ON LAGRANGE INTERPOLATION AT DISTURBED ROOTS OF UNITY

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ABSTRACT. Let $z_{nk} = e^{it_{nk}}$, $0 \leq t_{n0} < \cdots < t_{nn} < 2\pi$, f a function in the disc algebra A , and $\mu_n = \max\{|t_{nk} - 2k\pi/(n+1)|: 0 \leq k \leq n\}$. Denote by $L_n(f; \cdot)$ the polynomial of degree n that agrees with f at $\{z_{nk}: k = 0, \dots, n\}$. In this paper, we prove that for every p , $0 < p < \infty$, there exists a $\delta_p > 0$, such that $\|L_n(f; \cdot) - f\|_p = O(\omega(f; \frac{1}{n}))$ whenever $\mu_n \leq \delta_p/(n+1)$. It must be emphasized that δ_p necessarily depends on p , in the sense that there exists a family $\{z_{nk}: k = 0, \dots, n\}$ with $\mu_n = \delta_2/(n+1)$ and such that $\|L_n(f; \cdot) - f\|_2 = O(\omega(f; \frac{1}{n}))$ for all $f \in A$, but $\sup\{\|L_n(f; \cdot)\|_p: f \in A, \|f\|_\infty = 1\}$ diverges for sufficiently large values of p . In establishing our estimates, we also derive a Marcinkiewicz-Zygmund type inequality for $\{z_{nk}\}$.

1. INTRODUCTION

Let D be the open unit disc in the complex plane with closure $\overline{D}$ and boundary T . Also, let $z_{nk} = e^{it_{nk}}$, $0 \leq t_{n0} < \cdots < t_{nn} < 2\pi$, and for a function f defined on T , let $L_n(f; \cdot)$ be the Lagrange polynomial of degree n that interpolates f at $\{z_{nk}: k = 0, \dots, n\}$. If f is analytic on $\overline{D}$, then the following result is well known (cf. [17, Chapter 7]).

Theorem A. *For any f analytic on $\overline{D}$, a necessary and sufficient condition for*

$$\|L_n(f; \cdot) - f\|_\infty \rightarrow 0$$

is that the family $\{z_{nk}: k = 0, \dots, n\}$ is uniformly distributed on T .

Here and throughout, we use the usual notation:

$$\|f\|_p = \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} & \text{for } 0 < p < \infty, \\ \sup_{|z|=1} |f(z)| & \text{for } p = \infty. \end{cases}$$

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In this paper, we consider Lagrange interpolation of functions in the disc algebra A ; that is, functions which are analytic in D and continuous on $\bar{D}$. It is well known, however, that the above result does not hold for $f \in A$ in general (cf. [6, 16]). In fact, Vértesi [16] proved in 1982 that for *any* family $\{z_{nk}: k = 0, \dots, n\}$, there exists an $f_0 \in A$ such that

$$\limsup_{n \rightarrow \infty} |L_n(f_0; \cdot)| = \infty$$

almost everywhere on T . For this reason, we must consider convergence in $L_p = L_p(T)$, $0 < p < \infty$. In this direction, probably the earliest result is due to Lozinski [10] in 1941, as follows.

Theorem B. For $0 < p < \infty$ and $z_{nk} = \exp(i2k\pi/(n+1))$,

$$\|L_n(f; \cdot) - f\|_p \rightarrow 0$$

for all $f \in A$.

Related results on interpolation at the roots of unity have also been obtained by Walsh and Sharma [18], Sharma and Vértesi [13], Saff and Walsh [12], and Shen [14], and at Fejér points on a Jordan curve by Curtiss [4], Al'per and Kalinogorskaja [1], Shen and Zhong [15], and Chui and Shen [2]. In this paper, we consider sample points z_{nk} which are not necessarily the $(n+1)$ th roots of unity.

In order to give a sharp estimate of the order of convergence, we first establish the following Marcinkiewicz-Zygmund type inequality which is of independent interest. To facilitate our presentation, we need the following notation:

$\pi_n =$ the class of polynomials of degree at most n

and

$$(1.1) \quad \mu_n = \max_{0 \leq k \leq n} \left| t_{nk} - \frac{2k\pi}{n+1} \right|.$$

Theorem 1. For any p , $1 < p < \infty$, there exist positive constants δ_p and C_p , such that whenever

$$(1.2) \quad \mu_n \leq \frac{\delta_p}{n+1},$$

then

$$\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \leq \frac{C_p}{n+1} \sum_{k=0}^n |P_n(z_{nk})|^p$$

for all $P_n \in \pi_n$, where $z_{nk} = e^{it_{nk}}$.

As usual, $E_n(f)$ will denote the error of uniform approximation of f on T from π_n , namely:

$$E_n(f) = \inf_{q \in \pi_n} \|f - q\|_\infty.$$

The main theorem in this paper is the following.

Theorem 2. For any p , $0 < p < \infty$, there exist positive constants δ_p and C'_p , such that whenever (1.2) is satisfied, then

$$\|L_n(f; \cdot) - f\|_p \leq C'_p E_n(f)$$

for any $f \in A$.

We remark that for $1 < p < \infty$, the δ_p in the above two theorems are the same, and for $0 < p \leq 1$, we may choose $\delta_p = \delta_{p'}$ for any $p' > 1$. As a simple consequence of the above theorem, we have the following result.

