TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 338, Number 1, July 1993

LOCAL INTEGRABILITY OF MIZOHATA STRUCTURES

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ABSTRACT. In this work we study the local integrability of strongly pseudoconvex Mizohata structures of rank $n > 2$ (and co-rank 1). These structures are locally generated in an appropriate coordinate system $(t_1, \ldots, t_n, x)$ by flat perturbations of Mizohata vector fields $M_j = \frac{\partial}{\partial t_j} - it_j\frac{\partial}{\partial x}$, $j = 1, \ldots, n$. For this, we first prove the global integrability of small perturbations of the structure generated by $\frac{\partial}{\partial \bar{z}} + \sigma_1\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \theta_{n-1}} + \sigma_j\frac{\partial}{\partial z}$, $j = 2, \ldots, n$, defined over a manifold $\mathbf{C} \times S$, where S is simply connected.

1. INTRODUCTION

The local solvability of overdetermined and underdetermined systems of vector fields has been studied with some generality under the assumption that the structure generated by the vector fields is locally integrable and has co-rank equal to one [16, 4, 5]. It is thus natural to study the local integrability of structures of co-rank one. In this work we prove that a formally integrable (i.e. involutive) structure of co-rank one and rank $n > 2$ is locally integrable if it is strongly pseudoconvex, a result analogous to Kuranishi's embedding theorem for CR-structures as extended by Akahori [8, 1]. These structures, here called Mizohata structures, are locally generated in an appropriate coordinate system $(t_1, \ldots, t_n, x)$ by n vector fields

$$M_j = \frac{\partial}{\partial t_j} - it_j\frac{\partial}{\partial x} + \rho_j\frac{\partial}{\partial x}, \qquad j = 1, \ldots, n, \tag{1.1}$$

where ρ_j is flat at $t = 0$ [17], i.e., they are flat perturbations of Mizohata vector fields. The problem of local integrability is equivalent to the problem of finding new local coordinates where the structure is generated by the vector fields (1.1) with $\rho_j = 0$, $j = 1, \ldots, n$. For $n = 1$, Nirenberg [11] showed that it is possible to select ρ_1 so that the corresponding structure is not locally integrable whereas our result shows that there is always local integrability if $n > 2$; so, only the case $n = 2$ remains open.

To prove our result we follow, loosely speaking, the method of Kuranishi which consists in approximating the given structure by a sequence of locally integrable structures. This sequence is constructed by using a Nash-Moser

Received by the editors August 30, 1990 and, in revised form, April 18, 1991.

1991 *Mathematics Subject Classification*. Primary 35N10, 58G05.

Key words and phrases. overdetermined systems, localintegrability, Mizohata structures, differential complex.

The first author was partly supported by CNPq.

scheme.The success of this approach is linked to the existence of homotopy formulas at the level of one-forms for each one of the approximating structures. Furthermore, the homotopy operators involved should have adequate continuity properties in some scale of Banach spaces. In our proof, instead of dealing directly with the structure $\mathscr{M}$ generated by vector fields (1.1), we first introduce polar coordinates in t. This change of variables becomes singular precisely at the characteristic points of $\mathscr{M}$. This blows up the nonelliptic points of $\mathscr{M}$ and we are left with an elliptic structure $\mathscr{L}$ generated by vector fields

$$(1.2)\qquad \begin{aligned} L_1 &= \frac{\partial}{\partial \bar{z}} + \sigma_1 \frac{\partial}{\partial z}, \\ L_j &= \frac{\partial}{\partial \theta_{j-1}} + \sigma_j \frac{\partial}{\partial z}, \qquad j = 2, \dots, n, \end{aligned}$$

where $z = x + is = x + i(|t|^2/2)$ and the σ_j's are smooth functions of s, vanishing for $s \leq 0$. It is to the structure $\mathscr{L}$ that we apply the method of Kuranishi. The homotopy operators for the structures that approximate $\mathscr{L}$ have explicit integral expressions. In this respect, as well as in other tecnichal details, the proof runs closer to Webster's proof of the theorems of Kuranishi and Newlander-Nirenberg [19, 20, 17]. The price to pay for considering the better behaved elliptic structure $\mathscr{L}$ is that one has to prove integrability *globally* in $\theta \in S^{n-1}$. It is easy to show that global integrability for $\mathscr{L}$ implies local integrability for $\mathscr{M}$.

