COUNTABLE CLOSED LFC -GROUPS WITH p -TORSION

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ABSTRACT. Let LFC be the class of all locally FC -groups. We study the existentially closed groups in the class LFC_p of all LFC -groups H whose torsion subgroup $T(H)$ is a p -group. Differently from the situation in LFC , every existentially closed LFC_p -group is already closed in LFC_p , and there exist $2^{\aleph_0}$ countable closed LFC_p -groups G . However, in the countable case, $T(G)$ is up to isomorphism always a unique locally finite p -group with similar properties as the unique countable existentially closed locally finite p -group E_p .

1. INTRODUCTION

In the last few years, existentially closed groups were an area of intense and fruitful research. Recall that, for a class $\mathfrak{X}$ of groups, $G \in \mathfrak{X}$ is said to be *existentially closed* (e.c.) in $\mathfrak{X}$, if every finite system of equations and inequalities with coefficients in G , which has a solution in some $H \in \mathfrak{X}$ with $G \leq H$, already has a solution in G itself. Strongly related to this concept is the notion of closedness. An $\mathfrak{X}$ -group G is said to be *closed* in $\mathfrak{X}$, if whenever B and C are finitely generated $\mathfrak{X}$ -groups and whenever $H \in \mathfrak{X}$ satisfies

$$\begin{array}{ccccc} & & C & & \\ & \subseteq & & \subseteq & \\ B & & & & H, \\ & \subseteq & G & \subseteq & \end{array}$$

then id_B can be extended to an embedding of C into G . Obviously, every closed $\mathfrak{X}$ -group is e.c. in $\mathfrak{X}$. The converse is true for example in classes of locally finite groups, if the underlying language is finite. However, e.c. groups are not closed in general. In the class $\mathfrak{A}$ of all abelian groups, a group is e.c. if and only if it is divisible of infinite p -rank for all primes p , while closed groups must have infinite torsion-free rank too. In particular, there exists a countable infinity of (pairwise nonisomorphic) countable e.c. $\mathfrak{A}$ -groups, but just one countable closed $\mathfrak{A}$ -group.

In this paper we consider *FC-groups*, i.e., groups whose elements have only finitely many conjugates. We denote by LFC the class of all locally FC -groups, and by $L\mathfrak{F}$ the class of all locally finite groups. Obviously, $\mathfrak{A}$ and $L\mathfrak{F}$

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are contained in LFC . Elementary properties of FC - and of LFC -groups may be found in [19 and 4]. Note that the torsion elements in any LFC -group G form a subgroup $T(G) \in L\mathfrak{F}$ with $G/T(G) \in \mathfrak{A}$. The torsion-free rank of $G/T(G)$ is usually just called the *rank* $r(G)$ of G .

In [5 and 6], F. Haug has studied e.c. LFC -groups extensively. In the countable case, some of his results are as follows.

- (1.1) Let G be a countable e.c. LFC -group. Then
- (a) $G' = T(G)$, and G splits over $T(G)$;
 - (b) $T(G)$ is isomorphic to P. Hall's countable universal $L\mathfrak{F}$ -group ULF (which is the unique countable e.c. $L\mathfrak{F}$ -group (see [3 and 14]));
 - (c) $G/T(G)$ is divisible (and hence e.c. in the class of all torsion-free $\mathfrak{A}$ -groups).
- (1.2) For each $\rho \leq \omega$ there exists precisely one countable e.c. LFC -group G_ρ with rank ρ . Whenever $B \subseteq C$ are finitely generated FC -groups of rank $\leq \rho$ with $B \leq G_\rho$, then id_B can be extended to an embedding of C into G_ρ . In particular, G_ρ is closed in the class LFC^ρ of all LFC -groups of rank $\leq \rho$. (An application of [5, Lemma 1.5] and [4, Corollary 1.3] shows that every e.c. LFC^ρ -group is e.c. in LFC , whence G_ρ is actually the unique countable closed LFC^ρ -group.)

Now, let p be a fixed prime. The theory of e.c. $L\mathfrak{F}$ -groups has nice parallels in the class $L\mathfrak{F}_p$ of all locally finite p -groups. In particular, there exists a unique countable e.c. $L\mathfrak{F}_p$ -group E_p (see [15]). It is thus a quite natural continuation of Haug's investigations, to consider the class LFC_p of all LFC -groups G with $T(G) \in L\mathfrak{F}_p$. Do the statements corresponding to (1.1) and (1.2) hold for the class LFC_p with E_p in place of ULF ? In the present paper we will try to illuminate the structure of countable e.c. LFC_p -groups.

In fact, our results about e.c. LFC_p -groups suggest a more general treatment as in [13]. There it was shown that, for every fixed countable abelian p -group A , there exists a unique countable e.c. group E_A in the class $L\mathfrak{F}_{p,A}$ of all $L\mathfrak{F}_p$ -groups G with $A \leq Z(G)$. Since A is the Schur multiplier of E_A/A , this gave $2^{\aleph_0}$ isomorphism types of countable $L\mathfrak{F}_p$ -groups E_A/A similar to E_p . For example, every E_A/A is verbally complete and characteristically simple, has a unique chief series (of order type $(\mathbb{Q}, <)$), and normality is transitive in E_A/A .

In this paper, we will consider, for each fixed countable abelian p -group A , the class $LFC_{p,A}$ of all $G \in LFC_p$ with $A \leq Z(G)$. Here, the elements of A are constants in the underlying language. Note also that every $LFC_{p,A}$ -group G is contained in a closed $LFC_{p,A}$ -group of cardinality $\max\{\aleph_0, |G|\}$ (cf. [16, Proposition 1.2] and the remarks after Theorem 4.2). In contrast to (1.2) we will show that every countable e.c. $LFC_{p,A}$ -group is already closed in $LFC_{p,A}$ (and thus has infinite rank), and that there exist $2^{\aleph_0}$ countable closed $LFC_{p,A}$ -groups for each A . Corresponding results hold for the classes $LFC_{p,A}^\rho$ of $LFC_{p,A}$ -groups of rank $\leq \rho$ ($1 \leq \rho \in \omega$). Since the structure of closed $LFC_{p,A}^\rho$ -groups is less homogeneous, we will not study them in detail. Concerning the countable closed $LFC_{p,A}$ -groups G , it will turn out that they all have a unique isomorphism type T_A of torsion subgroup, and that (1.1) holds for $LFC_{p,A}$ with T_A in place of ULF , with the only exception that we

are not able to show that G splits over $T(G)$. As with E_A/A , the group T_A/A has Schur multiplier A , is verbally complete and characteristically simple, and normality is transitive in T_A/A . However, T_A is not isomorphic to E_A , since in place of the unique chief series of E_A/A , the group T_A/A has a unique normal series (of order type $(\mathbb{Q}, <)$) with infinite elementary-abelian factors. Every countable closed $LFC_{p,A}$ -group acts via conjugation on each of these factors in the same way as $K^\times$ acts on K^+ regularly, where K denotes the algebraic closure of $GF(p)$. We will also describe the automorphism group of T_A similarly as in [11, §6 and 1 and 5, §4].

Our techniques are completely different from F. Haug's ones. He employed the permutational product, and also the Černikov embedding of finitely generated FC -groups into direct products of free abelian and of finite groups. Both constructions cannot be used within the class LFC_p , because they increase the number of primes involved in the orders of torsion elements. Instead we modify wreath product embedding techniques as in [9] such that they can be applied to LFC -groups. Our restriction to countable groups derives from these techniques.

2. AT -SERIES AND CLOSEDNESS

Our notions of *normal series*, *induced series*, *order type* and *Dedekind completion* of a series coincide with those introduced in [9, pp. 208–209] and [11, §2]. Normal series of $LFC_{p,A}$ -groups G refining the series $1 \leq A \leq T(G) \leq G$ will play an important role in our investigations. Such a series we call an *AT -normal series* in G , if it has $G/T(G)$ and $A/1$ among its factors; the factors of such a series between A and $T(G)$ are called *AT -factors*. An *AT -chief series* is an AT -normal series whose AT -factors are chief factors in G ; such factors are called *AT -chief factors*.

Lemma 2.1. *Every AT -chief factor of an $LFC_{p,A}$ -group G is elementary-abelian and central in $T(G)$.*

Proof. Let M be a periodic minimal normal subgroup of the LFC_p -group G . From [8, 1.B.3] we know that M is elementary-abelian. Choose $m \in M - 1$ and $g \in T(G)$. Then $F = \langle m, m^g \rangle$ is finite. Choose a finitely generated FC_p -group H such that $\langle F, g \rangle \leq H \leq G$ and $F \leq \langle x^H \rangle$ for all $x \in F - 1$. Then every torsion chief factor K/L in H with $m \in K - L$ satisfies $F \leq K$ and $F \cap L = 1$. The torsion subgroup of H is a finite p -group, and hence $K/L \cap Z(T(H)/L) \neq 1$. This immediately yields $K/L \leq Z(T(H)/L)$, and hence $m^g \in mL$. But now $F \cap L = 1$ implies $m^g = m$. This shows that $M \leq Z(T(G))$. $\square$

Lemma 2.2. *Let M/N be an AT -chief factor of the $LFC_{p,A}$ -group G . Put $\overline{G} = G/C_G(M/N)$ and $\mathcal{K} = GF(p)$. Regard M/N as a $\mathcal{K}\overline{G}$ -module via conjugation. Then $K = \mathcal{K}\overline{G}/An(M/N)$ is a locally finite field, and for any $m_0 \in M - N$ a K -isomorphism $\varepsilon: K^+ \rightarrow M/N$ is given by $x\varepsilon = (m_0N)^x$ for all $x \in K$.*

Proof. From Lemma 2.1 we know that $\overline{G}$ is abelian. Hence there exists a canonical embedding of the ring K into $\text{Hom}_K(M/N, M/N)$, and the latter is a skew field by Schur's Lemma. Now let $G_0 \leq G$ be finitely generated with $m_0 \in G_0$. Since $\overline{G}$ is abelian, we have $C_G(M/N) = C_G(m_0N)$ and

$\overline{\overline{G}}_0 \cong G_0/C_{G_0}(m_0N)$. So $G_0 \in FC$ implies that $\overline{\overline{G}}_0$ is finite. Thus $\overline{\overline{G}}_0 + An(M/N)/An(M/N)$ is finite, commutative and multiplicatively closed; hence it must be a finite subfield of K . This shows that K is a locally finite field. Moreover ε is surjective because M/N is a simple K -module. Since ε is nontrivial and $\text{Ker } \varepsilon$ is an ideal in the field K , we obtain $\text{Ker } \varepsilon = \{0\}$. $\square$

Because locally finite fields are countable, Lemma 2.2 yields that every AT -chief factor of an $LFC_{p,A}$ -group is countable.

Chief factors of $L\mathfrak{F}_p$ -groups are central and cyclic of order p . Lemma 2.2 gives a first hint, that this is no longer true for AT -chief factors of $LFC_{p,A}$ -groups. We can make this more precise by using semidirect product constructions. If $\theta: G \rightarrow H$ is a group homomorphism, and if H acts on a group N , we let

$$N \rtimes_{\theta} G = \{(g, n) | g \in G, n \in N\} \quad \text{with group multiplication} \\ (g_1, n_1)(g_2, n_2) = (g_1g_2, n_1^{g_2\theta} n_2) \quad \text{for all } g_1, g_2 \in G, n_1, n_2 \in N.$$

Usually we identify N and G canonically with the corresponding subgroups of $N \rtimes_{\theta} G$. We suppress θ whenever $\theta = \text{id}$.

Lemma 2.3. *Let K be a locally finite field of characteristic p . Regard K^+ as regular $K^{\times}$ -module. Suppose that G is an $LFC_{p,A}$ -group and that there exists a homomorphism $\theta: G \rightarrow K^{\times}$ with $A \leq \text{Ker } \theta$. Then $H = K^+ \rtimes_{\theta} G$ is an $LFC_{p,A}$ -group.*

Proof. Let $H_0 \leq H$ be finitely generated. Since K^+ and $K^{\times}$ are locally finite, we have $K^+ \rtimes K^{\times} \in L\mathfrak{F}$. Therefore $H_0 \cap K^+$ is a finite p -group. Clearly, $H_0/H_0 \cap K^+ \cong H_0K^+/K^+$ is an FC_p -group. Now, extensions of finite groups by FC -groups are again FC -groups. Thus $H_0 \in FC_p$. This shows that $H \in LFC_p$. Moreover, $A \leq \text{Ker } \theta$ implies $[A, K^+] = 1$, whence $A \leq Z(H)$. $\square$

Corollary 2.4. *AT -chief factors are in general not central and not cyclic.*

Proof. Let $K \neq GF(p)$ be a locally finite field of characteristic p . Since $K^{\times}$ is periodic and locally cyclic, there exists a subgroup G of $A \times \mathbb{Q}^+$ with $A \leq G$, and an epimorphism $\theta: G \rightarrow K^{\times}$ with $A \leq \text{Ker } \theta$. Because $K^{\times}$ acts transitively on $K^+ - \{0\}$, we see that K^+A/A is a noncyclic noncentral AT -chief factor in the $LFC_{p,A}$ -group $K^+ \rtimes_{\theta} G$. $\square$

Corollary 2.4 shows that it is impossible to embed finitely generated FC_p -groups into direct products of free abelian and of finite p -groups, and that finitely generated FC_p -groups need not be residually finite p -groups.

In $LFC_{p,A}$ -groups, only elements of infinite order can induce p' -automorphisms on AT -chief factors. Now, for given $g \in G \in LFC_{p,A}$ with $o(g) = \infty$, Lemma 2.3 allows us to construct an $LFC_{p,A}$ -supergroup H of G such that g acts as a p' -automorphism on some AT -chief factor of H . Thus we can express by a finite system of equations and inequalities in H that g has infinite order. This provides the key for our

Theorem 2.5. *Every countable e.c. $LFC_{p,A}$ -group is closed in $LFC_{p,A}$. If A has finite p -rank, then every e.c. $LFC_{p,A}$ -group is closed in $LFC_{p,A}$.*

We denote the class of all finitely generated $LFC_{p,A}$ -groups by $FC_{p,A}^0$. Note that every $FC_{p,A}^0$ -group F has the form $F = \langle A, f_1, \dots, f_r \rangle$ for some

$f_1, \dots, f_r \in F$, since the elements of A are constants in the underlying language.

