DIVISORS ON SYMMETRIC PRODUCTS OF CURVES

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ABSTRACT. For a curve with general moduli, the Neron-Severi group of its symmetric products is generated by the classes of two divisors x and θ . In this paper we give bounds for the cones of effective and ample divisors in the $x\theta$ -plane.

1. INTRODUCTION

Let C be a smooth irreducible curve of genus g , $J(C)$ its Jacobian variety and C_d its d -fold symmetric product. Fixing a point P_0 on C , we define the maps

$$\begin{aligned} u_d: C_d &\rightarrow J(C) \quad \text{by} \quad u_d(D) = \mathcal{O}(D - dP_0), \\ i_{d-1}: C_{d-1} &\rightarrow C_d \quad \text{by} \quad i_{d-1}(D) = D + P_0. \end{aligned}$$

On $J(C)$ we denote by Θ , the theta divisor and by θ , its class in the Neron-Severi group. On C_d we denote by θ_d , or simply (again) θ , the class of $u_d^*(\theta)$ and by x_d , or simply x , the class of $i_{d-1}(C_{d-1})$ in C_d . For curves C with general moduli, it is known that the Neron-Severi group of the symmetric product C_d is generated by θ and x , see [A-C-G-H]. In this paper we give estimates for the cones of the effective and ample divisors on C_d , in the θ, x -plane.

By standard theory, we know the following things about the map u_d :

1. **ABEL'S THEOREM.** The fiber of the map u_d containing the divisor D , is exactly the set of divisors belonging to the complete series of D .

2. **JACOBI'S INVERSION THEOREM.** The map $u_g: C_g \rightarrow J(C)$ is onto.

3. **POINCARÉ'S FORMULA.** We denote by W_d the image of C_d in the Jacobian by the map u_d and by w_d its class. For $0 \leq d \leq g$ we have

$$w_d = \frac{\theta^{g-d}}{(g-d)!}.$$

In particular, for $d = g - 1$ we have $w_{g-1} = \theta$.

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4. If $d \leq g$, the map u_d is birational to its image, reflecting the fact that $h^0(C, D) = 1$, for D general point in C_d .
 If $g + 1 \leq d \leq 2g - 2$, the generic fiber of u_d has dimension $d - g$.
 If $2g - 1 \leq d$, the map is a $\mathbf{P}^{d-g}$ -fibration with tautological class x .

Lemma 1 (Intersections on C_d). *For $0 \leq r \leq d \leq g$ we have on C_d*

$$(1) \quad \theta^r x^{d-r} = \frac{g!}{(g-r)!}.$$

Proof. Indeed, $x = \text{class}(i_{d-1}(C_{d-1})) = \text{class}\{D + P_1, D \in C_{d-1}, P_1 \text{ a fixed point in } C\}$ and so,

$$x^{d-r} = \text{class}\{D + P_1 + \cdots + P_{d-r}, D \in C_r, P_1, \dots, P_{d-r} \text{ fixed in } C\}.$$

Therefore $u_{d*}x^{d-r} = w_r$. Since the map u_d is generically 1-1, projection formula implies that $\theta^r x^{d-r}$ on C_d , is equal to $\theta^r w_r$ on $J(C)$. Poincaré's Formula and the fact that $\theta^g = g!$ complete the proof. Q.E.D.

We denote by $H_{an}^{2n}(C_d, \mathbf{Q})$ the algebraic part of $H^{2n}(C_d, \mathbf{Q})$. Given an algebraic cycle Z in C_d , we define the maps

$$A_k: H_{an}^{2m}(C_d, \mathbf{Q}) \rightarrow H_{an}^{2m}(C_{d+k}, \mathbf{Q})$$

by

$$A_k(Z) = \{E \in C_{d+k}, E - D \geq 0 \text{ for some } D \in Z\}$$

and

$$B_k: H_{an}^{2m}(C_d, \mathbf{Q}) \rightarrow H_{an}^{2m-2k}(C_{d-k}, \mathbf{Q})$$

by

$$B_k(Z) = \{E \in C_{d-k}, D - E \geq 0 \text{ for some } D \in Z\}.$$

We have the standard formulas, see [A-C-G-H, p. 367].