Concerning the case of rank $n = 2$, Nagel and Rosay [10] proved recently nonexistence of homotopy formulas for CR-structures of hypersurface type, explaining why this case cannot be handled like the case of higher rank. This argument can be adapted to show that homotopy formulas do not exist either for locally integrable Mizohata structures of rank 2 [7].

The organization of this paper is as follows: in §2 we state the main theorem and introduce the elliptic structure $\mathscr{L}$, in §3 we prove the required homotopy formulas, in §4 we prove the continuity of the homotopy operators in Hölder norms, in §5 we briefly state the needed facts about smoothing operators, in the extremely long §6 we prove that the Nash-Moser scheme converges, thus proving the global integrability of $\mathscr{L}$ and, finally, in §7 we show that the integrability of $\mathscr{L}$ implies that of $\mathscr{M}$.

The authors are indebted to Professor Treves for teaching them about the work of Webster in his lectures at Recife as well as for pointing out reference [10].

2. Existence of a flat solution for a Mizohata structure

Let Ω be a paracompact manifold of class C^∞ and dimension N. We consider a subbundle $\mathscr{L}$ of

$$\mathbf{C}T\Omega = \mathbf{C} \otimes T\Omega.$$

Definition 2.1. $\mathscr{L}$ is a *formally integrable structure* if $[\mathscr{L}, \mathscr{L}] \subset \mathscr{L}$, that is, if the Lie bracket of two local sections of $\mathscr{L}$ is still a local section of $\mathscr{L}$.

We put $n = \dim \mathscr{L}$ and $m = \dim \mathscr{L}^\perp$.

The characteristic set of $\mathscr{L}$ is

$$C(\mathscr{L}) = \mathscr{L}^\perp \cap T^*\Omega$$

and the natural projection of $C(\mathscr{L})\backslash 0$ over Ω is a closed set called set of the *nonelliptic points.*

We now take $(p, \xi) \in C(\mathscr{L})$, $\xi \neq 0$. If $v, w \in \mathscr{L}_p$ we choose local sections L and M of $\mathscr{L}$ defined in a neigbourhood of p so that $L(p) = v$, $M(p) = w$, and we put

$$\Theta_{(p,\xi)}(v, w) = \frac{1}{2i}\xi([L, \overline{M}])(p).$$

This definition is independent of L and M. We define the *Levi form* of $\mathscr{L}$ at (p, ξ) by

$$v \mapsto \Theta_{(p,\xi)}(v, v), \qquad v \in \mathscr{L}_p.$$

$\Theta_{(p,\xi)}$ is a hermitian form and therefore it has a diagonal real representation.

Definition 2.2. $\mathscr{L}$ is *locally integrable* if, given $p \in \Omega$, there exists a neigbourhood U of p and C^∞ functions $z_k : U \to \mathbf{C}$, $k = 1, \ldots, m$, such that their differentials generate $\mathscr{L}^\perp$ over U.

Definition 2.3. Let $\mathscr{M}$ be a formally integrable structure of dimension n over a manifold Ω of dimension $n+1$. $\mathscr{M}$ is called a *Mizohata structure* if $C(\mathscr{M}) \neq 0$ and the Levi form associated to $\mathscr{M}$ is nondegenerated at every point of $C(\mathscr{M})$.

It is clear that Definition 2.2 is more restrictive than Definition 2.1. We will prove the equivalence of both definitions for a class of Mizohata structures. More precisely

Theorem 2.1. *Let $\mathscr{M}$ be a Mizohata structure over a manifold Ω of dimension $n+1$, $n > 2$. If $\mathscr{M}$ is a strongly pseudoconvex structure, that is, if all the eigenvalues of the Levi form associated to $\mathscr{M}$ are positive (or negative), then $\mathscr{M}$ is locally integrable.*

First, we need the following lemma.