Proof of Theorem 2.5. Let G be e.c. in $LFC_{p,A}$. Suppose that $B, C \in FC_{p,A}^0$ and $H \in LFC_{p,A}$ satisfy $B \leq C \leq H$ and $B \leq G \leq H$. Without loss we may assume that $H = \langle G, C \rangle$. Denote epimorphic images modulo torsion subgroups by bars. Choose $c_1, \dots, c_r \in C$ such that $\bar{C} = \langle \bar{c}_1 \rangle \times \dots \times \langle \bar{c}_r \rangle$. Since $\bar{G}$ is divisible by Theorem 5.2, we may assume that $\bar{c}_1, \dots, \bar{c}_\nu \in \bar{G}$ and $\bar{H} = \bar{G} \times \langle \bar{c}_{\nu+1} \rangle \times \dots \times \langle \bar{c}_r \rangle$ for some $\nu \leq r$. Let K be the algebraic closure of $GF(p)$. In the case when $p \neq 2$ put $q = p$, otherwise $q = 4$.

We construct a chain $H = H_0 \leq H_1 \leq \dots \leq H_r$ in $LFC_{p,A}$ with $\bar{H}_i = \bar{H}$ for $1 \leq i \leq r$ as follows. Suppose that H_{i-1} has been found. The pure subgroup $\bar{C}_i$ of $\bar{H}_{i-1}$ generated by $\{\bar{c}_i\}$ has a direct complement $\bar{D}_i$ in $\bar{H}_{i-1}$ which contains the pure subgroup of $\bar{H}_{i-1}$ generated by $\{\bar{c}_j | j \neq i\}$. Since $\bar{C}_i / \langle \bar{c}_i^{q-1} \rangle$ is periodic and locally cyclic, there exists a homomorphism $\varphi_i: \bar{C}_i \rightarrow K^\times$ with $GF(q)^\times = \langle \bar{c}_i \varphi_i \rangle$. Define $\theta: H_{i-1} \rightarrow K^\times$ via $h\theta = \bar{h}\varphi_i$ for all $h \in H_{i-1}$, where $\bar{\varphi}_i: \bar{H}_{i-1} \rightarrow K^\times$ is determined by $\bar{\varphi}_i|_{\bar{C}_i} = \varphi_i$ and $\bar{\varphi}_i|_{\bar{D}_i} \equiv 1$. Put $H_i = K^+ \rtimes_\theta H_{i-1}$ and $K_i = GF(q)^+ \leq K^+ \leq H_i$. Then K_i is a finite p -group, and $c_j \in C(K_i)$ for all $j \neq i$, while c_i induces an automorphism of order $q-1$ on K_i , which permutes the nontrivial elements in K_i transitively. Also, $H_i \in LFC_{p,A}$ with $\bar{H}_i = \bar{H}_{i-1}$ by Lemma 2.3.

Choose a transversal T of A in $T(C)$ with $1 \in T$. Then T is finite, since TA/A is the torsion subgroup of $C/A \in FC_p^0$. Consider the following finite systems of equations and inequalities with unknowns $(x_t | t \in T)$, $(y_1, \dots, y_r)$, and $(z_k | k \in K_i - 1, 1 \leq i \leq r)$.

$$\begin{aligned}
 (2.1) \quad & x_{t_1}x_{t_2} = x_{t_3}a \quad \text{whenever } t_1, t_2, t_3 \in T \text{ and } a \in A \text{ satisfy } t_1t_2 = t_3a; \\
 & y_i^{-1}x_{t_1}y_i = x_{t_2}a \quad \text{for } 1 \leq i \leq r \text{ whenever } t_1, t_2 \in T \text{ and } a \in A \text{ satisfy} \\
 & \quad c_i^{-1}t_1c_i = t_2a; \\
 & [y_1, y_2] = x_{t_1}a \quad \text{for } 1 \leq i, j \leq r \text{ whenever } t \in T \text{ and } a \in A \\
 & \quad \text{satisfy } [c_i, c_j] = ta.
 \end{aligned}$$

$$\begin{aligned}
 (2.2) \quad & y_i^{-1}z_{k_1}y_i = z_{k_2} \quad \text{for } 1 \leq i \leq r \text{ for all } k_1, k_2 \in K_i - 1 \text{ with } c_i^{-1}k_1c_i = k_2; \\
 & [y_i, z_k] = 1 \quad \text{for } 1 \leq i, j \leq r \text{ with } i \neq j \text{ and all } k \in K_j - 1; \\
 & \text{the group table } K_i \text{ in the unknowns } (z_k | k \in K_i - 1) \text{ for } 1 \leq i \leq r \\
 & \text{(including the inequalities } z_k \neq 1).
 \end{aligned}$$

Suppose that G contains a simultaneous solution $(u_t | t \in T)$, $(v_1, \dots, v_r)$, and $(w_k | k \in K_i - 1, 1 \leq i \leq r)$ to (2.1) and (2.2). Let $G_0 = \langle A, v_1, \dots, v_r, u_t | t \in T \rangle$. Define $\psi: C \rightarrow G_0$ via

$$\psi(c_1^{n_1} \dots c_r^{n_r} \cdot ta) = v_1^{n_1} \dots v_r^{n_r} \cdot u_t a \quad \text{for all } n_1, \dots, n_r \in \mathbb{Z}, t \in T, a \in A.$$

Then ψ is a homomorphism by virtue of (2.1).

Put $L_i = \langle w_k | k \in K_i - 1 \rangle$. From (2.2) we know that $L_i \cong K_i$, and that v_i induces an automorphism of order $q-1$ on L_i which permutes the elements from $L_i - 1$ transitively. In particular, $o(v_i) = \infty$, and there exists an AT -chief factor M_i/N_i in G such that $L_i \leq M_i$ and $N_i \cap L_i = 1$. Clearly $v_i \notin G_i =$

$C_{G_0}(M_i/N_i)$. Since $v_j \in C_{G_0}(L_i)$ for $j \neq i$, Lemma 2.2 yields that $v_j \in G_i$. Thus the canonical embedding

$$G_0 / \bigcap \{G_i | 1 \leq i \leq r\} \rightarrow \prod \{G_0/G_i | 1 \leq i \leq r\} = \prod \{\langle v_i G_i \rangle | 1 \leq i \leq r\}$$

is an isomorphism, and therefore $G_0 / \bigcap \{G_i | 1 \leq i \leq r\}$ has Prüfer rank r . (Note that $o(v_i G_i) = q - 1$ for all i .) This shows that $\overline{G}_0 = \langle \overline{v}_1 \rangle \times \cdots \times \langle \overline{v}_r \rangle$.

If $x = c_1^{n_1} \cdots c_r^{n_r} \cdot ta \in \text{Ker } \psi$, then $\overline{v}_1^{n_1} \cdots \overline{v}_r^{n_r} = 0$, and so $n_1 = \cdots = n_r = 0$. It follows that $u_t = a^{-1} \in \{u_t | t \in T\} \cap A$, and $\text{Ker } \psi$ will be trivial if we can ensure that $\{u_t | t \in T\} \cap A = 1$. If A has finite p -rank, then $A[p^n] = \{a \in A | o(a) | p^n\}$ is finite for all $n \in \omega$. In this case we supplement (2.1) and (2.2) by

$$(2.3) \quad x_t \neq a \quad \text{for all } t \in T - 1 \text{ and all } a \in A[o(t)].$$

Since ψ is a homomorphism, $o(u_t)$ divides $o(t)$. Thus (2.3) yields that $u_t \notin A$ for every $t \in T - 1$. In the other case, G is countable by assumption. Then we may assume without loss that H is countable, whence H_r is contained in a countable closed $LFC_{p,A}$ -group H_{r+1} . But Theorem 5.1 yields that $Z(H_{r+1}) = A$. In this case we supplement (2.1) and (2.2) by

$$(2.3) \quad [x_t, \xi_t] \neq 1 \quad \text{for all } t \in T - 1,$$

where $(\xi_t | t \in T - 1)$ are additional unknowns. This yields $u_t \notin Z(G) \geq A$ for all $t \in T - 1$.

Finally, let $B = \langle A, b_1, \dots, b_s \rangle$. Because of $B \leq C$ there exist words w_j in elements from $\{c_1, \dots, c_r\} \cup T$ such that

$$b_j = w_j a_j \quad \text{for } 1 \leq j \leq s \text{ and suitable } a_j \in A.$$

Let w_j^* be the word obtained from w_j by replacing c_i by y_i and t by x_t . Supplement (2.1)–(2.3) by

$$(2.4) \quad b_j = w_j^* a_j \quad \text{for } 1 \leq j \leq s.$$

Since the $LFC_{p,A}$ -supergroup H_r resp. H_{r+1} of the e.c. $LFC_{p,A}$ -group G contains a simultaneous solution to the systems (2.1)–(2.4), there does already exist a solution in G . As before, (2.1) gives rise to a homomorphism $\psi: C \rightarrow G$, which is injective because of (2.2) and (2.3), and which satisfies $\psi|B = \text{id}_B$ by virtue of (2.4). $\square$

By Theorem 5.1 there exists an uncountable closed $LFC_{p,A}$ -group with centre greater than A . Therefore, the method of proof of Theorem 2.5 cannot be applied to uncountable e.c. $LFC_{p,A}$ -groups G when A has infinite rank, since we cannot ensure that for every $t \in T - 1$ there exists an $LFC_{p,A}$ -supergroup H of G with $[t, h] \neq 1$ for some $h \in H$.

Question 2.6. *Do there exist uncountable e.c. $LFC_{p,A}$ -groups which are not closed in $LFC_{p,A}$ (when A has infinite p -rank)?*

Clearly, every closed $LFC_{p,A}$ -group contains a copy of every $FC_{p,A}^0$ -group. Together with Theorem 2.5 this immediately yields

Corollary 2.7. *Every countable e.c. $LFC_{p,A}$ -group has infinite rank. If A has finite p -rank, then every e.c. $LFC_{p,A}$ -group has infinite rank.*

This result sharply contrasts with the fact, that e.c. LFC -groups can have any prescribed rank [5, Lemma 2.2 and 6, §9].

3. MODIFIED STANDARD EMBEDDINGS

For the investigation of an e.c. group G in some class $\mathfrak{X}$ it is necessary to construct $\mathfrak{X}$ -supergroups of G . In LFC and $L\mathfrak{F}_p$ quite elegant constructions were made by means of the permutational product [4 and 11]. By a synthesis of both cases one would expect that it be possible to use the permutational product also within LFC_p . However, there are the following two difficulties. Firstly, the permutational product of torsion-free $\mathfrak{A}$ -groups is their central product over the amalgamated subgroup; therefore its torsion subgroup need not be a p -group. Secondly, the $L\mathfrak{F}_p$ -case suggests a choice of transversals with respect to AT -chief series in order to apply certain theorems of Gregorac [2]; however, these theorems require the centrality of chief factors in the *whole* group, whereas AT -chief factors are in general just central in the torsion subgroup. And a suitable generalization of Gregoracs theorems could not be found. Therefore, we try to modify the technique of standard embeddings into wreath products (see [7]), which has already been used advantageously for a large variety of group classes (see [9]), and which has turned out to be even more powerful than the permutational product in the study of *countable* e.c. $\mathfrak{X}$ -groups.

Let A be any group, and let B be a permutation group on a set Ω . The *unrestricted wreath product* $A \text{Wr}_{\Omega} B$ is the set $\{(b, f) | b \in B, f: \Omega \rightarrow A\}$ with group multiplication

$$(b_1, f_1) \cdot (b_2, f_2) = (b_1 b_2, f_1^{b_2} f_2)$$

where

$$(f_1^{b_2} f_2)(\omega) = f_1(\omega b_2^{-1}) \cdot f_2(\omega) \quad \text{for all } \omega \in \Omega.$$

$A \text{Wr}_{\Omega} B$ is a split extension of its *base group* $\{(1, f) | f: \Omega \rightarrow A\} \cong A^{\Omega}$ by its *top group* $\{(b, 1) | b \in B\} \cong B$. In the case when B is an arbitrary group, we can choose $\Omega = B$ and define a permutation action of B on Ω via $(\omega)b^{-1} = b \cdot \omega$ for all $\omega \in \Omega$, $b \in B$. In this case we write $A \text{Wr} B$ instead of $A \text{Wr}_{\Omega} B$. We usually identify canonically B with the top group and A with the 1-component $\{(1, f) | f(b) = 1 \text{ for all } b \in B - 1\}$ of $A \text{Wr} B$. The canonical embedding of A onto the *diagonal subgroup* $\{(1, f) | f \text{ constant}\}$ of $A \text{Wr} B$ is called the *diagonal embedding*.

Now, let $\theta: G \rightarrow H$ be a group homomorphism with $N = \text{Ker } \theta$. A *countermap* to θ is a map $\theta^*: H \rightarrow G$ satisfying

$$(g\theta h)\theta^* \theta = g\theta \cdot h\theta^* \theta \quad \text{for all } g \in G, h \in H.$$

See [7] for the existence of countermaps. Every countermap θ^* to θ gives rise to a *standard embedding* $\sigma: G \rightarrow W = N \text{Wr} H$ via

$$g\sigma = (g\theta, f_g) \quad \text{for all } g \in G,$$

where

$$f_g(h) = [(g\theta h)\theta^*]^{-1} \cdot g \cdot h\theta^* \quad \text{for all } h \in H.$$

Similar embeddings have been used in [9, Construction 4.1] to obtain $L\mathfrak{X}$ -supergroups of countable e.c. $L\mathfrak{X}$ -groups for certain classes $\mathfrak{X}$. In these applications it was required that wreath products of finitely generated $\mathfrak{X}$ -groups by $L\mathfrak{X}$ -groups were $L\mathfrak{X}$ -groups again. This is however not true for FC -groups. For example, C_p has infinitely many conjugates in the finitely generated subgroup $\langle C_p, C_\infty \rangle$ of $C_p \text{ Wr } C_\infty$. Therefore, we need some further specialization of standard embeddings. Our idea is as follows. In the case when G and H are finitely generated FC -groups (FC^0 -groups) and $N \leq T(G)$, choose the countermap $\theta^*: H \rightarrow G$ in such a way that, for some subgroup H_0 of finite index in H , each f_g is constant on the left cosets of H_0 in H . Then $\text{Im } \sigma$ is contained in

$$W_0 = \{(h, f) | h \in H, f \text{ is constant on each } \tilde{h}H_0 (\tilde{h} \in H)\} \leq W.$$

W_0 is a split extension of a direct product of finitely many copies of N by H . Since N is finite, it follows that $W_0 \in FC^0$.

