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THE FINITE PART OF SINGULAR INTEGRALS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. A divergent integral can sometimes be handled by assigning to it as its value the finite part in the sense of Hadamard. This is done by expanding the integral over the complement of a symmetric neighborhood of a singularity in powers of the radius, and throwing away the negative powers. In this paper the finite part of a singular integral of Cauchy type is defined, and this is then used to describe the boundary behavior of derivatives of a Cauchy-type integral. The finite part of a singular integral of Bochner-Martinelli type is studied, and an extension of the Plemelj jump formulas is shown to hold.

1. INTRODUCTION

We know that an integral of the type

$$\int_a^b \frac{f(x)dx}{(b-x)^{k+1/2}} \tag{1.1}$$

is divergent if $k \geq 1/2$. For k an integer ≥ 1, Hadamard [1952] derived an expression which he called the finite part of (1.1), and which, as he showed, possesses many important properties. His definition is

$$\begin{aligned} &\mathrm{FP}\int_a^b \frac{f(x)dx}{(b-x)^{k+1/2}} \\ &\quad= \lim_{\varepsilon\to 0}\left[\int_a^{b-\varepsilon} \frac{f(x)dx}{(b-x)^{k+1/2}} - \sum_{j=0}^{k-1} \frac{(-1)^j f^{(j)}(b)}{j!(k-j-1/2)\varepsilon^{k-j-1/2}}\right]. \end{aligned} \tag{1.2}$$

Fox [1957] considered a divergent integral

$$\int_a^b \frac{f(x)dx}{(x-u)^{k+1}}, \tag{1.3}$$

where $a < u < b$ and k is a nonnegative integer. His definition is

$$\mathrm{FP}\int_a^b \frac{f(x)dx}{(x-u)^{k+1}} = \lim_{\varepsilon\to 0}\left[\int_a^{u-\varepsilon} \frac{f(x)dx}{(x-u)^{k+1}} + \int_{u+\varepsilon}^b \frac{f(x)dx}{(x-u)^{k+1}} - H_k(u, \varepsilon)\right]. \tag{1.4}$$

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Here $H_k(u, \varepsilon) = 0$ if $k = 0$, and

$$H_k(u, \varepsilon) = \sum_{j=0}^{k-1} \frac{f^{(j)}(u)}{j!} \left(\frac{1 - (-1)^{k-j}}{(k-j)\varepsilon^{k-j}} \right),$$

if k is a positive integer. For a given function f the finite part is obtained here by subtracting negative powers, if any, of ε in an expansion of the integral over the interval $[a, b]$ minus the symmetric interval $[u - \varepsilon, u + \varepsilon]$. If $u = a$ or $u = b$, the finite part can also be defined, but it is somewhat less stable then for integer k.

So we shall call the right-hand side of (1.2) or (1.4) *the finite part* of the integral (1.1), (1.3) respectively; in the special case $k = 0$ in (1.4), when there are no negative powers of ε, it is also known as the *principal value*, or *Cauchy principal value*. We shall use the notation FP $\int$ and PV $\int$, respectively. See Hörmander [1983:70] for a general definition.

Similarly we can define the finite part of a singular integral

$$\oint_C \frac{f(z)dz}{(z-u)^{k+1}}, \qquad u \in C,$$

where C is a closed curve. We can get formulas corresponding to the Plemelj jump formulas.

This paper is aimed at studying singular integrals in $\mathbf{C}^n$. In §2 we define the finite part of a singular integral

$$\int_S \frac{f(\xi)}{(1 - v\overline{\xi}')^{n-\beta+k}} \, ds(\xi)$$

and obtain the corresponding Plemelj jump formulas which describe the boundary behavior of the higher derivatives of a Cauchy-type integral. In §3 we define the finite part of the singular integral of Bochner-Martinelli type

$$\int_\Omega f(z) \frac{z_k - w_k}{|z - w|^2} K_{2n-1}(z, w), \qquad w \in \Omega, \ k = 1, \dots, n,$$

and obtain similar theorems.

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2. Finite part of singular integrals on spheres

In several complex variables we do not have a kernel with all the good properties of the Cauchy kernel—both domain invariant and holomorphy. However, for a given domain there may exist good kernels, and in the case of the ball we can use

$$\frac{1}{\omega}(1 - z\overline{\xi}')^{-n} \, ds(\xi).$$

Here $\xi = (\xi_1, \dots, \xi_n)$, $z = (z_1, \dots, z_n)$ are points in $\mathbf{C}^n$, and we use matrix multiplication, the prime indicating the transpose of a matrix so that $z\overline{\xi}' = \sum z_j \overline{\xi}_j$ and $|z| = \sqrt{z\overline{z}'}$. We define the kernel for ξ on the unit sphere and z in the closed unit ball. We shall use the notations

$$B = \{z \in \mathbf{C}^n; \, z\overline{z}' < 1\}, \qquad S = \{z \in \mathbf{C}^n; \, z\overline{z}' = 1\}$$

for the open unit ball and its boundary, the unit sphere, respectively. Moreover $ds(\xi)$ denotes the area element of the unit sphere whose area is

$$\int_S ds(\xi) = \omega = \omega_{2n-1} = \frac{2\pi^n}{\Gamma(n)}.$$

Suppose f is an integrable function on S. Then

$$\frac{1}{\omega}\int_S \frac{f(\xi)ds(\xi)}{(1-z\overline{\xi}')^n} \tag{2.1}$$

exists for all $z \in B$, because in this case the integrand has no singularity at all. For $z \in S$ the integral exists for a particular point z if $|f(\xi)| \le C|\xi - z|^\alpha$ for some $\alpha > 0$. Indeed, without losing generality we may take $z = (1, 0, \dots, 0)$ and assume that $\operatorname{supp} f \subset \{\xi \in S; \xi_1 \neq 0\}$. Then

$$\begin{aligned}\frac{|f(\xi)|}{|1-z\overline{\xi}'|^n} &= \frac{|f(\xi)|}{|\overline{\xi}_1(\xi_1 - 1) + |\xi_2|^2 + \cdots + |\xi_n|^2|^n} \\ &\sim \frac{|f(\xi)|}{||1-\xi_1| + |\xi_2|^2 + \cdots + |\xi_n|^2|^n},\end{aligned}$$

so if we introduce local coordinates $(y, w) = (|1-\xi_1|, \xi_2, \dots, \xi_n) \in \mathbf{R} \times \mathbf{C}^{n-1}$ on the unit sphere near $(1, 0, \dots, 0)$, the singularity of $f(\xi)/(1-z\overline{\xi}')^n$ is not worse than that of the locally integrable function $(y + |w|^2)^{-n+\alpha/2}$.

Generally for every $z \in S$ and a Hölder-continuous function f, we can define the Cauchy principal value of (2.1) as

$$\mathrm{PV}\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-v\overline{\xi}')^n}ds(\xi) = \lim_{\varepsilon\to 0}\frac{1}{\omega}\int_{S\cap\{|1-v\overline{\xi}'|>\varepsilon\}} \frac{f(\xi)}{(1-v\overline{\xi}')^n}ds(\xi),$$

and we have a Plemelj jump formula, as shown in Lemmas 2.1 and 2.2; see Gong [1982].

Lemma 2.1. *Suppose f is a Hölder-continuous function on S, i.e., that there exist numbers α with $0 < \alpha \le 1$ and C such that for any ξ, $\eta \in S$ we have $|f(\xi) - f(\eta)| \le C|\xi - \eta|^\alpha$. Then the Cauchy principal value exists and we have that*

$$\mathrm{PV}\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-v\overline{\xi}')^n}ds(\xi) = \frac{1}{\omega}\int_S \frac{f(\xi)-f(v)}{(1-v\overline{\xi}')^n}ds(\xi) + \frac{1}{2}f(v), \qquad v \in S.$$

Lemma 2.2 (Plemelj jump formula). *Suppose f is a Hölder-continuous function on the unit sphere S. If z tends to v from the interior and satisfies*

$$\frac{|z-v|}{\min_{u\in S}|u-z|} = \frac{d(z, v)}{d(z, S)} \le M$$

for some constant M, then we have

$$\lim_{z\to v}\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-z\overline{\xi}')^n}ds(\xi) = \mathrm{PV}\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-v\overline{\xi}')^n}ds(\xi) + \frac{1}{2}f(v), \qquad v \in S.$$

Lemma 2.3. *Suppose f is a continuous function on some neighborhood of the unit sphere S and k times continuously differentiable, with support contained* *in the ball $B(p, 1/2)$ of center p and radius $1/2$, i.e., $B(p, 1/2) = \{z \in \mathbf{C}^n ; |z-p| < 1/2\}$, where $p = (1, 0, \dots, 0)$. Let L be the operator defined by*

$$(Lf)(\xi) = \frac{1}{\overline{\xi}_1}\left[\xi_1 \frac{\partial f}{\partial \xi_1}(\xi) - \overline{\xi}_1 \frac{\partial f}{\partial \overline{\xi}_1}(\xi) + f(\xi)\right],$$
$$L^j f = L(L^{j-1} f), \qquad L^0(f) = f.$$

Then (1) $(Lf)(\xi)$ *is a continuous function on a neighborhood of S and is $(k-1)$ times continuously differentiable, with* $\operatorname{supp} Lf \subset B(p, 1/2)$.

(2) *Suppose that all kth partial derivatives of f are Hölder continuous. Then all $(k-1)$th partial derivatives of Lf are Hölder continuous.*

(3) (*Partial integration*). *Let k be a positive integer and let $0 \leq \beta < 1$. For any $z \in \overline{B}$, $z_1 \neq 0$, and any surface $\Sigma \subset S$, $z \notin \overline{\Sigma}$, we have*

$$\begin{aligned}\int_\Sigma \frac{f(\xi)}{(1 - z\overline{\xi}')^{n-\beta+k}}\, ds(\xi) &= \frac{1}{(n-\beta+k-1)(n-\beta+k-2)\cdots(n-\beta)z_1^k} \int_\Sigma \frac{(L^k f)(\xi)}{(1-z\overline{\xi}')^{n-\beta}}\, ds(\xi) \\ &\quad - \sum_{j=0}^{k-1} \frac{A_{n,\beta,k,j}}{z_1^{j+1}} \int_{\partial\Sigma} \frac{(L^j f)(\xi) d\xi_2 \cdots d\xi_n d\overline{\xi}_2 \cdots d\overline{\xi}_n}{(1 - z\overline{\xi}')^{n-\beta+k-j-1}\overline{\xi}_1},\end{aligned}$$

where $A_{n,\beta,k,j}$ denotes the constant

$$A_{n,\beta,k,j} = \frac{1}{(n-\beta+k-1)\cdots(n-\beta+k-1-j)2^{n-1}i^n}. \tag{2.2}$$

In particular, when $z \in B$, $z_1 \neq 0$, we can take $\Sigma = S$ and obtain

$$\int_S \frac{f(\xi)}{(1-z\overline{\xi}')^{n-\beta+k}}\, ds(\xi) = \frac{1}{(n-\beta+k-1)(n-\beta+k-2)\cdots(n-\beta)z_1^k} \int_S \frac{(L^k f)(\xi)}{(1-z\overline{\xi}')^{n-\beta}}\, ds(\xi).$$

Proof. Since $B(p, \frac{1}{2}) \subset \{z \in \mathbf{C}^n ; z_1 \neq 0\}$, Lf is well defined on some neighborhood of S and (1) and (2) follow from the definition of the operator L. We shall prove (3).