Lemma 2 (Push-pull formulas for symmetric products).

$$(2) \quad A_k(x^\alpha \theta^\beta) = \sum_{i=0}^k \binom{\beta}{i} \binom{g - \beta + i}{i} \binom{d + k - \alpha - 2\beta}{k - i} i! x^{\alpha+i} \theta^{\beta-i}$$

and

$$(3) \quad B_k(x^\alpha \theta^\beta) = \sum_{j=0}^k \binom{\alpha}{k-j} \binom{\beta}{j} \binom{g - \beta + j}{j} j! x^{\alpha-k+j} \theta^{\beta-j}.$$

2. AMPLE AND EFFECTIVE DIVISORS

We will use often the following criterion for ampleness, see [Ha]:

Lemma 3 (Nakai-Moishezon). *Let D be a Cartier divisor on a variety X . Then D is ample on X , if and only if, for every subvariety Y in X of dimension r , we have that $D^r Y > 0$.*

In particular, when X is a smooth surface this says that the cone of effective and the cone of ample divisors are dual under the intersection pairing.

If X is a smooth surface, we have the following numerical criterion, for checking effectivity of a divisor D on X , see [Ha]:

Lemma 4. *Let X be a smooth surface, H an ample divisor on X and D a divisor with $D^2 > 0$ and $DH > 0$.*

Then, for n sufficiently large, nD is an effective divisor.

Proof. We prove first that for n large enough we have $h^2(X, nD) = 0$: Indeed, $h^2(X, nD) = h^0(X, K_X - nD)$, and if we assume that $h^2(X, nD) > 0$ for all n , then $K_X - nD$ is effective and so, $(K_X - nD)H > 0$ (Lemma 3) i.e. $K_X H > nDH$. Since $DH > 0$, taking n big enough—namely $n > K_X H$ —we get $K_X H > n$. A contradiction. Therefore there exists an n with $h^2(X, nD) = 0$ and the Riemann-Roch theorem completes the proof. Q.E.D.

In the higher dimensional case i.e. if X is a smooth variety of dimension d , the Riemann-Roch theorem gives that

$$\begin{aligned} h^0(X, D) - h^1(X, D) + h^2(X, D) - \cdots + (-1)^d h^d(X, D) \\ = \frac{c_1^d(D)}{d!} + (\text{terms containing strictly lower powers of } c_1). \end{aligned}$$

If H is again an ample divisor on X and D a divisor satisfying $D^r H^{d-r} > 0$ for all $0 \leq r \leq d$, then it is *not* known if

$$h^0(X, nD) > 0 \quad \text{for } n \text{ big enough}$$

In order to have such a conclusion, we have to impose extra condition, for example, the restriction of $\mathcal{O}(D)$ on H to be an ample divisor. We have

Theorem 1. *Let X be a d -dimensional variety, H an ample effective divisor on X and D a divisor with $D^d > 0$ and $\mathcal{O}(D)|_H$ is an ample line bundle on H .*

Then,

$$h^0(nD) > 0 \quad \text{for } n \text{ big enough.}$$

Proof. We use the following lemmas:

Lemma 5 (Serre). *Let X be a proper scheme over a Noetherian ring. If $\mathcal{N}$ is an invertible sheaf on X , then the following conditions are equivalent:*

- (i) $\mathcal{N}$ is ample.
- (ii) *For each coherent sheaf $\mathcal{E}$ on X , there exists an integer n_0 depending on $\mathcal{E}$ s.t. for each $i \geq 1$ and each $n \geq n_0$*

$$H^i(X, \mathcal{E} \otimes \mathcal{N}^n) = 0.$$

Lemma 6. *Let S be a proper scheme over a Noetherian ring, $\mathcal{L}$ an ample line bundle on S and $\mathcal{F}$ a line bundle generated by global sections. Then there exists an n_0 s.t.*

$$\forall n \geq n_0 \quad H^i(S, \mathcal{L}^n \otimes \mathcal{F}^k) = 0 \quad \forall k \geq 0, \quad \forall i \geq 1.$$

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Proof. We have an exact sequence $\bigoplus_{i=1}^r \mathcal{O}_S \rightarrow \mathcal{F} \rightarrow 0$. Tensoring by $\check{\mathcal{F}}$ we get $\bigoplus_{i=1}^r \check{\mathcal{F}} \rightarrow \mathcal{O}_S \rightarrow 0$. Hence (Koszul):

$$0 \rightarrow (\check{\mathcal{F}})^r \rightarrow \bigoplus_r (\check{\mathcal{F}})^{r-1} \rightarrow \bigoplus_r (\check{\mathcal{F}})^{r-2} \rightarrow \cdots \rightarrow \bigoplus_r \check{\mathcal{F}} \rightarrow \mathcal{O}_S \rightarrow 0.$$

And so, tensoring by $\mathcal{F}^r$ we get

$$0 \rightarrow \mathcal{O}_S \rightarrow \bigoplus_r \mathcal{F} \rightarrow \bigoplus_{\binom{r}{2}} (\mathcal{F})^2 \rightarrow \dots \rightarrow \bigoplus_r (\mathcal{F})^{r-1} \rightarrow (\mathcal{F})^r \rightarrow 0.$$