Corollary 1. Under the hypotheses stated in Theorem 2,

$$\|L_n(f; \cdot) - f\|_p \leq C_{p,r} n^{-r} \omega\left(f^{(r)}; \frac{1}{n}\right)$$

for any $f \in A$ with $f(e^{it}) \in C^r[0, 2\pi]$, where $\omega(g; t)$ is the uniform modulus of continuity of g and $C_{p,r}$ is an absolute constant, depending only on p and r .

We remark that in condition (1.2) the constant δ_p must necessarily depend on p . In §4, we will construct a family $\{z_{nk}: k = 0, \dots, n\}$ with $\mu_n \leq \delta/(n+1)$, where $\delta > 0$, for which

$$\|L_n(f; \cdot) - f\|_2 \leq C_2 E_n(f)$$

for all $f \in A$, but

$$\sup\{\|L_n(f; \cdot)\|_p: f \in A, \|f\|_\infty = 1\} \rightarrow \infty.$$

We also remark that a necessary condition for $\|L_n(f; \cdot) - f\|_p \rightarrow 0$, where $f \in A$ and $0 < p < \infty$, is that the family $\{z_{nk}: k = 0, \dots, n\}$ is uniformly distributed on T . This will be proved in §5. (Recall from Theorem A that this is also a necessary condition for $\|L_n(f; \cdot) - f\|_\infty \rightarrow 0$.)

2. PRELIMINARY RESULTS

We first derive a result in harmonic analysis which is perhaps of independent interest. Let $\varphi \in L^1(T)$. Recall that $\varphi \in \text{BMO}$ with norm $\|\varphi\|_*$ if

$$\|\varphi\|_* := \sup_I \frac{1}{|I|} \int_I |\varphi(e^{i\theta}) - \varphi_I| d\theta < \infty,$$

where the supremum is taken over all arcs I on T with length $|I|$ and

$$\varphi_I = \frac{1}{|I|} \int_I \varphi(e^{i\theta}) d\theta.$$

The following is a well-known result due to John-Nirenberg (cf. [8, Chapter 6]).

Theorem C. There exist positive constants m and M such that for any $\varphi \in \text{BMO}$, any arc $I \subset T$, and any $\lambda > 0$,

$$\frac{|\{e_{i\theta} \in I: |\varphi(e^{i\theta}) - \varphi_I| > \lambda\}|}{|I|} \leq M \exp\left\{\frac{-m\lambda}{\|\varphi\|_*}\right\}.$$

We remark that the constants m and M can be chosen to be $m = \frac{1}{4e}$ and $M = \sqrt{e}$.

The following result on the BMO norm can be found in [9].

Theorem D. *There exists an absolute constant C_* such that for any $g(e^{i\theta}) \in BMO$ with $g(z)$ analytic in $|z| > 1$ and bounded at ∞ ,*

$$\|g(e^{i\theta})\|_* \leq C_* \inf\{\|g - h\|_\infty : h \in H^\infty\}.$$

For any $\delta > 0$ and $1 < p < \infty$, a nonnegative function w defined on T is called an A_p -weight relative to δ if

$$(2.1) \quad \sup_I \left(\frac{1}{|I|} \int_I w(e^{i\theta}) d\theta \right) \left(\frac{1}{|I|} \int_I (w(e^{i\theta}))^{-\frac{1}{p-1}} d\theta \right)^{p-1} < \delta.$$

For A_p -weights, the following result is due to Muckenhoupt [11].

Theorem E. *Let $1 < p < \infty$ and $w(e^{i\theta})$ be an A_p -weight relative to some $\delta > 0$. Then for any $g \in L^p(T)$, its Cauchy transform*

$$(2.2) \quad (Hg)(z) := \frac{1}{2\pi i} \int_T \frac{g(\zeta)}{\zeta - z} d\zeta, \quad |z| < 1,$$

satisfies

$$(2.3) \quad \int_0^{2\pi} |(Hg)(e^{i\theta})|^p w(e^{i\theta}) d\theta \leq C_H \int_0^{2\pi} |g(e^{i\theta})|^p w(e^{i\theta}) d\theta,$$

where C_H is an absolute constant depending only on δ and p .

Of course, $(Hg)(e^{i\theta})$ in inequality (2.3) is the almost everywhere radial limit of $(Hg)(re^{i\theta})$.

We have the following result.

Lemma 1. *For any p , $1 < p < \infty$, there exists an $\varepsilon_p > 0$, such that for all $\varphi \in BMO$ with $\|\varphi\|_* < \varepsilon_p$, $|e^\varphi|$ is an A_p -weight relative to $\delta = 2$.*