On the unit sphere S we have

$$\overline{\xi}_1 d\xi_1 + \cdots + \overline{\xi}_n d\xi_n + \xi_1 d\overline{\xi}_1 + \cdots + \xi_n d\overline{\xi}_n = 0.$$

On the subset of S where $\xi_1 \neq 0$ we have, if $j \geq 2$,

$$d\overline{\xi}_1 \equiv -\frac{\xi_j}{\xi_1} d\overline{\xi}_j \bmod (d\xi_1, \dots, d\xi_n, d\overline{\xi}_2, \dots, [d\overline{\xi}_j], \dots, d\overline{\xi}_n).$$

Here and in the sequel the notation $[d\overline{\xi}_j]$ means that $d\overline{\xi}_j$ shall be omitted.

The area element can be expressed as follows:

$$\begin{aligned} ds(\xi) &= \frac{1}{2^{n-1}i^n}\sum_{j=1}^{n}(-1)^{j-1}\overline{\xi}_j d\xi_1\cdots d\xi_n d\overline{\xi}_1\cdots[d\overline{\xi}_j]\cdots d\overline{\xi}_n \\ &= \frac{1}{2^{n-1}i^n\xi_1}\sum_{j=1}^{n}\xi_j\overline{\xi}_j d\xi_1\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n \\ &= \frac{1}{2^{n-1}i^n\xi_1}d\xi_1\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n . \end{aligned}$$

Because of $\operatorname{supp} f \subset B(p, 1/2)$ and

$$d\xi_1 \equiv -\xi_1 d\overline{\xi}_1/\overline{\xi}_1 \quad \operatorname{mod}(d\xi_2, \dots, d\xi_n, d\overline{\xi}_2, \dots, d\overline{\xi}_n),$$

we get for any positive number m:

$$\begin{aligned} &\int_\Sigma \frac{f(\xi)}{(1-z\overline{\xi}')^{m+1}}\,ds(\xi) = \int_{\Sigma\cap B(p,1/2)} \frac{f(\xi)}{(1-z\overline{\xi}')^{m+1}}\,ds(\xi) \\ &= \frac{1}{2^{n-1}i^n}\int_{\Sigma\cap B(p,1/2)}\frac{f(\xi)}{\xi_1(1-z\overline{\xi}')^{m+1}}d\xi_1\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n \\ &= \frac{1}{2^{n-1}i^n m}\int_{\Sigma\cap B(p,1/2)}\frac{f(\xi)}{\xi_1}\left(-\frac{\xi_1}{z_1\overline{\xi}_1}\right)d\left[\frac{1}{(1-z\overline{\xi}')^m}\right]d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n \\ &= \frac{1}{2^{n-1}i^n m z_1}\int_{\Sigma\cap B(p,1/2)}\frac{1}{(1-z\overline{\xi}')^m}\,d\left(\frac{f(\xi)}{\overline{\xi}_1}\right)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n \\ &\quad - \frac{1}{2^{n-1}i^n m z_1}\int_{\partial(\Sigma\cap B(p,1/2))}\frac{1}{(1-z\overline{\xi}')^m}\left(\frac{f(\xi)}{\overline{\xi}_1}\right)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n \\ &= \frac{1}{mz_1}\int_{\Sigma\cap B(p,1/2)}\frac{1}{(1-z\overline{\xi}')^m\overline{\xi}_1}\left\{\xi_1\frac{\partial f}{\partial\xi_1}(\xi) - \overline{\xi}_1\frac{\partial f}{\partial\overline{\xi}_1}(\xi) + f(\xi)\right\} \\ &\quad \cdot\frac{1}{2^{n-1}i^n\xi_1}d\xi_1\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n \\ &\quad - \frac{1}{2^{n-1}i^n m z_1}\int_{\partial(\Sigma\cap B(p,1/2))}\frac{1}{(1-z\overline{\xi}')^m}\left(\frac{f(\xi)}{\overline{\xi}_1}\right)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n \\ &= \frac{1}{mz_1}\int_{\Sigma\cap B(p,1/2)}\frac{(Lf)(\xi)}{(1-z\overline{\xi}')^m}\,ds(\xi) \\ &\quad - \frac{1}{2^{n-1}i^n m z_1}\int_{\partial(\Sigma\cap B(p,1/2))}\frac{1}{(1-z\overline{\xi}')^m}\left(\frac{f(\xi)}{\overline{\xi}_1}\right)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n \\ &= \frac{1}{mz_1}\int_\Sigma\frac{(Lf)(\xi)}{(1-z\overline{\xi}')^m}\,ds(\xi) \\ &\quad - \frac{1}{2^{n-1}i^n m z_1}\int_{\partial\Sigma}\frac{1}{(1-z\overline{\xi}')^m}\left(\frac{f(\xi)}{\overline{\xi}_1}\right)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n . \end{aligned}$$

Now (3) follows by iteration.

In view of Lemma 2.3 it is reasonable to give the following definition.

Definition 2.4. For any function f which is $k-1$ times continuously differentiable on some neighborhood of S, $k \geq 1$, and any number β with $0 \leq \beta < 1$ we define *the finite part* of a singular integral

$$\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)$$

by

$$\begin{aligned}\mathrm{FP}\,&\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)\\ &=\lim_{\varepsilon\to 0}\left\{\frac{1}{\omega}\int_{S\cap\{|1-\overline{\xi}_1|\geq\varepsilon\}}\frac{f(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)\right.\\ &\qquad\left.+\frac{1}{\omega}\sum_{j=0}^{k-1}A_{n,\beta,k,j}\int_{S\cap\{|1-\overline{\xi}_1|=\varepsilon\}}\frac{L^j f(\xi)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n}{(1-\overline{\xi}_1)^{n-\beta+k-j-1}\overline{\xi}_1}\right\},\end{aligned}$$

provided the limit exists. Here $A_{n,\beta,k,j}$ are the constants defined by (2.2).

Theorem 2.5. (1) *Suppose* f *is a continuous function on some neighborhood of* S *and is continuously differentiable up to order* k *with Hölder-continuous derivatives. Then the finite part in Definition* 2.4 *exists.*

(2) *There exist constants* c_j, $j=0,\ldots,k-1$, *such that*

$$\begin{aligned}\sum_{j=0}^{k-1}&A_{n,\beta,k,j}\int_{S\cap\{|1-\overline{\xi}_1|=\varepsilon\}}\frac{(L^j(f))(\xi)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n}{(1-\overline{\xi}_1)^{n-\beta+k-j-1}\overline{\xi}_1}\\ &=\sum_{j=0}^{k-1}\frac{c_j}{\varepsilon^{k-j-\beta}}+O(\varepsilon^{\beta}),\end{aligned}$$

provided that f *is* $2k$ *times differentiable. Therefore the finite part of a singular integral is obtained by subtracting the negative powers of* ε *from an expansion of the integral over the complement of a symmetric neighborhood of the singularity in the case* $0<\beta<1$.

Proof. (1) Take an open covering $(U_j)_{1\leq j\leq m}$ of S such that $p\in U_1$ and $U_1\subset B(p,1/2)$, and a partition of unit $(f_j)_{1\leq j\leq m}$ subordinated to $(U_j)_{1\leq j\leq m}$. Then we have $\sum_{j=1}^m f_j=1$, $\operatorname{supp}(f_1 f)\subset U_1$, and $\operatorname{supp}(f_j f)\subset \mathbf{C}^n\backslash B(p,\sigma)$, where $2\leq j\leq m$ and σ is a positive number smaller than $1/2$.

Let $f'=\sum_{j=2}^m f_j f$; then $\operatorname{supp} f'\subset\mathbf{C}^n\backslash B(p,\sigma)$. If ε is small enough, we have that

$$\begin{aligned}\int_{S\cap\{|1-\overline{\xi}_1|>\varepsilon\}}&\frac{f(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)=\int_{S\cap\{|1-\overline{\xi}_1|\geq\varepsilon\}}\frac{\sum_{j=1}^m f_j f}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)\\ &=\int_{S\cap\{|1-\overline{\xi}_1|\geq\varepsilon\}}\frac{f'(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)+\int_{S\cap\{|1-\overline{\xi}_1|\geq\varepsilon\}}\frac{f_1(\xi)f(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)\\ &=\int_S\frac{f'(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)+\int_{S\cap\{|1-\overline{\xi}_1|\geq\varepsilon\}}\frac{f_1(\xi)f(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\,ds(\xi)\\ &=I_1+I_2\end{aligned}$$

where

$$I_1 = \int_S \frac{f'(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\, ds(\xi),$$

and

$$I_2 = \int_{S\cap\{|1-\overline{\xi}_1|\ge\varepsilon\}} \frac{f_1(\xi)f(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\, ds(\xi).$$

Here I_1 is independent of ε so $\lim I_1$ exists.