Lemma 2.1 [17]. *Let $\mathscr{M}$ be a Mizohata structure of dimension n over a manifold Ω. There are a system of coordinates defined in a neigbourhood U of an arbitrary nonelliptic point $p \in \Omega$, a nondegenerated quadratic form Q in $\mathbf{R}^n$ and functions ρ_j, $j = 1, \ldots, n$, defined in U, such that*

(i) *ρ_j is flat at $t = 0$, $j = 1, \ldots, n$.*

(ii) *$\mathscr{M}$ is generated over U by*

$$M_j = \frac{\partial}{\partial t_j} - i\frac{\partial Q}{\partial t_j}\frac{\partial}{\partial x} + \rho_j\frac{\partial}{\partial x}, \qquad j = 1, \ldots, n.$$

Moreover, $\mathscr{M}$ is locally integrable in a neighbourhood of p if and only if there are coordinates $(x, t_1, \ldots, t_n)$ and a quadratic form Q as above such that $\rho_j \equiv 0$, $j = 1, \ldots, n$.

When $\mathscr{M}$ is strongly pseudoconvex, we can take

$$Q(t_1, \ldots, t_n) = \frac{1}{2}(t_1^2 + \cdots + t_n^2)$$

and so we can assume that $p = 0$ is a nonelliptic point of $\mathscr{M}$, $\Omega = \mathbf{R}^{n+1}$ and $\mathscr{M}$ is (globally) generated by the vector fields

$$M_j = \frac{\partial}{\partial t_j} - it_j\frac{\partial}{\partial x} + \rho_j\frac{\partial}{\partial x}, \qquad j = 1, \ldots, n, \tag{2.1}$$

where ρ_j are flat at $t = 0$.

Now, we use polar coordinates for $t \in \mathbf{R}^n$:

$$r = \sqrt{t_1^2 + \cdots + t_n^2}, \qquad \theta \in S^{n-1}.$$

We fix a local chart $(\theta_1, \dots, \theta_{n-1})$ of S^{n-1}. Then $\mathscr{M}$ is generated by

$$\begin{aligned} \widetilde{M}_1 &= \frac{\partial}{\partial r} - ir\frac{\partial}{\partial x} + \tilde{\rho}_1 \frac{\partial}{\partial x}, \\ \widetilde{M}_j &= \frac{\partial}{\partial \theta_{j-1}} + \tilde{\rho}_j \frac{\partial}{\partial x}, \qquad j = 2, 3, \dots, n, \end{aligned} \tag{2.2}$$

where $\tilde{\rho}_j$ are flat at $r = 0$.

We now take

$$s = \frac{r^2}{2}, \quad \theta = \theta \quad \text{and} \quad x = x. \tag{2.3}$$

When $r > 0$, (2.3) defines a change of coordinates and we can write (2.2) as

$$\begin{aligned} M_1^\# &= -i\sqrt{2s}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial s}\right) + \tilde{\rho}_1(x, \sqrt{2s}, \theta)\frac{\partial}{\partial x}, \\ M_j^\# &= \frac{\partial}{\partial \theta_{j-1}} + \tilde{\rho}_1(x, \sqrt{2s}, \theta)\frac{\partial}{\partial x}, \qquad j = 2, \dots, n. \end{aligned} \tag{2.4}$$

When $r = 0$, (2.3) is not a change of coordinates, so (2.2) and (2.4) do not have the same solutions because

$$r \mapsto s = \frac{r^2}{2}$$

becomes singular at $r = 0$, but we will see that a solution of (2.4) will yield a solution of (2.2).

Now, since

$$\frac{\tilde{\rho}_1(x, \sqrt{2s}, \theta)}{\sqrt{2s}} \qquad \text{and} \qquad \tilde{\rho}_j(x, \sqrt{2s}, \theta)$$

are also flat at $t = 0$, we can extend them to be zero when $s \leq 0$ and therefore we can study the structure generated by

$$\begin{aligned} L_1 &= \frac{\partial}{\partial \bar{z}} + \sigma_1 \frac{\partial}{\partial z}, \\ L_j &= \frac{\partial}{\partial \theta_{j-1}} + \sigma_j \frac{\partial}{\partial z}, \qquad j = 2, \dots, n, \end{aligned} \tag{2.5}$$

where $z = x + is$ and σ_j, $j = 1, \dots, n$, are C^∞ functions vanishing for $s = \operatorname{Im}(z) \leq 0$.