Proof. Since $\|\operatorname{Re} \varphi\|_* \leq \|\varphi\|_*$, we may assume, without loss of generality, that φ is a real-valued function. Set $\psi = -\varphi/(p-1)$ and $w = e^\varphi$. Then we have

$$(2.4) \quad \begin{aligned} & \left(\frac{1}{|I|} \int_I w d\theta \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} d\theta \right)^{p-1} \\ &= \left(\frac{1}{|I|} \int_I e^{\varphi - \varphi_I} d\theta \right) \left(\frac{1}{|I|} \int_I e^{\psi - \psi_I} d\theta \right)^{p-1}, \end{aligned}$$

where it follows from Theorem C that

$$\begin{aligned} \frac{1}{|I|} \int_I e^{\varphi - \varphi_I} d\theta &= \frac{1}{|I|} \int_0^\infty |\{z \in I: e^{\varphi(z) - \varphi_I} > \lambda\}| d\lambda \\ &\leq \frac{1}{|I|} \int_0^\infty |\{z \in I: |\varphi(z) - \varphi_I| > \ln \lambda\}| d\lambda \\ &\leq 1 + \frac{1}{|I|} \int_1^\infty |\{z \in I: |\varphi(z) - \varphi_I| > \ln \lambda\}| d\lambda \\ &\leq 1 + \int_1^\infty M \exp\left\{-\frac{m \ln \lambda}{\|\varphi\|_*}\right\} d\lambda = 1 + \frac{M\|\varphi\|_*}{m - \|\varphi\|_*}. \end{aligned}$$

Similarly, we also have

$$\frac{1}{|I|} \int_I e^{\psi - \psi_I} d\theta \leq 1 + \frac{M\|\psi\|_*}{m - \|\psi\|_*} = 1 + \frac{M\|\varphi\|_*}{(p-1)m - \|\varphi\|_*}.$$

Combining these two estimates in (2.4), we have

$$(2.5) \quad \left(\frac{1}{|I|} \int_I w \, d\theta \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \, d\theta \right)^{p-1} \\ \leq \left(1 + \frac{M\|\varphi\|_*}{m - \|\varphi\|_*} \right) \left(1 + \frac{M\|\varphi\|_*}{(p-1)m - \|\varphi\|_*} \right)^{p-1} < 2$$

for all $\varphi \in \text{BMO}$ with sufficiently small $\|\varphi\|_*$. $\square$

Next, to go from T to D , we introduce the function

$$(2.6) \quad \omega_n(z) = \prod_{k=0}^n \left(z - \left(1 - \frac{1}{n+1} \right) z_{nk} \right),$$

where $z_{nk} = e^{it_{nk}}$. Our key lemma for establishing Theorem 1 is the following.

Lemma 2. *Let $1 < p < \infty$ and μ_n be defined as in (1.1). Then there exists a positive constant δ_p such that whenever $\mu_n \leq \delta_p/(n+1)$, $|\omega_n(e^{i\theta})|^p$ is an A_p -weight relative to 2^{p+1} .*

Proof. We introduce three more polynomials:

$$(2.7) \quad \begin{cases} \omega_n^*(z) = \prod_{k=0}^n \left(z - \left(1 + \frac{1}{n+1} \right) z_{nk} \right), \\ \tilde{\omega}_n(z) = z^{n+1} - \left(1 - \frac{1}{n+1} \right)^{n+1}, \\ \tilde{\omega}_n^*(z) = z^{n+1} - \left(1 + \frac{1}{n+1} \right)^{n+1}. \end{cases}$$

Clearly, $\ln\left(\frac{\omega_n(z)}{\tilde{\omega}_n(z)}\right)$ is in BMO on T , analytic in $|z| > 1$, and bounded at ∞ , so that it follows from Theorem D that

$$(2.8) \quad \left\| \ln \left(\frac{\omega_n(e^{i\theta})}{\tilde{\omega}_n(e^{i\theta})} \right) \right\|_* \leq C_* \left\| \ln \left(\frac{\omega_n(z)}{\tilde{\omega}_n(z)} \right) - \ln \left(\frac{\omega_n^*(z)}{\tilde{\omega}_n^*(z)} \right) \right\|_\infty \\ = C_* \max_{\theta} \left| \sum_{k=0}^n \ln \frac{(e^{i\theta} - (1 - \frac{1}{n+1})e^{it_{nk}})(e^{i\theta} - (1 + \frac{1}{n+1})e^{i\frac{2k\pi}{n+1}})}{(e^{i\theta} - (1 + \frac{1}{n+1})e^{it_{nk}})(e^{i\theta} - (1 - \frac{1}{n+1})e^{i\frac{2k\pi}{n+1}})} \right| \\ = C_* \max_{\theta} \left| \sum_{k=0}^n \ln \left(1 + \frac{2e^{i\theta}(e^{it_{nk}} - e^{i\frac{2k\pi}{n+1}})}{(n+1)(e^{i\theta} - (1 + \frac{1}{n+1})e^{it_{nk}})(e^{i\theta} - (1 - \frac{1}{n+1})e^{i\frac{2k\pi}{n+1}})} \right) \right|.$$

To estimate the quantity

$$(2.9) \quad I_n(\theta) := \frac{2e^{i\theta}(e^{it_{nk}} - e^{i\frac{2k\pi}{n+1}})}{(n+1)(e^{i\theta} - (1 + \frac{1}{n+1})e^{it_{nk}})(e^{i\theta} - (1 - \frac{1}{n+1})e^{i\frac{2k\pi}{n+1}})}$$

on $[0, 2\pi]$, it is sufficient to consider $0 \leq \theta \leq \frac{\pi}{n+1}$. We separate the estimation of the denominator of $I_n(\theta)$ in (2.9) into two cases:

