let h_1, h_2 be the obvious generators of $CH^*(\mathbf{P}^t \times \mathbf{P}^t)$; an easy exercise shows that, under the homomorphism of Chow rings

$$p_-^*: CH^*(G_t) \rightarrow CH^*(\mathbf{P}^t \times \mathbf{P}^t),$$

$p_-^* \sigma_{20} = h_1^2 + h_2^2 + h_1 h_2$ and $p_-^* \sigma_{11} = h_1 h_2$. On the other hand one computes $\tilde{X}_{\tilde{L}} \sim (2t-2)(h_1^{t-2} h_2^2 + h_1^2 h_2^{t-2}) + 2t(h_1 h_2)^{t-1}$. Hence $\sigma_{11} \cdot X_{L'} = \frac{1}{2} p_-^* \sigma_{11} \cdot \tilde{X}_{\tilde{L}} = t$, $\sigma_{20} \cdot X_{L'} = \frac{1}{2} p_-^* \sigma_{20} \cdot \tilde{X}_{\tilde{L}} = 3t - 2$.

Let G_t^* be the dual grassmannian of G_t and $g: G_t^* \rightarrow \mathbf{P} = \mathbf{P}H^0(\mathcal{O}_{G_t}(1))$ the map sending $M \in G_t^*$ in the codimension 1 Schubert cycle $\sigma_M = \{l \in G_t/l \cap M \neq \emptyset\}$. g is just the Plücker embedding of G_t^* .

Observe that σ_M is ruled by a pencil of codimension 2 Schubert cycles of class σ_{11} : the elements of this pencil are the grassmannians of lines of the hyperplanes through M .

Assume σ_M does not contain $X_{L'}$: since $\sigma_{11} \cdot X_{L'} = t$ the hyperplane section $L' = \sigma_M \cdot X_{L'}$ is a t -gonal curve. Let $r: H^0(\mathcal{O}_{G_t}(1)) \rightarrow H^0(\mathcal{O}_X(L'))$ be the restriction map; of course r induces a linear projection $p: \mathbf{P} \rightarrow \mathbf{P}H^0(\mathcal{O}_X(L'))$ of center $|\mathcal{I}_X(1)|$, ($\mathcal{I}_X$ = ideal of $X_{L'}$ in G_t). We expect that, for general $X_{L'}$, p/G_t is a birational morphism. However: assume p/G_t is generically finite, then

(4.10) **Corollary.** $|\mathcal{O}_X(L')|$ contains a codimension 1 family of t -gonal curves.

For any polarization L on X let us define

(4.11) $\mathcal{E}_L(s) = \{E/E = \text{isolated elliptic curve}, E \cdot L = s, h^1(\mathcal{O}_X(L' - 2E)) = 0\}$ ($L' = L + K_X$), then

(4.12) **Theorem.** Let L be a polarization on X as in (4.2), $L^2 = 4t - 2 \geq 10$. Assume $\mathcal{E}_L(t) \neq \emptyset$, then the following conditions are equivalent:

- (1) The pair (X, L) fits in a Reye diagram of index t ,
- (2) $h^1(\mathcal{O}_X(E + E' - L)) = 1$ for all $E \in \mathcal{E}_L(t)$,
- (3) $h^1(\mathcal{O}_X(E + E' - L)) = 1$ for some $E \in \mathcal{E}_L(t)$,
- (4) $L \sim 2E + R$ with $E = \text{isolated elliptic curve}$, $R = \text{indecomposable nodal cycle}$, $E \cdot R = t$.

Proof. (1) $\Rightarrow$ (2): let $E \in \mathcal{E}_L(t)$, first we show $h^0(\mathcal{O}_X(L' - E - E')) > 0$. By assumption $X_{L'}$ fits in diagram (4.4), therefore, with the same notations used there, we have that $\tilde{E} = (p_-/\tilde{X})^*(E)$ is a degree $2t$ elliptic curve which is invariant with respect to the involution $I_t(x, y) = (y, x)$. Let $\tilde{E}_i$ be the projection of $\tilde{E}$ on the i th factor of $\mathbf{P}^t \times \mathbf{P}^t$ then $\deg(\tilde{E}_i) = t$ and $\tilde{E}_i$ is contained in a hyperplane. This means $\tilde{E} \subset \{x_0 = y_0 = 0\}$ where $(x_0: \dots: x_t) \times (y_0: \dots: y_t)$ are suitable coordinates on $\mathbf{P}^t \times \mathbf{P}^t$. The same holds for $\tilde{E}' = (p_-/\tilde{X})^*(E')$. Assume we have again $\tilde{E}' \subset \{x_0 = y_0 = 0\}$. Let $H = \{x_0 y_0 = 0\}$; note that, by (4.3)(2), $\tilde{X}$ is not in H and that H is $+1$ -invariant with respect to I_t . Therefore $H \cdot \tilde{X}$ would be the pull back of an element of $|\mathcal{O}_X(L)|$ and $h^0(\mathcal{O}_X(L - E - E')) > 0$. Since $h^0(\mathcal{O}_X(L - E - E')) = h^1(\mathcal{O}_X(L' - 2E)) = 0$ this is impossible. Hence, up to changing coordinates, we can assume $\tilde{E} \subset \{x_0 = y_0 = 0\}$, $\tilde{E}' \subset \{x_1 = y_1 = 0\}$. Finally consider $\tilde{\sigma} = \{x_0 y_1 - x_1 y_0 = 0\}$ and assume $\tilde{X} \subset \tilde{\sigma}$. Then the rational function on $\tilde{X}$: $\tilde{f} = x_1/x_0 = y_1/y_0$

is $+1$ -invariant with respect to the fixed-point-free involution of $\tilde{X}$. On the other hand $\tilde{f} = (p_-/\tilde{X})^* f$ where $\text{div}(\tilde{f}) = E - E'$. As is well known the pull back of f is -1 -invariant, hence $\tilde{X}$ is not in $\tilde{\sigma}$. Note that $\tilde{\sigma} = p_-^*(\sigma)$, where σ is a hyperplane section of G_t (more precisely: a codimension 1 Schubert cycle). Hence $X \cdot \sigma = R + E + E'$, $h^0(\mathcal{O}_X(L' - E - E')) > 0$. To show $h^1(\mathcal{O}_X(E + E' - L)) = 1$ first observe that $h^0(\mathcal{O}_X(R)) = h^1(\mathcal{O}_X(E + E' - L))$. Moreover $h^1(\mathcal{O}_X(-R)) = h^1(\mathcal{O}_X(L' - 2E)) = 0 \Rightarrow h^0(\mathcal{O}_R) = 1 \Rightarrow R$ is connected. Then, since $p_a(R) = 0$, $h^0(\mathcal{O}_X(R)) = 1$.