The structure $\mathscr{L}$ generated by (2.5) is an elliptic structure over $\mathbf{C} \times S^{n-1}$; therefore it is locally integrable [15], but this depends on a localization in $\theta \in S^{n-1}$ which is not enough to cover a full neighbourhood of the origin in the original coordinates.

An invariant way of treating this problem is the following: we fix a canonical chart z of $\mathbf{C}$ and we consider a compact manifold S instead of S^{n-1}. We also consider the structure

$$\mathscr{A} = \langle dz \rangle^\perp \tag{2.6}$$

defined over $\mathbf{C}\times S$. We want to study the perturbated structure $\mathscr{L}$ given by

$$\mathscr{L}=\langle w\rangle^{\perp}, \quad w=dz-\sigma_1 d\bar{z}-\sigma, \tag{2.7}$$

where σ is a smooth 1-form in S which depends on z as a parameter, σ_1 is a C^∞ function defined in $\mathbf{C}\times S$ and both are flat at $\mathrm{Im}(z)=0$. When $S=S^{n-1}$, we get a structure as in (2.5).

We now consider the complex associated to $\mathscr{L}$:

$$\begin{aligned} C^\infty(\mathbf{C}\times S)&\xrightarrow{\delta_0} C^\infty(\mathbf{C}\times S, \wedge^1\mathscr{L}^*(\mathbf{C}\times S)) \\ &\xrightarrow{\delta_1} C^\infty(\mathbf{C}\times S, \wedge^2\mathscr{L}^*(\mathbf{C}\times S))\xrightarrow{\delta_2}\cdots, \end{aligned} \tag{2.8}$$

$$\bigwedge^k\mathscr{L}^*(\mathbf{C}\times S)=\frac{\bigwedge^k(\mathbf{C}\times S)}{\mathscr{J}\cap\bigwedge^k(\mathbf{C}\times S)}, \qquad k=1,\dots,N, \tag{2.9}$$

where $\mathscr{J}$ is the ideal generated by the sections of $\mathscr{L}^\perp$ in $\sum_{k=0}^N\bigwedge^k(\mathbf{C}\times S)$ and δ_k is the operator induced by the exterior derivative in the space (2.9).

Using the decomposition

$$\mathbf{C}T^*(\mathbf{C}\times S)=\langle w\rangle\oplus\langle d\bar{z}\rangle\oplus\mathbf{C}T^*S\ , \tag{2.10}$$

we can write

$$\delta_0 f=\left(\frac{\partial f}{\partial\bar{z}}+\sigma_1\frac{\partial f}{\partial z}\right)d\bar{z}+d_S f+\frac{\partial f}{\partial z}\sigma \tag{2.11}$$

where d_S is the exterior derivative of S.

Indeed,

$$df=\left(\frac{\partial f}{\partial\bar{z}}+\sigma_1\frac{\partial f}{\partial z}\right)d\bar{z}+\left(d_S f+\sigma\frac{\partial f}{\partial z}\right)+\omega\frac{\partial f}{\partial z}. \tag{2.12}$$

So, the equations

$$L_j f=0, \qquad j=1,\dots,n, \tag{2.13}$$

where L_j are given by (2.5), can be invariantly defined by

$$\delta_0 f=0, \tag{2.14}$$

taking $S=S^{n-1}$.

3. A homotopy formula for $\mathscr{A}=\langle dz\rangle^\perp$

Let S be a simply connected compact orientable manifold of dimension $n-1\geq 2$. We fix a Riemannian metric in S.

We consider in $\mathbf{C}\times S$ the globally integrable structure $\mathscr{A}$ given by (2.6). We will construct a homotopy formula for the complex associated to $\mathscr{A}$:

(3.1)

$$\begin{aligned} C^\infty(\mathbf{C}\times S)&\xrightarrow{\delta_0^{\mathscr{A}}} C^\infty\left(\mathbf{C}\times S, \bigwedge^1\mathscr{A}^*(\mathbf{C}\times S)\right)\xrightarrow{\delta_1^{\mathscr{A}}} C^\infty\left(\mathbf{C}\times S, \bigwedge^2\mathscr{A}^*(\mathbf{C}\times S)\right) \\ &\xrightarrow{\delta_2^{\mathscr{A}}}\cdots\bigwedge^k\mathscr{A}^*(\mathbf{C}\times S)=\frac{\bigwedge^k(\mathbf{C}\times S)}{\langle dz\rangle} \end{aligned}$$

at the first stage.