(i) For $k = 0$ and $0 \leq \theta \leq \frac{\pi}{n+1}$, we have

$$\left| e^{i\theta} - \left(1 + \frac{1}{n+1}\right) e^{it_{nk}} \right| \geq \left(1 + \frac{1}{n+1}\right) - 1 = \frac{1}{n+1}.$$

(ii) For $k \geq 1$ and $0 \leq \theta \leq \frac{\pi}{n+1}$, we assume, without loss of generality, that $\delta_p \leq \frac{1}{30}$, so that

$$\left| t_{nk} - \frac{2k\pi}{n+1} \right| \leq \frac{1}{30(n+1)}, \quad k = 1, \dots, n.$$

Hence, it follows that, for $k \geq 1$,

$$\begin{aligned} \left| e^{i\theta} - \left(1 + \frac{1}{n+1}\right) e^{it_{nk}} \right| &\geq |e^{i\theta} - e^{it_{nk}}| - \frac{1}{n+1} \geq \frac{2}{\pi} |\theta - t_{nk}| - \frac{1}{n+1} \\ &\geq \frac{2}{\pi} \left(\left| \theta - \frac{2k\pi}{n+1} \right| - \left| \frac{2k\pi}{n+1} - t_{nk} \right| \right) - \frac{1}{n+1} \\ &\geq \frac{2}{\pi} \left(\frac{(2k-1)\pi}{n+1} - \frac{1}{30(n+1)} \right) - \frac{1}{n+1} \\ &\geq \frac{4k}{3(n+1)}. \end{aligned}$$

Combining the estimates in (i) and (ii), we have

$$\left| e^{i\theta} - \left(1 + \frac{1}{n+1}\right) e^{it_{nk}} \right| \geq \frac{2(k+1)}{3(n+1)}, \quad k = 0, \dots, n.$$

The same lower bound also applies to the quantity

$$\left| e^{i\theta} - \left(1 - \frac{1}{n+1}\right) e^{i\frac{2k\pi}{n+1}} \right|.$$

Consequently, we have the following estimate of $I_n(\theta)$ defined in (2.9):

$$\begin{aligned} |I_n(\theta)| &\leq \frac{9}{2} \cdot \frac{n+1}{(k+1)^2} |e^{it_{nk}} - e^{i\frac{2k\pi}{n+1}}| \\ &\leq \frac{9}{2} \cdot \frac{n+1}{(k+1)^2} \left| t_{nk} - \frac{2k\pi}{n+1} \right| \leq \frac{9}{2} \cdot \frac{n+1}{(k+1)^2} \mu_n. \end{aligned}$$

Recalling that $\delta_p \leq \frac{1}{30}$, we have, for $\mu_n \leq \frac{\delta_p}{n+1}$,

$$(2.10) \quad |I_n(\theta)| \leq \frac{9}{2} \frac{n+1}{(k+1)^2} \mu_n \leq \frac{3}{20} < \frac{2}{3}, \quad k = 0, \dots, n.$$

Note that for $|\zeta| \leq \frac{2}{3}$, we have $|\ln(1 + \zeta)| \leq 2|\zeta|$, so that by using $\zeta = I_n(\theta)$ and estimate (2.10), the result in (2.8) yields

$$\left\| \ln \left(\frac{\omega_n(e^{i\theta})}{\tilde{\omega}_n(e^{i\theta})} \right) \right\|_* \leq 9C_*(n+1)\mu_n \sum_{K=0}^n \frac{1}{(k+1)^2} < 18C_*(n+1)\mu_n.$$

Hence, it follows from Lemma 1 that if

$$(2.11) \quad \mu_n \leq \frac{\varepsilon_p}{18C_*p(n+1)},$$

then $\left| \frac{\omega_n(e^{i\theta})}{\tilde{\omega}_n(e^{i\theta})} \right|^p$ is an A_p -weight relative to $\delta = 2$. However, it is clear that since

$$2^{-p} \leq |\tilde{\omega}_n(e^{i\theta})|^p \leq 2^p,$$

$|\omega_n(e^{i\theta})|^p$ is also an A_p -weight relative to $\delta = 2^{p+1}$. In view of (2.11), this completes the proof of Lemma 2 by choosing

$$(2.12) \quad \delta_p = \min\left(\frac{\varepsilon_p}{18C_*p}, \frac{1}{30}\right). \quad \square$$

In what follows, we need a result on H^p -interpolation. As usual, a sequence $\{\zeta_j\}$, $j = 1, 2, \dots$, in D is said to be δ -uniformly separated, where $\delta > 0$, if

$$(2.13) \quad \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left| \frac{\zeta_j - \zeta_k}{1 - \bar{\zeta}_k \zeta_j} \right| \geq \delta > 0 \quad \text{all } k.$$

The following result can be found in [5, Chapter 9, p. 149].

Theorem F. Let $0 < p < \infty$ and $\{\zeta_j\}$ be a δ -uniformly separated sequence in D . Then for any sequence of complex numbers $\{a_j\}$ satisfying

$$\sum_{j=1}^{\infty} |a_j|^p (1 - |\zeta_j|^2) < \infty,$$

there exists a function $g \in H^p$, such that

- (i) $g(\zeta_j) = a_j$, $j = 1, 2, \dots$, and
- (ii)

$$\frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^p d\theta \leq C_{p,\delta} \sum_{j=1}^{\infty} |a_j|^p (1 - |\zeta_j|^2)$$

where $C_{p,\delta}$ is an absolute constant depending only on p and δ .