Since $f_1 f$ satisfies the condition in Lemma 2.3 we have

$$I_2 = -\sum_{j=0}^{k-1} A_{n,\beta,k,j}\int_{S\cap\{|1-\overline{\xi}_1|=\varepsilon\}} \frac{(L^j(f_1 f))(\xi)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n}{(1-\overline{\xi}_1)^{n-\beta+k-j-1}\overline{\xi}_1}$$
$$+\frac{1}{(n-\beta+k-1)\cdots(n-\beta+1)(n-\beta)}\int_{S\cap\{|1-\overline{\xi}_1|\ge\varepsilon\}} \frac{(L^k(f_1 f))(\xi)}{(1-\overline{\xi}_1)^{n-\beta}}\, ds(\xi).$$

If ε is small enough, we have $S\cap\{|1-\overline{\xi}_1|\le\varepsilon\}\subset B(p,\sigma)$. Therefore we see that

$$f_1|_{S\cap\{|1-\overline{\xi}_1|\le\varepsilon\}} = 1,$$

and

$$L^j(f_1 f)|_{S\cap\{|1-\overline{\xi}_1|\le\varepsilon\}} = L^j f|_{S\cap\{|1-\overline{\xi}_1|\le\varepsilon\}}.$$

Now

$$I_2 = \frac{1}{(n-\beta+k-1)\cdots(n-\beta+1)(n-\beta)}\int_{S\cap\{|1-\overline{\xi}_1|\ge\varepsilon\}} \frac{(L^k(f_1 f))(\xi)}{(1-\overline{\xi}_1)^{n-\beta}}\, ds(\xi)$$
$$-\sum_{j=0}^{k-1} A_{n,\beta,k,j}\int_{S\cap\{|1-\overline{\xi}_1|=\varepsilon\}} \frac{(L^j f)(\xi)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n}{(1-\overline{\xi}_1)^{n-\beta+k-j-1}\overline{\xi}_1},$$

so

$$\mathrm{FP}\,\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-\overline{\xi}_1)^{n-\beta+k}}\, ds(\xi)$$
$$= \lim_{\varepsilon\to 0}\left\{\frac{1}{\omega}\int_{S\cap\{|1-\overline{\xi}_1|\ge\varepsilon\}} \frac{f(\xi)}{(1-\overline{\xi}_1)^{n-\beta}}\, ds(\xi)\right.$$
$$\left. +\frac{1}{\omega}\sum_{j=0}^{k-1} A_{n,\beta,k,j}\int_{S\cap\{|1-\overline{\xi}_1|=\varepsilon\}} \frac{(L^k f)(\xi)d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n}{(1-\overline{\xi}_1)^{n-\beta+k-j-1}\overline{\xi}_1}\right\}$$
$$= \frac{1}{\omega}I_1 + \lim_{\varepsilon\to 0}\frac{1}{\omega}\frac{1}{(n-\beta+k-1)\cdots(n-\beta)}\int_{S\cap\{|1-\overline{\xi}_1|\ge\varepsilon\}} \frac{(L^k(f_1 f))(\xi)}{(1-\overline{\xi}_1)^{n-\beta}}\, ds(\xi).$$

Because all partial derivatives of f are Hölder continuous, all partial derivatives of $f_1 f$ are Hölder continuous. According to Lemma 2.3, $L^k(f_1 f)$ is Hölder continuous and so the above limit exists, which is a Cauchy principal in the case $\beta = 0$; a convergent integral in case $0<\beta<1$.

(2) Let $S_\varepsilon = S\cap\{|1-\overline{\xi}_1| = \varepsilon\}$. Let

$$\varphi_j(\varepsilon) = \int_{S_\varepsilon} \frac{g(\xi)}{(1-\overline{\xi}_1)^{n-\beta+j-1}\overline{\xi}_1}\, d\xi_2\cdots d\xi_n d\overline{\xi}_2\cdots d\overline{\xi}_n,$$

where g is any $2j$ times differentiable function. We shall prove that there exists a function φ which is real analytic at the point $\varepsilon = 0$ such that

$$\varphi_j(\varepsilon) = \frac{\varphi(\varepsilon)}{\varepsilon^{j-\beta}} + O(\varepsilon^\beta).$$

Expand $g(\xi)/\bar{\xi}_1$ around the point p as follows

$$\frac{g(\xi)}{\bar{\xi}_1} = \sum_{s+t+|K|<2j} C_{s,t,K}(1-\bar{\xi}_1)^s(1-\xi_1)^t\xi^K + O(|\xi - p|^{2j}),$$

where $K = (0, 0, k_2, k_{\bar{2}}, \dots, k_n, k_{\bar{n}})$ and $\xi^K = \xi_2^{k_2}\bar{\xi}_2^{k_{\bar{2}}}\cdots\xi_n^{k_n}\bar{\xi}_n^{k_{\bar{n}}}$. Since for all $\xi \in S_\varepsilon$ we have

$$|\xi - p|^2 = 2\,\mathrm{Re}(1-\bar{\xi}_1) \le 2|1-\bar{\xi}_1|,$$

we see that

$$\int_{S_\varepsilon} \frac{|\xi - p|^{2j}}{|1-\bar{\xi}_1|^{n-\beta+j-1}}\, d\xi_2\cdots d\xi_n d\bar{\xi}_2\cdots d\bar{\xi}_n \\ \le C\int_{S_\varepsilon} \frac{d\xi_2\cdots d\xi_n d\bar{\xi}_2\cdots d\bar{\xi}_n}{|1-\bar{\xi}_1|^{n-\beta-1}} \le C\varepsilon^\beta.$$

Let $1-\bar{\xi}_1 = \varepsilon e^{i\theta}$ on S_ε. In terms of the new coordinates $(\theta, \xi_2, \dots, \xi_n)$, S_ε has the following representation

$$S_\varepsilon = \{(1-\varepsilon e^{-i\theta}, \xi_2, \dots, \xi_n) \in \mathbf{C}^n;\ |\tilde{\xi}|^2 = \varepsilon(e^{i\theta} + e^{-i\theta} - \varepsilon)\},$$

where we write $\tilde{\xi}$ for the vector $(\xi_2, \dots, \xi_n)$; $|\tilde{\xi}|^2 = |\xi_2|^2 + \cdots + |\xi_n|^2$. We now sum over all s, t and K such that $0 \le s < j$, $-n-2j \le t \le -n+j$, $|K| < 2j$; this gives

$$\begin{aligned}
\varphi_j(\varepsilon) &= \sum C_{stK}\frac{\varepsilon^s}{\varepsilon^{n-\beta+j-1}}\int_{|\tilde{\xi}|^2=\varepsilon(e^{i\theta}+e^{-i\theta}-\varepsilon)} e^{i(t-\beta)\theta}\xi^K d\xi_2\cdots d\xi_n d\bar{\xi}_2\cdots d\bar{\xi}_n + O(\varepsilon^\beta)\\
&= \sum C_{stK}\frac{\varepsilon^s}{\varepsilon^{j-\beta}}\int_{|\tilde{\xi}|^2=e^{i\theta}+e^{-i\theta}-\varepsilon} e^{i(t-\beta)\theta}(\varepsilon^{1/2}\xi)^K d\xi_2\cdots d\xi_n d\bar{\xi}_2\cdots d\bar{\xi}_n + O(\varepsilon^\beta)\\
&= \sum C_{stK}\frac{\varepsilon^s}{\varepsilon^{j-\beta}}\int_{|\tilde{\xi}|^2\le e^{i\theta}+e^{-i\theta}-\varepsilon} e^{i(t-\beta)\theta}(\varepsilon^{1/2}\xi)^K d\theta d\xi_2\cdots d\xi_n d\bar{\xi}_2\cdots d\bar{\xi}_n + O(\varepsilon^\beta)\\
&= \sum C_{stK}\frac{\varepsilon^s}{\varepsilon^{j-\beta}}\int_{-\arccos\varepsilon}^{\arccos\varepsilon} e^{i(t-\beta)\theta}d\theta\int_{|\tilde{\xi}|^2\le e^{i\theta}+\varepsilon^{-i\theta}-\varepsilon}(\varepsilon^{1/2}\xi)^K d\xi_2\cdots d\xi_n d\bar{\xi}_2\cdots d\bar{\xi}_n + O(\varepsilon^\beta).
\end{aligned}$$

When $|K|$ is an odd number we have

$$\int_{|\tilde{\xi}|^2\le e^{i\theta}+e^{-i\theta}-\varepsilon}(\varepsilon^{1/2}\xi)^K d\xi_2\cdots d\xi_n d\bar{\xi}_2\cdots d\bar{\xi}_n = 0$$

because of the symmetry of the ball. When $|K|$ is an even integer, say $2k$, we have

$$\int_{-\arccos\varepsilon}^{\arccos\varepsilon} e^{i(t-\beta)\theta}d\theta\int_{|\tilde{\xi}|^2\le e^{i\theta}+e^{-i\theta}-\varepsilon}\varepsilon^k\xi^{2k}d\xi_2\cdots d\xi_n d\bar{\xi}_2\cdots d\bar{\xi}_n \\ = C\varepsilon^k\int_{-\arccos\varepsilon}^{\arccos\varepsilon} e^{i(t-\beta)\theta}(e^{i\theta}+e^{-i\theta}-\varepsilon)^{n-1+k}d\theta,$$

which is a real-analytic function of ε.

Theorem 2.6. *Let the hypotheses be the same as those of Theorem 2.5. The Plemelj jump formulas hold, i.e., if z tends to p from the interior and satisfies $d(z, p)/d(z, S) \leq M$, where M is a constant, then we have*

(i)

$$\lim_{z\to p}\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-z\overline{\xi}')^{n+k}}\,ds(\xi)$$
$$= \mathrm{FP}\,\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-\overline{\xi}_1)^{n+k}}\,ds(\xi) + \frac{1}{2}\frac{1}{(n+k-1)\cdots(n+1)n}(L^k f)(p).$$

(ii) *Let $K = (k_1, \dots, k_n)$ and $k = k_1 + \cdots + k_n$, where $k_1, \dots, k_n$ are nonnegative integers. Then*

$$\lim_{z\to p}\frac{\partial^k}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-z\overline{\xi}')^{n}}\,ds(\xi)$$
$$= \frac{n\cdots(n+k-1)}{\omega}\,\mathrm{FP}\int_S \frac{(\overline{\xi}_1^{k_1}\cdots\overline{\xi}_n^{k_n})f(\xi)}{(1-\overline{\xi}_1)^{n+k}}\,ds(\xi) + \frac{1}{2}L^k[(\overline{z}_1^{k_1}\cdots\overline{z}_n^{k_n})f(z)]|_{z=p}.$$

In particular, if $K \neq (k_1, 0, \dots, 0)$, we have

$$\tfrac{1}{2}L^k[(\overline{z}_1^{k_1}\cdots\overline{z}_n^{k_n})f(z)]|_{z=p} = 0,$$

and

$$\lim_{z\to p}\frac{\partial^k}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-z\overline{\xi}')^{n}}\,ds(\xi)$$
$$= \frac{n\cdots(n+k-1)}{\omega}\,\mathrm{FP}\int_S \frac{(\overline{\xi}_1^{k_1}\cdots\overline{\xi}_n^{k_n})f(\xi)}{(1-\overline{\xi}_1)^{n+k}}\,ds(\xi).$$