More precisely, we will construct operators K_1 and K_2 such that

$$K_2\delta_1^{\mathscr{A}} F + \delta_0^{\mathscr{A}} K_1 F = F \quad \text{in } \Delta \times S \tag{3.2}$$

for all $F \in C^\infty(\mathbf{C} \times S, \bigwedge^1 \mathscr{A}^*(\mathbf{C} \times S))$, where Δ is the unit ball of $\mathbf{C}$.

According to the Hodge theorem, it is possible to find operators K_1^S and K_2^S such that

$$d_S K_1^S + K_2^S d_S = I \tag{3.3}$$

because S is simply connected. Here d_S is the exterior derivative in S,

$$K_1^S = d_S^* G_1 \qquad \text{and} \qquad K_2^S = d_S^* G_2 \tag{3.4}$$

where G_i is the Green operator for the Laplacean in S, $i = 1, 2$.

If $F \in C^\infty(\mathbf{C} \times S, \bigvee^j \mathscr{A}^*(\mathbf{C} \times S))$ then F has the following unique decomposition:

$$F = w + \phi \wedge d\overline{z}, \tag{3.5}$$

$w \in C^\infty(\mathbf{C}, C^\infty(S, \bigwedge^j S))$ and $\phi \in C^\infty(\mathbf{C}; C^\infty(S, \bigwedge^{j-1} S))$.

So,

$$\delta_j^{\mathscr{A}} F = \delta_j^{\mathscr{A}}(w + \phi \wedge d\overline{z}) = d_S w + (-1)^j \frac{\partial w}{\partial \overline{z}} \wedge d\overline{z} + d_S \phi \wedge d\overline{z} . \tag{3.6}$$

Now, we define

$$Tf(z) = \frac{1}{2\pi i} \int\int_{|\tau|<1} \frac{f(\tau)}{\tau - z} d\tau \wedge d\overline{\tau}, \qquad z \in \Delta. \tag{3.7}$$

We know that

$$\frac{\partial}{\partial \overline{z}}(Tf)(z) = f(z), \qquad z \in \Delta. \tag{3.8}$$

Then we define

$$K_1 F = K_1^S(\omega - d_S T\phi) + T\phi \tag{3.9}$$

for $F = \omega + \phi d\overline{z} \in C^\infty(\mathbf{C} \times S; \bigwedge^1 \mathscr{A}^*(\mathbf{C} \times S))$, and

$$K_2 F = K_1^S \phi d\overline{z} + K_2^S \omega \tag{3.10}$$

for $F = \omega + \phi \wedge d\overline{z} \in C^\infty(\mathbf{C} \times S, \bigwedge^2 \mathscr{A}^*(\mathbf{C} \times S))$.

Since K_1^S comutes with the Laplacean of S, it also comutes with d_S and with $\partial/\partial\overline{z}$. By (3.6), (3.9), (3.3), and (3.8),

$$\begin{aligned}
\delta_0^{\mathscr{A}}(K_1 F) &= d_S K_1 F + \frac{\partial}{\partial \overline{z}}(K_1 F) d\overline{z} \\
&= d_S K_1^S(\omega - d_S T\phi) + d_S T\phi \\
&\quad + \left[K_1^S \frac{\partial \omega}{\partial \overline{z}} - K_1^S d_S \frac{\partial}{\partial \overline{z}}(T\phi) + \frac{\partial}{\partial \overline{z}}(T\phi)\right] d\overline{z} \\
&= \omega - d_S T\phi - K_2^S d_S(\omega - d_S T\phi) \\
&\quad + d_S T\phi + \left[K_1^S \left(\frac{\partial \omega}{\partial \overline{z}} - d_S \phi\right) + \phi\right] d\overline{z} \\
&= \omega - K_2^S d_S \omega - K_1^S \left(d_S \phi - \frac{\partial \omega}{\partial \overline{z}}\right) + \phi d\overline{z}
\end{aligned} \tag{3.11}$$

in $\Delta \times S$, because $d_S{}^2 = 0$. So, (3.2) is proved.