We remark that the above theorem holds for any finite sequence, provided that δ is independent of the length of this sequence. In order to apply the above theorem, we need the following.

Lemma 3. Let μ_n satisfy (1.2). Then the sequence $\{(1 - \frac{1}{n+1})e^{it_{nk}}\}$, $k = 0, \dots, n$, is δ -uniformly separated for some $\delta > 0$ independent of n .

Proof. To simplify the notation, set

$$\rho = \left(1 - \frac{1}{n+1}\right) \quad \text{and} \quad \zeta_k = \rho z_{nk} = \left(1 - \frac{1}{n+1}\right)e^{it_{nk}},$$

where $z_{nk} = e^{it_{nk}}$. Then

$$(2.14) \quad \begin{aligned} \left| \frac{\zeta_j - \zeta_k}{1 - \bar{\zeta}_k \zeta_j} \right|^2 &= \frac{2\rho^2(1 - \cos(t_{nj} - t_{nk}))}{1 - 2\rho^2 \cos(t_{nj} - t_{nk}) + \rho^4} \\ &= \frac{4\rho^2 \sin^2\left(\frac{t_{nj} - t_{nk}}{2}\right)}{(1 - \rho^2)^2 + 4\rho^2 \sin^2\left(\frac{t_{nj} - t_{nk}}{2}\right)}. \end{aligned}$$

On the other hand, from the hypothesis, we have

$$\begin{aligned}
 |t_{nj} - t_{nk}| &\geq \left| \frac{2j\pi}{n+1} - \frac{2k\pi}{n+1} \right| - \left| t_{nj} - \frac{2j\pi}{n+1} \right| - \left| t_{nk} - \frac{2k\pi}{n+1} \right| \\
 (2.15) \quad &\geq \frac{2\pi|j-k|}{n+1} - \frac{2\delta_p}{n+1} \\
 &\geq \frac{2\pi}{n+1} \left(|j-k| - \frac{1}{4} \right) > \frac{\pi}{n+1} |j-k|
 \end{aligned}$$

for $j \neq k$, by recalling from (2.12) that $\delta_p \leq \frac{1}{30} < \frac{\pi}{4}$. Hence, since $\sin \theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$ and, as a function of x ,

$$\frac{4\rho^2 x}{(1-\rho^2)^2 + 4\rho^2 x}$$

is monotonically increasing for $x \geq 0$, we have for $0 \leq t_{nj} - t_{nk} \leq \pi$ or $-\pi \leq t_{nj} - t_{nk} \leq 0$, by applying (2.15):

$$\left| \frac{\zeta_j - \zeta_k}{1 - \bar{\zeta}_k \zeta_j} \right|^2 \geq \frac{4\rho^2 \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi}{2(n+1)}\right)^2 (j-k)^2}{(1-\rho^2)^2 + 4\rho^2 \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi}{2(n+1)}\right)^2 (j-k)^2}.$$

Consequently, in view of the fact that each term is less than 1, we may conclude from (2.14) that

$$(2.16) \quad \prod_{\substack{j=0 \\ j \neq k}}^n \left| \frac{\zeta_j - \zeta_k}{1 - \bar{\zeta}_k \zeta_j} \right|^2 \geq \prod_{l=1}^n \left(\frac{4\rho^2 \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi}{2(n+1)}\right)^2 l^2}{(1-\rho^2)^2 + 4\rho^2 \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi}{2(n+1)}\right)^2 l^2} \right)^2.$$

To estimate the lower bound in (2.16), we recall that $\rho = (1 - \frac{1}{n+1})$, so that

$$(1-\rho^2)^2 \leq \frac{4}{(n+1)^2} \leq \frac{8\rho^2}{(n+1)^2},$$

and (2.16) yields

$$\prod_{\substack{j=0 \\ j \neq k}}^n \left| \frac{\zeta_j - \zeta_k}{1 - \bar{\zeta}_k \zeta_j} \right| \geq \prod_{l=1}^n \frac{l^2}{2+l^2} \geq \prod_{l=1}^{\infty} \left(1 - \frac{2}{2+l^2} \right) =: \delta,$$

where $\delta > 0$, since $\sum_{l=1}^{\infty} 2/(2+l^2) < \infty$. $\square$

3. PROOF OF THE THEOREMS

We are now ready to prove Theorem 1. For any polynomial $P_n \in \pi_n$, we use the notation

$$(3.1) \quad P_n^*(z) = P_n \left(\left(1 - \frac{1}{n+1} \right)^{-1} z \right),$$

so that

$$(3.2) \quad P_n^* \left(\left(1 - \frac{1}{n+1} \right) e^{it_{nk}} \right) = P_n(z_{nk}),$$

where $z_{nk} = e^{it_{nk}}$, and

$$(3.3) \quad \begin{aligned} \int_{|z|=1} |P_n(z)|^p |dz| &\leq \int_{|z|=1+\frac{1}{n}} |P_n(z)|^p |dz| \\ &= \left(1 + \frac{1}{n}\right) \int_{|z|=1} |P_n^*(z)|^p |dz|. \end{aligned}$$