(2) $\Rightarrow$ (3): obvious.

(3) $\Rightarrow$ (4): $h^1(\mathcal{O}_X(E + E' - L)) = 1 \Rightarrow L' - E - E' \sim R$, with R effective, $p_a(R) = 0$, $E \cdot R = t$. The condition $h^0(\mathcal{O}_R) = 1$ can be shown as above.

(4) $\Rightarrow$ (1) is Proposition 4.5.

(4.13) **Definition.** Let L be a polarization on X as in (4.2). Assume (i) $\mathcal{E}_L(t) \neq \emptyset$, (ii) the equivalent conditions of Theorem (4.13) are satisfied. Then we say that $L' = L + K_X$ is a Reye polarization of index t and $X_{L'}$ a Reye congruence of index t .

5. REYE CONSTRUCTIONS

The original construction of Reye congruences in G_3 was given as follows:

(5.1) Let W be a general 3-dimensional linear system of quadrics of $\mathbf{P}^3$ and

$$X = \{l \in G_3 / l \text{ is included in two quadrics of } W\}$$

then X is a Reye congruence and, conversely, all Reye congruences arise in this way. At the end we want to point out that (5.1) generalizes again to Reye congruences in G_t :

(5.2) **Theorem.** Let X be a Reye congruence of index t . Then there exists a linear system W of quadrics of $\mathbf{P}^t$ such that

(i) $\dim W = \binom{t}{2}$;

(ii) $X = \{l \in G_t \cap \mathbf{P}^{2t-1} / \text{codim}(W_l, W) = 2\}$;

where $W_l = \{Q \in W / l \subset Q\}$, $\mathbf{P}^{2t-1} = \text{linear span of } X \text{ in the Plücker space of } G_t$.

To show (5.2) we first consider the following situation: $E = \text{elliptic curve}$; $\mathcal{O}_E(L_1)$, $\mathcal{O}_E(L_2) = \text{line bundles of degree } t \geq 3 \text{ on } E$. Assume $L_1 \approx L_2$ and consider

$$E \xrightarrow{\gamma_1 \times \gamma_2} \mathbf{P}^{t-1} \times \mathbf{P}^{t-1} \xrightarrow{\sigma} \mathbf{P}^{t^2-1}$$

with $\sigma = \text{Segre inclusion}$ and $\gamma_j = \text{map associated to } |L_j|$. Let $E' = \sigma \cdot (\gamma_1 \times \gamma_2)(E)$, then

(5.3) **Lemma.** Let Λ be the $(2t-1)$ -space spanned by E' in the ambient space of $\mathbf{P}^{t-1} \times \mathbf{P}^{t-1}$ then $E' = \Lambda \cap \mathbf{P}^{t-1} \times \mathbf{P}^{t-1}$.

Proof. For brevity we will leave some details as an exercise. Consider the multiplication map $\mu: H^0(\mathcal{O}_E(L_1)) \otimes H^0(\mathcal{O}_E(L_2)) \rightarrow H^0(\mathcal{O}_E(L_1 + L_2))$. Since μ is surjective $\dim \text{Ker}(\mu) = t^2 - 2t$ and E' is contained in $t^2 - 2t$ linearly independent hyperplanes of $\mathbf{P}^{t^2-1}$. Let Λ be the intersection of them and

$$Z = \Lambda \cap \mathbf{P}^{t-1} \times \mathbf{P}^{t-1}.$$

We have to show that $E' = Z$: this is done by induction on t .

$t = 3$: in this case $\Lambda = \mathbf{P}^5$, E' = sextic curve. Note that the Segre variety $\mathbf{P}^2 \times \mathbf{P}^2$ is a 4-fold of degree 6 in $\mathbf{P}^8$. Assume $E' \subset Z$, then, by Bézout theorem and its corollaries, $\dim Z \geq 2$ and there exists an irreducible component Y of Z which contains properly E' . Let h_j ($j = 1, 2$) be the obvious generators of $CH^*(\mathbf{P}^2 \times \mathbf{P}^2)$, $\pi_j: \mathbf{P}^2 \times \mathbf{P}^2 \rightarrow \mathbf{P}^2$ the canonical projections. If $\dim Y = 2$ then $\deg(Y) \leq 5$; assume $\dim(Y) = 2$, $\deg(Y) = 5$ or $\dim(Y) = 3$ then $Y \sim h_j^2 + 2h_1h_2$ or $Y \sim h_j$. In both cases one can check that $\pi_j(Y)$ is a line. Since $\pi_j(E') \subset \pi_j(Y)$ it follows that $\pi_j(E')$ does not span $\mathbf{P}^2$: a contradiction. Hence $\dim(Y) = 2$, $\deg(Y) = 4$; this implies $Y \sim h_1^2 + h_2^2 + h_1h_2$, Y = Veronese surface. Hence Y is the graph of a projective isomorphism $f: \mathbf{P}^2 \rightarrow \mathbf{P}^2$ such that $f(\pi_1(E')) = \pi_2(E')$. Since $L_1 \approx L_2$ this is again a contradiction. Therefore $E' = \Lambda \cap \mathbf{P}^2 \times \mathbf{P}^2$.