By Lemma 5 we can choose n_0 so that $\mathbf{H}^i(S, \mathcal{L}^n \otimes \mathcal{F}^k) = 0$ for all $i \geq 1$, $n \geq n_0$, $k = 0, 1, \dots, r$. Then this n_0 works: suppose the claim is true for $k \leq l$. Tensoring the above exact sequence by $\mathcal{L}^n \otimes \mathcal{F}^{l+1-r}$, all sheaves have zero $\mathbf{H}^i$'s, $i \geq 1$, $n \geq n_0$, except maybe the last one which is $\mathcal{L}^n \otimes \mathcal{F}^{l+1}$. Therefore the last one has also zero $\mathbf{H}^i$'s, $i \geq 1$, $n \geq n_0$. Q.E.D.

Going back to the proof of Theorem 1, we use first the following fact: “A line bundle L is ample on H iff L_{red} is ample on H_{red} ”. By this fact, we can replace H by mH without changing hypothesis on D , and so, we can assume that H is in fact *very* ample on X . We denote by L the line bundle $\mathcal{O}(D)$. We have that $L_1 = L \otimes \mathcal{O}_H$ is ample on H . Let H_1 be the restriction of $\mathcal{O}(H)$ to H ; then, H_1 is generated by global sections. Also L_1 is ample on H and so, by the above Lemma 6, there exists an integer n_0 s.t.

$$(4) \quad \mathbf{h}^i(H, L_1^n \otimes H_1^k) = 0 \quad \text{for all } i \geq 1, n \geq n_0, k \geq 0.$$

On the other hand H is ample on X and so, given the coherent sheaf L^n there exists by Lemma 5 an integer m_n s.t. $\mathbf{H}^i(X, L^n \otimes H^{m_n}) = 0$. Consider now the exact sequence:

$$0 \rightarrow \mathcal{O}_X(L^n \otimes H^{(l-1)}) \rightarrow \mathcal{O}_X(L^n \otimes H^l) \rightarrow \mathcal{O}_H(L_1^n \otimes H_1^l) \rightarrow 0.$$

The corresponding long exact sequence gives

$$\begin{aligned} 0 &\rightarrow \mathbf{H}^0(X, L^n \otimes H^{(l-1)}) \rightarrow \mathbf{H}^0(X, L^n \otimes H^l) \rightarrow \mathbf{H}^0(H, L_1^n \otimes H_1^l) \\ &\rightarrow \mathbf{H}^1(X, L^n \otimes H^{(l-1)}) \rightarrow \mathbf{H}^1(X, L^n \otimes H^l) \rightarrow \mathbf{H}^1(H, L_1^n \otimes H_1^l) \\ &\rightarrow \mathbf{H}^2(X, L^n \otimes H^{(l-1)}) \rightarrow \mathbf{H}^2(X, L^n \otimes H^l) \rightarrow \mathbf{H}^2(H, L_1^n \otimes H_1^l) \\ &\dots \\ &\rightarrow \mathbf{H}^k(X, L^n \otimes H^{(l-1)}) \rightarrow \mathbf{H}^k(X, L^n \otimes H^l) \rightarrow \mathbf{H}^k(H, L_1^n \otimes H_1^l) \\ &\dots \end{aligned}$$

and so, we get for each $n \geq n_0$ that

$$\mathbf{h}^i(X, L^n \otimes H^{(l-1)}) = \mathbf{h}^i(X, L^n \otimes H^l) \quad \text{for all } i \geq 2, l \geq 1.$$

Therefore, for each $n \geq n_0$ we have that

$$\mathbf{h}^i(X, L^n) = \mathbf{h}^i(X, L^n \otimes H) = \dots = \mathbf{h}^i(X, L^n \otimes H^{m_n}) = 0 \quad \text{for all } i \geq 2.$$

For each $n \geq n_0$ the Riemann-Roch theorem gives

$$\begin{aligned} \mathbf{h}^0(X, L^n) &= \mathbf{h}^1(X, L^n) + \frac{\mathbf{c}_1^d(L^n)}{d!} \\ &\quad + (\text{terms containing strictly lower powers of } \mathbf{c}_1) \end{aligned}$$

and so, since $\mathbf{c}_1^d(L) > 0$, we get that there exists an n big enough s.t. $\mathbf{h}^0(X, L^n) > 0$. Q.E.D.