From hypothesis (1.2) on μ_n , we can apply Lemma 3 and conclude from Theorem F that there exists a function $g \in H^p$ which satisfies

$$(3.4) \quad g\left(\left(1 - \frac{1}{n+1}\right)e^{it_{nk}}\right) = P_n^*\left(\left(1 - \frac{1}{n+1}\right)e^{it_{nk}}\right) = P_n(z_{nk}),$$

$k = 0, \dots, n$, by using (3.2), and

$$(3.5) \quad \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^p d\theta \leq \frac{2C_{p,\delta}}{n+1} \sum_{k=0}^n |P_n(z_{nk})|^p,$$

by observing that

$$1 - \left|\left(1 - \frac{1}{n+1}\right)e^{it_{nk}}\right|^2 \leq \frac{2}{n+1}.$$

Hence, to complete the proof of Theorem 2, it is sufficient to show that

$$(3.6) \quad \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \leq C \int_0^{2\pi} |g(e^{i\theta})|^p d\theta$$

for some absolute constant C . Now, from (3.4), it is well known (cf. [17, Chapter 3]) that

$$(3.7) \quad g(z) - P_n^*(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\omega_n(z)}{\omega_n(\zeta)} \frac{g(\zeta)}{\zeta - z} d\zeta$$

for $|z| < 1$, where ω_n is defined in (2.6). By using the notation in (2.2), we have

$$\left(\frac{g - P_n^*}{\omega_n}\right)(z) = \left(H\left(\frac{g}{\omega_n}\right)\right)(z), \quad |z| < 1.$$

Also, for $1 < p < \infty$ and under the condition (1.2), Lemma 2 allows us to conclude that $|\omega_n(e^{i\theta})|^p$ is an A_p -weight. Hence, from Theorem E, it follows that

$$\begin{aligned} \int_0^{2\pi} |g(e^{i\theta}) - P_n^*(e^{i\theta})|^p d\theta &= \int_0^{2\pi} \left| \left(H\left(\frac{g}{\omega_n}\right)\right)(e^{i\theta}) \right|^p |\omega_n(e^{i\theta})|^p d\theta \\ &\leq C_H \int_0^{2\pi} |g(e^{i\theta})|^p d\theta, \end{aligned}$$

so that, by Minkowski's inequality,

$$(3.8) \quad \int_0^{2\pi} |P_n^*(e^{i\theta})|^p d\theta \leq \left(1 + C_H^{\frac{1}{p}}\right)^p \int_0^{2\pi} |g(e^{i\theta})|^p d\theta.$$

Finally, by applying (3.3) and (3.8), we arrive at (3.6) with $C \leq 2(1 + C_H^{\frac{1}{p}})^p$. This completes the proof of Theorem 1. $\square$

To prove Theorem 2, we first remark that the case $0 < p \leq 1$ can be reduced to the case $1 < p < \infty$ by a simple application of Hölder's inequality (cf. [15]). Hence, we now assume that $1 < p < \infty$ and that (1.2) is satisfied. Let $f \in A$ and $\hat{P}_n$ be its best uniform approximant from π_n ; that is, $\hat{P}_n \in \pi_n$ and

$$(3.9) \quad \|\hat{P}_n - f\|_\infty = E_n(f).$$

Since $L_n(\hat{P}_n; \cdot) = \hat{P}_n$, we have

$$(3.10) \quad \|L_n(f; \cdot) - f\|_p \leq \|\hat{P}_n - f\|_p + \|L_n(f - \hat{P}_n; \cdot)\|_p.$$

Also, since $L_n(f - \hat{P}_n; z_{nk}) = f(z_{nk}) - \hat{P}_n(z_{nk})$, it follows from Theorem 1 that

$$\|L_n(f - \hat{P}_n; \cdot)\|_p^p \leq \frac{C_p}{n+1} \sum_{k=0}^n |f(z_{nk}) - \hat{P}_n(z_{nk})|^p.$$

Hence, an application of (3.9) and (3.10) yields:

$$\|L_n(f; \cdot) - f\|_p \leq \left(1 + C_p^{\frac{1}{p}}\right) E_n(f).$$

This completes the proof of Theorem 2 with $C'_p = 1 + C_p^{\frac{1}{p}}$. $\square$

4. DEPENDENCE OF δ_p ON p

In this section, we construct a family $\{z_{nk}: k = 0, \dots, n\}$ on T with $\mu_n \leq \frac{\delta_2}{n+1}$ for some constant $\delta_2 > 0$ such that

$$(4.1) \quad \|L_n(f; \cdot) - f\|_2 \leq C'_2 E_n(f)$$

for all $f \in A$, but

$$(4.2) \quad \sup\{\|L_n(f; \cdot)\|_p: f \in A, \|f\|_\infty = 1\} \rightarrow \infty$$

for all sufficiently large values of p .