$t > 3$: let $p = (p_1, p_2) \in E'$, $\phi_i: \mathbf{P}^{t-1} \rightarrow \mathbf{P}^{t-2}$ the projection from p_i and

$$\phi = \phi_1 \times \phi_2: \mathbf{P}^{t-1} \times \mathbf{P}^{t-1} \rightarrow \mathbf{P}^{t-2} \times \mathbf{P}^{t-2}.$$

The fundamental locus of ϕ is $\{p_1\} \times \mathbf{P}^{t-1} \cup \mathbf{P}^{t-1} \times \{p_2\}$. Let Γ be the linear span of this set,

$$\gamma: \mathbf{P}^{t^2-1} \rightarrow \mathbf{P}^{(t-1)^2-1}$$

the projection from Γ ; then $\sigma \cdot \phi = \gamma/\mathbf{P}^{t-1} \times \mathbf{P}^{t-1}$ (with σ = Segre embedding of $\mathbf{P}^{t-2} \times \mathbf{P}^{t-2}$). Now observe the following:

- (i) $\phi/E' = \psi_1 \times \psi_2$ with ψ_i = morphism associated to $|L_i - p|$;
- (ii) γ/Λ is the projection from l = tangent line to E' at p and $\Gamma \cap \Lambda = l$.

Let $\Lambda' = \gamma(\Lambda)$, $E'' = \phi(E')$; then Λ' is the linear span of E'' and, by (i) and the induction

$$E'' = \Lambda' \cap \mathbf{P}^{t-2} \times \mathbf{P}^{t-2}.$$

Therefore, to complete the proof of our statement, it remains only to show that

$$(5.4) \quad \forall q \in E' \quad \overline{(\gamma/\Lambda)^*(\phi(q))} \cdot \overline{\phi^*(\phi(q))} = p + q$$

(the overline denoting Zariski closure). Now $\overline{(\gamma/\Lambda)^*(\phi(q))} = \alpha_q$ = plane containing l and q ; while $\overline{\phi^*(\phi(q))} = l_1 \times l_2$, with l_i = line through p_i . The Segre map embeds $l_1 \times l_2$ in $\mathbf{P}^{t^2-1}$ as a smooth quadric surface S_q . Let P_q be the 3-space of S_q , l_q the line joining p to q ; then $l_q \subseteq \alpha_q \cap S_q$. Clearly (5.4) holds if and only if: (1) l_q is not in S_q , (2) α_q is not in P_q (i.e. $l_q = \alpha_q \cap P_q$).

Assume $l_q \subset S_q$ then l_q is $\{p_1\} \times l_1$ or $l_1 \times \{p_2\}$ and $\pi_1(l_q)$ or $\pi_2(l_q)$ is a point; since l_q is bisecant to E' and $\pi_i: E' \rightarrow \mathbf{P}^{t-1}$ is an embedding we get a contradiction. Hence (1) holds.

Assume $l \subset P_q$; by (ii) $l \subset \Gamma \cap P_q$ which is just the plane of the lines $\{p_1\} \times l_2$, $l_1 \times \{p_2\}$ (i.e. the tangent plane to S_q at p). Hence the image of l by the tangent map $d(\pi)_p$ is l_i . This implies l_i = tangent line to $\pi_i(E')$ at p_i and $p = q$. Therefore $p \neq q \Rightarrow l$ not in $P_q \Rightarrow \alpha_q$ not in P_q and (2) holds for $p \neq q$. After some tedious remarks one can show (2) also in the case $p = q$.

(5.5) *Remark.* With some more effort one can show the same for any divisor of canonical type.

Now let X be an Enriques surface, $L' = L + K_X$ a Reye polarization of index t on it; by (4.4) the pair (X, L) defines the Reye diagram

$$\begin{array}{ccccc}
 \mathbf{P}^- & \xrightarrow{p_-} & \mathbf{P}^\otimes & \xrightarrow{p_+} & \mathbf{P}^+ \\
 \cup & & \cup & & \cup \\
 G_t & \xleftarrow{p_-} & \mathbf{P}^t \times \mathbf{P}^t & \xrightarrow{p_+} & \Sigma_t \\
 \cup & & \cup & & \cup \\
 X_{L'} & \xrightarrow{p_-/\tilde{X}} & \tilde{X} & \xrightarrow{p_+/\tilde{X}} & X_L
 \end{array}$$

we have

(5.6) **Lemma.** *Let $\tilde{\Lambda}$ be the linear span of $\tilde{X}$ in $\mathbf{P}^\otimes$, then $\tilde{X} = \tilde{\Lambda} \cap \mathbf{P}^t \times \mathbf{P}^t$.*

Proof. Let $\pi_j: \mathbf{P}^t \times \mathbf{P}^t \rightarrow \mathbf{P}^t$ be the canonical projections ($j = 1, 2$), $X_j = \pi_j(\tilde{X})$. By (4.12) $L \sim 2E + R$, with E = isolated elliptic curve, R = indecomposable nodal cycle and $E \cdot R = t$. As in the proof of (4.5) consider $|\tilde{E}| = |\pi^*E|$, $R_1 + R_2 = \pi^*R$: we know that, for the fixed points free involution i on $\tilde{X}$, $i = I_t/\tilde{X}$, where I_t is the involution on $\mathbf{P}^t \times \mathbf{P}^t$ sending (x, y) in (y, x) ; moreover $\mathcal{O}_{X_j}(1) \cong \mathcal{O}_{\tilde{X}}(\tilde{E} + R_j)$. Therefore we can fix projective coordinates $(x_0 \cdots x_t) \times (y_0 \cdots y_t)$ on $\mathbf{P}^t \times \mathbf{P}^t$ such that $i = I_t/\tilde{X}$ and the divisor associated to the rational function x_0/x_1 on X_1 is $F_0 - F_1$, ($F_0, F_1 \in |\tilde{E}|$). Since x_0/x_1 is anti-invariant we have $-x_0/x_1 = i^*(x_0/x_1) = y_0/y_1$, so that $\tilde{X} \subset H$ where the equation of H in $\mathbf{P}^t \times \mathbf{P}^t$ is $x_0y_1 + x_1y_0 = 0$. H contains the pencil of divisors

$$H_z = \{z_0x_0 + z_1x_1 = z_0y_0 - z_1y_1 = 0\}$$

with $z = (z_0, z_1) \in \mathbf{P}^1$. Observe that $H_z \cap \tilde{X} \supset \tilde{E}_z$ for some $\tilde{E}_z \in |\tilde{E}|$ and that $H_z = \mathbf{P}^{t-1} \times \mathbf{P}^{t-1}$. Then consider $\mathcal{O}_{\tilde{E}_z}(L_j) = \mathcal{O}_{\tilde{E}_z}(\tilde{E} + R_j)$: the embedding of $\tilde{E}_z$ in H_z is exactly the one described above in Lemma (5.3); moreover, applying standard exact sequences, one easily shows $\mathcal{O}_{\tilde{E}_z}(L_1) \neq \mathcal{O}_{\tilde{E}_z}(L_2)$. Therefore, by Lemma (5.4), we obtain

$$\tilde{E}_z = \Lambda_z \cap H_z,$$

Λ_z being the linear span of $\tilde{E}_z$. Finally $\tilde{E}_z = \tilde{\Lambda} \cap H_z$ ($\forall z$) $\Rightarrow \tilde{X} = \tilde{\Lambda} \cap H \Rightarrow \tilde{X} = \tilde{\Lambda} \cap \mathbf{P}^t \times \mathbf{P}^t$ (because H is a hyperplane section of $\mathbf{P}^t \times \mathbf{P}^t$).