3. THE CLASS OF THE DIAGONAL AND OF $\Gamma_n(g_d^r)$'s

We recall some theory from [A-C-G-H]. Consider the diagonal map

$$\phi_a = \phi: C_{d-2} \times C \rightarrow C_d$$

defined by

$$\phi(D, p) = D + 2p.$$

The image of this map is the diagonal Δ_d in C_d . A special case of Proposition 5.1 on p. 358 in [A-C-G-H] gives that

Lemma 7 (MacDonald). *The class δ_d of the diagonal Δ_d in C_d is given by*

$$(5) \quad \delta_d = 2((d + g - 1)x - \theta).$$

We denote now by g_d^r a base point free linear system of degree d and dimension r on C . Given such a g_d^r , then for each n with $r < n \leq d$, we can construct in C_n the following cycle

$$\Gamma_n(g_d^r) = \{D \in C_n \text{ s.t. } D \leq E \text{ for some } E \in g_d^r\}.$$

The standard way to calculate the class $\gamma_n(g_d^r)$ of the above cycle is given by the following lemma, see [A-C-G-H, Lemma 3.2, p. 342].

Lemma 8. *For integers $d \geq n > r$ the class $\gamma_n(g_d^r)$ in C_n is given by*

$$\gamma_n(g_d^r) = \sum_{k=0}^{n-r} \binom{d-g-r}{k} \frac{x^k \theta^{n-r-k}}{(n-r-k)!}.$$

In the particular case where $n = r + 1$ and so, $\Gamma_{r+1}(g_d^r)$ is a divisor in C_{r+1} , we can find the class as following:

We denote by $C^{\times(r+1)}$ the $(r+1)$ th Cartesian product of C , by $f_1, \dots, f_{r+1}$ the class of the coordinate planes and by δ_C the class of the sum of the diagonals in the product. Also we define $\gamma_C = \pi^*(\gamma_{r+1}(g_d^r))$, where $\pi: C^{\times(r+1)} \rightarrow C_{r+1}$ the canonical map. We have the following relations:

$$\pi^*x = f \stackrel{\text{def}}{=} f_1 + \dots + f_{r+1}, \quad \pi^*\delta_{r+1} = 2\delta_C, \quad \delta_{r+1} = 2((g+r)x - \theta).$$

Given a g_d^r on C we have a canonical map $\phi: C \rightarrow \mathbf{P}^r = \mathbf{P}$ and an induced (product) map $\Phi: C^{\times(r+1)} \rightarrow \mathbf{P}^{\times(r+1)}$. We denote by $\delta_{\mathbf{P}}$ the class of the sum of the diagonals in $\mathbf{P}^{\times(r+1)}$. Observe that

$$\Phi^*(\delta_{\mathbf{P}}) = \delta_C + \gamma_C.$$

We have $\delta_{\mathbf{P}} = f_1^{\mathbf{P}} + \dots + f_{r+1}^{\mathbf{P}}$, where $f_i^{\mathbf{P}}$'s are the classes of the coordinate planes in $\mathbf{P}^{\times(r+1)}$. Therefore, $\Phi^*(\delta_{\mathbf{P}}) = \Phi^*(f_1^{\mathbf{P}} + \dots + f_{r+1}^{\mathbf{P}}) = d(f_1 + \dots + f_{r+1}) = df$ and so, $\gamma_C = df - \delta_C$.

Now, $\pi^*(\gamma_{r+1}(g_d^r)) = \gamma_C = df - \delta_C = d\pi^*x - \frac{1}{2}\pi^*(\delta_{r+1})$ and so, $\gamma_{r+1}(g_d^r) = dx - \delta_{r+1}/2$. Using relation (5) we conclude that

$$(6) \quad \gamma_{r+1}(g_d^r) = \theta - (g - d + r)x.$$

4. FIRST BOUNDS FOR THE CONES

We examine the case $d \leq g$. If D is an effective divisor on C_d , then $u_d^* \theta^{d-1} \cdot D = \theta^{d-1} \cdot u_{d*} D$, where u_d the Abel-Jacobi map. Since θ is an ample

divisor on $J(C)$, we get by Lemma 3, that $\theta^{d-1} \cdot u_{d*} D \geq 0$, where equality holds iff $u_{d*} D = 0$. This gives the first naive bound for the effective cone in C_d :

Suppose that D is divisor with class $a\theta - bx$, $a, b > 0$ i.e. it "lies" in the fourth quarter of the θ, x -plane. We define slope m of D to be $m = \frac{b}{a}$.

If D is effective, then by the above discussion we have $(\theta - mx)\theta^{d-1} \geq 0$ which implies

$$\frac{g!}{(g-d)!} - m \frac{g!}{(g-d+1)!} \geq 0 \quad \text{i.e.} \quad m \leq g - d + 1.$$

If D is ample, Lemma 3 implies that $(\theta - mx)^d > 0$. Equivalently

$$\sum_{k=0}^d \binom{d}{k} \theta^k m^{d-k} x^{d-k} > 0,$$

i.e.

$$(7) \quad \sum_{k=0}^d \binom{d}{k} m^{d-k} \frac{g!}{(g-k)!} > 0.$$

Since for $m = 0$ this is positive, we must have $m < (\text{min. posit. root of (7)})$.