In the following constructions we will make extensive use of the fact, that in FC^0 -groups the centre has finite index. The rank $r(G)$ of an FC^0 -group G is equal to its Hirsch number and thus also to the torsion-free rank of $Z(G)$. By an *independent system* of an $\mathfrak{A}$ -group we always mean an independent system of torsion-free elements. For any periodic $\mathfrak{A}$ -group A , we denote by LFC_A the class of all LFC -groups with $A \leq Z(G)$.

Construction 3.1. Suppose that we are given a homomorphism $\theta: G \rightarrow H$ of countable LFC -groups with $A \leq Z(G)$ and $A \leq N = \text{Ker } \theta \leq T(G)$, and that $\{H_n | n \in \omega\}$ is an ascending chain of FC^0 -groups with union H . Then there exists an ascending chain of FC^0 -groups $\{G_n | n \in \omega\}$ with union G such that $H_n \cap G\theta = G_n\theta$ for all n .

Proof. Let $N = \{v_m | m \in \omega\}$ and choose elements $g_m \in G$ such that $H_n \cap G\theta = \langle g_0\theta, \dots, g_{k_n}\theta \rangle$ for suitable k_n . Put $G_n = \langle v_0, \dots, v_n, g_0, \dots, g_{k_n} \rangle$. $\square$

Choose recursively a torsion-free $Z_n \leq Z(G_n)$ with $Z_n\theta \leq Z(H_n)$, such that $r(Z_0) = r(G_0)$, and also $r(Z_{n+1}) = r(G_{n+1}) - r(G_n)$ and $Z_{n+1} \cap G_n = 1$.

Proof. Let S be a subset of $Z(G_{n+1})$ which supplements a maximal independent system of $Z_n \cap Z(G_{n+1})$ to one of $Z(G_{n+1})$. Then Z_{n+1} may be generated by a suitable power of S . $\square$

It follows that $Z_{n+1}\theta \cap H_n = Z_{n+1}\theta \cap H_n \cap G\theta = Z_{n+1}\theta \cap G_n\theta = 1$.

Proof. Let $z \in Z_{n+1}$ and $g \in G_n$ with $z\theta = g\theta$. Then $g = zv$ for some $v \in N \cap G_{n+1} \leq T(G_{n+1})$. Thus, $g^{o(v)} = z^{o(v)} \in Z_{n+1} \cap G_n = 1$. But Z_{n+1} is torsion-free, whence $z = 1$. $\square$

Choose recursively a torsion-free $\hat{Z}_n \leq Z(H_n)$ with $Z_n\theta \leq \hat{Z}_n$, such that $r(\hat{Z}_0) = r(H_0)$, and also $r(\hat{Z}_{n+1}) = r(H_{n+1}) - r(H_n)$ and $\hat{Z}_{n+1} \cap H_n = 1$.

Proof. Supplement a maximal independent system of $(Z_{n+1}\theta \times \hat{Z}_n) \cap Z(H_{n+1})$. $\square$

Clearly, $Z_n \cdots Z_0$ and $\hat{Z}_n \cdots \hat{Z}_0$ have finite index in G_n resp. H_n . Note also that $\theta|_{Z_n \cdots Z_0}$ is injective as $Z_n \cdots Z_0$ is torsion-free and $N \leq T(G)$. Choose

$C_n \leq Z_n \cdots Z_0$ with $C_n\theta = Z(H_n) \cap (Z_n \cdots Z_0 \cap Z(G_n))\theta$. Then

$$\begin{aligned} C_{n+1}\theta &= Z(H_{n+1}) \cap [Z_{n+1}(Z_n \cdots Z_0 \cap Z(G_{n+1}))]\theta \\ &= Z_{n+1}\theta \cdot (Z(H_{n+1}) \cap [Z_n \cdots Z_0 \cap Z(G_{n+1}))\theta) \\ &\leq Z_{n+1}\theta \cdot (Z(H_n) \cap [Z_n \cdots Z_0 \cap Z(G_n)]\theta) = (Z_{n+1}C_n)\theta, \end{aligned}$$

whence

$$(3.1) \quad Z_{n+1} \leq C_{n+1} \leq Z_{n+1}C_n.$$

Choose a torsion-free $X_0 \leq \hat{Z}_0$ with $X_0 \cap C_0\theta = 1$ and $r(X_0) = r(\hat{Z}_0) - r(C_0\theta)$, and then recursively a torsion-free $X_{n+1} \leq \hat{Z}_{n+1}X_n \cap Z(H_{n+1})$ such that $X_{n+1} \cap C_{n+1}\theta = 1$ and $r(X_{n+1}) = r(\hat{Z}_{n+1}D_n) - r(C_{n+1}\theta)$, where $D_n = C_n\theta \times X_n$.

Proof. Note that $C_{n+1}\theta \leq (Z_{n+1}C_n)\theta \leq \hat{Z}_{n+1}D_n = \hat{Z}_{n+1} \times X_n \times C_n\theta$. In order to find X_{n+1} , supplement a maximal independent system of $C_{n+1}\theta$ to one of $\hat{Z}_{n+1}D_n$. From (3.1) and $r(C_{n+1}) = r(Z_{n+1}) + r(C_n)$ we know that $C_{n+1}\theta$ contains a maximal independent system of $(Z_{n+1}C_n)\theta$. Therefore we can even get $X_{n+1} \leq \hat{Z}_{n+1} \times X_n$. $\square$

An easy induction shows that $r(D_n) = r(H_n)$. Moreover,

$$(3.2) \quad D_n \cap G\theta = C_n\theta \quad \text{and} \quad D_{n+1} \leq \hat{Z}_{n+1}D_n.$$

Proof. $|X_n \cap G_n\theta| = |X_n \cap G_n\theta : X_n \cap C_n\theta| \leq |G_n\theta : C_n\theta| < \infty$, since $r(G_n\theta) = r(G_n) = r(C_n) = r(C_n\theta)$. But X_n is torsion-free. So $X_n \cap G_n\theta = 1$ and $D_n \cap G_n\theta = C_n\theta$. Furthermore, $D_{n+1} = C_{n+1}\theta \cdot X_{n+1} \leq Z_{n+1}\theta \cdot C_n\theta \cdot \hat{Z}_{n+1} \cdot X_n \leq \hat{Z}_{n+1}D_n$. $\square$

Now, let $W = (T(G) \rtimes G) \text{Wr} H$ where G acts on $T(G)$ via conjugation. We will specify a certain LFC $_{A\delta}$ -subgroup $\widehat{W}$ of W , where $\delta: A \rightarrow W$ is given by $a\delta = (1, f_a)$ for all $a \in A$ and $f_a \equiv (1, a)$. To this end, put $\hat{Z}^n = \langle \hat{Z}_k | k \geq n+1 \rangle$, and fix a right transversal T_n of $\hat{Z}^n H_n$ in $\hat{Z}^{n+1} H_{n+1}$ with $T_n \subseteq H_{n+1}$. Then $T^n = \bigcup \{T_n \cdots T_k | k \geq n\}$ is a right transversal of $\hat{Z}^n H_n$ in H . Clearly,

$$(3.3) \quad h \cdot \hat{Z}^{n+1} X_{n+1} T^{n+1} \subseteq \check{h} \cdot \hat{Z}^n X_n T^n \quad \text{and} \quad h \cdot \hat{Z}^{n+1} D_{n+1} T^{n+1} \subseteq \check{h} \cdot \hat{Z}^n D_n T^n$$

whenever $h \in H_{n+1}$ satisfies $h = z\check{h}t$ with $z \in \hat{Z}^n$, $\check{h} \in H_n$, $t \in T_n$.

Let

$$W_n = \{(h, f) \in W | h \in H_n, \text{Im } f \subseteq T(G_n) \rtimes G_n, f \text{ is constant on each of the sets } \check{h} \cdot \hat{Z}^n D_n T^n (\check{h} \in H_n), \text{ the } G_n\text{-component of } f \text{ is constant modulo } T(G_n)\}.$$

Then W_n is a subgroup of W . Moreover, $W_n \leq W_{n+1}$ by (3.3). And $\widehat{W} = \bigcup \{W_n | n \in \omega\} \in \text{LFC}_{A\delta}$.

Proof. Clearly, $A\delta \leq Z(\widehat{W})$. It remains to show that $W_n \in FC$. Let

$$\Omega_n = \{(h, f) \in W_n | \text{Im } f \subseteq T(G_n) \rtimes T(G_n) \text{ and } h \in T(H_n)\}.$$

Since for every $(h, f) \in W_n$ the G_n -component of f is constant modulo $T(G_n)$, we obtain that $\Omega_n \trianglelefteq W_n$. If $(h, f) \in \Omega_n$, then f is constant on each of the finitely many sets $\check{h} \cdot \hat{Z}^n D_n T^n (\check{h} \in H_n)$. Hence Ω_n is a finite

normal subgroup of W_n . Moreover, $W_n/\Omega_n \cong H_n/T(H_n) \times G_n/T(G_n)$ is a torsion-free $\mathfrak{A}$ -group, whence $W_n \in FC$. $\square$

Finally note that $\widehat{W} \in LFC_{p, A\delta}$ whenever $G \in LFC_{p, A}$ and $H \in LFC_p$.

Construction 3.2. Adopt the notation introduced in Construction 3.1, and suppose in addition that $G_0 \leq T(G)$. For each $n \in \omega$, choose a right transversal R_n of $G_n\theta \cdot X_n$ in H_n , and let S_n be a transversal of $C_n \cdot (N \cap G_n)$ in G_n . Then $S_n\theta$ is a transversal of $C_n\theta$ in $G_n\theta$. Define $\theta_n^*: H \rightarrow G_n$ via

$$(zxc\theta s\theta rt)\tilde{\theta}_n^* = cs \quad \text{for all } z \in \widehat{Z}^n, x \in X_n, c \in C_n, s \in S_n, r \in R_n, t \in T^n.$$

$\tilde{\theta}_n^*$ is a countermap to $\theta|_{G_n}$ because $\widehat{Z}^n X_n R_n T^n$ is a right transversal of $G_n\theta$ in H (cf. [7]). Moreover, we have that

$$(3.4) \quad \tilde{\theta}_n^* \text{ is constant on each of the sets } h \cdot \widehat{Z}^n X_n T^n \ (h \in H_n), \text{ and}$$

$$(3.5) \quad (c\theta h)\tilde{\theta}_n^* = c \cdot h\tilde{\theta}_n^* \quad \text{for all } c \in C_n, h \in H.$$

Define $\theta_n^*: H \rightarrow G_n$ recursively via

$$\theta_0^* = \tilde{\theta}_0^* \quad \text{and} \quad (g\theta y)\theta_{n+1}^* = (g\theta y)\theta_n^* \cdot y\theta_n^{*-1} \cdot y\tilde{\theta}_{n+1}^* \\ \text{for all } g \in G_n, y \in \widehat{Z}^n X_n R_n T^n.$$

Straightforward calculations yield that θ_n^* is also a countermap to $\theta|_{G_n}$, and that the map $\omega_n: H \rightarrow G_{n+1}$ given by $h\omega_n = h\theta_n^{*-1} \cdot h\theta_{n+1}^*$ is constant on each coset $G_n\theta \cdot h$ ($h \in H$). It follows from (3.3) that θ_n^* satisfies (3.4) too. Furthermore, θ_n^* satisfies (3.5).