Let $\mathcal{I}_{\tilde{X}}$ be the ideal of $\tilde{X}$ in $\mathbf{P}^t \times \mathbf{P}^t$, J the ideal of $\tilde{E}_z$ in H_z ; consider the natural restriction map

$$r: H^0(\mathcal{I}_{\tilde{X}}(1)) \rightarrow H^0(\mathcal{I}(1));$$

By Lemma (5.3), to show $\tilde{E}_z = \tilde{\Lambda} \cap H_z$, it suffices to show that r is surjective. For this let $\mathcal{I}_z$ be the ideal of H_z in $\mathbf{P}^t \times \mathbf{P}^t$; observe that $H^1(\mathcal{I}_{\tilde{X}}(1)) = H^1(\mathcal{I}_z(1)) = 0$ because $\tilde{X}$ and H_z are linearly normal. For the same reason

$H^1(\mathcal{O}_{\tilde{X}}(\tilde{L} - \tilde{E})) = 0$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & K & \rightarrow & H^0(\mathcal{I}_{\tilde{X}}(1)) & \xrightarrow{r} & H^0(\mathcal{I}(1)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & H^0(\mathcal{I}_z(1)) & \rightarrow & H^0(\mathcal{O}_{\mathbf{P}^t \times \mathbf{P}^t}(1)) & \rightarrow & H^0(\mathcal{O}_{H_z}(1)) \rightarrow 0 \\
 & & \downarrow r' & & \downarrow & & \downarrow \\
 0 & \rightarrow & H^0(\mathcal{O}_{\tilde{X}}(\tilde{L} - \tilde{E})) & \rightarrow & H^0(\mathcal{O}_{\tilde{X}}(\tilde{L})) & \rightarrow & H^0(\mathcal{O}_{\tilde{E}_z}(1)) \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

the second and third rows (columns) of which are exact. By the snake lemma r is surjective iff r' is surjective. One easily computes $h^0(\mathcal{I}_z(1)) = 2t + 1$, $h^0(\mathcal{O}_{\tilde{X}}(\tilde{L} - \tilde{E})) = 2t$. Let $s = x_0 y_1 + x_1 y_0$, since $\text{div}(s) = H$, s belongs to K .

Let s' be another element of K : as for any element of $H^0(\mathcal{I}_z(1))$ we can write $s' = (z_0 x_0 + z_1 x_1)A + (z_0 y_0 - z_1 y_1)B$ (with $A(B) =$ linear form in $(y_0 \cdots y_t)$ (in $(x_0 \cdots x_t)$). Let $H' = \text{div}(s')$; then H_z moves in a pencil $\{H'_t, t \in \mathbf{P}^1\}$ on H' having the same properties of the pencil $\{H_z\}$ of H . In particular, changing t by z , we have $\tilde{E}_z \subset H'_z \forall z$. Let $h_{1,z}, h_{2,z}$ be the equations of H'_z , then, for the two projections of $\tilde{E}_z$, we have $\pi_1(\tilde{E}_z) \subset \{h_{1,z} = 0\} \cap \{z_0 x_0 + z_1 x_1 = 0\}$ and $\pi_2(\tilde{E}_z) \subset \{h_{2,z} = 0\} \cap \{z_0 y_0 - z_1 y_1 = 0\}$. Now there is a unique hyperplane containing $\pi_j(\tilde{E}_z)$ because

$$h^0(\mathcal{O}_{X_j}(1) \otimes \mathcal{O}_{X_j}(-\tilde{E}_x)) = h^0(\mathcal{O}_{X_j}(R)) = 1.$$

Therefore $H_z = H'_z \forall z$, $H = H'$ and $s' = cs$ for some constant c . Hence $\dim(K) = 1$ and the proof is complete.

Finally we can give a

Proof of (5.2). By (5.6) $\tilde{X} = \tilde{\Lambda} \cap \mathbf{P}^t \times \mathbf{P}^t$. Let $\Lambda_+ = p_+(\tilde{\Lambda})$, $\Lambda_- = p_-(\tilde{\Lambda})$. By the Reye diagram $X_{L'} \subset \Lambda_- \cap G_t \subset \mathbf{P}^-$, $X_L \subset \Lambda_+ \cap \Sigma_t \subset \mathbf{P}^+$ and $\dim(\Lambda_-) = \dim(\Lambda_+) = 2t - 1$. We denote by $\mathcal{W}$ the linear system of hyperplanes of $\mathbf{P}^+$ containing Λ_+ . Clearly $\dim(\mathcal{W}) = \dim(\mathbf{P}^+) - \dim(\Lambda_-) - 1 = \binom{t}{2}$.

Recall, (3.4), that $\mathbf{P}^+ = \mathbf{P}(\text{Sym}^2 V^*) = \mathbf{P}H^0(\mathcal{O}_{\mathbf{P}(V)}(2))$ so that $\mathcal{W}$ is a linear system of quadrics of $\mathbf{P}^+ = \mathbf{P}(V)$. Now identify (canonically) $\text{Sym}^2(V^*)$ to the vector space of symmetric bilinear maps on $V \times V$; then denote by $\widehat{\mathcal{W}}$ the subvector space whose projectivization is $\mathcal{W}$. Let $x \in \mathbf{P}^t \times \mathbf{P}^t$, x not in the diagonal. x corresponds (up to scalars) to a pair (u_1, u_2) of linearly independent vectors of V ; moreover the line $u = p_-(x)$ is the projectivization of the vector space $U = \langle u_1, u_2 \rangle$.