For a divisor D with class $ax - b\theta$, $a, b > 0$, i.e. it "lies" in the second quarter of the θ, x -plane we define the slope $\bar{m}$ of D to be $\bar{m} = \frac{a}{b}$. Similar argument gives that if D is effective then $\bar{m} \geq g - d + 1$, and if D is ample then $\bar{m}$ satisfies a similar relation as in (7). For example if D is ample then for

$$d = 2 \text{ we have } m < g - \sqrt{g} \text{ and } \bar{m} > g + \sqrt{g}.$$

5. EFFECTIVE AND AMPLE CONES FOR C_2

By the previous discussion we have the first bounds for the effective and ample cones for C_2 . On the other hand using the Lemma 4 we know that every

FIGURE 1. Bounds for the cones in C_2

class between the thick lines in Figure 1 is effective. Since the ample and effective cones are dual, it is enough to describe the effective cone. We make the observation:

Observation. If D is an irreducible effective divisor with slope m (resp. $\bar{m}$) between

$$g - \sqrt{g} < m \leq g - 1 \quad (\text{resp. } g + \sqrt{g} > \bar{m} \geq g - 1),$$

then it is unique with this property.

Indeed, if D' is another irreducible effective divisor with slope m' (resp. $\bar{m}'$) in the above range then we get $DD' < 0$, a contradiction.

If we are able to find such a divisor, then this describes the cone. In the second quarter such a divisor exists, namely the diagonal: Recall that $\delta_2 = 2((g+1)x - \theta)$ and so $\bar{m} = g + 1$ i.e. $g + \sqrt{g} > \bar{m} > g - 1$. Therefore the slope of the effective cone in the second quarter is given by

$$(9) \quad \bar{m}_{ef} = g + 1$$

and (by duality) of the ample cone by

$$(10) \quad \bar{m}_a = 2g.$$

(Note that a divisor with slope $\bar{m}_a$, has positive intersection with all the effective divisors in the first or the fourth quarter.)

Also, if C is of genus 2, then it is hyperelliptic and the corresponding g_2^1 gives an effective class in C_2 belonging in the fourth quarter with slope $m = 1$, see formula (6). By the above observation we get that $m_{ef} = 1$ and by duality $m_a = 0$. For general genus, we have the following:

Theorem 2. *Let C be a curve of genus $g \geq 3$ with general moduli. For the slopes of the cones in C_2 , in the fourth quarter of the θ, x -plane, we have*

1. *If g is a square, then $m_{ef} = m_a = g - \sqrt{g}$.*
2. *If g is not a square and $g > 3$, then $g - [\sqrt{g}] + 1 \geq m_{ef} \geq g - \sqrt{g}$ and so*

$$g - \sqrt{g} \geq m_a \geq g - \frac{g}{[\sqrt{g}] - 1}.$$

3. *If $g = 3$, then $m_{ef} = \frac{4}{3}$ and $m_a = \frac{6}{5}$.*

Proof. We use degenerations to special curves: From the formula (6) we have that the class of $\gamma_2(g_d^1)$ in C_2 is given by

$$\gamma_2(g_d^1) = \theta - (g - d + 1)x.$$

Therefore if $d < \sqrt{g} + 1$, the slope of the divisor is

$$m^0 = g - d + 1 \geq g - \sqrt{g}.$$

Take a smooth curve C^0 with a g_d^1 , $d < \sqrt{g} + 1$. If in addition, we choose the curve to be “general” having such a g_d^1 , then the corresponding divisor $\Gamma_2(g_d^1)$ is irreducible in C_2 . Since C^0 is special, the $H_{an}^*(C^0, \mathbb{Q})$ may not be generated by x, θ . Consider in $H_{an}^*(C^0, \mathbb{Q})$ the plane Π spanned by x, θ . By the previous analysis, since $m^0 > g - \sqrt{g}$ the intersection of the effective cone with the plane Π is given by the slopes $\bar{m}_{ef}^0 = g + 1$ and $m_{ef}^0 = g - d + 1$.

Since $\mathcal{M}_g$ is connected, we can find a flat family of smooth curves $\mathcal{C} \rightarrow \Delta$, Δ a disk, with central fiber C^0 and the other fiber curves with general moduli.

FIGURE 2. The genus 3 case

Let $\Phi: \mathcal{E}_2 \rightarrow \Delta$ be the flat family with fibers the 2-symmetric products of the fibers of $\mathcal{E} \rightarrow \Delta$.