Proof. (i) Let f', f_1 be the same as in the proof of Theorem 2.5. Then, according to Lemma 2.3, we have

$$\int_S \frac{f(\xi)}{(1-z\overline{\xi}')^{n+k}}\,ds(\xi) = \int_S \frac{f'(\xi)}{(1-z\overline{\xi}')^{n+k}}\,ds(\xi) + \int_S \frac{f_1 f(\xi)}{(1-z\overline{\xi}')^{n+k}}\,ds(\xi)$$
$$= \int_S \frac{f'(\xi)}{(1-z\overline{\xi}')^{n+k}}\,ds(\xi) + \frac{1}{n(n+1)\cdots(n+k-1)z_1^k}\int_S \frac{L^k(f_1 f)}{(1-z\overline{\xi}')^{n}}\,ds(\xi).$$

According to the proof of Theorem 2.5 we know that

$$\lim_{z\to p}\frac{1}{\omega}\int_S \frac{f(\xi)}{(1-z\overline{\xi}')^{n+k}}\,ds(\xi) = \frac{1}{\omega}\int_S \frac{f'(\xi)}{(1-\overline{\xi}_1)^{n+k}}\,ds(\xi)$$
$$+ \lim_{z\to p}\frac{1}{n(n+1)\cdots(n+k-1)\omega}\int_S \frac{L^k(f_1 f)(\xi)}{(1-z\overline{\xi}')^{n}}\,ds(\xi)$$
$$= \frac{1}{\omega}\int_S \frac{f'(\xi)}{(1-\overline{\xi}_1)^{n+k}}\,ds(\xi) + \frac{1}{n(n+1)\cdots(n+k-1)\omega}\,\mathrm{PV}\int_S \frac{L^k(f_1 f)}{(1-\overline{\xi}_1)^{n}}\,ds(\xi)$$
$$+ \frac{1}{2n(n-1)\cdots(n+k-1)}L^k(f_1 f)(p)$$
$$= \frac{1}{\omega}\,\mathrm{FP}\int_S \frac{f(\xi)}{(1-\overline{\xi}_1)^{n+k}}\,ds(\xi) + \frac{1}{2n(n-1)\cdots(n+k-1)}L^k(f)(p).$$

(ii) Since we have the following relation:

$$\frac{\partial^k}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\frac{1}{\omega}\int_S\frac{f(\xi)}{(1-z\overline{\xi}')^n}\,ds(\xi)$$

$$=\frac{(n+k-1)\cdots n}{\omega}\int_S\frac{(\overline{\xi}_1^{k_1}\cdots\overline{\xi}_n^{k_n})f(\xi)}{(1-z\overline{\xi}')^{n+k}}\,ds(\xi),$$

we can deduce (ii) from (i).

We now consider the finite part at an arbitrary point on S by reduction to the case already discussed. For any point $v\in S$, take a unitary matrix U such that $vU=p=(1,0,\dots,0)$. Let $\xi=w\overline{U}'$. Then we have

$$\int_{S\cap\{|1-v\overline{\xi}'|\ge\varepsilon\}}\frac{f(\xi)}{(1-v\overline{\xi}')^{n-\beta+k}}\,ds(\xi)$$

$$=\int_{S\cap\{|1-w_1|\ge\varepsilon\}}\frac{f(w\overline{U}')}{(1-\overline{w}_1)^{n-\beta+k}}\,ds(w).$$

Definition 2.7. *The finite part* of a singular integral

$$\int_S\frac{f(\xi)}{(1-v\overline{\xi}')^{n-\beta+k}}\,ds(\xi)$$

is defined by

$$\mathrm{FP}\int_S\frac{f(\xi)}{(1-v\overline{\xi}')^{n-\beta+k}}\,ds(\xi)=\mathrm{FP}\int_S\frac{f(w\overline{U}')}{(1-\overline{w}_1)^{n-\beta+k}}\,ds(w).$$

The above definition is independent of the choice of the matrix U. If f is k times continuously differentiable and the derivatives are Hölder continuous, then the finite part at the point $v\in S$ exists. For points z with $z\overline{z}'<1$, write $zU=\eta$. We obtain

$$\lim_{z\to v}\int_S\frac{f(\xi)}{(1-z\overline{\xi}')^{n+k}}\,ds(\xi)=\lim_{\eta\to p}\int_S\frac{f(w\overline{U}')}{(1-\eta\overline{w}')^{n+k}}\,ds(w)$$

$$=\mathrm{FP}\int_S\frac{f(w\overline{U}')}{(1-\overline{w}_1)^{n+k}}\,ds(w)+\frac{\omega}{2(n+k-1)\cdots n}L^k(f(w\overline{U}'))\Big|_{w=p}$$

$$=\mathrm{FP}\int_S\frac{f(\xi)}{(1-v\overline{\xi}')^{n+k}}\,ds(\xi)+\frac{\omega}{2(n+k-1)\cdots n}L^k(f(w\overline{U}'))\Big|_{w=p}.$$

3. Finite part of singular integrals of Bochner-Martinelli type

Let D be a domain in $\mathbf{C}^n\cong\mathbf{R}^{2n}$ whose boundary $\partial D=\Omega$ is of class C^3. Write $z_k=u_k+iu_{n+k}$, $k=1,\dots,n$. We define a Bochner-Martinelli type integral by

$$F(w)=\int_\Omega f(z)K_{2n-1}(z,w),\qquad w\in\mathbf{C}^n\backslash\Omega,$$

where

$$K_{2n-1}(z,w)=C_n\sum_{j=1}^n(-1)^{j-1}\frac{\overline{z}_j-\overline{w}_j}{|z-w|^{2n}}dz_1\cdots dz_n d\overline{z}_1\cdots[d\overline{z}_j]\cdots d\overline{z}_n.$$

Here C_n is a constant:

$$C_n = \frac{(n-1)!}{(2\pi i)^n}.$$

Lu and Zhong [1957] have studied the boundary value of the Bochner-Martinelli type integral and the corresponding Plemelj jump formulas. Their results are the following two lemmas.

Lemma 3.1. *Let f be a Hölder-continuous function on $\partial D = \Omega$ and let $a \in \Omega$. Then the principal value*

$$\mathrm{PV} \int_{\Omega} f(z) K_{2n-1}(z, a) = \lim_{\varepsilon \to 0} \int_{\Omega \setminus B(a, \varepsilon)} f(z) K_{2n-1}(z, a)$$

exists, where $B(a, \varepsilon) = \{z \in \mathbf{C}^n ;\ |z - a| < \varepsilon\}$.

Lemma 3.2. *Let f be as in Lemma* 3.1. *Then for every $a \in \Omega$ we have*

$$F_i(a) = \mathrm{PV} \int_{\Omega} f(z) K_{2n-1}(z, a) + \frac{1}{2} f(a),$$
$$F_e(a) = \mathrm{PV} \int_{\Omega} f(z) K_{2n-1}(z, a) - \frac{1}{2} f(a),$$

where $F_i(a)$ and $F_e(a)$ are the limits of $F(w)$ as w tends to a in D and in $\mathbf{C}^n \setminus (\Omega \cup D)$, respectively.

Now assume that $0 \in \Omega$ and that in a neighborhood U of the origin we have

$$\Omega \cap U = \{z \in U ;\ r(u_1, \dots, u_{2n}) = 0\},$$

for a function $r \in C^3(U)$ satisfying

$$\left.\frac{\partial r}{\partial u_1}\right|_{u=0} = 1, \qquad \left.\frac{\partial r}{\partial u_2}\right|_{u=0} = \cdots = \left.\frac{\partial r}{\partial u_{2n}}\right|_{u=0} = 0.$$

In other words, the tangent plane of Ω at the origin is defined by $u_1 = 0$. Since $\partial r / \partial u_1|_{u=0} \neq 0$, we can write the equation $r(u_1, \dots, u_{2n}) = 0$ as $u_1 - h(u_2, \dots, u_{2n}) = 0$, for a suitable function h of class C^3 and satisfying $h(0, \dots, 0) = 0$, $\partial h / \partial u_2|_{u=0} = \cdots = \partial h / \partial u_{2n}|_{u=0} = 0$.

Lemma 3.3. *Define $\Omega_{\varepsilon, \eta} = \{z \in \Omega ;\ \varepsilon < |z| < \eta\}$ for $0 < \varepsilon < \eta$. Fix η. Then*

$$\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon, \eta}} \frac{u_j \, du_2 \cdots du_{2n}}{|z|^{2n}}, \qquad j = 1, 2, \dots, 2n; \tag{3.1}$$

exists. Moreover, if $w = (w_1, \dots, w_n) = (-\varepsilon^2, 0, \dots, 0)$, then

$$\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon, \eta}} \left| \frac{\overline{z}_j - \overline{w}_j}{|z - w|^{2n}} - \frac{\overline{z}_j}{|z|^{2n}} \right| du_2 \cdots du_{2n} = 0, \qquad j = 1, 2, \dots, n; \tag{3.2}$$

$$\lim_{\varepsilon \to 0} \int_{\Omega \cap B(0, \varepsilon)} \frac{\overline{z}_j - \overline{w}_j}{|z - w|^{2n}} du_2 \cdots du_{2n} = \begin{cases} \dfrac{\pi^n}{(n-1)!}, & j = 1; \\ 0, & j = 2, \dots, n; \end{cases} \tag{3.3}$$

$$\lim_{\varepsilon \to 0} \int_{\Omega \cap B(0, \varepsilon)} |z|^{\alpha} \frac{|\overline{z}_j - \overline{w}_j|}{|z - w|^{2n}} du_2 \cdots du_{2n} = 0, \qquad j = 1, 2, \dots, n; \tag{3.4}$$

(3.5)
$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon,\eta}}\left|\frac{|z|^\alpha(\bar z_j-\bar w_j)}{|z-w|^{2n}}-\frac{|z|^\alpha \bar z_j}{|z|^{2n}}\right| du_2\cdots du_{2n}=0, \qquad j=1,2,\dots,n.$$

Here α denotes any positive constant.

Proof. We shall use spherical coordinates

$$u_1=u_1,\ u_2=\rho\cos\theta_1,\ \dots,\ u_{2n-1}=\rho\sin\theta_1\cdots\sin\theta_{2n-3}\cos\theta_{2n-2}, \tag{3.6}$$
$$u_{2n}=\rho\sin\theta_1\cdots\sin\theta_{2n-3}\sin\theta_{2n-2}.$$

Proof of (3.1). We shall only prove the case of $j=2$. The proof for $j=3,\dots,2n$ is similar. For $j=1$, the function is even integrable.