On the other hand, we can put (3.9) in the following form

$$K_1 F = K_1^S \omega + \frac{1}{\sigma(S)} \int_S T\phi \, d\sigma \tag{3.12}$$

for $F = \omega + \phi d\bar{z} \in C^\infty(\mathbf{C} \times S, \bigwedge^1 \mathscr{A}^*(\mathbf{C} \times S))$, where $d\sigma$ is the volume element in S. Indeed, we consider $v = d_S{}^* G_1(\omega)$, $\omega = d_S f$. Then (3.3) implies $\omega = d_S d_S^* G_1 \omega$ and $d_S(v - f) = 0$.

Hence from (3.4),

$$K_1^S d_S f - f = d_S^* G_1 d_S f - f = c, \tag{3.13}$$

c constant. But

$$\int_S v \, d\sigma = \langle v, 1 \rangle_{L^2(S)} = \langle d_S{}^* G_1 \omega, 1 \rangle_{L_2(S)} = 0 \tag{3.14}$$

then (3.12) follows from (3.9).

4. Hölder regularity of the homotopy operators

Let K be a compact convex subset of $\mathbf{R}^l$. We define

$$|u|_\alpha = \sup_{x, y \in K} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \tag{4.1}$$

for $u \in C^0(K)$ and $0 < \alpha \leq 1$.

If $u \in C^k(K)$ and $k < \alpha \leq k + 1$, k positive integer, we put

$$|u|_\alpha = \sum_{|\beta| = k} |\partial^\beta u|_{\alpha - k} \tag{4.2}$$

and

$$||u||_\alpha = |u|_\alpha + |u|_0, \quad |u|_0 = \sup_K |u|. \tag{4.3}$$

We can define $C^\alpha(\Omega)$ when Ω is a compact manifold using coordinates and a partition of unity as usual. We can also define the spaces $C^\alpha(\Omega; E)$ of sections of a bundle E over Ω. Given $\Omega \subset \mathbf{C} \times S$ possibly noncompact, we want to define the space $C^\alpha(\Omega, \bigwedge^j T^*(\mathbf{C} \times S))$. For this, we fix a chart z of $\mathbf{C}$ and using it we can define a norm which we denote by

$$|| \; ||_{\substack{\alpha, z \\ \Omega}} \tag{4.4}$$

If we fix a chart z of $\mathbf{C}$, the norms coming from different coverings and partitions of unity on S are equivalent.

We want to study the Hölder regularity of the homotopy operators K_1 and K_2 constructed before. For this, we fix a chart z of $\mathbf{C}$ and we take

$$F = \omega + \phi d\bar{z} \in C^\alpha \left(\mathbf{C} \times S, \bigwedge^1 \mathscr{A}^*(\mathbf{C} \times S) \right) \tag{4.4}$$

then F belongs to C^α if and only if ω and ϕ belong to C^α.

Now, we claim that there exists a constant $C = C(\alpha) > 0$ such that

$$||K_1 F||_{\substack{\alpha, z \\ \Delta \times S}} \leq C ||F||_{\substack{\alpha, z \\ \Delta \times S}} \tag{4.5}$$

for every $F \in C^\alpha(\mathbf{C}\times S, \bigwedge^1 \mathscr{A}^*(\mathbf{C}\times S))$, $m < \alpha < m+1$, m integer positive (Δ is the unit ball of $\mathbf{C}$). Indeed, we fix a covering $\{U_j, \theta_j\}$ of S by coordinate neighbourhoods and a subordinate partition of unity $\{\chi_j\}$. Then

$$\|K_1 F\|_{\substack{\alpha,z\\ \Delta\times S}} \le \|K_1^S \omega\|_{\substack{\alpha,z\\ \Delta\times S}} + \frac{1}{\sigma(S)}\left\|\int_S T\phi\, d\sigma\right\|_{C^\alpha(\Delta)} \tag{4.6}$$

and so

$$\|K_1 F\|_{\substack{\alpha,z\\ \Delta\times S}} \le \|K_1^S \omega\|_{\substack{\alpha,z\\ \Delta\times S}} + C\|\phi\|_{\substack{\alpha,z\\ \Delta\times S}} \tag{4.7}$$

because

$$\frac{1}{\pi z} * \cdot$$

maps $C^\alpha(\Delta)$ into $C^{1+\alpha}(\Delta)$ [2] and the regularity in the variables of S remains the same.