Proof. Let $c \in C_{n+1}$ and $h \in H$. Then (3.1) yields $c = \tilde{z}\tilde{c}$ for some $\tilde{z} \in Z_{n+1}$, $\tilde{c} \in C_n$. Moreover, $h = g\theta \cdot y$ for some $g \in G_n$, $y \in \widehat{Z}^n X_n R_n T^n$. But then

$$\begin{aligned} (c\theta h)\theta_{n+1}^* &= ((\tilde{z}\tilde{c}g)\theta y)\theta_{n+1}^* = ((\tilde{c}g\tilde{z})\theta y)\theta_{n+1}^* \\ &= (\tilde{c}\theta g\theta \tilde{z}\theta y)\theta_n^* \cdot (\tilde{z}\theta y)\theta_n^{*-1} \cdot (\tilde{z}\theta y)\tilde{\theta}_{n+1}^* \\ &= \tilde{c} \cdot (g\theta \tilde{z}\theta y)\theta_n^* \cdot (\tilde{z}\theta y)\theta_n^{*-1} \cdot (\tilde{z}\theta y)\tilde{\theta}_{n+1}^* \quad \text{by induction} \\ &= \tilde{c} \cdot (g\theta y)\theta_n^* \cdot y\theta_n^{*-1} \cdot \tilde{z} \cdot y\tilde{\theta}_{n+1}^* \quad \text{by (3.4) and (3.1)/(3.5)} \\ &= \tilde{z}\tilde{c} \cdot (g\theta y)\theta_{n+1}^* = c \cdot h\theta_{n+1}^*. \quad \square \end{aligned}$$

Now, the standard embedding $G_0 \rightarrow W$ determined by θ_0^* can be extended to an embedding $\sigma: T \rightarrow W$, given by

$$g\sigma = (g\theta, f_g) \quad \text{for all } g \in G_n, \text{ where}$$

$$f_g(h) = ((g\theta h)\theta_0^{*-1} \cdot (g\theta h)\theta_n^* \cdot h\theta_n^{*-1} \cdot h\theta_0^*, h\theta_0^{*-1} \cdot h\theta_n^* \cdot (g\theta h)\theta_n^{*-1} \cdot g \cdot h\theta_0^*) \\ \in (N \cap G_n) \rtimes G_n \text{ for all } h \in H.$$

(This is the embedding σ of [9, Construction 4.1].) We list some properties of σ .

$$(3.6) \quad \text{If } g \in N, \text{ then } g\sigma = (1, f_g) \text{ where } f_g(h) = (1, g^{h\theta_0^*}) \text{ for all } h \in H. \text{ In particular, } \sigma|_A = \delta. \text{ And if we choose } G_0 \leq N \text{ and } S_0 = \{1\}, \text{ then } \sigma|_N \text{ is the diagonal embedding.}$$

$$(3.7) \quad \text{If } g \in G_n, \text{ then } f_g \text{ is constant on each of the sets } h \cdot \widehat{Z}^n D_n T^n \ (h \in H_n), \text{ and the } G_n\text{-component of } f_g \text{ constantly equals } g \text{ modulo } T(G). \text{ In particular, } \text{Im } \sigma \leq \widehat{W}.$$

Proof. Because of (3.3)/(3.4) the $T(G_n)$ -component f_g^r of f_g is constant on each $h \cdot \widehat{Z}^n X_n T^n$. Now, let $c \in C_n$. Because of $G_0 \leq T(G)$ we have $C_0 = 1$, whence $c\theta \in C_n\theta \leq \widehat{Z}^0$ by (3.1). Thus (3.5) and (3.4) yield

$$\begin{aligned} f_g^r(c\theta h) &= ((c\theta h)\theta_n^* \cdot (g\theta c\theta h)\theta_n^{*-1} \cdot g)^{(c\theta h)\theta_0^*} \\ &= (c \cdot h\theta_n^* \cdot (g\theta h)\theta_n^{*-1} \cdot c^{-1} \cdot g)^{h\theta_0^*} = f_g^r(h). \end{aligned}$$

By the same arguments, the G_n -component f_g^1 of f_g is constant on each $h \cdot \widehat{Z}^n D_n T^n$ too. Furthermore, $\text{Im } \theta_0^* \subseteq G_0 \leq T(G)$. So $f_g^1(h) \cdot T(G) = (g\theta h)\theta_n^* \cdot h\theta_n^{*-1} \cdot T(G) = g \cdot T(G_n)$ by definition of a countermap. $\square$

Construction 3.3. *Adopt the notation introduced in Constructions 3.1 and 3.2. Suppose in addition that $N = \text{Ker } \theta$ is a minimal normal subgroup of G . Then, by Lemma 2.1, we have that $T(G) \leq C_G(N)$. This allows it to modify the embedding $\sigma: G \rightarrow \widehat{W}$ as follows.*

Denote epimorphic images modulo $T(G)$ by bars. Let $W^0 = (N \rtimes \overline{G})\text{Wr } H$ and

$$\begin{aligned} W_n^0 &= \{(h, f) \in W^0 \mid h \in H_n, \text{Im } f \subseteq (N \cap G_n) \rtimes \overline{G}_n, \\ &\quad f \text{ is constant on each of the sets } \check{h} \cdot \widehat{Z}^n D_n T^n \ (\check{h} \in H_n), \\ &\quad \text{the } \overline{G}_n\text{-component of } f \text{ is constant}\}. \end{aligned}$$

Then $\widehat{W}^0 = \bigcup \{W_n^0 \mid n \in \omega\} \in \text{LFC}$ (resp. $\widehat{W}^0 \in \text{LFC}_p$ whenever $G, H \in \text{LFC}_p$). Define $\sigma^0: G \rightarrow \widehat{W}^0$ by

$$g\sigma^0 = (g\theta, f_g^0) \quad \text{for all } g \in G,$$

where

$$f_g^0(h) = (\overline{g}, h\theta_n^* \cdot (g\theta h)\theta_n^{*-1} \cdot g) \quad \text{for all } h \in H.$$

Then σ^0 is also an embedding with the properties that $\sigma|N$ is the diagonal embedding, and that f_g^0 is constant on each of the sets $h \cdot \widehat{Z}^n D_n T^n$ ($h \in H_n$) whenever $g \in G_n$. Usually we suppress the 0's.

Embeddings σ as in Constructions 3.2 and 3.3 we call *modified standard embeddings (ms-embeddings)*. We will need a technical lemma concerning ms-embeddings which corresponds to [9, Corollary 4.4].

Lemma 3.4. *Let $(h, f) \in W_n$ with $h \in H_n - D_n$. If B_n denotes the base group of W_n , then $T(B_n)' \leq [((h, f), T(B_n)), T(B_n)] \leq \langle (h, f)^{T(B_n)} \rangle$.*

Proof. Let $\widehat{R}_n$ be a right transversal of $D_n\langle h \rangle$ in H_n . Follow the proof of [9, Lemma 4.3(b)] with $\widehat{Z}^n D_n \widehat{R}_n T^n$ in place of T . (Note that $[((b, f), x_{1j}), x_{2j}] = [x_{1j}, x_{2j}]$ actually holds for all j in the proof of [9, Lemma 4.3(b)], whence the assumption $f(b') = 1$ for all $b' \in B - T$ is superfluous). $\square$

4. MANY COUNTABLE CLOSED STRUCTURES

In this section we will show that there exist $2^{\aleph_0}$ countable closed $\text{LFC}_{p,A}$ -groups. We need a preparatory lemma similar to [11, Lemma 2.2]. Note that for $U \leq G \in \text{LFC}_{p,A}$, every AT -normal series with elementary-abelian AT -factors in G induces an AT -normal series with elementary-abelian AT -factors in U in the sense of [11, p. 163].

Lemma 4.1. *Let Σ_U be an AT -chief series in the $FC_{p,A}^0$ -group U . Then there exists an $FC_{p,A}^0$ -supergroup V of U such that every AT -normal series with elementary-abelian AT -factors in V induces Σ_U in U .*

Proof. Since every $FC_{p,A}^0$ -group is a central product of A with some FC_p^0 -group over a finite subgroup of A , we may assume without loss that A is finite. Let $1 \leq A = U_n < U_{n-1} < \dots < U_0 = T(U) \leq U$ be the series Σ_U . Recursively we will construct for $0 \leq l \leq n$ an FC_p^0 -supergroup V_l of U/U_l such that

$$(4.1) \quad hU_l \in \langle gU_l^{V_l} \rangle' \quad \text{for all } h \in U_k, \ g \in U_0 - U_k \text{ and } 1 \leq k \leq l-1.$$

To this end put $V_0 = U/U_0$ and assume that V_{l-1} has been found for some $l \in \{1, \dots, n\}$. Let $\hat{U} = U/U_l$, and denote the canonical epimorphism $\hat{U} \rightarrow U/U_{l-1} \leq V_{l-1}$ by θ . Identify $\hat{U}_{l-1}$ with the diagonal subgroup of $H = (\hat{U}_{l-1} \text{Wr } C_p) \text{Wr } C_{p^2}$. Because of $\exp(\hat{U}_{l-1}) = p$ we have $Z(H) = \hat{U}_{l-1} \leq H''$ (see [17, Theorem 4.1]).

Now, as in the 0th step of Construction 3.2, there exists a torsion-free central subgroup D of finite index in V_{l-1} and a countermap $\theta^*: V_{l-1} \rightarrow \hat{U}$ to θ such that the corresponding standard embedding

$$\sigma: \hat{U} \rightarrow \hat{U}_{l-1} \text{Wr } V_{l-1} \leq H \text{Wr } V_{l-1}$$

maps $\hat{U}$ into the FC_p^0 -group

$$V_l = \{(v, f) \in H \text{Wr } V_{l-1} \mid \text{Im } f \text{ is constant on each coset } \tilde{v} \cdot D \ (\tilde{v} \in V_{l-1})\}.$$

Let Ω be the base group of V_l . Fix $h \in U_k$ and $g \in U_0 - U_k$ for some $k \in \{1, \dots, l-1\}$. By our recursion, (4.1) yields that $\hat{h}\sigma \in \langle \hat{g}\sigma^{V_{l-1}} \rangle' \cdot w$ for some $w \in \Omega''$. Moreover, $w \in \langle \hat{g}\sigma^{V_l} \rangle'$ by Lemma 3.4. Hence $\hat{h}\sigma \in \langle \hat{g}\sigma^{V_l} \rangle'$. Suppress σ . This completes the recursion.

A further application of the above argument, with A in place of $\hat{U}_{l-1}$, and with $(A \text{Wr } C_q) \text{Wr } C_{q^2}$ in place of H where $q = \exp(A)$, yields an $FC_{p,A}^0$ -supergroup V of U such that

$$h \in \langle g^V \rangle' \quad \text{for all } h \in U_k, \ g \in U_0 - U_k \text{ and } 1 \leq k \leq n.$$

Now, let M/N be an elementary-abelian AT -factor in V . Choose k minimal with respect to $U_k \leq N$. If $M \cap U = U_k$, then $N \cap U = U_k$, and we are done. Suppose that $M \cap U > U_k$. Every $g \in (M \cap U) - U_k$ satisfies $g \in U_{j-1} - U_j$ for some $j \leq k$. On the other hand $j \geq k$, since (4.1) yields $U_j \leq \langle g^V \rangle' \leq M' \leq N$. Therefore, $g \in U_{k-1} - U_k$, and $U_k \leq N \cap U < M \cap U \leq U_{k-1}$. Since U_{k-1}/U_k is a chief factor in U , we obtain $U_k = N \cap U < M \cap U = U_{k-1}$. $\square$

In order to show the main result of this section we will use a result of B. Maier [16]. This makes it necessary to introduce the notion of a controller. Let $\mathfrak{X}$ be a class of groups. An $\mathfrak{X}$ -supergroup V of $U \in \mathfrak{X}$ is an $\mathfrak{X}$ -controller for U , if the following holds.

$$(4.2) \quad \begin{aligned} &\text{Whenever } G \text{ and } H \text{ are } \mathfrak{X}\text{-supergroups of } V, \text{ then there exist} \\ &\text{an } \mathfrak{X}\text{-group } W \text{ and embeddings } \sigma: G \rightarrow W \text{ and } \tau: H \rightarrow W \\ &\text{such that } \sigma|U = \tau|U. \end{aligned}$$

Theorem 4.2. *Let $U \in FC_{p,A}^0$ with $U \neq T(U) \neq A$. Then U has no $FC_{p,A}^0$ -controller.*

Now, every FC^0 -group is embeddable into the direct product of a free abelian group of finite rank and a finite group (see [19, Theorem 1.7]). Since each of these products has only countably many finitely generated subgroups, we obtain that there exist only countably many FC^0 -groups. Every $FC_{p,A}^0$ -group is the central product of A with some FC^0 -group over a finite subgroup of A , whence there exist only countably many $FC_{p,A}^0$ -groups. Therefore, [16, Theorem 4.1] applies, and so Theorem 4.2 immediately yields

Corollary 4.3. *There exist $2^{\aleph_0}$ pairwise nonisomorphic countable closed $LF C_{p,A}$ -groups for every fixed countable abelian p -group A .*

Proof of Theorem 4.2. Assume that there exists an $FC_{p,A}^0$ -controller V for U . Let Σ_V be an AT -chief series in V . Then $V_1 \cap U > V_2 \cap U = A$ for some AT -chief factor V_1/V_2 of Σ_V . Denote epimorphic images modulo torsion subgroups by bars. Fix $u \in U - T(U)$, and let $\bar{V} = \langle \bar{v}_1 \rangle \times \cdots \times \langle \bar{v}_r \rangle$. By Rado's Lemma [18] we may assume that $\bar{u} = \bar{v}_1^m$ for some $m \in \omega$. Fix $u_0 \in (U \cap V_1) - V_2$, put $\mathcal{K} = GF(p)$, and let $\bar{\bar{V}} = \bar{V}/C_{\bar{V}}(V_1/V_2)$. Lemma 2.2 yields that $K = \mathcal{K}\bar{\bar{V}}/An(V_1/V_2)$ is a finite field, and that a K -isomorphism $\varepsilon: K^+ \rightarrow V_1/V_2$ is given by $x\varepsilon = (u_0V_2)^x$ for all $x \in K$. Choose a prime $q \neq p$ not dividing $|K^\times|$. Let $\hat{V} = \langle \bar{v}_0 \rangle \times \langle \bar{v}_2 \rangle \times \cdots \times \langle \bar{v}_r \rangle \geq \bar{V}$ with $\bar{v}_0^q = \bar{v}_1$.

We will construct $FC_{p,A}^0$ -supergroups G and H of V with the following properties.

$$(4.3) \quad \bar{V} \leq \hat{V} \leq \bar{G} \cap \bar{H}.$$

$$(4.4) \quad \text{There exist } AT\text{-chief series } \Sigma_G \text{ in } G \text{ and } \Sigma_H \text{ in } H \text{ which induce } \Sigma_V \text{ in } V.$$

$$(4.5) \quad \text{If } V_1/V_2 \text{ is induced from the } AT\text{-chief factors } G_1/G_2 \text{ of } \Sigma_G \text{ and } H_1/H_2 \text{ of } \Sigma_H, \text{ then}$$

$$o(\bar{v}_0 \cdot C_{\bar{G}}(G_1/G_2)) = q \cdot o(\bar{v}_0 \cdot C_{\bar{H}}(H_1/H_2)).$$

Let us first show that the existence of G and H leads to a contradiction. From Lemma 4.1 we obtain $G \leq \hat{G} \in FC_{p,A}^0$ and $H \leq \hat{H} \in FC_{p,A}^0$ such that every AT -normal series with elementary-abelian AT -factors in $\hat{G}$ resp. $\hat{H}$ induces Σ_G in G resp. Σ_H in H . Since V is an $FC_{p,A}^0$ -controller for U , there exist $Y \in FC_{p,A}^0$ and embeddings $\sigma: \hat{G} \rightarrow Y$ and $\tau: \hat{H} \rightarrow Y$ with $\sigma|U = \tau|U$. Suppress σ and τ .