Observe that, by Lemma (5.6), $x \in \tilde{X}$ if and only if $u \in \Lambda_-$ and $q(u_1, u_2) = 0$ for all symmetric bilinear maps $q \in \widehat{\mathcal{W}}$. Let $\widehat{\mathcal{W}}_U = \{q \in \widehat{\mathcal{W}} / U \text{ is isotropic for } q\}$: it is a standard exercise in linear algebra showing that $\text{codim}(\widehat{\mathcal{W}}_U, \widehat{\mathcal{W}}) \leq 2$ if and only if there exist vectors $u_1, u_2 \in U / q(u_1, u_2) = 0 \forall q \in \widehat{\mathcal{W}}$. Hence $X_{L'} = p_-(\tilde{X}) = \{u \in G_t \cap \Lambda_- / \text{codim}(\mathcal{W}_u, \mathcal{W}) \leq 2\}$. Assume $\text{codim}(\mathcal{W}_u, \mathcal{W}) \leq 1$ (for some $u \in X_{L'}$); then $\mathcal{W}$ has a based point p on u . Equivalently (p, p) is contained in each hyperplane of $\mathbf{P}^\otimes$ which contains $\tilde{\Lambda}$ and is $+1$ -invariant

with respect to I_t . On the other hand, since it is in the diagonal of $\mathbf{P}^t \times \mathbf{P}^t$, (p, p) is in all -1 hyperplanes of I_t . Hence $(p, p) \in \tilde{\Lambda}$ and, by Lemma 5.6, $(p, p) \in \tilde{X}$: this is a contradiction because $I_t/\tilde{X}$ is base-point-free. Hence $\text{codim}(W_u, W) = 2$ and the proof is complete.

6. EXISTENCE OF REYE CONGRUENCES OF INDEX $t \geq 3$

Now we want to check for which values of t the Reye construction is possible. Let X be a nodal Enriques surface,

$$\mathcal{R}(X) = \{R \subset X / R = \text{indecomposable nodal cycle}\};$$

$$\mathcal{E}(X) = \{E \subset X / E = \text{isolated elliptic curve}\}, \text{ and}$$

$$(6.1) \quad m: \mathcal{R}(X) \times \mathcal{E}(X) \rightarrow \mathbf{N}$$

the intersection product. By (4.12) X admits a Reye polarization of index t if and only if $t \in \text{Im}(m)$. In this section we show that actually $\text{Im}(m) = \mathbf{N}$ on a general nodal X . To make precise what we mean by general we give the following

(6.2) **Definition** [10, vol. II]. A nodal Enriques surface X is said to be generic if all the elements of $\mathcal{R}(X)$ have the same numerical equivalence class modulo $2L$, $L = \text{Num}(X)$.

(6.3) **Definition.** Let X be a nodal Enriques surface, C a Reye polarization on X , we will say that C is good if $C \cdot R \geq 4 \quad \forall R \in \mathcal{R}(X)$.

Note that C is very ample because there is no $R \in \mathcal{R}(X)$ such that $R \cdot C = 0$. Moreover [10, vol. II]:

(6.4) X is generic if and only if X admits a good Reye polarization.

Finally it is known [9, 10] that a general (in sense of moduli) nodal X is generic.

Let C be a good Reye polarization on X ; E_i, E'_i the twenty plane cubics in the smooth Reye model X_C ; it is well known (cf. proof of (3.11)) that

$$(6.5) \quad C - E_i - E'_i \sim R_i$$

with $R_i = \text{indecomposable nodal cycle}$, $C \cdot R_i = 4$. Actually, since C is good and $h^0(\mathcal{O}_X(C - R_i)) = 1$, R_i is a rational normal quartic curve in $\mathbf{P}^4$. Our program is to fix one of these curves, e.g.

$$(6.6) \quad R = R_1$$

and construct a sequence $\{F_t, t \geq 1\}$ of isolated elliptic curves such that $F_t \cdot R = t$. First we construct such a sequence numerically: in the lattice $L = \text{Num}(X)$ we consider all the sets $\{e_1 \cdots e_{10}\}$ of isotropic vectors satisfying $e_i \cdot e_j = 1 - \delta_{ij}$, $\frac{1}{3} \sum e_i \in L$ as in (1.1). Let $c = \frac{1}{3} \sum e_i$, we will say that c is a *Fano vector*. Let $l_{ijk} = c - e_i - e_j - e_k$, (with $i, j, k = \text{distinct elements of } \{1 \cdots 10\}$); we define

$$(6.7) \quad s_{ijk}: L \rightarrow L$$

by $s_{ijk}(v) = v + (v \cdot l_{ijk})l_{ijk}$. Since $l_{ijk}^2 = -2$ s_{ijk} is a reflection and $s_{ijk}(s_{ijk}(v)) = v$. Let $c' = s_{ijk}(c)$. Observe that c' has the following properties:

$$(6.8) \quad c' = 2c - e_i - e_j - e_k.$$

Let $e'_m = s_{ijk}(e_m)$, then $\{e'_1 \cdots e'_{10}\}$ is again a set of isotropic vectors satisfying (1.1), moreover

$$(6.9) \quad \begin{aligned} e'_m &= e_m, \quad m \neq i, j, k, \quad e'_i = c - e_j - e_k, \\ e'_j &= c - e_i - e_k, \quad e'_k = c - e_i - e_j. \end{aligned}$$

In particular $c' = s_{ijk}(c) = \frac{1}{3} \sum e'_i$ is still a Fano vector. Finally, $\forall v \in L$, consider the intersection numbers

$$d = v \cdot c, \quad v_i = v \cdot e_i, \quad d' = v \cdot c', \quad v'_i = e'_i \cdot v;$$

by definition of s_{ijk} we have

$$(6.10) \quad \begin{aligned} d' &= 2d - v_i - v_j - v_k, \quad v'_i = d - v_j - v_k, \quad v'_j = d - v_i - v_k, \\ v'_k &= d - v_i - v_j, \quad v'_m = v_m \quad (m \neq i, j, k). \end{aligned}$$