According to Theorem 1, there exists a δ_2 with $\frac{2\pi}{n+1} > \delta_2 > 0$ such that if we select

$$(4.3) \quad z_{nk} = \begin{cases} \exp\left(i \frac{2k\pi + \delta_2}{n+1}\right) & \text{for } 0 \leq k \leq [\frac{n}{2}], \\ \exp\left(i \frac{2k\pi}{n+1}\right) & \text{for } [\frac{n}{2}] < k \leq n, \end{cases}$$

then (4.1) is satisfied. Let $\lambda_n(z) = \prod_{k=0}^n (z - z_{nk})$. Then we may also write

$$(4.4) \quad \lambda_n(z) = (z^{n+1} - 1) \prod_{k=0}^{[\frac{n}{2}]} \frac{z - z_{nk}}{z - e^{i \frac{2k\pi}{n+1}}}.$$

Hence, for $\frac{5}{8}n \leq j \leq \frac{7}{8}n$, we have

$$\begin{aligned} |\lambda'_n(z_{nj})| &= (n+1) \left| z_{nj}^n \prod_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{z_{nj} - z_{nk}}{z_{nj} - e^{i\frac{2k\pi}{n+1}}} \right| \\ &\leq (n+1) \prod_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(1 + \left| \frac{1 - e^{i\frac{\delta_2}{n+1}}}{1 - e^{i\frac{2(k-j)\pi}{n+1}}} \right| \right) \\ &\leq (n+1) \left(1 + \frac{\delta_2}{2 \sin \frac{\pi}{8}} \right)^{\lfloor \frac{n}{2} \rfloor + 1} \leq c_1 n \end{aligned}$$

for some constant $c_1 > 0$. This gives

$$(4.5) \quad \sum_{k=0}^n \frac{1}{|\lambda'_n(z_{nk})|} \geq \sum_{\frac{5}{8}n \leq k \leq \frac{7}{8}n} \frac{1}{|\lambda'_n(z_{nk})|} \geq \frac{1}{4c_1}.$$

On the other hand, for $\zeta_0 = e^{-i\frac{\pi}{2(n+1)}}$, we have, from (4.4),

$$(4.6) \quad \left| \frac{-i-1}{\lambda_n(\zeta_0)} \right| = \left| \frac{\zeta_0^{n+1} - 1}{\lambda_n(\zeta_0)} \right| = \prod_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left| 1 - \frac{e^{i\frac{2k\pi}{n+1}} - z_{nk}}{\zeta_0 - z_{nk}} \right|.$$

It is obvious that there exists a constant $\varepsilon > 0$ such that whenever $0 \leq k \leq \varepsilon n$, we have

$$\left| \arg \frac{e^{i\frac{2k\pi}{n+1}} - z_{nk}}{\zeta_0 - z_{nk}} \right| \leq \frac{\pi}{4}.$$

Hence, it follows that

$$\begin{aligned} (4.7) \quad \prod_{0 \leq k \leq \varepsilon n} \left| 1 - \frac{e^{i\frac{2k\pi}{n+1}} - z_{nk}}{\zeta_0 - z_{nk}} \right| &\leq \prod_{0 \leq k \leq \varepsilon n} \left| 1 - \frac{e^{i\frac{2k\pi}{n+1}} - z_{nk}}{\zeta_0 - z_{nk}} \right| e^{i\frac{\pi}{4}} \\ &\leq c_2 \prod_{1 \leq k \leq \varepsilon n} \left(1 - \frac{c_3 \delta_2}{k} \right) \leq c_2 (\varepsilon n)^{-c_4 \delta_2} \end{aligned}$$

for some absolute positive constants c_2 , c_3 , and c_4 . In addition,

$$\begin{aligned} (4.8) \quad \prod_{\varepsilon n < k \leq \lfloor \frac{n}{2} \rfloor} \left| 1 - \frac{e^{i\frac{2k\pi}{n+1}} - z_{nk}}{\zeta_0 - z_{nk}} \right| &\leq \prod_{\varepsilon n < k \leq \lfloor \frac{n}{2} \rfloor} \left(1 + \left| \frac{e^{i\frac{2k\pi}{n+1}} - z_{nk}}{\zeta_0 - z_{nk}} \right| \right) \\ &\leq \prod_{\varepsilon n < k \leq \lfloor \frac{n}{2} \rfloor} \left(1 + \frac{c_5 \delta_2}{k} \right) \leq c_6 \end{aligned}$$

for some absolute constants c_5 and c_6 . By combining the information from (4.6), (4.7), and (4.8), we obtain

$$(4.9) \quad |\lambda_n(\zeta_0)| \geq c_7 n^{c_4 \delta_2}$$

for some $c_7 > 0$. Hence, using (4.5) and (4.9), we arrive at

$$(4.10) \quad \sum_{k=0}^n \left| \frac{\lambda_n(\zeta_0)}{(\zeta_0 - z_{nk}) \lambda'_n(z_{nk})} \right| \geq \frac{c_7}{8c_1} n^{c_4 \delta_2}.$$

To get rid of the absolute value, let

$$\arg \frac{\lambda_n(\zeta_0)}{(\zeta_0 - z_{nk})\lambda'_n(z_{nk})} = \theta_{nk}.$$

By a lemma in [3], there exists a function $f_n \in \mathcal{A}$ satisfying $\|f_n\|_\infty = 1$ and $f_n(z_{nk}) = e^{-i\theta_{nk}}$, $k = 0, \dots, n$. Hence, (4.10) yields

$$(4.11) \quad |L_n(f_n; \zeta_0)| = \left| \sum_{k=0}^n f_n(z_{nk}) \frac{\lambda_n(\zeta_0)}{(\zeta_0 - z_{nk})\lambda'_n(z_{nk})} \right| \geq \frac{c_7}{8c_1} n^{c_4\delta_2}.$$