Consider an AT -chief series Σ_Y in Y . Clearly $\Sigma_Y \cap G = \Sigma_G$ and $\Sigma_Y \cap H = \Sigma_H$, whence $\Sigma_Y \cap V = \Sigma_V$ by (4.4). Let Y_1/Y_2 be the AT -chief factor in Σ_Y inducing V_1/V_2 in V . Then Y_1/Y_2 induces G_1/G_2 in G and H_1/H_2 in H . Considering everything modulo Y_2 we may assume that $Y_2 = G_2 = H_2 = V_2 = 1$. Clearly, $C_{\bar{X}}(X_1) = C_{\bar{X}}(u_0)$ holds for G , H , and Y in place of X , because X_1 is a minimal normal subgroup in X , and because $\bar{X}/C_{\bar{X}}(X_1)$ is abelian.

Thus, the diagram of canonical homomorphisms

commutes. Now, $\overline{v}_0$ is the unique q th root of $\overline{u}$ in the torsion-free abelian group $\overline{Y}$. Hence $\overline{v}_0 \cdot C_{\overline{G}}(G_1) \equiv \overline{v}_0 \cdot C_{\overline{Y}}(Y_1) \equiv \overline{v}_0 \cdot C_{\overline{H}}(H_1)$, which contradicts (4.5).

It remains to construct G and H . As in the proof of Lemma 4.1 we may assume that A is finite. Then V_2 is finite. Suppose that we can find FC_p^0 -supergroups $\tilde{G}$ and $\tilde{H}$ of V/V_2 satisfying (4.3)–(4.5) correspondingly. Then, by the 0th step of Construction 3.2 there exist, for the canonical homomorphisms $V \rightarrow V/V_2 \leq \tilde{G}$ resp. $V \rightarrow V/V_2 \leq \tilde{H}$, standard embeddings of V into $FC_{p,A}^0$ -subgroups $\tilde{G}$ of $V_2 \text{Wr} \tilde{G}$ resp. $\tilde{H}$ of $V_2 \text{Wr} \tilde{H}$. Since every $v \in V_2$ is mapped onto $(1, f_v) \in V_2 \text{Wr} \tilde{G}$, where $\text{Im } f_v$ is contained in the V -conjugacy class of v , there exists an AT -chief series in $V_2 \text{Wr} \tilde{G}$ and hence in $\tilde{G}$, which induces Σ_V in V such that G inherits the properties (4.3)–(4.5) from $\tilde{G}$. Similarly, H satisfies (4.3)–(4.5). Therefore we may assume from now on that $V_2 = 1$.

By the 0th step of Construction 3.2 there exists, for the canonical homomorphism $V/V_1 \rightarrow \overline{V} \leq \widehat{V}$, a standard embedding μ of V/V_1 into an FC_p^0 -subgroup W of $T(V)/V_1 \text{Wr} \widehat{V}$ with $\overline{W} = \widehat{V}$. As before, there is a T -chief series in W which induces the series $(\Sigma_V/V_1)\mu$ in $(V/V_1)\mu$.

Let $L = K(\alpha)$ for some primitive q th root of unity α . Since q does not divide $|K^\times|$, there exists $z \in K$ such that $z^q = \overline{v}_1 + An(V_1)$. Extend the canonical homomorphism $\theta: \overline{V} \rightarrow \overline{V} \leq K^\times$ to homomorphisms $\theta_1: \widehat{V} \rightarrow K^\times$ and $\theta_2: \widehat{V} \rightarrow L^\times$ via

$$\overline{v}_0 \theta_1 = z \quad \text{and} \quad \overline{v}_0 \theta_2 = z \cdot \alpha.$$

Put $P = K^+ \rtimes_{\theta_1} \widehat{V}$ and $Q = L^+ \rtimes_{\theta_2} \widehat{V}$. Since K^+ is an irreducible $K^\times$ -module, K^+ is a minimal normal subgroup of P . Similarly, because L is the smallest subring of L containing $\text{Im } \theta_2$, we see that L^+ is a minimal normal subgroup of Q .

By Construction 3.3 there is a countermap μ^* to the composition $\overline{\mu}$ of the canonical epimorphism $V \rightarrow V/V_1$ and of μ such that, for some torsion-free central subgroup D of finite index in W , the ms-embedding $\sigma: V \rightarrow P \text{Wr} W$ given by $v\sigma = (v\overline{\mu}, f_v)$ where $f_v(w) = (\overline{v}, (w\mu^* \cdot [(v\overline{\mu} \cdot w)\mu^*]^{-1} \cdot v)\varepsilon^{-1})$, satisfies

$$V\sigma \leq G = \{(w, f) \in P \text{Wr} W \mid f \text{ is constant on each of the cosets } \tilde{w} \cdot D \\ (\tilde{w} \in W), \text{ and the } \widehat{V}\text{-component of } f \text{ is constant}\} \in FC_p^0.$$

G inherits the property (4.3) from W : we may identify $\widehat{V}$ canonically with $\{(\bar{v}, 1) \cdot \bar{v}\delta \cdot T(G) | \bar{v} \in \widehat{V}\}$, where $\delta: P \rightarrow W$ denotes the diagonal embedding. (Observe that

$$T(G) = \{(w, f) \in G | w \in T(W), \text{ and the } \widehat{V}\text{-component of } f \text{ is trivial}\}$$

and that $\bar{v}\bar{\sigma} = (\bar{v}, 1) \cdot \bar{v}\delta \cdot T(G)$ for all $v \in V$.)

Now $\sigma|V_1 = \varepsilon^{-1}\delta$, and $G_1 = K^+\delta$ is a minimal normal subgroup in G . Therefore, the normal series $1 \leq G_1 \leq T(G) \leq G$ can be refined to a T -normal series in G which induces $\Sigma_V\sigma$ in $V\sigma$. Put $x = (\bar{v}_0, 1) \cdot \bar{v}_0\delta$. Then $\bar{x} = \bar{v}_0$ in $\bar{G}$, and

$$|\{u_0\sigma^{(x^k)} | k \in \mathbb{Z}\}| = |\{(u_0\varepsilon^{-1})\bar{v}_0^k | k \in \mathbb{Z}\}\delta| = |\{u_0\varepsilon^{-1} \cdot z^k | k \in \mathbb{Z}\}| = o(z),$$

whence $o(\bar{v}_0 \cdot C_{\bar{G}}(G_1)) = o(z)$ by Lemma 2.2. Similarly, we can find an ms-embedding $\tau: V \rightarrow H \leq Q \text{Wr} W$ where H is an FC_p^0 -group with the properties (4.3) and (4.4) such that $o(\bar{v}_0 \cdot C_{\bar{H}}(H_1)) = o(z \cdot \alpha) = o(z) \cdot o(\alpha) = q \cdot o(z)$. $\square$

Observe, that the groups G and H in the proof of Theorem 4.2 will have the same rank as V , if we replace G by

$$\{(w, f) \in P \text{Wr} W | f \text{ is constant on each of the cosets } \tilde{w} \cdot D \text{ } (\tilde{w} \in W),$$

$$\text{and the } \widehat{V}\text{-component of } f \text{ constantly equals } \bar{w}\},$$

and similarly H . Therefore we can even show that there exist $2^{\aleph_0}$ countable closed groups in the class of all $LFC_{p,A}$ -groups of rank $\leq \rho$ (for each fixed $\rho \in \omega - \{0\}$).

5. SOME PROPERTIES

In this section we will collect properties of countable closed $LFC_{p,A}$ -groups G which correspond to those which hold for countable e.c. LFC -groups [5, §§1–2] resp. the countable e.c. $L\mathfrak{F}_{p,A}$ -group E_A [13 and 11], and we will determine the action of G on its AT -chief factors.

Theorem 5.1. (a) *Let G be a countable closed $LFC_{p,A}$ -group. Then $C_G(T(G)) = A$. In particular, $Z(G) = A = Z(T(G))$.*

(b) *Let $\widehat{A}$ be an abelian p -group with $A \leq \widehat{A}$. Then there exists a closed $LFC_{p,A}$ -group G of cardinality $\max\{|\widehat{A}|, \aleph_1\}$ with $\widehat{A} \leq Z(G)$.*

Proof. (a) Assume that there exists $x \in C_G(T(G)) - A$. Put $\widehat{G} = G \times C_p$ and $C_p = \langle c \rangle$. Let $\{G_n | n \in \omega\}$ be an ascending chain of FC^0 -groups with union $\widehat{G}$. Following Construction 3.1 with the canonical epimorphism $\widehat{G} \rightarrow \bar{G} = \widehat{G}/(A \times C_p)$ in place of $\theta: G \rightarrow H$ we obtain for each $n \in \omega$ a partition $\bar{G} = \Omega_{n,0} \dot{\cup} \dots \dot{\cup} \Omega_{n,k_n}$ such that $\bar{G}_n$ permutes the $\Omega_{n,i}$ regularly by left multiplication, and such that the groups

$$W_n = \{(\tilde{g}, f) \in (A \times C_p) \cap G_n \text{Wr } \bar{G} | g \in G_n, \text{ and } f \text{ is constant on each } \Omega_{n,i}\}$$

form an ascending chain of FC_p -subgroups of $(A \times C_p)\text{Wr } \bar{G}$. Thus, Construction 3.2 yields an ms-embedding σ of $\widehat{G}$ into the $LFC_{p,A\sigma}$ -group $W = \bigcup\{W_n | n \in \omega\} \leq (A \times C_p)\text{Wr } \bar{G}$. (Because of $A \times C_p \leq Z(\widehat{G})$ we can replace $(A \times C_p) \rtimes \widehat{G}$ by $A \times C_p$ in the base group.)

Without loss we may assume that $\tilde{x} \in \tilde{G}_0$ permutes the $\Omega_{0,i}$ nontrivially (see Construction 3.1). Let $y = (1, f) \in W_0$ where

$$f(\tilde{g}) = \begin{cases} c & \text{if } \tilde{g} \in \Omega_{0,0}, \\ 1 & \text{else.} \end{cases}$$

Then $[x\sigma, y] \neq 1$ and $y^p = 1$. Since G is e.c. in $LFC_{p,A}$, there does already exist some $z \in G$ such that $[x, z] \neq 1$ and $z^p = 1$, in contradiction to the choice of x .

(b) Follow the construction in the proof of [4, Theorem 2.2] with $\hat{A}$ in place of A , and p^n in place of $n!$. Then $\bar{G} = B \rtimes E \in LFC_{p,A}$. Since every infinite $LFC_{p,A}$ -group is contained in a closed $LFC_{p,A}$ -group of the same cardinality, it suffices to show that $\bar{G}$ satisfies $\hat{A} \leq A(\bar{G})$ (see proof of [4, Theorem 2.2]).

Fix $a \in \hat{A}$ and $g: \bar{G} \rightarrow \omega - \{0\}$. Then there exists $m \in \omega - \{0\}$ and $\xi < \omega_1$ such that $\{\beta < \xi | g(d(\beta)) = m\}$ is infinite. Put $t = g(e(\xi, a))$. Let $tm = p^k l$ with $(p, l) = 1$. Then there exists $\beta < \xi$ with $g(d(\beta)) = m$ and $s = \varphi_\xi(\beta) \geq k$. Put $u = p^{s-k}$. Then

$$[d(\beta)^{g(d(\beta))}, e(\xi, a)^{g(e(\xi, a))}]^u = (z(s, a) \cdot c(s, a))^{tmu} = (z(s, a) \cdot c(s, a))^{p^l} = a^l.$$

It follows that $a^l \in A_g(\bar{G})$. But now $(p, l) = 1$ yields $a \in \langle a^l \rangle \leq A_g(\bar{G})$. $\square$

In the following, G will always denote a countable closed $LFC_{p,A}$ -group. The proof below of the divisibility of $G/T(G)$ however only requires that G is e.c. in $LFC_{p,A}$ (this was used in the proof of Theorem 2.5 above).

Theorem 5.2. *$T(G)$ is verbally complete, and $G/T(G)$ is divisible. In particular, $G' = T(G)$, and $G/T(G) \cong \mathbb{Q}^{(\omega)}$.*

Proof. Fix $h \in T(G)$ and any word $w(x_1, \dots, x_\nu) \neq 1$. Let $F = \langle f_1, \dots, f_\nu \rangle$ be a finite p -group such that $w = w(f_1, \dots, f_\nu) \in Z(F)$ and $o(w) = o(h)$ (see [3, Lemma 7]). Identify w with h . Let $\delta: G \rightarrow G \text{Wr } F / \langle h \rangle$ be the diagonal embedding. Put $V = T(G) \text{Wr } F / \langle h \rangle$. Because of $h \in Z(F)$, any Krasner-Kaloujnine-embedding $\sigma: F \rightarrow V$ extends $\delta|_{\langle h \rangle}$. Let $\hat{V} = V \rtimes G$, where $g \in G$ acts on V via conjugation by $g\delta$. Suppress δ and σ . Applying Construction 3.1 to the canonical epimorphism $\theta: G \rightarrow \bar{G} = G/T(G)$ (with $G_0 = 1$) we obtain for each $n \in \omega$ a partition $\bar{G} = \Omega_{n,0} \dot{\cup} \dots \dot{\cup} \Omega_{n,k_n}$ such that $\bar{G}_n$ permutes the $\Omega_{n,i}$ regularly by left multiplication, and such that the groups

$$W_n = \{(\bar{g}, f) \in \hat{V} \text{Wr } \bar{G} | g \in G_n, \text{Im } f \subseteq (T(G_n) \text{Wr } F / \langle h \rangle) \rtimes G_n,$$

$$f \text{ is constant on each } \Omega_{n,i},$$

$$\text{and the } G_n\text{-component of } f \text{ is constant modulo } T(G_n)\}$$

form an ascending chain of FC -subgroups of $\hat{V} \text{Wr } \bar{G}$. Thus, Construction 3.2 yields an ms-embedding τ of G into the $LFC_{p,A\epsilon}$ -group $W = \bigcup \{W_n | n \in \omega\} \leq \hat{V} \text{Wr } \bar{G}$, where $\epsilon: V \rightarrow \hat{V} \text{Wr } \bar{G}$ denotes the diagonal embedding. Because of $G_0 = 1$ we have $\tau|_{T(G)} = \epsilon|_{T(G)}$. In particular, $f_1\epsilon, \dots, f_\nu\epsilon$ is a solution to $w(x_1, \dots, x_\nu) = h\tau$ in $T(W)$. Since G is e.c. in $LFC_{p,A}$, this shows that the equation $w(x_1, \dots, x_\nu) = h$ has a solution in $T(G)$. Thus, $T(G)$ is verbally complete.