By using spherical coordinates we have

$$\int_{\Omega_{\varepsilon,\eta}}\frac{u_2du_2\cdots du_{2n}}{|z|^{2n}}=\int_0^\pi\sin^{2n-3}\theta_1\cos\theta_1 d\theta_1\int_0^\pi\sin^{2n-4}\theta_2 d\theta_2$$
$$\cdots\int_0^\pi\sin\theta_{2n-3}d\theta_{2n-3}\int_0^{2\pi}d\theta_{2n-2}\int_{\varepsilon^2<\psi^2+\rho^2<\eta^2}\frac{\rho^{2n-1}d\rho}{(\rho^2+\psi(\rho,\theta)^2)^n},$$

where $\psi(\rho,\theta)=h(\rho\cos\theta_1,\dots,\rho\sin\theta_1\cdots\sin\theta_{2n-2})$.

It is clear that

$$\psi(0,\theta)=0,\qquad \frac{\partial\psi}{\partial\rho}(0,\theta)=0;$$

we can therefore assume $\psi(\rho,\theta)=\rho^2\varphi(\rho,\theta)$ for some $\varphi(\rho,\theta)$ which is of class C^3 outside $\rho=0$ and continuous at $\rho=0$.

Consider

$$g(\theta)=\int_{\varepsilon^2<\psi^2+\rho^2<\eta^2}\frac{\rho^{2n-1}d\rho}{(\rho^2+\psi(\rho,\theta)^2)^n}.$$

Let $t=\sqrt{\rho^2+\psi^2}$. Then

$$\psi(\rho,\theta)=\rho^2\varphi(\rho,\theta)=t^2\varphi_1(t,\theta),$$

where $\varphi_1(t,\theta)$ is C^3 outside $t=0$ and continuous at $t=0$. We get

$$\begin{aligned}
g(\theta)&=\int_\varepsilon^\eta\frac{(t^2-\psi^2)^{n-1}}{t^{2n-1}}dt-\int_\varepsilon^\eta\frac{(t^2-\psi^2)^{n-1}}{t^{2n}}\psi\frac{\partial\psi}{\partial t}dt\\
&=\int_\varepsilon^\eta\frac{(1-t^2\varphi_1^2)^{n-1}}{t}dt-\int_\varepsilon^\eta(1-t^2\varphi_1^2)^{n-1}\varphi_1\frac{\partial\psi}{\partial t}dt\\
&=\int_\varepsilon^\eta\frac{1}{t}dt+\int_\varepsilon^\eta\sum_{p=1}^{n-1}(-1)^pC_{n-1}^p t^{2p-1}\varphi_1^{2p}dt-\int_\varepsilon^\eta(1-t^2\varphi_1^2)^{n-1}\varphi_1\frac{\partial\psi}{\partial t}dt\\
&=\log\eta-\log\varepsilon+\int_\varepsilon^\eta\sum_{p=1}^{n-1}(-1)^pC_{n-1}^p t^{2p-1}\varphi_1^{2p}dt-\int_\varepsilon^\eta(1-t^2\varphi_1^2)^{n-1}\varphi_1\frac{\partial\psi}{\partial t}dt,
\end{aligned}$$

$C_{n-1}^p=(n-1)!/(p!(n-p-1)!)$ denoting the binomial coefficients.

Using the known result that $\int_0^\pi \sin^{2n-3}\theta_1 \cos\theta_1 d\theta_1 = 0$, we get

$$\begin{aligned}
\int_{\Omega_{\varepsilon,\eta}} & \frac{u_2 du_2 \cdots du_{2n}}{|z|^2} \\
&= \int_0^\pi \sin^{2n-3}\theta_1 \cos\theta_1 d\theta_1 \int_0^\pi \sin^{2n-4}\theta_2 d\theta_2 \cdots \\
&\qquad\qquad \int_0^\pi \sin\theta_{2n-3} d\theta_{2n-3} \int_0^{2\pi} g(\theta) d\theta_{2n-3} \\
&= \int_0^\pi \sin^{2n-3}\theta_1 \cos\theta_1 d\theta_1 \cdots \\
&\qquad \int_0^{2\pi} \left\{ \int_\varepsilon^\eta \sum_{p=1}^{n-1} (-1)^p C_{n-1}^p t^{2p-1} \varphi_1^{2p}\, dt \right. \\
&\qquad\qquad \left. - \int_\varepsilon^\eta (1 - t^2\varphi_1^2)^{n-1} \varphi_1 \frac{\partial \psi}{\partial t}\, dt \right\} d\theta_{2n-2}.
\end{aligned}$$

The integrals in braces are continuous functions of ε. So when $\varepsilon \to 0$, the limit exists.

Proof of (3.2). Applying spherical coordinates we know that

$$\begin{aligned}
\int_{\Omega_{\varepsilon,\eta}} & \left| \frac{u_2}{|z-w|^{2n}} - \frac{u_2}{|z|^{2n}} \right| du_2 \cdots du_{2n} \\
&= \int_0^\pi \sin^{2n-3}\theta_1 |\cos\theta_1| d\theta_1 \int_0^\pi \sin^{2n-4}\theta_2 d\theta_2 \cdots \\
&\qquad \int_0^\pi \sin\theta_{2n-3} d\theta_{2n-3} \int_0^{2\pi} d\theta_{2n-2} \\
&\quad \cdot \int_\varepsilon^\eta \left[\frac{1}{[t^2 + \varepsilon^4 + 2\varepsilon^2 t^2 \varphi_1(t,\eta)]^n} - \frac{1}{t^{2n}} \right] t^{2n-1} \\
&\quad \cdot \left[(1 - t^2\varphi_1^2)^{n-1} - (1 - t^2\varphi_1^2)^{n-1} t\varphi_1 \frac{\partial \psi}{\partial t} \right] dt.
\end{aligned} \tag{3.7}$$

Notice that $(\varepsilon^4 + 2\varepsilon^2 t^2 \varphi_1)/t^2 = O(\varepsilon^2)$, $t \in [\varepsilon, \eta]$, and that

$$\begin{aligned}
\frac{1}{[t^2 + \varepsilon^4 + 2\varepsilon^2 t^2 \varphi_1(t,\eta)]^n} &= \frac{1}{t^{2n}(1 + (\varepsilon^4 + 2\varepsilon^2 t^2 \varphi_1)/t^2)^n} = \frac{1}{t^{2n}} + O\left(\frac{\varepsilon^2}{t^{2n}}\right).
\end{aligned}$$

Applying the above estimate to (3.7), we obtain

$$\int_{\Omega_{\varepsilon,\eta}} \left| \frac{u_2}{|z-w|^{2n}} - \frac{u_2}{|z|^{2n}} \right| du_2 \cdots du_{2n} \le C\varepsilon^2 \int_\varepsilon^\eta \frac{1}{t}\, dt.$$

Similarly we can prove that

$$\int_{\Omega_{\varepsilon,\eta}} \left| \frac{u_j}{|z-w|^{2n}} - \frac{u_j}{|z|^{2n}} \right| du_2 \cdots du_{2n} \le C\varepsilon^2 \int_\varepsilon^\eta \frac{1}{t}\, dt, \qquad j = 3, \ldots, 2n,$$

which implies (3.2) for the case $j = 2, \ldots, n$. Since $u_1 = o(|u_2| + \cdots + |u_{2n}|)$, on $\Omega_{\varepsilon, \eta}$, we see that

$$\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon, \eta}} \left| \frac{u_1}{|z-w|^{2n}} - \frac{u_1}{|z|^{2n}} \right| du_2 \cdots du_{2n} = 0.$$

On the other hand we have the following estimates

$$\int_{\Omega_{\varepsilon, \eta}} \frac{\varepsilon^2}{|z-w|^{2n}} du_2 \cdots du_{2n} \leq C\varepsilon^2 \int_\varepsilon^\eta \frac{1}{t^2} dt.$$

Therefore we obtain (3.2) when $j = 1$.

Proof of (3.3). Applying the spherical coordinates (3.6) we know that

$$\begin{aligned}
&\int_{\Omega \cap B(0, \varepsilon)} \frac{u_2}{|z-w|^{2n}} du_2 \cdots du_{2n} \\
&\quad = \int_0^\pi \sin^{2n-3}\theta_1 \cos\theta_1 d\theta_1 \int_0^\pi \sin^{2n-4}\theta_2 d\theta_2 \cdots \\
&\qquad\qquad \int_0^\pi \sin\theta_{2n-3} d\theta_{2n-3} \int_0^{2\pi} d\theta_{2n-2} \\
&\quad \cdot \int_0^\varepsilon \frac{1}{[t^2 + \varepsilon^4 + 2\varepsilon^2 t^2 \varphi_1(t, \eta)]^n} t^{2n-1} \\
&\qquad\qquad \cdot \left[(1 - t^2\varphi_1^2)^{n-1} - (1 - t^2\varphi_1^2)^{n-1} t\varphi_1 \frac{\partial \psi}{\partial t} \right] dt.
\end{aligned} \tag{3.8}$$

Notice that for $t \in [0, \varepsilon]$, $2\varepsilon^2 t^2 \varphi_1(t, \theta)/(t^2 + \varepsilon^4) = O(\varepsilon^2)$. So we have

$$\frac{1}{t^2 + \varepsilon^4 + 2\varepsilon^2 t^2 \varphi_1(t, \eta)} = \frac{1}{(t^2 + \varepsilon^4)}(1 + O(\varepsilon^2)).$$

Applying the above estimate and $\int_0^\pi \sin^{2n-3}\theta_1 \cos\theta_1 d\theta_1 = 0$ to (3.8), we obtain

$$\begin{aligned}
&\left| \int_{\Omega \cap B(0, \varepsilon)} \frac{u_2}{|z-w|^{2n}} du_2 \cdots du_{2n} \right| \leq C\varepsilon^2 \int_0^\varepsilon \frac{t^{2n-1}}{(t^2 + \varepsilon^4)^n} dt \\
&\quad \leq C\varepsilon^2 \int_0^\varepsilon \frac{t}{t^2 + \varepsilon^4} dt = \frac{C\varepsilon^2}{2} \int_0^\varepsilon \frac{d(t^2)}{t^2 + \varepsilon^4} \\
&\quad = \frac{C\varepsilon^2}{2} \log(1 + 1/\varepsilon^2) \to 0.
\end{aligned}$$

Similarly we can prove that

$$\left| \int_{\Omega \cap B(0, \varepsilon)} \frac{u_j}{|z-w|^{2n}} du_2 \cdots du_{2n} \right| \leq C\varepsilon^2 \int_0^\varepsilon \frac{dt^2}{t^2 + \varepsilon^4}, \qquad j = 2, \ldots, 2n,$$

which implies (3.3) when $j = 2, \ldots, n$.