Suppose that the general curve has an irreducible divisor with slope m , $g - 1 \geq m > g - \sqrt{g}$. By the above observation, this is unique and so, it gives rise to an effective divisor in $\mathcal{E}_2 \setminus \Phi^{-1}(0)$. Degenerating to the central fiber C_2^0 , we get an effective divisor D^0 . Since the degeneration “preserves” the algebraic equivalence we have that D^0 belongs to Π and so, since it is effective, the slope m satisfies $g - d + 1 \geq m \geq g - \sqrt{g}$.

Therefore if g is a square i.e. $g = k^2$, then choosing $d = k - 1 = \sqrt{g} - 1$ we get that m has to satisfy $m \leq g - \sqrt{g}$.

Since for slopes smaller than $g - \sqrt{g}$ we are in the effective cone by Lemma 4, we conclude that

$$m_{ef} = m_a = g - \sqrt{g}.$$

If g is not a square, choosing $d = \lceil \sqrt{g} \rceil$ we have

$$g - \sqrt{g} \leq m_{ef} \leq g - \lceil \sqrt{g} \rceil + 1.$$

The estimation of m_a comes from the duality of the ample and effective cone. Of course as $g \rightarrow \infty$ we have $m_{ef} \sim m_a \sim g - \sqrt{g}$. Q.E.D.

The case $g = 3$. In this case we can obtain for the general curve an irreducible divisor with slope $\frac{4}{3}$. (See Figure 2.) It is known that a general smooth curve C of genus 3, can be represented as a smooth plane quartic. We construct the following divisor in C_2 : For each point P in C consider the tangent at that point. This intersects the curve at two additional points Q, R . Take now in C_2 the divisor D consisting of all the sums $Q + R$ (with P moving on C). In order to calculate the class of D , we find its intersections with x and δ_2 .

Intersection with x . The degree of the dual curve is 12. Fixing a point Q on C , we ask for the number of tangents to the curve passing through Q (excluding the tangent to the curve at Q). These are 10. So $Dx = 10$.

Intersection with δ_2 . This is twice the number of bitangents to C , so $D\delta_2 = 28 \times 2 = 56$.

Therefore if $D \sim a\theta - bx$ then, $Dx = 10$ i.e. $3a - b = 10$ and $D\theta = 56$ i.e. $2(a\theta - bx)(4x - \theta) = 56$ so $6a - b = 28$. This gives $a = 6$, $b = 8$, i.e. $m_{ef} = \frac{4}{3}$, $m_a = \frac{6}{5}$.

6. ONE SIDE SLOPE OF EFFECTIVE CONE IN C_d

Theorem 3. *The boundary of the effective cone in C_d , in the second quarter of the θ, x -plane, is given by the class δ_d of the diagonal Δ_d*

$$\delta_d = 2((d + g - 1)x - \theta) \quad \text{i.e.} \quad \overline{m}_{ef} = d + g - 1.$$

Proof (Induction on d). For $d = 2$ it has been proved. Assuming it is true for d , we prove it for $d + 1$, i.e. we prove that in C_{d+1} , $\overline{m}_{ef} = d + g$.

Suppose that there exists an irreducible divisor D on C_{d+1} with class $mx_{d+1} - \theta_{d+1}$ and $m < d + g$ (we add subindices for avoiding confusion). Fixing a point in C , we define the canonical embedding $i_k: C_k \rightarrow C_{d+1}$. Note that $i_k^*(x_{d+1}) = x_k$ and $i_k^*(\theta_{d+1}) = \theta_k$. The image of C_d is an ample divisor on C_{d+1} , see [A-C-G-H, p. 310] and so, $i_d^*(D)$ is effective nonzero on C_d . Therefore the slope must satisfy $m \geq d + g - 1$.

Since Δ_{d+1} , D are irreducible the intersection $\Delta_{d+1}D$ is nonempty effective. Indeed, D , Δ are not disjoint. Otherwise,

$$m(d + g)x_{d+1}^2 - (m + d + g)\theta_{d+1}x_{d+1} + \theta_{d+1}^2 = 0.$$

Applying i_2^* we get $m(d + g)x_2^2 - (m + d + g)\theta_2x_2 + \theta_2^2 = 0$. Then formula (1) implies $md = (d + 1)g$. Since $i_2^*(\Delta_{d+1})$ contains Δ_2 , Δ_2 must be disjoint from $i_2^*(D) = D_2$. Therefore, $\Delta_2D_2 = 0$ i.e. $(mx_2 - \theta_2)\delta_2 = 0$ i.e. $m = 2g$, and the above relation becomes $2gd = (d + 1)g$ i.e. $d = 1$ a contradiction.