Assume that there exists a prime q and some $\bar{z} \in \bar{G}$ which has no q th root in $\bar{G}$. Put $C_q = \langle c \rangle$ and $W = G \text{Wr } C_q$. Let $V = \langle G\delta, cz \rangle$ where $\delta: G \rightarrow W$

denotes the diagonal embedding. Then $A\delta \leq Z(V)$ and $(cz)^q = z\delta$. Since G is e.c. in $LFC_{p,A}$, this will be a contradiction if we can prove that $V \in LFC_p$.

Let $V_0 = \langle G_0\delta, cz \rangle$ for some $G_0 \leq G$ with $z \in G_0 \in FC_p^0$. Then $(G_0\delta)' \leq (T(G_0))\delta \leq \Omega = \{(1, f) \in W | \text{Im } f \subseteq T(G_0)\}$ and $[g\delta, cz] = [g\delta, z] \in \Omega$ for all $g \in G_0$. Hence, $V_0' \leq \langle \Omega^{V_0} \rangle = \Omega$. Since Ω is a finite p -group, it remains to show that V_0/Ω is torsion-free abelian (then $V_0 \in FC_p$). Clearly, V_0/Ω is abelian. Thus, V_0/Ω is the central product of the torsion-free abelian groups $G_0\delta \cdot \Omega/\Omega \cong G_0/T(G_0)$ and $\langle cz \rangle\Omega/\Omega$ over $\langle z\delta \rangle\Omega/\Omega$. Since $G_0\delta \cdot \Omega/\Omega$ contains no q th root of $z\delta \cdot \Omega$, it follows that V_0/Ω is also torsion-free. $\square$

In view of F. Haug's results one is tempted to conjecture that G is a split extension of $T(G)$ by $\mathbb{Q}^{(\omega)}$. However, we did not even find a way to answer

Question 5.3. *Can every embedding $\mathbb{Z} \rightarrow G$ be extended to an embedding $\mathbb{Q} \rightarrow G$?*

Theorem 5.4. *Let M/N be an AT-chief factor in G . (a) If V is a nontrivial set of words, then $N \leq V(\langle g^{T(G)} \rangle)$ and $M = \langle g^G \rangle$ for every $g \in M - N$. In particular, G has a unique AT-chief series.*

(b) *Every $K \trianglelefteq T(G)$ satisfies $K \leq A$ or $A \leq K$, and every $K \trianglelefteq G$ satisfies $K \leq T(G)$ or $T(G) \leq K$.*

(c) *If $g \in M - N$, then gN contains elements of order p , and any such two are conjugate in $T(G)$.*

(d) *The normal series induced in $T(G)/A$ by the unique AT-chief series in G has order-type $(\mathbb{Q}, <)$.*

Proof. (a) Use §3 and follow the proof of [9, Theorem 4.7].

(b) If $h \in T(G)$ and $g \in G - T(G)$, then an application of Construction 3.2 and Lemma 3.4 similar as in (a), with $\theta: G \rightarrow G/T(G)$, $G_0 = 1$, $G_1 = \langle g \rangle$ and $\hat{Z}_1 = Z_1\theta = C_1\theta = D_1 = \langle g\theta^2 \rangle \not\leq \langle g\theta \rangle$, yields that $h \in \langle g^G \rangle$.

(c) Use §3 and follow the proof of [9, Theorem 4.10(b)].

(d) Follow the proof of [9, Theorem 4.11(a)/(b)] to show that the order-type in question is dense without a maximal element. Now, suppose that $T(G)/A$ has a minimal normal subgroup M/A . The group $G \times C_p$ is contained in a closed $LFC_{p, A \times C_p}$ -group H , and $C_p \leq \langle g^H \rangle''$ for every $g \in M - A$. Since G is closed in $LFC_{p,A}$, we obtain that $C_p \leq \langle g^G \rangle'' \leq M'' = 1$, a contradiction. $\square$

Lemma 5.5. *If U is a finite subgroup of G , then every conjugacy class of elements in $T(G)$ contains an element which centralizes U .*

Proof. Extend the conjugacy action of G on $T(G)$ to an action of G on $T(G)\text{Wr } C_p$ via $(c, f)^g = (c, f^g)$ where $f^g(d) = f(d)^g$ for all $d \in C_p$. Use Construction 3.2 to find an embedding σ of G into an $LFC_{p, A\sigma}$ -subgroup W of $((T(G)\text{Wr } C_p) \rtimes G)\text{Wr } G/T(G)$ which contains the diagonal subgroup, and such that $\sigma|T(G)$ is the diagonal embedding. Then the diagonal subgroup contains the required elements (see [10, Lemma 2.2]). $\square$

Theorem 5.6. (a) *Let M/N be an AT-chief factor in G . If $\mathcal{K} = GF(p)$ and $\overline{\overline{G}} = G/C_G(M/N)$, then $\overline{\overline{G}}$ is isomorphic to the multiplicative group of the algebraic closure of $\mathcal{K}$ via the canonical embedding $\overline{\overline{G}} \rightarrow \mathcal{K}\overline{\overline{G}}/An(M/N)$. In particular, M/N is infinite, $\overline{\overline{G}}$ acts transitively on $M/N - \{1\}$, and any two elements of order p in $M - N$ are conjugate in G .*

(b) If M_1/N_1 and M_2/N_2 are AT-chief factors in G , if $g \in G - T(G)$, and if n is any p' -number, then there exists an AT-chief factor M/N between M_1/N_1 and M_2/N_2 such that g acts as an automorphism of order n on M/N . In particular, $T(G)$ is the intersection of all $C_G(M/N)$ where M/N is an AT-chief factor in G .

(c) Let $M_1/N_1, \dots, M_r/N_r$ be AT-chief factors in G . Let K be the algebraic closure of $GF(p)$. Because of (a) we can identify $K^\times$ with $G/C_G(M_i/N_i)$ for each i . Fix K -isomorphisms $\varepsilon_i: K^+ \rightarrow M_i/N_i$ as in Lemma 2.2. Then, for any $k_1, \dots, k_r \in K^\times$, there exists $g \in G$ such that $g \cdot C_G(M_i/N_i) = k_i$.

Proof. (a) Since $\overline{G}$ is divisible by Theorem 5.2, it suffices to find for every prime $q \neq p$ an element of order q in $\overline{G}$. Adopt the notation introduced in Lemma 2.2. Let L be the algebraic closure of K . Then the p' -component of $\mathbb{Q}^+/\mathbb{Z}$ is isomorphic to $L^\times$, and this gives rise to an epimorphism $\mu: \mathbb{Q}^+ \rightarrow L^\times$. Choose $x \in \mathbb{Q}$ with $o(x\mu) = q$. Let $\nu: \overline{G} = G/T(G) \rightarrow L^\times$ be the composition of the canonical homomorphisms $\overline{G} \rightarrow \overline{G}$ and $\overline{G} \rightarrow K^\times \leq L^\times$. Define $\varphi: \mathbb{Q} \times \overline{G} \rightarrow L^\times$ via $(z, \overline{g})\varphi = z\mu \cdot \overline{g}\nu$. Consider L^+ as regular $L^\times$ -module. Identify K^+ with M/N via ε and suppress ε .

Let $\{G_n/N | n \in \omega\}$ be an ascending chain of FC_p^0 -groups with union G/N , where $G_0 = N$. Let $\{L_n | n \in \omega\}$ be an ascending chain of finite subfields of L such that $x\mu \in L_0$, $|L_n| \geq p^n$, $(M \cap G_n)/N \leq L_n^+$, and such that the preimage U_n of $L_n^\times$ under φ contains $\overline{G}_n$. Applying Construction 3.1 to the canonical epimorphism $\theta: G/N \rightarrow G/M$ we obtain for each $n \in \omega$ a partition $G/M = \Omega_{n,0} \cup \dots \cup \Omega_{n,k_n}$ such that $G_n M/M$ permutes the $\Omega_{n,i}$ regularly by left multiplication, and such that the groups

$$\begin{aligned} W_n = \{& (gM, f) \in (L^+ \rtimes_\varphi (\mathbb{Q} \times \overline{G})) \text{Wr } G/M | g \in G_n, \text{Im } f \subseteq L_n^+ \rtimes_\varphi U_n, \\ & f \text{ is constant on each } \Omega_{n,i}, \\ & \text{and the } (\mathbb{Q} \times \overline{G})\text{-component of } f \text{ is constant}\} \end{aligned}$$

form an ascending chain of FC -subgroups of $(L^+ \rtimes_\varphi (\mathbb{Q} \times \overline{G})) \text{Wr } G/M$. Thus, Construction 3.3 yields an ms-embedding σ of G/N into the $LF C_p$ -group $W = \bigcup \{W_n | n \in \omega\} \leq (L^+ \rtimes_\varphi (\mathbb{Q} \times \overline{G})) \text{Wr } G/M$. Identify $L^+ \rtimes_\varphi (\mathbb{Q} \times \overline{G})$ canonically with the diagonal subgroup of $(L^+ \rtimes_\varphi (\mathbb{Q} \times \overline{G})) \text{Wr } G/M$, and let $\Omega = L_0^+ \rtimes_\varphi U_0 \leq W$. Because of $G_0 = N$ we have that $\sigma|M/N$ is the diagonal embedding.

Now, let η be the composition of the canonical epimorphism $G \rightarrow G/N$ and σ . Apply Construction 3.2 to $\eta: G \rightarrow W$ to obtain an embedding τ of G into an $LF C_{p, A\tau}$ -subgroup V of $(T(G) \rtimes G) \text{Wr } W$. By Theorem 5.2 and Lemma 3.4 we have $y = (m_0\eta, 1) \in m_0\tau \cdot \langle m_0\tau^V \rangle'$. Let F be an $FC_{p, A\tau}^0$ -subgroup of V such that $\langle m_0\tau, (\omega, 1) | \omega \in \Omega \rangle \leq F$ and $y \in m_0\tau \cdot \langle m_0\tau^F \rangle'$. Since G is closed in $LF C_{p, A}$, there exists an embedding $\lambda: F \rightarrow G$ with $\tau\lambda|_A = \text{id}$. Let $J = \langle (\omega, 1) | \omega \in \Omega \rangle \lambda$. Then $J_0 = \langle (\omega, 1) | \omega \in \Omega_0^+ \rangle \lambda$ is a minimal normal subgroup in J , and $y\lambda \in m_0N$ enforces that $J_0 \cap N = 1$ and $J_0 \leq M$. Thus, $(x, 1)\lambda$ induces an automorphism of order q on $J_0 N/N$ and hence on M/N .

(b) Let L be the algebraic closure of $GF(p)$. Since $\overline{G} = G/T(G)$ is divisible, there exists an epimorphism $\varphi: \overline{G} \rightarrow L^\times$ such that $o(\overline{g}\varphi) = n$. Consider L^+ as regular $L^\times$ -module, and put $\hat{L} = L^+ \rtimes_\varphi \overline{G}$. Identify L^+ canonically

with the diagonal subgroup of $L^+ \text{Wr } C_p$, and extend the action of $L^\times$ on L^+ to an action of $L^\times$ on $L^+ \text{Wr } C_p$ as in the proof of Lemma 5.5. Then $\tilde{L} \leq \tilde{L} = (L^+ \text{Wr } C_p) \rtimes_\varphi \bar{G}$.

Suppose that $M_2 \leq N_1$. By Theorem 5.4(d) there exist AT -chief factors M_i^*/N_i^* such that $M_2 \leq N_2^* \leq M_2^* \leq N_1^* \leq M_1^* \leq N_1$. Choose $m_i \in M_i^* - N_i^*$ with $o(m_i) = p$ (Theorem 5.4(c)). Use Construction 3.3 with $\text{id}: G/N_1^* \rightarrow G/N_1^*$ in place of $\theta: G \rightarrow H$ to find an embedding σ of G/N_1^* into an LFC_p -subgroup V of $\tilde{L} \text{Wr } G/N_1^*$, which contains the diagonal subgroup, and such that $gN_1^*\sigma = (gN_1^*, f_{gN_1^*})$ where $f_{gN_1^*} \equiv (\bar{g}, 1)$. Identify $\tilde{L}$ canonically with the diagonal subgroup of $\tilde{L} \text{Wr } G/N_1^*$. Fix $l_0 = 1 \in L^+$. Since $L^+ \leq (L^+ \text{Wr } C_p)'$, Lemma 3.4 yields that $l_0 \in \langle m_1 N_1^* \sigma^V \rangle$. Let L_0 be a finite subfield of L containing $\bar{g}\varphi$, and let $\bar{U}_0$ be the preimage of $L_0^\times$ under φ . Put $\Omega = L_0^+ \rtimes_\varphi \bar{U}_0 \leq \tilde{L} \leq V$.

Now, apply Construction 3.2 to extend the standard embedding $\tau_0: \langle m_1 \rangle \rightarrow 1 \text{Wr } V$ given by $m_1 \tau_0 = (m_1 N_1^* \sigma, 1)$ to an embedding τ of G into an $LFC_{p, A\tau}$ -subgroup W of $(T(G) \rtimes G) \text{Wr } V$. Then Lemma 3.4 yields that $m_2 \tau \in \langle (l_0, 1)^W \rangle$ and $(gN_1^* \sigma, f) \in g\tau \cdot \langle m_1 \tau^W \rangle$ where $f \equiv (g, 1)$.