Since $\partial h / \partial u_j(0) = 0$ for $j = 2, \ldots, 2n$ we see that

$$|u_1| \leq C(|u_2|^2 + \cdots + |u_n|^2)$$

on $\Omega_{0,\eta}$. So we obtain the inequality

$$\left|\int_{\Omega\cap B(0,\varepsilon)}\frac{u_1}{|z-w|^{2n}}\,du_2\cdots du_{2n}\right| \\ \leq\left|\int_{\Omega\cap B(0,\varepsilon)}\frac{|z|^2}{|z-w|^{2n}}du_2\cdots du_{2n}\right|\leq C\int_0^\varepsilon\frac{t^{2n}}{(t^2+\varepsilon^4)^n}\,dt,$$

where the last inequality is obtained as in the proof of (3.2) when $j=1$. Similarly we see that

$$\int_{\Omega\cap B(0,\varepsilon)}\frac{\varepsilon^2}{|z-w|^{2n}}\,du_2\cdots du_{2n}$$
$$=\int_0^\pi\sin^{2n-3}\theta_1 d\theta_1\cdots\int_0^\pi\sin\theta_{2n-3}d\theta_{2n-3}\int_0^{2\pi}d\theta_{2n-2}\int_0^\varepsilon\frac{\varepsilon^2 t^{2n-2}}{(t^2+\varepsilon^4)^n}\,dt+o(1)$$
$$=\int_0^\pi\sin^{2n-3}\theta_1 d\theta_1\cdots\int_0^\pi\sin\theta_{2n-3}d\theta_{2n-3}\int_0^{2\pi}d\theta_{2n-2}\int_0^\infty\frac{t^{2n-2}}{(t^2+1)^n}\,dt+o(1)$$
$$=(2n-1)b_{2n-1}\int_0^\infty\frac{t^{2n-2}}{(t^2+1)^n}\,dt+o(1).$$

Here

$$b_{2n-1}=\frac{2^n\pi^{n-1}}{(2n-1)!!}$$

is the volume of the unit ball in $\mathbf{R}^{2n-1}$ and the value of the integral

$$I_n=\int_0^\infty\frac{t^{2n-2}}{(t^2+1)^n}\,dt$$

is easily found to be

$$I_n=\frac{\pi}{2}\frac{(2n-3)!!}{(2n-2)!!}=\frac{\pi(2n-3)!!}{2^n(n-1)!},$$

so that the last expression becomes $\pi^n/(n-1)!+o(1)$. Combining with the above estimates we get that

$$\lim_{\varepsilon\to0}\int_{\Omega\cap B(0,\varepsilon)}\frac{\overline{z}_1-\overline{w}_1}{|z-w|^{2n}}\,du_2\cdots du_{2n} \\ =(2n-1)b_{2n-1}\int_0^\infty\frac{t^{2n-2}}{(t^2+1)^n}\,dt=\frac{\pi^n}{(n-1)!}.$$

Proof of (3.4). Similarly as in the previous step we can prove the estimates

$$\int_{\Omega\cap B(0,\varepsilon)}|z|^\alpha\frac{\overline{z}_j-\overline{w}_j}{|z-w|^{2n}}\,du_2\cdots du_{2n}\leq\begin{cases}C\varepsilon^2\displaystyle\int_0^\varepsilon\frac{t^{2n+\alpha-2}}{(t^2+\varepsilon^4)^n}\,dt, & j=1,\\ C\displaystyle\int_0^\varepsilon\frac{t^{2n+\alpha-1}}{(t^2+\varepsilon^4)^n}\,dt, & j=2,\dots,n,\end{cases}$$

which is what we have to prove.

Proof of (3.5). As in the proof of (3.2) we get

$$\int_{\Omega_{\varepsilon,\eta}} \left| \frac{|z|^\alpha(\overline{z}_j - \overline{w}_j)}{|z-w|^{2n}} - \frac{|z|^\alpha \overline{z}_j}{|z|^{2n}} \right| du_2 \cdots du_{2n} \le \varepsilon^2 \int_\varepsilon^\eta \frac{t^\alpha}{t^2} dt$$

which is what we want.

Theorem 3.4. *Suppose $\partial h/\partial u_i$, $i = 2, \dots, 2n$, are Hölder continuous on Ω and f is a continuously differentiable function on Ω such that $\partial f/\partial u_j$, $j = 1, \dots, 2n$, are Hölder continuous. Then the finite part*

$$\mathrm{FP} \int_\Omega f(z) \frac{\overline{z}_k}{|z|^2} K_{2n-1}(z, 0)$$

defined as

$$\lim_{\varepsilon \to 0} \left[\int_{\Omega \setminus B(0,\varepsilon)} f(z) \frac{\overline{z}_k}{|z|^2} K_{2n-1}(z, 0) - \frac{B_k f(0)}{\varepsilon} \right],$$

exists, where

$$B_k = \begin{cases} 2^{n-1} i^n C_n b_{2n-1} = \dfrac{2^{n-1}(n-1)!}{\pi(2n-1)!!}, & k = 1; \\ 0, & k = 2, \dots, n; \end{cases}$$

here b_{2n-1} is the volume of the unit ball of dimension $2n-1$ and C_n is the constant in the Bochner-Martinelli kernel.

Proof. We prove only the case of $k = 1$; other cases can be proved similarly. By Stokes' formula we get

$$\begin{aligned} &\int_{\Omega \setminus B(0,\varepsilon)} f(z) \frac{\overline{z}_1}{|z|^{2n+2}} \sum_{j=1}^n (-1)^{j-1} \overline{z}_j dz_1 \cdots dz_n d\overline{z}_1 \cdots [d\overline{z}_j] \cdots d\overline{z}_n \\ &= -\frac{1}{n} \int_{\Omega \setminus B(0,\varepsilon)} f(z) d\left[\frac{\sum_{j=1}^n (-1)^{j-1} \overline{z}_j dz_2 \cdots dz_n d\overline{z}_1 \cdots [d\overline{z}_j] \cdots d\overline{z}_n}{|z|^{2n}} \right] \\ &= -\frac{1}{n} \int_{\partial(\Omega \cap B(0,\varepsilon))} f(z) \frac{\sum_{j=1}^n (-1)^{j-1} \overline{z}_j dz_2 \cdots dz_n d\overline{z}_1 \cdots [d\overline{z}_j] \cdots d\overline{z}_n}{|z|^{2n}} \\ &\quad + \frac{1}{n} \int_{\Omega \setminus B(0,\varepsilon)} df(z) \frac{\sum_{j=1}^n (-1)^{j-1} \overline{z}_j dz_2 \cdots dz_n d\overline{z}_1 \cdots [d\overline{z}_j] \cdots d\overline{z}_n}{|z|^{2n}}. \end{aligned}$$

Let

$$\psi(\varepsilon) = -\frac{C_n}{n\varepsilon^{2n}} \int_{\partial(B(0,\varepsilon) \cap \Omega)} f(z) \sum_{j=1}^n (-1)^{j-1} \overline{z}_j dz_2 \cdots dz_n d\overline{z}_1 \cdots [d\overline{z}_j] \cdots d\overline{z}_n.$$

Then

$$\begin{aligned} &\int_{\Omega \setminus B(0,\varepsilon)} f(z) \frac{\overline{z}_1}{|z|^2} K_{2n-1}(z, 0) - \psi(\varepsilon) \\ &= \frac{1}{n} \int_{\Omega \setminus B(0,\varepsilon)} \frac{\partial f}{\partial z_1} K_{2n-1}(z, 0) \\ &\quad + \frac{C_n}{n} (-1)^{n-1} \int_{\Omega \setminus B(0,\varepsilon)} \frac{1}{|z|^{2n}} \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j} \overline{z}_j dz_2 \cdots dz_n d\overline{z}_1 \cdots d\overline{z}_n \\ &= I_1 + I_2. \end{aligned}$$

According to Lemma 3.1, $\lim_{\varepsilon\to 0} I_1$ exists.

$$I_2 = \frac{C_n}{n}(-1)^{n-1}\left\{\int_{\Omega\setminus B(0,\eta)} + \int_{\Omega_{\varepsilon,\eta}}\right\} \frac{1}{|z|^{2n}}\sum_{j=1}^{n}\frac{\partial f}{\partial \overline{z}_j}\overline{z}_j dz_2\cdots dz_n d\overline{z}_1\cdots d\overline{z}_n$$

$$= I_1' + I_1'.$$

If η is small enough, $\lim_{\varepsilon\to 0} I_1'$ exists because I_1' is independent of ε.

$$I_2' = \frac{(-2i)^{n-1}C_n}{n}\int_{\Omega_{\varepsilon,\eta}} \frac{1}{|z|^{2n}}\sum_{j=1}^{n}\frac{\partial f}{\partial \overline{z}_j}\overline{z}_j d\overline{z}_1 du_2\cdots du_n du_{n+2}\cdots du_{2n}.$$

Notice that

$$d\overline{z}_1 = du_1 - i du_{n+1} = \left(\frac{\partial h}{\partial u_{n+1}} - i\right) du_{n+1} \pmod{du_2\cdots du_n du_{n+2}\cdots du_{2n}}$$

$$I_2' = \frac{(-1)^{n-1}(-2i)^{n-1}C_n}{n}\int_{\Omega_{\varepsilon,\eta}} \frac{1}{|z|^{2n}}\left(\sum_{j=1}^{n}\frac{\partial f}{\partial \overline{z}_j}\overline{z}_j\right)\left(\frac{\partial h}{\partial u_{n+1}} - i\right) du_2\cdots du_{2n}$$

$$= \frac{(-1)^{n-1}(-2i)^{n-1}C_n}{n}\int_{\Omega_{\varepsilon,\eta}} \frac{1}{|z|^{2n}}$$

$$\cdot\sum_{j=1}^{n}\left[\frac{\partial f}{\partial \overline{z}_j}(z)\left(\frac{\partial h}{\partial u_{n+1}} - i\right) - \left(\frac{\partial f}{\partial \overline{z}_j}(0)\right)\left(\frac{\partial h}{\partial u_{n+1}}(0) - i\right)\right]\overline{z}_j du_2\cdots du_{2n}$$

$$+ \frac{1}{n}(-1)^{n-1}(-2i)^{n-1}C_n\sum_{j=1}^{n}\frac{\partial f}{\partial \overline{z}_j}(0)\left(\frac{\partial h(0)}{\partial u_{n+1}} - i\right)\int_{\Omega_{\varepsilon,\eta}}\frac{1}{|z|^{2n}}\overline{z}_j du_2\cdots du_{2n}.$$