Now we consider the following functions on L :

$$(6.11) \quad f = s_{178} \cdot s_{256} \cdot s_{234}, \quad g = s_{178} \cdot s_{256} \cdot s_{234} \cdot s_{1910}$$

and

$$(6.12) \quad h_t = g^{t-1} \cdot f, \quad t \geq 1.$$

Then we define in L the elements

$$(6.13) \quad c_t = h_t(c) = \frac{1}{3} \sum e'_i, \quad f_t = e'_{10}, \quad r = c - 2e_1$$

($e'_i = h_t(e_i)$). Let $v_i = r \cdot e_i$, one immediately computes $v_1 = 3$, $v_i = 1$, $i \geq 2$. Then, applying repeatedly the previous formulae (6.10), one computes with some pain:

$$(6.14) \quad \begin{aligned} r \cdot e'_1 &= 2t + 2, \quad r \cdot e'_2 = 2t + 1, \quad r \cdot e'_3 = \cdots = r \cdot e'_8 = t + 1, \\ r \cdot e'_9 &= r \cdot e'_{10} = t \quad \text{and} \quad c_t \cdot r = 4t + 3. \end{aligned}$$

In particular it follows $r \cdot f_t = t$. Since $f_t^2 = 0$, $f_t \cdot e'_1 = 1$, f_t has the numerical properties of the class of an isolated elliptic curve on X . On the other hand $r^2 = -2$ so that, numerically, r can be the class of an indecomposable nodal cycle. Now we shall show that, for a given (nodal) Enriques surface X , one can construct in $\text{Num}(X)$ elements r, f_t which are represented by curves $R \in \mathcal{R}(X)$, $F_t \in \mathcal{E}(X)$. Let

(6.15) **Proposition.** *Let X_C be a smooth Fano model; E_i, E'_i its twenty plane cubics. Fix three distinct $i, j, k \in \{1 \cdots 10\}$ and consider the linear system*

$$|H| = |2C - E_i - E_j - E_k|.$$

Assume there is no line in X_C , then H is a very ample Fano polarization.

Proof. Let $L = \text{Num}(X)$, $s_{ijk}L \rightarrow L$ be the reflection considered in (6.7), c, h the numerical classes of C, H ; note that s_{ijk} is an isometry of L and that, by definition, $s_{ijk}(c) = h$. Therefore H is constructed as in (1.2) from ten isotropic vectors satisfying (1.1). By (1.3) $|H|$ irreducible $\Rightarrow \phi_H$ is a birational morphism. Since H has positive self-intersection the irreducibility of $|H|$ is equivalent to $h^1(\mathcal{O}_X(H)) = 0$ [10]. Consider the exact sequence

$$(6.16) \quad 0 \rightarrow \mathcal{O}_X(H - C) \rightarrow \mathcal{O}_X(H) \rightarrow \mathcal{O}_C(H) \rightarrow 0.$$

Since $C \cdot (H - C) = 1$ and X_C does not contain lines, it follows that $h^0(\mathcal{O}_X(H - C)) = 0$. Since $C \cdot (C - H) = -1$, it follows that

$$h^0(\mathcal{O}_X(K_X + C - H)) = h^2(\mathcal{O}_X(H - C)) = 0.$$

Then, by Riemann-Roch, $h^1(\mathcal{O}_X(H - C)) = 0$. Furthermore $\mathcal{O}_C(H)$ is a degree 11 line bundle on a genus 6 curve so that $h^1(\mathcal{O}_C(H)) = 0$. Hence, passing to the long exact sequence associated to (6.16), we obtain $h^1(\mathcal{O}_X(H)) = 0$ and $|H|$ irreducible. By (1.3) X_H has at most rational double points.

Let $l, m \in \{1 \cdots 10\}$, $l \neq m$, $F_{lm} = C - E_l - E_m$, and consider the exact sequence

$$(6.17) \quad 0 \rightarrow \mathcal{O}_X(F_{lm} - E_n) \rightarrow \mathcal{O}_X(F_{lm}) \rightarrow \mathcal{O}_{E_n}(F_{lm}) \rightarrow 0$$

($n \neq l, m$). Observe that, with exactly the same proof used for $H - C$, $h^0(\mathcal{O}_X(F_{lm} - E_n)) = h^1(\mathcal{O}_X(F_{lm} - E_n)) = 0$. Then, since $h^0(\mathcal{O}_{E_n}(F_{lm})) = 1$, it follows that F_{lm} is an isolated curve in X_C . The degree of F_{lm} is 4 ($C \cdot F_{lm} = 4$) and the arithmetic genus 1 ($F_{lm}^2 = 0$). We can show that F_{lm} is a nef divisor: since it is isolated F_{lm} is not nef if and only if $D \cdot F_{lm} < 0$ for some component $D \subset F_{lm}$. Now F_{lm} is a quartic curve and X_C does not contain lines; therefore either F_{lm} is irreducible and nef because $F_{lm}^2 = 0$ or $F_{lm} = D_1 + D_2$ with D_i smooth conic. In this latter case $D_i^2 = -2$ so that $(D_1 + D_2)^2 = 0 \Rightarrow D_1 \cdot D_2 = 2 \Rightarrow D_i \cdot F_{lm} = 0$; hence F_{lm} is nef. Now assume H is not very ample, then X_H has a rational double point and $H \cdot R = 0$ for some indecomposable nodal cycle R . Observe that

$$(6.18) \quad H \sim F_{xy} + F_{zs} + E_s, \quad s \neq z,$$

where (x, y, z) is any permutation of (i, j, k) ; then $R \cdot H = 0 \Rightarrow R \cdot (F_{xy} + F_{zs} + E_s) = 0$ and, since E_s is also nef, $F_{xy} \cdot R = F_{zs} \cdot R = E_s \cdot R = 0$. In particular, $E_s \cdot R = 0 \quad \forall s = 1 \cdots 10$: on the other hand $C \sim \frac{1}{3} \sum E_s$ so that $C \cdot R = 0$: a contradiction because C is very ample. Hence H is very ample too.