Let F be an $FC_{p, A\tau}^0$ -subgroup of W such that $\langle g\tau, m_1 \tau, m_2 \tau, (\omega, 1) | \omega \in \Omega \rangle \leq F$ and also $m_2 \tau \in \langle (l_0, 1)^F \rangle$, $(l_0, 1) \in \langle m_1 \tau^F \rangle$, and $(gN_1^* \sigma, f) \in g\tau \cdot \langle m_1 \tau^F \rangle$. Since G is closed in $LFC_{p, A}$, there exists an embedding $\lambda: F \rightarrow G$ such that $\tau\lambda|_{\langle A, m_1, m_2, g \rangle} = \text{id}$. Let M/N be the unique AT -chief factor in G with $x = (l_0, 1)\lambda \in M - N$. Because of $m_2 \in \langle x^G \rangle$ and $x \in \langle m_1^G \rangle$ we have that $M_2 \leq N_2^* \leq N \leq M \leq M_1^* \leq N_1$ (Theorem 5.4(a)). Because of $y = (gN_1^* \sigma, f)\lambda \in g \cdot \langle m_1^G \rangle \subseteq \bar{g}$ we have that g acts on M/N as y does. Let $J = \langle (\omega, 1) | \omega \in \Omega \rangle \lambda$. Then $J_0 = \langle (\omega, 1) | \omega \in L_0^+ \rangle \lambda$ is a minimal normal subgroup in J , and $x \in M - N$ enforces that $J_0 \cap N = 1$ and $J_0 \leq M$. Thus, y induces an automorphism of order q on $J_0 N/N$ and hence on M/N .

(c) Suppose that $M_{i+1} \leq N_i$ for $1 \leq i \leq r-1$. Proceeding recursively, we may assume that there exists $h \in G$ with $h \cdot C_G(M_i/N_i) = k_i$ for $1 \leq i \leq r-1$. Choose a finite subfield $K_0 \leq K$ such that $k_i \in K_0$ for all i . An iterated application of Theorem 5.4(c) and Lemma 5.5 yields embeddings $\mu_i: K_0^+ \rightarrow G$ such that $k\mu_i N_i = k\varepsilon_i$ for all $k \in K_0$ and all i , and such that $[K_0\mu_i, K_0\mu_j] = 1$ for all i, j . Note that $\langle K_0\mu_i | 1 \leq i \leq r-1 \rangle \cap M_r = 1$, and that $\langle K_0\mu_i | 1 \leq i \leq r \rangle \cap N_r = 1$.

Applying Construction 3.3, $\langle K_0\mu_i | 1 \leq i \leq r-1 \rangle N_r/N_r \rightarrow 1 \text{Wr } G/M_r$, the canonical embedding, can be extended to an embedding σ of G/N_r into an LFC_p -subgroup V of $(M_r/N_r \rtimes \bar{G}) \text{Wr } G/M_r$ which contains the diagonal subgroup, and such that $\sigma|_{M_r/N_r}$ is the diagonal embedding. Choose $g_r \in G$ such that $g_r \cdot C_G(M_r/N_r) = k_r$. Put $v = (hM_r, f) \in V$ where $f \equiv (\bar{g}_r, 1)$. Then v acts on $L_i = (K_0\mu_i N_r/N_r)\sigma$ as right multiplication with $k_i\mu_i N_r \sigma$. Put $V_0 = \langle v, L_1, \dots, L_r \rangle$.

Now use Construction 3.2 with $\sigma: G/N_r \rightarrow V$ in place of $\theta: G \rightarrow H$ to extend the canonical embedding $\langle K_0\mu_i | 1 \leq i \leq r \rangle \rightarrow 1 \text{Wr } V$ to an embedding τ of G into an $LFC_{p, A\tau}$ -subgroup W of $(T(G) \rtimes G) \text{Wr } V$. Let W_0 be an $FC_{p, A\tau}^0$ -subgroup of W with $(v_0, 1) \in W_0$ for all $v_0 \in V_0$. Since G is closed in $LFC_{p, A}$, there exists an embedding $\lambda: W_0 \rightarrow G$ with $\tau\lambda|_{\langle A, K_0\mu_1, \dots, K_0\mu_r \rangle} = \text{id}$. Now $g = (v, 1)\lambda$ is the desired element. $\square$

It follows from Theorem 5.6(a) that $T(G)/A$ is not isomorphic to E_A/A , where E_A denotes the unique countable e.c. $L\mathfrak{F}_{p,A}$ -group, since every abelian factor of E_A/A has order p (see [13]).

Theorem 5.7. *Let $K \trianglelefteq G$ with $A < K < T(G)$ and $K \neq \langle g^G \rangle$ for all $g \in G$.*

(a) *For every $FC_{p,A}^0$ -subgroup F of $T(G)$ there exists an embedding $\sigma: F \rightarrow K$ with $\sigma|_{K \cap F} = \text{id}$.*

(b) *Every normal subgroup of K is normal in $T(G)$, and $T(G)$ induces via conjugation locally inner automorphisms on K .*

(c) *$T(G)$ splits over K , and every finite p -subgroup U of $T(G)$ with $U \cap K = 1$ is contained in a complement to K in $T(G)$.*

Proof. (a) Let F_0 be a finite subgroup of F such that $F = AF_0$. Because of $Z(T(G)) = A$ there exists a finite group $F_0^* \leq T(G)$ such that $F_0 \leq F_0^*$ and $C_{F_0}(F_0^*) = A \cap F_0$. Use §3 and follow the proof of [9, Theorem 4.8(a)] to obtain an embedding $\tau: F_0^* \rightarrow K$ with $\tau|_{F_0^* \cap K} = \text{id}$. Then $A \cap F_0 \tau \leq C_{F_0 \tau}(F_0^* \tau) = (A \cap F_0) \tau = A \cap F_0$. Therefore, $\sigma: F \rightarrow K$ given by $(af)\sigma = a \cdot f\tau$ for all $a \in A$, $f \in F_0$ is the desired embedding.

(b) Follow the proof of [9, Theorem 4.8(b)/(c)].

(c) Use §3 and follow the proofs of [10, Theorems 4.1/4.2]. $\square$

Theorem 5.8. *Normality is transitive in $T(G)$, and for every proper subnormal subgroup S of G there exists an AT -chief factor M/N in G such that $N < S < M$.*

Proof. Use §3 and follow the proof of [9, Theorem 4.11(f)] to show that every subnormal subgroup S of $T(G)$ satisfies either $S \leq A$, or $S = K$ for some normal torsion subgroup K in G which does not occur in any AT -chief factor, or $N \leq S \leq M$ for some AT -chief factor M/N in G . Because of $M/N \leq Z(T(G)/N)$ this implies that normality is transitive in $T(G)$.

Now, let $S_2 \trianglelefteq S_1 \trianglelefteq G$. If $S_2 \leq T(G)$, then we are done. Suppose now that S_2 is not contained in $T(G)$. Then $T(G) \leq S_1$ by Theorem 5.4(b). Let $g \in S_2 - T(G)$. Applying Construction 3.2 as in the proof of Theorem 5.4(b), we obtain from Lemma 3.4 that $T(G) \leq [[g, T(G)], T(G)] \subseteq S_2$, whence $S_2 \trianglelefteq G$. $\square$

6. THE TORSION SUBGROUP

The aim of this section is to show the uniqueness of torsion subgroup in countable closed $LFC_{p,A}$ -groups via an algebraic characterization in terms of injectivity. As before, G will always denote a countable closed $LFC_{p,A}$ -group.

Theorem 6.1. *Let $\Sigma_{T(G)} = \{(M_q, N_q) | q \in \mathbb{Q} \cup \{-\infty\}\}$ be the series in $T(G)$ induced from the unique AT -chief series in G . Let H be a countable $L\mathfrak{F}_{p,A_0}$ -group with an A_0T -normal series $\Sigma_H = \{(K_j, L_j) | j \in J \cup \{-\infty\}\}$, which has central elementary-abelian A_0T -factors. Let $\alpha: J \cup \{-\infty\} \rightarrow \mathbb{Q} \cup \{-\infty\}$ be an order-preserving injection, and for every $j \in J$, let $\beta_j: K_j/L_j \rightarrow M_{j\alpha}/N_{j\alpha}$ be an embedding. (Note that $\beta_{-\infty}: A_0 \rightarrow A$.) If $\sigma_0: U \rightarrow G$ is an embedding of some finite subgroup $U \leq H$ satisfying $(K_j \cap U)\sigma_0 = M_{j\alpha} \cap U\sigma_0$, $(L_j \cap U)\sigma_0 = N_{j\alpha} \cap U\sigma_0$, and inducing β_j on $(K_j \cap U)L_j/L_j$ for every $j \in J$, then σ_0 can be extended to an embedding $\sigma: H \rightarrow G$ satisfying $K_j\sigma = M_{j\alpha} \cap H\sigma$, $L_j\sigma = N_{j\alpha} \cap H\sigma$, and inducing β_j on K_j/L_j for every $j \in J$.*

Proof. Use §3 and Lemma 5.5, and follow the proofs of [10, Theorems 3.1/3.2] (see also [1, Satz]). $\square$

Theorem 6.1 shows that $T(G)$ embeds every countable $L\mathfrak{F}_{p,A_0}$ -group ($A_0 \leq A$) in every possible way. In combination with a back-and-forth argument, Theorem 6.1 immediately yields

Corollary 6.2. *If G and H are countable closed $LFC_{p,A}$ -groups, then $T(G) \cong T(H)$.*

We denote the unique isomorphism type of torsion subgroup of the countable closed $LFC_{p,A}$ -groups by T_A ; we will also write T_p instead of T_1 . Note that Theorem 6.1 characterizes T_A , i.e., a countable $L\mathfrak{F}_{p,A}$ -group T is isomorphic to T_A if and only if it has an AT -normal series Σ of order type $(\mathbb{Q} \cup \{-\infty\}, <)$ with infinite central elementary-abelian factors such that Theorem 6.1 holds with T in place of $T(G)$ and with Σ in place of $\Sigma_{T(G)}$. In the following, $\Sigma = \{(M_q, N_q) | q \in \mathbb{Q} \cup \{-\infty\}\}$ will always denote the distinguished AT -normal series in T_A .

Theorem 6.3. (a) *An isomorphism $\alpha: U \rightarrow V$ between finite subgroups of T_A is induced by conjugation in T_A if and only if α induces the identity on each $(M_q \cap U)N_q/N_q$ ($q \in \mathbb{Q} \cup \{-\infty\}$).*

(b) *Let K be the algebraic closure of $GF(p)$. Because of Theorem 5.6(a) we may identify $K^\times$ with each $G/C_G(M_q/N_q)$ ($q \in \mathbb{Q}$). Fix K -isomorphisms $K^+ \rightarrow M_q/N_q$ as in Lemma 2.2. Then an isomorphism $\alpha: U \rightarrow V$ between finite subgroups of T_A is induced by conjugation in G , if and only if*

(1) *there exist $k_q \in K^\times$ such that α acts on $(U \cap M_q)N_q/N_q$ as k_q does, and*

(2) *α induces the identity on $A \cap U$.*

Proof. (a) If $U \leq A$, then we are done. Otherwise, let M/N be the unique AT -chief factor in G with $U \cap N < U \cap M = U$. Choose $U \cap N \leq V < U$ with $|U : V| = p$. As in the proof of [11, Theorem 6.1], we may assume that $\alpha|_V = \text{id}$. Fix $w \in U - V$.

Apply Construction 3.2 to the canonical epimorphism $\theta: G \rightarrow G/N$, with $G_0 = V$ and $G_1 = \langle U, w\alpha \rangle$. This yields an embedding σ of G into an $LFC_{p,A\sigma}$ -subgroup W of $(T(G) \rtimes G)\text{Wr } G/N$ such that $\sigma: V \rightarrow ((V \cap N) \rtimes 1)\text{Wr } G/N$. Let $\hat{R}$ be a right transversal of $U\theta$ in $G_1\theta$. In the notation of Construction 3.2, put $T = \hat{Z}^1 V\theta \cdot \hat{R} \cdot T^1$ (observe that $D_1 = 1$). Then T is a right transversal of $\langle w\theta \rangle$ in G/N , and as in the proof of [11, Theorem 6.1] the element $\tilde{g} = (1, s) \in T(W_1)$ given by

$$\begin{aligned} s((w\theta)^r t) &= f_{w^r}(t) \cdot f_{w^r\alpha}(t)^{-1} = (1, (w^r \cdot (w\alpha)^{-r})^{((w\theta)^r t)\theta_0^*}) \\ &= (1, w^r \cdot (w\alpha)^{-r}) \quad \text{for all } t \in T, \quad 0 \leq r \leq p-1, \end{aligned}$$

satisfies $w\sigma^{\tilde{g}} = w\alpha\sigma$.

It remains to show that $[V\sigma, \tilde{g}] = 1$. Let $v\sigma = (v\theta, f_v) \in W_0$ for $v \in V$. Since $U\theta$ is abelian, our choice of T ensures that $[(v\theta, 1), \tilde{g}] = 1$. Moreover, $[(1, f_v), \tilde{g}] = 1$ as in the proof of [11, Theorem 6.1].

(b) Let α be given. From Theorem 5.6(c) we obtain $g \in G$ such that $g \cdot C_G(M_q/N_q) = k_q$ for all $q \in \mathbb{Q}$ with $M_q \cap U > N_q \cap U$. But now, $g^{-1} \cdot \alpha$ induces the identity on each $(M_q \cap U)N_q/N_q$ ($q \in \mathbb{Q} \cup \{-\infty\}$), whence there exists $t \in T(G)$ such that $t = g^{-1} \cdot \alpha$. $\square$

Theorem 6.4. (a) $N_q \cong T_A$ and $T_A/N_q \cong T_E$ for every $q \in \mathbb{Q}$, where E denotes the countably infinite elementary-abelian p -group.