Therefore $B_1(D\Delta_{d+1})$, see §1 for definition of B_1 , lies in the effective cone of C_d , i.e.

$$(11) \quad \text{slope of } B_1(D\Delta_{d+1}) \geq d + g - 1.$$

Now,

$$\text{class}(D\Delta_{d+1}) = m(d + g)x_{d+1}^2 - (m + d + g)x_{d+1}\theta_{d+1} + \theta_{d+1}^2.$$

By Lemma 2 we have

$$B_1(x_{d+1}^2) = 2x_d, \quad B_1(x_{d+1}\theta_{d+1}) = \theta_d + gx_d, \quad B_1(\theta_{d+1}^2) = 2(g - 1)\theta_d.$$

Therefore,

$$B_1(D\Delta_{d+1}) = (2md + mg - dg - g^2)x_d - (m + d - g + 2)\theta_d.$$

Since $m \geq d + g - 1$, both coefficients are positive and so,

$$\text{slope of } B_1(D\delta_{d+1}) = \frac{2md + mg - dg - g^2}{m + d - g + 2}.$$

Relation (11) implies that

$$\frac{2md + mg - dg - g^2}{m + d - g + 2} \geq d + g - 1$$

i.e. $m(d + 1) \geq dg + d^2 + d + 3g - 2$. Since $m < d + g$ we get $(d + g)(d + 1) > dg + d^2 + d + 3g - 2$ or $1 > g$, a contradiction. Q.E.D.

7. BOUNDS FOR THE EFFECTIVE CONE IN C_r , $r \geq 3$

We start with C_3 . Let D be a divisor in C_3 with class $\theta - mx$, $m \geq 0$, i.e. it “lies” in the fourth quarter. Since x is the class of $i_2(C_2)$ in C_3 and

$$\frac{d}{dm}(\theta - mx)^3 = -3(\theta - mx)^2x,$$

we conclude that

$$\frac{d}{dm}(\theta - mx)^3 = 0 \text{ in } C_3 \Leftrightarrow (\theta - mx)^2 = 0 \text{ in } C_2,$$

i.e. when $m = g + \sqrt{g}$ or $m = g - \sqrt{g}$.

FIGURE 3. The graph of $(\theta - mx)^3$

The graph of the $(\theta - mx)^3$ considered as function of m is given by Figure 3.

Theorem 4. We denote by r_1 the root of $(\theta - mx)^3 = 0$ closest to 0. Then for $m < r_1$, the class $\theta - mx$ is effective.

Proof. Note that this equation has three positive roots, see Figure 3 above. Approximately as $g \rightarrow \infty$, r_1 goes to $g - \sqrt{3g}$. Say D a divisor with class $\theta - mx$ where $m < r_1$. By Theorem 2 the restriction of D to C_2 is ample. Since $D^3 > 0$, Theorem 1, applied to $H = C_2$, gives the result. Q.E.D.

It is difficult to continue the above method for higher r 's, since we do not have a good estimate for the ample cone in C_3 . For these cases we have

Theorem 5. Let C be a smooth curve of genus $g \geq 1$ with general moduli. For $r \geq 3$ we have the following estimates for the effective and ample cone of C_r in the fourth quarter of the θ, x -plane

1. If $r \geq g + 1$ then the boundary of the effective and ample cone is given by the θ line.

2. If $3 \leq r \leq g$ then we have the following bounds for the effective cone.

Bound from inside: for each rational m with

$$0 \leq m \leq \max([\frac{g}{r}], g - (r - 1)\sqrt{g})$$

there is an effective divisor with slope m .

Bound from outside: for each m with $m > g - [\sqrt{r - 1}\sqrt{g}]$ there is no effective divisor with slope m .

Remarks. 1. For $r = g$ the slope of the boundary of the effective cone is equal to 1. Indeed in this case $[\frac{g}{r}] = 1 = g - [\sqrt{r - 1}\sqrt{g}]$.

2. For $r \leq \sqrt{g}$ we have that

$$\max([\frac{g}{r}], g - (r - 1)\sqrt{g}) = g - (r - 1)\sqrt{g}.$$

For $r \geq \sqrt{g}$ we have that

$$\max([\frac{g}{r}], g - (r - 1)\sqrt{g}) = [\frac{g}{r}].$$

Proof. The proof of the first part of the theorem is easy: For $r \geq g + 1$ we have that $\theta^r = u_r^*(\theta)^r = 0$ and so, the class θ is not ample. On the other hand, since θ is ample on the Jacobian and x is ample on C_r , see [A-C-G-H, p. 310], projection formula and Lemma 3, imply that the class $\theta + \varepsilon x$ is ample, for each $\varepsilon > 0$. Therefore the bound for the ample cone is given by the line θ . Also any divisor with class $\theta - \sigma x$ cannot be effective since there is an $\varepsilon > 0$ small enough with $(\theta + \varepsilon x)^{r-1}(\theta - \sigma x) < 0$; a contradiction by Lemma 3. Therefore the bound of the effective cone is given by the θ line too.