Finally, by one of the two Marcinkiewicz-Zygmunds inequalities (cf. [20, p. 30]) and (4.11), we have, for $p \geq 1$,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |L_n(f_n; e^{i\theta})|^p d\theta &\geq \frac{c_p''}{n+1} \sum_{k=0}^n |L_n(f_n; \zeta_0 e^{i\frac{2k\pi}{n+1}})|^p \\ &\geq \frac{c_p''}{n+1} |L_n(f_n; \zeta_0)|^p \geq c_8 n^{pc_4\delta_2-1} \end{aligned}$$

for some absolute constant c_8 . Hence, using the fact that $\|f_n\|_\infty = 1$, we arrive at (4.2) for $p > \frac{1}{c_4\delta_2}$.

5. FINAL REMARKS

I. An assumption such as (1.2) on the distribution of $\{z_{nk}: k = 0, \dots, n\}$ on T is necessary for $\|L_n(f; \cdot) - f\|_p \rightarrow 0$ for all $f \in \mathcal{A}$. We already know from Theorem A that for $p = \infty$, this family must necessarily be uniformly distributed on T . In the following, we will show that the uniform distribution of $\{z_{nk}: k = 0, \dots, n\}$ on T is also necessary for $\|L_n(f; \cdot) - f\|_p \rightarrow 0$ for all $f \in \mathcal{A}$. Let

$$\eta_n(z) = \prod_{k=0}^n (z - z_{nk}).$$

It is well known (cf. [7, Chapter 2]) that the uniform distribution of the family $\{z_{nk}: k = 0, \dots, n\}$ on T is equivalent to

$$(5.1) \quad \lim_{n \rightarrow \infty} \frac{|\eta_n(z)|^{\frac{1}{n+1}}}{|z|} = 1, \quad |z| > 1,$$

where the convergence is uniform on every compact subset in $|z| > 1$. Assume that (5.1) does not hold. Then it is known (cf. [17, p. 161]) that there is some ζ_0 , $|\zeta_0| > 1$, and a sequence of integers $n_k \rightarrow \infty$ such that $|\eta_{n_k}(\zeta_0)|^{\frac{1}{n_k+1}} < 1$ for all k , so that

$$(5.2) \quad |\eta_{n_k}(\zeta_0)| < (1 - \varepsilon_0)^{n_k+1}$$

for some $\varepsilon_0 > 0$ and all k . Consider the function $f_0(z) = 1/(\zeta_0 - z)$ which is in $\mathcal{A}$. Then it follows from the formula

$$f_0(z) - L_n(f_0; z) = \frac{\eta_n(z)}{\eta_n(\zeta_0)(\zeta_0 - z)}$$

that

$$|f_0(0) - L_{n_k}(f_0; 0)| = \frac{1}{|\eta_{n_k}(\zeta_0)||\zeta_0|},$$

which tends to ∞ in view of (5.2). Hence, by [5, Theorem 1.5],

$$\|L_n(f_0; \cdot) - f_0\|_p \neq 0.$$

II. Recall that two of the main tools in establishing Theorem 1, and hence Theorem 2, are the H^p -interpolation result stated in Theorem F and the integral representation formula in equality (3.7). These two results, however, can be generalized to multiple nodes z_{nk} . In addition, if each z_{nk} , $k = 0, \dots, n$, has the same multiplicity, then Lemma 2 also applies, since $|\omega_n(e^{i\theta})|^\alpha$ is always an A_p -weight for any $\alpha > 0$. Hence, for any nonnegative integer q , by setting $N = (q+1)(n+1) - 1$, a simple modification of our proof of Theorem 1 yields the inequality

$$\frac{1}{2\pi} \int_0^{2\pi} |P_N(e^{i\theta})|^p d\theta \leq \frac{C_p}{n+1} \sum_{k=0}^n \sum_{j=0}^q \frac{|P_N^{(j)}(z_{nk})|^p}{(n+1)^{jp}}$$

for all $P_N \in \pi_N$ provided that (1.2) is satisfied, where $1 < p < \infty$. Of course, a different value of δ_p is required. Consequently, Theorem 2 can be easily extended to Hermite or Hermite-Fejér interpolation with the same estimates. In [20], this result was established for the roots of unities using another method.

Of course, the analogous problems for nodes z_{nk} with different multiplicities still remain open.

III. A seemingly very difficult problem is to determine the largest δ_p in the condition (1.2) for the validity of Theorems 1 and 2.

IV. A more interesting problem is to find a necessary and sufficient condition on the distribution of $\{z_{nk}: k = 0, \dots, n\}$ so that $\|L_n(f; \cdot) - f\|_p \rightarrow 0$ for all $f \in A$, where $0 < p < \infty$. Recall that (1.2) is a sufficient condition and the uniform distribution on T is a necessary condition. We remark that an example can be constructed to show that the uniform distribution of $\{z_{nk}: k = 0, \dots, n\}$ on T is not sufficient for L_p convergence, $0 < p < \infty$.

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