Now let us fix

$$(6.19) \quad \begin{aligned} X &= \text{good nodal Enriques surface,} \\ C &= \text{good Reye polarization on } X, \\ E_i, E'_i &= \text{the 20 plane cubics in } X_C. \end{aligned}$$

As above let e_i = numerical class of E_i , $c = \frac{1}{3} \sum e_i$ = class of C , $r = c - 2e_1$ = class of the rational quartic $R \subset X_C$ as in (6.6). Consider as in (6.13) the Fano vectors $c_t = \frac{1}{3} \sum e'_i = h_t(c)$ and $f_t = e'_{10}$, then

(6.20) **Proposition.** (i) c_t is the numerical equivalence class of a very ample Fano polarization C_t ;

(ii) X_{C_t} does not contain lines.

Proof. By induction on t ($t = 1$). By definition

$$h_1 = f = s_{178} \cdot s_{256} \cdot s_{234}.$$

Let $c = \frac{1}{3} \sum e_i$ be as above, we construct from c the Fano vectors

$$\begin{aligned} a &= s_{234}(c) = \frac{1}{3} \sum e_i^a, & b &= s_{256}(a) = \frac{1}{3} \sum e_i^b, \\ c_1 &= s_{178}(b) = \frac{1}{3} \sum e_i^1 = h_1(c). \end{aligned}$$

Let $r = c - 2e_1$, and consider the intersection numbers

$$x_i = r \cdot e_i, \quad a_i = r \cdot e_i^a, \quad b_i = r \cdot e_i^b, \quad y_i = r \cdot e_i^1$$

and recall that $x_1 = 3$, $x_i = 1$, $i \geq 2$ (cf. (6.13)). Then, applying the formulae (6.10) to s_{234} , s_{256} , s_{178} , it is easy to compute

$$(6.21) \quad \begin{aligned} (a_1 \cdots a_{10}) &= (3, 2, 2, 2, 1, 1, 1, 1, 1, 1), & a \cdot r &= 5, \\ (b_1 \cdots b_{10}) &= (3, 3, 2, 2, 2, 2, 1, 1, 1, 1), & b \cdot r &= 6, \\ (y_1 \cdots y_{10}) &= (4, 3, 2, 2, 2, 2, 2, 2, 1, 1), & c_1 \cdot r &= 7. \end{aligned}$$

a is the class of $A = 2C - E_2 - E_3 - E_4$. Since C is good A is very ample by Proposition (6.15). Let $L \subset X_A$ be a line, l its numerical class. Since X is good $l - r$ is divisible by 2 in $\text{Num}(X)$; then $l \cdot e_i^a = a_i \bmod 2$ and, in particular, $l \cdot e_i^a$ is odd for $i \neq 2, 3, 4$. On the other hand $l \cdot e_i^a \geq 0$ because e_i^a represents a nef divisor, (since A is very ample $2e_i^a$ corresponds to an elliptic pencil, cf. (1.4)). Then we compute $2 < \frac{1}{3} \sum e_i^a \cdot l = a \cdot l = 1$. Hence X_A cannot contain lines. Now consider b : b is the class of $B = 2A - E_2^a - E_3^a - E_6^a$ which is again a very ample Fano polarization by Proposition (6.15). Applying exactly the same arguments used for X_A one shows that X_B does not contain lines. So, by (6.15) again, c_1 is the class of the very ample Fano polarization $C_1 = 2B - E_1^b - E_7^b - E_8^b$.

To complete the first step of induction we must show that X_{C_1} does not contain lines: assume L is a line (of numerical class l) in X_{C_1} , let $l_i = e_i^1 \cdot l$. Since X is good $l_i = y_i \bmod 2$. Then, using (6.21) and $l \cdot c_1 = 1$, one obtains

$$(l_1 \cdots l_{10}) = (0, 1, 0, 0, 0, 0, 0, 0, 1, 1).$$

Note that $s_{178}^2 = \text{id}$ so that $s_{178}(e_i^1) = e_i^b$. Let $l'_i = e_i^b \cdot l$; then, applying formulae (6.10), it is not difficult to compute $(l'_1 \cdots l'_{10}) = (1, 1, 0, 0, 0, 0, 1, 1, 1, 1)$ so that $b \cdot l = 3$: since $b \cdot r = 6$ and X is good this is a contradiction. Hence there is no line in X_{C_1} .

($t > 1$) This time we start with $c_t = \frac{1}{3} \sum e_i^t$ and we assume $c_t = \text{class of } C_t$, where X_{C_t} is a smooth Fano model not containing lines. Recall that $c_{t+1} = h_t(c) = g(c_t)$ with $g = s_{178} \cdot s_{256} \cdot s_{243} \cdot s_{1910}$ (cf. (6.13)). Then consider the Fano vectors:

$$\begin{aligned} d &= \frac{1}{3} \sum e_i^d = s_{1910}(c_t), & m &= \frac{1}{3} \sum e_i^m = s_{234}(d), \\ n &= \frac{1}{3} \sum e_i^n = s_{256}(m), & c_{t+1} &= \frac{1}{3} \sum e_i^{t+1} = s_{178}(n), \end{aligned}$$

and the intersection numbers

$$u_i = e_i^t \cdot r, \quad d_i = e_i^d \cdot r, \quad m_i = e_i^m \cdot r, \quad n_i = e_i^n \cdot r, \quad v_i = e_i^{t+1} \cdot r$$

(r as above). From (6.14) we know that

$$u_1 = 2t + 2, \quad u_2 = 2t + 1, \quad u_3 = \cdots = u_8 = t + 1, \quad u_9 = u_{10} = t$$

and that $c_t \cdot r = 4t + 3$. Therefore, using the formulae (6.10), we can explicitly

compute

$$\begin{aligned}
 (d_1 \cdots d_{10}) &= (2t+3, 2t+1, t+1, \dots, t+1), \\
 (m_1 \cdots m_{10}) &= (2t+3, 2t+2, t+2, t+2, t+1, \dots, t+1), \\
 (6.22) \quad (n_1 \cdots n_{10}) &= (2t+3, 2t+3, t+2, t+2, \\
 &\quad t+2, t+2, t+1, \dots, t+1), \\
 (v_1 \cdots v_{10}) &= (2t+4, 2t+3, t+2, \dots, t+2, t+1, t+1).
 \end{aligned}$$