To prove the second part of the theorem we use again degenerations to special curves. We start with a lemma:

Lemma 9. *If a curve C has a “general” g_d^{r-1} , $r \geq 3$ (i.e. without base points and not composed with an involution), then for any $d \geq r$ the divisor $\Gamma_r(g_d^{r-1})$ in C_r is irreducible.*

Proof. This is an application of the fact that the monodromy acts as the full symmetric group on the generic divisor of g_d^r . Q.E.D.

Let us now do some calculations. Recall from relation (6), that the class of $\gamma_r(g_d^{r-1})$ is given by $\theta - (g - d + r - 1)x$. Formula (1) gives that

$$x^k \theta^{r-k} = \frac{g!}{(g-r+k)!} \quad \text{and} \quad x^{k+1} \theta^{r-k-1} = \frac{g!}{(g-r+k+1)!}.$$

Using Lemma 8 we have for $d+1 \geq 2r$ that

$$(12) \quad \gamma_r(g_{d-r+2}^1) = \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{x^k \theta^{r-k-1}}{(r-k-1)!}.$$

Therefore intersection number $I = \gamma_r(g_d^{r-1}) \cdot \gamma_r(g_{d-r+2}^1)$ is

$$\begin{aligned} I &= \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{g!}{(g-r+k)!(r-k-1)!} \\ &\quad - (g-d+r-1) \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{g!}{(g-r+k+1)!(r-k-1)!} \\ &= g \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \binom{g-1}{r-k-1} \\ &\quad - (g-d+r-1) \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \binom{g}{r-k-1} \\ &= g \binom{d-r}{r-1} - (g-d+r-1) \binom{d-r+1}{r-1} \\ &= A(d^2 - 2d(r-1) - (r-1)(g-r-1)) \quad (A \text{ a positive constant}). \end{aligned}$$

This implies that

$$(13) \quad I \leq 0 \Leftrightarrow r-1 - \sqrt{r-1}\sqrt{g} \leq d \leq r-1 + \sqrt{r-1}\sqrt{g}.$$

Take now a smooth curve C^0 having a “general” $g_{d_0}^{r-1}$ with $d_0 = r+1 + [\sqrt{r-1}\sqrt{g}]$ (note that $d_0 \geq r$). We claim that the irreducible divisor $D = \Gamma_r(g_{d_0}^{r-1})$ with class $\gamma_r(g_{d_0}^{r-1}) = \theta - (g - [\sqrt{r-1}\sqrt{g}])x$ gives the bound for the effective cone in the θ, x -plane for $(C^0)_r$. Indeed, note first that the divisor D is covered by a family of curves $\mathcal{E}_{d_0-r+2}$ with class $\gamma_r(g_{d_0-r+2}^1)$. These curves correspond to the various g_d^1 ’s obtained by the one-parameter family of hyperplane sections of the image of the curve in $\mathbf{P}^{r-1}$, through collections of $r-2$ fixed points on this curve. Suppose that there exists another irreducible effective divisor D' with slope m' strictly greater than the slope of D . Then relation (13) implies that $D' \cdot \gamma_r(g_{d_0-r+2}^1) < 0$ and so, since D' is irreducible we get that all the members of $\mathcal{E}_{d_0-r+2}$ are contained in D' . But since the divisor

D' is “covered” by the family $\mathcal{E}_{d_0-r+2}$ this implies that D is contained in D' . A contradiction. The rest of the proof for the bound from outside goes, using degenerations after a possible base change, as the proof of Theorem 2.

To prove the case for the bound from inside, we use the maps A_k defined in the introduction of this paper. From formula (2) we have

$$(14) \quad A_{r-2}(x_r) = (r-1)x_2 \quad \text{and} \quad A_{r-2}(\theta_r) = \theta_2 + g(r-2)x_2.$$

Since in C_2 the slope m with $m < g - \sqrt{g}$ is “effective”, pulling back by the A_{r-2} map we get in C_r that

$$(15) \quad A_{r-2}(\theta_r - (g - \sqrt{g})x_r) = \theta_2 - (g - (r-1)\sqrt{g})x_2.$$

On the other hand we have that for $d = g+r-1 - [\frac{g}{r}]$, a general curve has a g_d^{r-1} (this is the minimum d for which the Brill-Noether number ρ is nonnegative). Using formula (6) we obtain the following class of an effective divisor in C_r :

$$(16) \quad \gamma_r(g_d^{r-1}) = \theta - [\frac{g}{r}]x$$

and this concludes the proof. Q.E.D.

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