(b) If $K \triangleleft T_A$ with $A < K$ does not satisfy $N_q \leq K \leq M_q$ for some $q \in \mathbb{Q}$, then $K \cong T_A$ and $T_A/K \cong T_p$.

Proof. Because of Theorem 5.7(c), the groups in question satisfy the characterizations of T_A resp. T_E, T_p given by Theorem 6.1. $\square$

Question 6.5. In the situation of Theorem 6.4(b), is G/K a closed LFC_p -group?

As in the case of countable e.c. $L\mathfrak{F}_{p,A}$ -groups, we can show that different groups A yield different factors T_A/A :

Theorem 6.6. $H_n(T_A, \mathbb{Z}) = 0$ for all $n \geq 1$. In particular, T_A/A has Schur multiplier A .

For the proof of Theorem 6.6 we use a result of K. Varadarajan [20] which can be applied correspondingly to show that every e.c. $L\mathfrak{F}_{p,A}$ -group G satisfies $H_n(G, \mathbb{Z}) = 0$ for all $n \geq 1$. To this end we need the following definition. A group G is said to be *pseudo-mitotic*, if for every finitely generated subgroup H of G there exist embeddings $\psi_1, \psi_2: H \rightarrow G$ and an element $g \in G$ such that

- (a) $h\psi_2 = h \cdot h\psi_1$ for all $h \in H$,
- (b) $[H, H\psi_1] = 1$, and
- (c) $h\psi_2 = (h\psi_1)^g$ for all $h \in H$.

Proof of Theorem 6.6. Since every pseudo-mitotic group G satisfies $H_n(G, \mathbb{Z}) = 0$ for all $n \geq 1$ [20], we only need to show that T_A is pseudo-mitotic. Let $H \leq T_A$ be finite, and let G be a countable closed $LFC_{p,A}$ -group with $T_A = T(G)$. Without loss we may assume that H is not contained in A . Identify $T(G)$ with the diagonal subgroup of $V = T(G) \text{Wr } H$, and extend the conjugacy action of G on $T(G)$ to an action of G on V as in the proof of Lemma 5.5. Denote by $\tau: H \rightarrow V$ the canonical embedding of H onto the top group of V . Clearly, $[H, H\tau] = 1$, and from Theorem 5.2 and [13, (3.2.3)] we have that $H \leq \langle h\tau^V \rangle'$ for all $h \in H - 1$.

Use Construction 3.2 to find an embedding σ of G into an $LFC_{p,A\sigma}$ -subgroup W of $(V \rtimes G) \text{Wr } G/T(G)$ which contains the diagonal subgroup, and such that $\sigma|_{T(G)}$ is the diagonal embedding. Since G is e.c. in $LFC_{p,A}$, we obtain an embedding $\psi_1: H \rightarrow G$ such that $[H, H\psi_1] = 1$ and

$$(6.1) \quad H \leq \langle h\psi_1^G \rangle' \quad \text{for all } h \in H - 1.$$

Define $\psi_2: H \rightarrow G$ via $h\psi_2 = h \cdot h\psi_1$. Since H is not contained in A , (6.1) yields that $A \cap H\psi_1 = 1$, and that $H \leq N$ where M/N denotes the AT -chief factor in G with $1 = N \cap H\psi_1 < M \cap H\psi_1$. Therefore, ψ_2 is an embedding, and $\psi_1^{-1}\psi_2: H\psi_1 \rightarrow H\psi_2$ induces the identity on each $(M_q \cap H\psi_1)N_q/N_q$ ($q \in \mathbb{Q} \cup \{-\infty\}$) and is thus induced by conjugation in T_A (Theorem 6.3). $\square$

7. AUTOMORPHISMS

In this section we will show that the structure of $\text{Aut}(T_A/A)$ is similar to that of $\text{Aut}(E_A/A)$ and $\text{Aut}(ULF)$ (cf. [1 and 5, §4]). As before, G denotes a countable closed $LFC_{p,A}$ -group with $T(G) = T_A$.

Theorem 7.1. *Consider the diagram*

$$\begin{array}{ccc} \text{Aut}(G) & \xrightarrow{\varphi} & \text{Aut}(G/A) \\ \eta \downarrow & & \downarrow \psi \\ \text{Aut}(T_A) & \xrightarrow{\chi} & \text{Aut}(T_A/A) \end{array}$$

of canonical homomorphisms. Then χ and ψ are embeddings, and $\text{Ker } \eta \cong \text{Ker } \varphi \cong \text{Hom}(\mathbb{Q}^{(\omega)}, D(A))$, where $D(A)$ denotes the divisible radical of A . In particular, η and φ are embeddings if A contains no Prüfer p -subgroup.

Proof. Clearly, $g^{-1} \cdot g\alpha \in A$ for all $g \in G$, $\alpha \in \text{Ker } \varphi$. Therefore, straightforward calculations show that an isomorphism $\tau: \text{Ker } \varphi \rightarrow \text{Hom}(G, A) \cong \text{Hom}(\mathbb{Q}^{(\omega)}, D(A))$ is given by $[g](\alpha\tau) = g^{-1} \cdot g\alpha$. (Note that $A \leq T_A = G' \leq \text{Ker}(\alpha\tau)$.)

Since $C_{G/A}(T_A/A) = 1$ by Theorem 5.4, it follows as in [5, Lemma 4.1] that $\text{Ker } \psi = 1$, and that $g^{-1} \cdot g\alpha \in A$ for all $g \in G$, $\alpha \in \text{Ker } \eta$. As above, we obtain $\text{Ker } \eta \cong \text{Hom}(\mathbb{Q}^{(\omega)}, D(A))$. Correspondingly, $\text{Ker } \chi \cong \text{Hom}(T_A, A)$ is trivial as T_A is perfect. $\square$

In the same way, an assertion on [1, p. 202] can be improved: The canonical homomorphism $\text{Aut}(E_A) \rightarrow \text{Aut}(E_A/A)$ is an embedding.

Now, put

$$\text{Stab}(\Sigma) = \{\alpha \in \text{Aut}(T_A) \mid \alpha \text{ centralizes every } M_q/N_q \ (q \in \mathbb{Q} \cup \{-\infty\})\},$$

and let $L \text{ Inn}(T_A)$ be the group of all locally inner automorphisms of T_A . Denote by $A(\mathbb{Q})$ the group of all order-preserving permutations of $\mathbb{Q}$, and by $GL(\aleph_0, p)$ the group of all automorphisms of the $\aleph_0$ -dimensional $GF(p)$ -vector space V . Fix isomorphisms $\gamma_q: M_q/N_q \rightarrow V$ ($q \in \mathbb{Q}$). Then an embedding

$$\phi: \text{Aut}(T_A)/\text{Stab}(\Sigma) \rightarrow \text{Aut}(A) \times [GL(\aleph_0, p)\text{Wr}_{\mathbb{Q}}A(\mathbb{Q})]$$

is given by

$$(\alpha \cdot \text{Stab}(\Sigma))\phi = (\alpha|_A, (\bar{\alpha}, f_\alpha)),$$

where

$$(M_{q\bar{\alpha}}, N_{q\bar{\alpha}}) = (M_q\alpha, N_q\alpha) \quad \text{and} \quad f_\alpha(q) = \gamma_{q\bar{\alpha}^{-1}}^{-1} \cdot \alpha \cdot \gamma_q \quad \text{for all } q \in \mathbb{Q}.$$

Theorem 7.2. (a) $\text{Stab}(\Sigma) = L \text{ Inn}(T_A)$, and $L \text{ Inn}(T_A)$ contains $2^{\aleph_0}$ automorphisms of order m for every p -number m , and for $m = \infty$.

(b) $\phi: \text{Aut}(T_A)/\text{Stab}(\Sigma) \rightarrow \text{Aut}(A) \times [GL(\aleph_0, p)\text{Wr}_{\mathbb{Q}}A(\mathbb{Q})]$ is actually an isomorphism. In particular, T_A/A is characteristically simple.

Proof. (a) follows from Theorem 6.3, and as in the proof of [9, Theorem 5.1].

(b) Follow the lines of proof of [11, Theorem 6.2] or [1, Korollar], using Theorem 6.1. $\square$

Corollary 7.3. *Let U and V be finite subgroups of T_A with*

$$\Sigma \cap U = \{(M_{q_i} \cap U, N_{q_i} \cap U) \mid 0 \leq i \leq r\}.$$

Then an isomorphism α between U and V is induced by an automorphism of T_A , if and only if

- (1) $\alpha|_{A \cap U}$ can be extended to an automorphism of A , and

(2) the map $\lambda: \{q_0, \dots, q_r\} \rightarrow \mathbb{Q}$ given by $(M_{q_i\lambda} \cap V, N_{q_i\lambda} \cap V) = ((M_{q_i} \cap U)\alpha, (N_{q_i} \cap U)\alpha)$ can be extended to an order-preserving permutation of $\mathbb{Q}$.

As in [5, §4], for groups $E \leq H$ we put

$$F_E(H) = \{\alpha \in \text{Aut}(H) \mid \alpha \text{ is locally induced by conjugation with an element from } E\}$$

and

$$F \text{ Inn}(H) = \{\alpha \in \text{Aut}(H) \mid \alpha \in F_E(H) \text{ for some finitely generated subgroup } E \text{ of } H\}.$$

Theorem 7.4. *If E is an $FC_{p,A}^0$ -subgroup of G , then $F_{E/A}(G/A)$ is isomorphic to the universal profinite completion of E/A .*

Proof. Let $\{G_n \mid n \in \omega\}$ be an ascending chain of $FC_{p,A}^0$ -subgroups of G with $G_0 = E$ and union G . As in the proof of [5, Theorem 4.4] it suffices to find for every $m \in \omega$ some $n \in \omega$ such that $C_{E/A}(G_n/A) \leq (E/A)^m$. Choose $E_0 \in FC^0$ with $E = E_0 A$. Let $\{1 = e_0, \dots, e_r\}$ be a transversal of E_0^m in E_0 . Without loss we may assume that E_0^m is a torsion-free subgroup of $Z(E_0)$.

Put $\tilde{G} = G \times C_{p^2}$ and $C_{p^2} = \langle c \rangle$. Let $\{\tilde{G}_n \mid n \in \omega\}$ be an ascending chain of FC^0 -groups with $\tilde{G}_0 = 1$, $\tilde{G}_1 = E_0 \times C_{p^2}$, and union $\tilde{G}$. Following Construction 3.1 with the canonical epimorphism $\theta: \tilde{G} \rightarrow H = \tilde{G}/(A \times C_{p^2})$ and with $Z_1 = C_1 = E_0^m$, $\hat{Z}_1 = D_1 = E_0^m \theta$, we obtain for each $n \in \omega$ a partition $H = \Omega_{n,0} \dot{\cup} \dots \dot{\cup} \Omega_{n,k_n}$ such that $\tilde{G}_n \theta$ permutes the $\Omega_{n,i}$ regularly by left multiplication, and such that the groups

$$W_n = \{(h, f) \in (A \times C_{p^2}) \cap \tilde{G}_n \text{ Wr } H \mid h \in \tilde{G}_n \theta, \text{ and } f \text{ is constant on each } \Omega_{n,i}\}$$

form an ascending chain of FC -subgroups of $(A \times C_{p^2}) \text{ Wr } H$. In the case when $n = 1$, the $\Omega_{n,i}$ are precisely the sets $(e_i E_0^m \theta) \cdot T$, where T is a right transversal of $E_0 \theta$ in H . Now, Construction 3.2 yields an ms-embedding σ of $\tilde{G}$ into the $LFC_{p,A\sigma}$ -group $W = \bigcup \{W_n \mid n \in \omega\} \leq (A \times C_{p^2}) \text{ Wr } H$ with $g\sigma = (g\theta, f_g)$ for all $g \in G$. (Because of $A \times C_{p^2} \leq Z(\tilde{G})$ we can replace $(A \times C_{p^2}) \rtimes \tilde{G}$ by $A \times C_{p^2}$ in the base group.)

Let $y = (1, f) \in W_1$, where

$$f(h) = \begin{cases} c & \text{if } h \in E_0^m \theta \cdot T, \\ 1 & \text{else.} \end{cases}$$

Then $[y, E_0^m \sigma] = 1$, and $[[y, e_i \sigma], e_i \sigma] \neq 1$ for $1 \leq i \leq r$. Since G is e.c. in $LFC_{p,A}$ (and since E_0^m is finitely generated), we obtain an element $x \in G$ such that $[x, E_0^m] = 1$ and $[[x, e_i], e_i] \neq 1$ for $1 \leq i \leq r$. The latter implies that $[x, e_i] \notin Z(G) = A$ (by Theorem 5.1). Choose $n \in \omega$ with $x \in G_n$. Then $C_{E/A}(G_n/A) \leq (E/A)^m$. $\square$

Note that the above method also yields an alternative proof of [5, Theorem 4.4].

Theorem 7.5. $\mathbb{R}^+ \lesssim \text{Aut}(T_A/A)$.

Proof. Let G be a countable closed $LFC_{p,A}$ -supergroup of $A \times \mathbb{Q}^+$. Then $\mathbb{Q}^+ \lesssim G/A$, and so we may follow the proof of [6, Korollar 5.9] with Theorem

7.4 in place of [6, Satz 5.7], to show that $\mathbb{R}^+ \lesssim \text{Aut}(G/A)$. But $\text{Aut}(G/A) \lesssim \text{Aut}(T_A/A)$ by Theorem 7.1. $\square$

Theorem 7.6. $F \text{ Inn}(G/A)$ is an LFC_p -group with

$$T(F \text{ Inn}(G/A)) = F \text{ Inn}(G/A)' = T(\text{Inn}(G/A)) \cong T_A/A.$$

Proof. Copy the proof of [5, Theorem 4.5]. $\square$

Finally, let us note that the following can be proved from Theorems 6.1 and 7.2(b) in the same way as the corresponding result in [12] for E_p .

Theorem 7.7. Let K be any field. Then the augmentation ideal of KT_p is the unique proper ideal in KT_p which is invariant under the basis transformations of KT_p induced by $\text{Aut}(T_p)$.

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