Now the proof goes as in the case $t = 1$: let L be any element of $\mathcal{R}(X)$, l its class in $\text{Num}(X)$. By (6.15) $D = 2C_t - E_1^t - E_9^t - E_{10}^t$ is a very ample Fano polarization of class d . Let $l_i^d = l \cdot e_i^d$, then $l_i^d \geq 0$ because e_i^d represents a nef divisor. Since X is good one computes from (6.22): t even $\Rightarrow (l_1^d, \dots, l_{10}^d) = (1, \dots, 1) \bmod 2 \Rightarrow l \cdot d \geq 3$; t odd $\Rightarrow (l_1^d, \dots, l_{10}^d) = (1, 1, 0, \dots, 0) \bmod 2 \Rightarrow l \cdot d \geq 2$. This implies that there is no line in X_D and that $M = 2D - E_2^d - E_3^d - E_4^d$ is a very ample Fano polarization representing m . Now we just go on in the same way: let $l_i^m = l \cdot e_i^m$; for the same reasons as above we have $l_i^m \geq 0$ and

$$\begin{aligned}
 (l_1^m \cdots l_{10}^m) &= (1, 0, 0, 0, 1, 1, 1, 1, 1, 1) \bmod 2, \quad (t \text{ even}), \\
 (l_1^m \cdots l_{10}^m) &= (1, 0, 1, 1, 0, 0, 0, 0, 0, 0) \bmod 2, \quad (t \text{ odd}).
 \end{aligned}$$

This implies that $l \cdot m = 1$ if and only if t is odd and

$$(l_1^m \cdots l_{10}^m) = (1, 0, 1, 1, 0, 0, 0, 0, 0, 0).$$

But now, if such an l exists, we have $l \cdot s_{234}(e_2^m) = l \cdot (m - e_3^m - e_4^m) = -1$; impossible because $s_{234}(e_2^m) = e_2^d = \text{class of plane cubic in } X_D = \text{class of a nef divisor}$. Therefore $N = 2M - E_2^m - E_5^m - E_6^m$ is a very ample Fano polarization representing n . Let $l_i^n = l \cdot e_i^n$ then $l_i^n \geq 0$ and we have this time

$$\begin{aligned}
 (l_1^n \cdots l_{10}^n) &= (1, 1, 0, 0, 0, 0, 1, 1, 1, 1) \bmod 2, \quad (t \text{ even}), \\
 (l_1^n \cdots l_{10}^n) &= (1, 1, 1, 1, 1, 1, 0, 0, 0, 0) \bmod 2, \quad (t \text{ odd}).
 \end{aligned}$$

Clearly $l \cdot n \geq 2$ so that X_N does not contain lines. Finally we obtain from this the very ample Fano polarization $C_{t+1} = 2N - E_1^n - E_7^n - E_8^n$ of class c_{t+1} . Let $l_i = c_{t+1} \cdot l$, then

$$\begin{aligned}
 (l_1 \cdots l_{10}) &= (0, 1, 0, 0, 0, 0, 0, 0, 1, 1) \bmod 2, \quad (t \text{ even}), \\
 (l_1 \cdots l_{10}) &= (1, 1, 1, 1, 1, 1, 0, 0, 0, 0) \bmod 2, \quad (t \text{ odd}).
 \end{aligned}$$

If t is odd $l \cdot c_{t+1} \geq 2$ and $X_{C_{t+1}}$ does not contain lines; if t is even it is completely clear how to complete the proof using the previous arguments.

Finally we can show

(6.22) **Theorem.** *Let X be a good nodal Enriques surface, $m: \mathcal{E}(X) \times \mathcal{R}(X) \rightarrow \mathbb{N}$ the intersection map considered in (6.1). Then*

- (1) m is surjective;
- (2) X admits a Reye polarization of index t for each $t \geq 3$.

Proof. Fix on X a good Reye polarization C and its numerical class $c = \frac{1}{3} \sum e_i$. Then, as in (6.13), reconstruct from c the Fano vectors

$$c_t = \frac{1}{3} \sum e_i^t \quad (t \geq 1)$$

and consider also $r = c - 2e_1$, $f_t = e_{10}^t$. By (6.6) r is represented by an element of $\mathcal{R}(X)$ (a rational quartic curve R in X_C). By (6.14) $r \cdot f_t = t$. By Proposition (6.20) c_t is the class of a very ample Fano polarization C_t . Hence, by (1.4), f_t is represented by $F_t =$ plane cubic curve in X_{C_t} = isolated elliptic curve on X . Therefore $F_t \in \mathcal{E}(X)$ and $m(F_t, R) = t$, $t \geq 1$. On the other hand, it is a standard fact that for a given $R \in \mathcal{R}(X)$ there exists $E \in \mathcal{E}(X)$ such that $E \cdot R = 0$. Therefore m is surjective. Finally, by (4.12), the surjectivity of m implies (2).

(6.23) *Remark* (Cremona transformations of $\mathbf{P}^5$). We want to explain without proofs the true geometric construction underlying the numerical arguments used in the section. Fix three distinct planes π_i, π_j, π_k in $\mathbf{P}^5$ such that: (i) there is no hyperplane containing all of them, (ii) any two of them intersect exactly in one point. Then consider the linear system Σ of the quadrics containing $\pi_i \cup \pi_j \cup \pi_k$: Σ is 5-dimensional and defines a birational transformation $s_{ijk}: \mathbf{P}^5 \rightarrow \mathbf{P}^5$, this is called a standard Cremona transformation of $\mathbf{P}^5$. Under s_{ijk} each π_m ($m = i, j, k$) is blown up to a hyperplane, while the hyperplanes containing any two of π_i, π_j, π_k are contracted to three new planes satisfying (i), (ii). Taking them and constructing the corresponding Cremona transformation we obtain the inverse of s_{ijk} . Let $X_C \subset \mathbf{P}^5$ be a smooth Fano model, X_C not contained in a quadric, (note that this is always possible up to replacing C by $C + K_X$). Assume that three plane cubics of X_C , e.g. E_i, E_j, E_k , are contained in π_i, π_j, π_k ; then s_{ijk}/X_C is the morphism associated to $|2C - E_i - E_j - E_k|$ and, at least if there is no line in X_C , $s_{ijk}(X_C)$ is a new Fano model by (6.18). Therefore, thinking more geometrically, the reflections $s_{ijk}: \text{Num}(X) \rightarrow \text{Num}(X)$ we have used throughout this section could be considered as Cremona transformations of $\mathbf{P}^5$ applied to a suitable Fano model.

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