Because $\partial f/\partial \overline{z}_j$ and $\partial h/\partial u_{n+1}$ are Hölder continuous on Ω, the limit of the first term exists when $\varepsilon \to 0$. The limit of last term exists by (3.1) in Lemma 3.3. On the other hand, by assumption $\partial f/\partial u_j$, $j = 1, 2, \ldots, 2n$, are Hölder continuous, so we have the following identity:

$$\psi(\varepsilon) = -\frac{C_n}{n\varepsilon^{2n}}\int_{\partial(B(0,\varepsilon)\cap\Omega)} f(z)\sum_{k=1}^{n}(-1)^{k-1}\overline{z}_k dz_2\cdots dz_n d\overline{z}_1\cdots[d\overline{z}_k]\cdots d\overline{z}_n$$

$$= -\frac{C_n}{n\varepsilon^{2n}}\int_{\partial(B(0,\varepsilon)\cap\Omega)}\left[f(0) + \sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}(0)z_j + \frac{\partial f}{\partial \overline{z}_j}(0)\overline{z}_j\right)\right]$$

$$\cdot\sum_{k=1}^{n}(-1)^{k-1}\overline{z}_k dz_2\cdots dz_n d\overline{z}_1\cdots[d\overline{z}_k]\cdots d\overline{z}_n + o(1)$$

$$= \frac{(-1)^n C_n f(0)}{\varepsilon^{2n}}\int_{B(0,\varepsilon)\cap\Omega} dz_2\cdots dz_n d\overline{z}_1\cdots d\overline{z}_n + o(1)$$

$$- \frac{C_n}{n\varepsilon^{2n}}\frac{\partial f}{\partial z_1}(0)\sum_{k=1}^{n}(-1)^{k-1}\int_{B(0,\varepsilon)\cap\Omega}\overline{z}_k dz_1\cdots dz_n d\overline{z}_1\cdots[d\overline{z}_k]\cdots d\overline{z}_n$$

$$+ \frac{(-1)^n C_n}{\varepsilon^{2n}}\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}(0)\int_{B(0,\varepsilon)\cap\Omega} z_j dz_2\cdots dz_n d\overline{z}_1\cdots d\overline{z}_n$$

$$+ \frac{(-1)^n C_n(n+1)}{n\varepsilon^{2n}}\sum_{j=1}^{n}\frac{\partial f}{\partial \overline{z}_j}(0)\int_{B(0,\varepsilon)\cap\Omega}\overline{z}_j dz_2\cdots dz_n d\overline{z}_1\cdots d\overline{z}_n.$$

The last identity is obtained by Stokes' formula.

By the method used in proving Lemma 3.3 we can prove that the last three terms tend to zero as ε tends to zero.

$$\begin{aligned}
&\int_{B(0,\varepsilon)\cap\Omega} dz_2\cdots dz_n d\bar{z}_1\cdots d\bar{z}_n \\
&\quad= \int_{B(0,\varepsilon)\cap\Omega} (-1)^{n-1}(du_1 - i du_{n+1})(-2i)^{n-1} du_2\cdots du_n du_{n+2}\cdots du_{2n} \\
&\quad= (-2i)^{n-1}(-i)\int_{B(0,\varepsilon)\cap\Omega} du_2 du_3\cdots du_{2n} + (-2i)^{n-1} \\
&\qquad\cdot \int_{B(0,\varepsilon)\cap\Omega} \frac{\partial h}{\partial u_{n+1}} du_2 du_3\cdots du_{2n} \\
&\quad= (-2i)^{n-1}(-i)\int_{B(0,\varepsilon)\cap\Omega} du_2 du_3\cdots du_{2n} + o(\varepsilon^{2n}) \\
&\quad= (-2i)^{n-1}(-i)\int_0^\pi \sin^{2n-3}\theta_1 d\theta_1 \cdots \\
&\qquad \int_0^\pi \sin\theta_{2n-3} d\theta_{2n-3}\int_0^{2\pi} d\theta_{2n-2}\int_{\psi^2+\rho^2\le\varepsilon^2} \rho^{2n-2} d\rho + o(\varepsilon^{2n}) \\
&\quad= (-2i)^{n-1}(-i)\int_0^\pi \sin^{2n-3}\theta_1 d\theta_1 \cdots \\
&\qquad \int_0^\pi \sin\theta_{2n-3} d\theta_{2n-3}\int_0^{2\pi} d\theta_{2n-2}\int_0^\varepsilon t^{2n-2} dt + o(\varepsilon^{2n}).
\end{aligned}$$

Therefore we see that

$$\psi(\varepsilon) = \frac{i}{\varepsilon}(2i)^{n-1} C_n b_{2n-1} f(0) + o(1).$$

The proof is finished.

Theorem 3.5. *Let f and D be as in Theorem 3.4. Define*

$$F(w) = \int_\Omega f(z) K(z, w).$$

Then the limits of $(\partial F/\partial w_k)(w)$, $k = 1, \ldots, n$, as w tends to 0 along the inner normal and the exterior normal of Ω at the point 0, exist and are given by the following expressions:

$$\left(\frac{\partial F}{\partial w_k}\right)_i (0) = \mathrm{FP}\int_\Omega n f(z)\frac{\bar{z}_k}{|z|^2} K(z, 0) + \frac{1}{2}\frac{\partial f}{\partial z_k}(0) + \frac{1}{2} A_k \frac{\partial f}{\partial \bar{z}_k}(0);$$

$$\left(\frac{\partial F}{\partial w_k}\right)_\varepsilon (0) = \mathrm{FP}\int_\Omega n f(z)\frac{\bar{z}_k}{|z|^2} K(z, 0) - \frac{1}{2}\frac{\partial f}{\partial z_k}(0) - \frac{1}{2} A_k \frac{\partial f}{\partial \bar{z}_k}(0),$$

where

$$A_k = \begin{cases} -(2i)^n (2n-1) C_n b_{2n-1}\int_0^\infty \dfrac{t^{2n-2} dt}{(t^2+1)^n} = -1; & k = 1, \\ 0; & k = 2, \ldots, n. \end{cases}$$

Here b_{2n-1} is the volume of the unit ball of dimension $2n-1$ and C_n is the constant in the Bochner-Martinelli kernel.

Proof. We only prove the first equality for the case $k = 1$. For the other cases the proof is similar.

$$\frac{\partial F}{\partial w_1}(w) = nC_n \int_\Omega f(z) \frac{\overline{z}_1 - \overline{w}_1}{|z-w|^{2n+2}} \sum_{j=1}^{n} (-1)^{j-1} (\overline{z}_j - \overline{w}_j) dz_1 \cdots dz_n d\overline{z}_1 \cdots [d\overline{z}_j] \cdots d\overline{z}_n$$

$$= -C_n \int_\Omega f(z) d\left[\frac{1}{|z-w|^{2n}} \sum_{j=1}^{n} (-1)^{j-1} (\overline{z}_j - \overline{w}_j) dz_2 \cdots dz_n d\overline{z}_1 \cdots [d\overline{z}_j] \cdots d\overline{z}_n\right]$$

$$= C_n \int_\Omega \frac{\partial f(z)}{\partial z_1} \frac{1}{|z-w|^{2n}} \sum_{j=1}^{n} (-1)^{j-1} (\overline{z}_j - \overline{w}_j) dz_1 \cdots dz_n d\overline{z}_1 \cdots [d\overline{z}_j] \cdots d\overline{z}_n$$

$$+ (-1)^{n-1} C_n \int_\Omega \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} \frac{1}{|z-w|^{2n}} (\overline{z}_j - \overline{w}_j) dz_2 \cdots dz_n d\overline{z}_1 \cdots d\overline{z}_n$$

$$= \int_\Omega \frac{\partial f}{\partial z_1} K(z, w)$$

$$+ (-1)^{n-1} C_n \int_\Omega \frac{1}{|z-w|^{2n}} \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} (\overline{z}_j - \overline{w}_j) dz_2 \cdots dz_n d\overline{z}_1 \cdots d\overline{z}_n .$$

By the proof of Theorem 3.4 we know that

$$\mathrm{FP} \int_\Omega f(z) \frac{\overline{z}_1}{|z|^2} K(z, 0) = \lim_{\varepsilon \to 0} \frac{1}{n} \int_{\Omega \setminus B(0,\varepsilon)} \frac{\partial f}{\partial z_1} K(z, 0)$$

$$+ \lim_{\varepsilon \to 0} \frac{C_n}{n} (-1)^{n-1} \int_{\Omega \setminus B(0,\varepsilon)} \frac{1}{|z|^{2n}} \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} \overline{z}_j dz_2 \cdots dz_n d\overline{z}_1 \cdots d\overline{z}_n .$$

So we only need to prove the following inequality

(3.9)

$$\lim_{w \to 0} \int_\Omega \frac{\partial f}{\partial z_1} K(z, w)$$

$$+ \lim_{w \to 0} (-1)^{n-1} C_n \int_\Omega \frac{1}{|z-w|^{2n}} \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} (\overline{z}_j - \overline{w}_j) dz_2 \cdots dz_n d\overline{z}_1 \cdots d\overline{z}_n$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega \setminus B(0,\varepsilon)} \frac{\partial f}{\partial z_1} K(z, 0) + \frac{1}{2} \frac{\partial f}{\partial z_1}(0)$$

$$+ \lim_{\varepsilon \to 0} C_n (-1)^{n-1} \int_{\Omega \setminus B(0,\varepsilon)} \frac{1}{|z|^{2n}} \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} \overline{z}_j dz_2 \cdots dz_n d\overline{z}_1 \cdots d\overline{z}_n$$

$$+ \frac{1}{2} A_1 \frac{\partial f}{\partial \overline{z}_1}(0) .$$

It follows from Lemma 3.2 that

$$\lim_{w\to 0}\int_\Omega \frac{\partial f}{\partial z_1}K(z,w)=\lim_{\varepsilon\to 0}\int_{\Omega\setminus B(0,\varepsilon)}\frac{\partial f}{\partial z_1}K(z,0)+\frac{1}{2}\frac{\partial f}{\partial z_1}(0).$$

So we only need to prove that

$$\begin{aligned}&\lim_{w\to 0}(-1)^{n-1}C_n\int_\Omega \frac{1}{|z-w|^{2n}}\sum_{j=1}^n\frac{\partial f}{\partial \overline{z}_j}(\overline{z}_j-\overline{w}_j)dz_2\cdots dz_n d\overline{z}_1\cdots d\overline{z}_n\\ &=\lim_{\varepsilon\to 0}(-1)^{n-1}C_n\int_{\Omega\setminus B(0,\varepsilon)}\frac{1}{|z|^{2n}}\sum_{j=1}^n\frac{\partial f}{\partial \overline{z}_j}\overline{z}_j dz_2\cdots dz_n d\overline{z}_1\cdots d\overline{z}_n\\ &\quad+\frac{1}{2}A_1\frac{\partial f}{\partial \overline{z}_1}(0).\end{aligned}\tag{3.10}$$

Take w on the inner normal. Its coordinates are $w_1=-\varepsilon^2$, $w_2=0,\ldots,$ $w_n=0$. Let

$$\begin{aligned}\Delta_j(\varepsilon)&=(-1)^{n-1}C_n\int_\Omega\frac{1}{|z-w|^{2n}}\frac{\partial f}{\partial \overline{z}_j}(\overline{z}_j-\overline{w}_j)dz_2\cdots dz_n d\overline{z}_1\cdots d\overline{z}_n\\ &\quad-(-1)^{n-1}C_n\int_{\Omega\setminus B(0,\varepsilon)}\frac{1}{|z|^{2n}}\frac{\partial f}{\partial \overline{z}_j}\overline{z}_j dz_2\cdots dz_n d\overline{z}_1\cdots d\overline{z}_n\\ &=(2i)^{n-1}C_n\int_\Omega\frac{1}{|z-w|^{2n}}\left[\frac{\partial f}{\partial \overline{z}_j}(z)\left(\frac{\partial h}{\partial u_{n+1}}(z)-i\right)\right.\\ &\qquad\qquad\left.-\frac{\partial f}{\partial \overline{z}_j}(0)\left(\frac{\partial h}{\partial u_{n+1}}(0)-i\right)\right]\\ &\qquad\qquad\cdot(\overline{z}_j-\overline{w}_j)du_2\cdots du_{2n}\\ &\quad-(2i)^{n-1}C_n\int_{\Omega\setminus B(0,\varepsilon)}\frac{1}{|z|^{2n}}\left[\frac{\partial f}{\partial \overline{z}_j}(z)\left(\frac{\partial h}{\partial u_{n+1}}(z)-i\right)\right.\\ &\qquad\qquad\left.-\frac{\partial f}{\partial \overline{z}_j}(0)\left(\frac{\partial h}{\partial u_{n+1}}(0)-i\right)\right]\overline{z}_j du_2\cdots du_{2n}\\ &\quad+(2i)^{n-1}C_n\frac{\partial f}{\partial \overline{z}_j}(0)\left(\frac{\partial h}{\partial u_{n+1}}(0)-i\right)\int_\Omega\frac{\overline{z}_j-\overline{w}_j}{|z-w|^{2n}}du_2\cdots du_{2n}\\ &\quad-(2i)^{n-1}C_n\frac{\partial f}{\partial \overline{z}_j}(0)\left(\frac{\partial h}{\partial u_{n+1}}(0)-i\right)\int_{\Omega\setminus B(0,\varepsilon)}\frac{\overline{z}_j}{|z|^{2n}}du_2\cdots du_{2n}.\end{aligned}$$

The function $\partial h/\partial u_{n+1}-i$ is Hölder continuous. Therefore we see that when $j=2,\ldots,n$,

$$\begin{aligned}|\Delta_j(\varepsilon)|&\le C\int_{\Omega\setminus B(0,\varepsilon)}|z|^\alpha\left|\frac{\overline{z}_j}{|z-w|^{2n}}-\frac{\overline{z}_j}{|z|^{2n}}\right|du_2\cdots du_{2n}\\ &\quad+C\int_{\Omega\cap B(0,\varepsilon)}\frac{|z|^\alpha|\overline{z}_j|}{|z-w|^{2n}}du_2\cdots du_{2n}\\ &\quad+C\int_{\Omega\setminus B(0,\varepsilon)}\left|\frac{\overline{z}_j}{|z-w|^{2n}}-\frac{\overline{z}_j}{|z|^{2n}}\right|du_2\cdots du_{2n}\\ &\quad+C\left|\int_{\Omega\cap B(0,\varepsilon)}\frac{\overline{z}_j}{|z-w|^{2n}}du_2\cdots du_{2n}\right|.\end{aligned}$$

By applying Lemma 3.3 we get

$$\Delta_j(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0, \ j = 2, \dots, n. \tag{3.11}$$

The case $j = 1$ remains to be considered:

$$\begin{aligned}
\left|\Delta_1(\varepsilon) - \frac{1}{2}A_1\frac{\partial f}{\partial \overline{z}_1}(0)\right| &\le C\int_{\Omega\setminus B(0,\varepsilon)} |z|^\alpha \left|\frac{\overline{z}_1 - \overline{w}_1}{|z-w|^{2n}} - \frac{\overline{z}_1}{|z|^{2n}}\right| du_2\cdots du_{2n} \\
&+ C\int_{\Omega\cap B(0,\varepsilon)} \frac{|z|^\alpha|\overline{z}_1 - \overline{w}_1|}{|z-w|^{2n}} du_2\cdots du_{2n} \\
&+ C\int_{\Omega\setminus B(0,\varepsilon)} \left|\frac{\overline{z}_1 - \overline{w}_1}{|z-w|^{2n}} - \frac{\overline{z}_1}{|z|^{2n}}\right| du_2\cdots du_{2n} \\
&+ C\left|\int_{\Omega\cap B(0,\varepsilon)} \frac{\overline{z}_1 - \overline{w}_1}{|z-w|^{2n}} du_2\cdots du_{2n} - \frac{\pi^n}{(n-1)!}\right|.
\end{aligned}$$

Again by applying Lemma 3.3 we get

$$\Delta_1(\varepsilon) \to \frac{1}{2}A_1\frac{\partial f}{\partial \overline{z}_1}(0) \quad \text{as } \varepsilon \to 0. \tag{3.12}$$

Now (3.10) follows from (3.11) and (3.12).

Now let us consider the general case. In the sequel we assume that Ω is the boundary of a domain D and is of class C^3. Let $\alpha \in \Omega$ be a point. Assume that Ω is defined locally by an equation $r(z) = 0$ in a neighborhood of a point a, $dr|_{z=a} \neq 0$, so that its tangent plane at the point a is

$$\operatorname{Re}\sum_{j=1}^n \left(\frac{\partial r}{\partial z_j}\right)_{z=a} (z_j - a_j) = 0.$$

We make a unitary transformation

$$z - a = z^* U, \qquad z_k - a_k = \sum_{j=1}^n z_j^* U_{jk},$$

where the U_{jk} satisfy

$$\sum_{j=1}^n U_{kj}\left(\frac{\partial r}{\partial z_j}\right)_{z=a} = \begin{cases} c > 0, & k = 1; \\ 0, & k = 2, \dots, n. \end{cases}$$

If Ω is transformed to Ω^*, the equation for the tangent plane of Ω^* at $z^* = 0$ is $c(z_1^* + \overline{z}_1^*) = 0$.

Let $z_k^* = v_k + iv_{n+k}$, $k = 1, \dots, n$. Then the tangent plane of Ω^* at the point $z^* = 0$ has the equation $v_1 = 0$.

Because $K_{2n-1}(z, w)$ is invariant under the aforementioned unitary transformation, the finite part of the integral of f over Ω at point a can be defined as the following

$$\begin{aligned}
&\mathrm{FP}\int_\Omega f(z)\frac{\overline{z}_k - \overline{a}_k}{|z-a|^2}K_{2n-1}(z, a) \\
&= \mathrm{FP}\int_{\Omega^*} \sum_{j=1}^n \overline{U}_{jk} f(z^*U - a)\frac{\overline{z}_j^*}{|z^*|^2}K_{2n-1}(z^*, 0).
\end{aligned}$$

Now let us take

$$U_{1j} = \frac{(\partial r/\partial \overline{z}_j)_{z=a}}{\sum_{k=1}^n |(\partial r/\partial z_k)_{z=a}|^2}.$$

Let $F(w) = \int_\Omega f(z)K(z, w)$ be the same as in Theorem 3.5. We know that

$$\begin{aligned}\left(\frac{\partial F}{\partial w_k}\right)(w) &= n\int_\Omega f(z)\frac{\overline{z}_k - \overline{w}_k}{|z-w|^2}K(z, w)\\ &= n\int_{\Omega^*} f(z^*U - a)\frac{\sum_{j=1}^n \overline{U}_{jk}(\overline{z}_j^* - \overline{w}_j^*)}{|z^* - w^*|^2}K_{2n-1}(z^*, w^*).\end{aligned}$$

By using Theorem 3.5 we see that

$$\begin{aligned}\left(\frac{\partial F}{\partial w_k}\right)_i(a) &= \mathrm{FP}\int_\Omega nf(z)\frac{\overline{z}_k - \overline{z}_k^0}{|z-a|^2}K(z, a)\\ &\quad + \frac{1}{2}\sum_{j=1}^n \overline{U}_{jk}\frac{\partial f(z^*U)}{\partial z_j^*}(0) + \frac{1}{2}A_1\overline{U}_{1k}\frac{\partial f(z^*U)}{\partial \overline{z}_1^*}(0)\\ &= \mathrm{FP}\int_\Omega nf(z)\frac{\overline{z}_k - \overline{z}_k^0}{|z-a|^2}K(z, a)\\ &\quad + \frac{1}{2}\frac{\partial f(z)}{\partial z_k}(a) - \frac{1}{2}\overline{U}_{1k}\sum_{j=1}^n \overline{U}_{1j}\frac{\partial f(z)}{\partial \overline{z}_j}(a).\end{aligned}$$

So we have proved the first one of the following formulas. The second one can be proved similarly.

$$\begin{aligned}\left(\frac{\partial F}{\partial w_k}\right)_i(a) &= \mathrm{FP}\int_\Omega nf(z)\frac{\overline{z}_k - \overline{a}_k}{|z-a|^2}K(z, a)\\ &\quad + \frac{1}{2}\frac{\partial f}{\partial z_k}(a) - \frac{1}{2}\overline{U}_{1k}\sum_{j=1}^n \overline{U}_{1j}\frac{\partial f}{\partial \overline{z}_j}(a).\\ \left(\frac{\partial F}{\partial w_k}\right)_e(a) &= \mathrm{FP}\int_\Omega nf(z)\frac{\overline{z}_k - \overline{a}_k}{|z-a|^2}K(z, a)\\ &\quad - \frac{1}{2}\frac{\partial f}{\partial z_k}(a) + \frac{1}{2}\overline{U}_{1k}\sum_{j=1}^n \overline{U}_{1j}\frac{\partial f}{\partial \overline{z}_j}(a).\end{aligned}$$

Here $(\partial F/\partial w_k)_i(a)$ and $(\partial F/\partial w_k)_e(a)$ are the limits of $(\partial F/\partial w_k)(w)$, $k = 1, \dots, n$, as w tends to a along the inner normal and the exterior normal of Ω at the point a, respectively.

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