From now on, the letter C will be used to denote several universal constants.

$$\begin{aligned}\|K_1^S\omega\|_{\substack{\alpha,z\\ \Delta\times S}} &= \sup_{\Delta\times S}|K_1^S\omega(z,\theta)|\\ &+\sum_j \sum_{|\beta|+|\gamma|=m} \sup_{\Delta\times S}\frac{|\partial_z^\beta\partial_\theta^\gamma\chi_j K_1^S\phi(z,\theta_j)-\partial_z^\beta\partial_\theta^\gamma\chi_j K_1^S\phi(z',\theta_j')|}{(|z-z'|+|\theta_j-\theta_j'|)^{\alpha-m}}.\end{aligned}\tag{4.8}$$

Since

$$\begin{aligned}&\frac{|\partial_\theta^\gamma\chi_j(K_1^S\partial_z^\beta\omega)(z,\theta_j)-\partial_\theta^\gamma\chi_j(K_1^S\partial_z^\beta\omega)(z,\theta_j')|}{|\theta_j-\theta_j'|^{\alpha-m}}\\ &\le C\|K_1^S\partial_z^\beta\omega(z,.)\|_{C^{|\gamma|+\alpha-m}(S)}\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}&\left|\partial_\theta^\gamma\chi_j K_1^S\left(\frac{\partial_z^\beta\omega(z,\theta_j')-\partial_z^\beta\omega(z',\theta_j')}{|z-z'|^{\alpha-m}}\right)\right|\\ &\le C\left\|K_1^S\left(\frac{\partial_z^\beta\omega(z,.)-\partial_z^\beta\omega(z',.)}{|z-z'|^{\alpha-m}}\right)\right\|_{C^{|\gamma|}(S)}\end{aligned}\tag{4.10}$$

using the fact that K_1^S is a continuous linear operator from $C^\eta(S)$ into $C^{\eta+1}(S)$, $\eta > 0$, $\eta \notin \mathbf{N}$, we get (4.5). A proof of this continuity can be found in [2].

Similarly, we can find another $C = C(\alpha) > 0$ such that

$$\|K_2F\|_{\substack{\alpha,z\\ \Delta\times S}} \le C\|F\|_{\substack{\alpha,z\\ \Delta\times S}}, \qquad F\in C^\alpha\left(\mathbf{C}\times S, \bigwedge^2\mathscr{A}^*(\mathbf{C}\times S)\right). \tag{4.11}$$

We must note that K_1 and K_2 do not gain a derivative as in the Newlander-Nirenberg Theorem but we do not need to shrink the domain $\Delta\times S$ to get the estimates (4.5) and (4.11).

Now, we know that there is a continuous linear extension operator [14]:

$$\varepsilon : C^\alpha(\Delta\times S)\to C^\alpha(\mathbf{C}\times S) \tag{4.12}$$

Then we take

$$\widetilde{K}_1 = \eta\cdot(\varepsilon_1\circ K_1), \quad \widetilde{K}_2 = \eta\cdot(\varepsilon_2\circ K_2), \tag{4.13}$$

where η is a smooth function supported in Δ_2, the ball of radius 2 at the origin of $\mathbf{C}$ and ε_i is an extention operator acting in $C^\alpha(\Delta \times S, \bigwedge^i \mathscr{A}^*(\mathbf{C} \times S))$, $i = 1, 2$, as in (4.12).

So, for $F \in C^\infty(\mathbf{C} \times S, \bigwedge^1 \mathscr{A}^*(\mathbf{C} \times S))$,we still have

$$\delta_0^{\mathscr{A}} \widetilde{K}_1 F + \widetilde{K}_2 \delta_1^{\mathscr{A}} F = F \quad \text{in } \Delta \times S. \tag{4.14}$$

Now supp $\widetilde{K}_1 F$, supp $\widetilde{K}_2 F \subset \Delta_2 \times S$, and the inequalities (4.5) and (4.11) are also valid for $\widetilde{K}_1$ and $\widetilde{K}_2$, respectively.

5. Smoothing operators

Let K be a compact subset of $\mathbf{C}$. We consider a covering $\{U_j, \theta_j\}$ of S by coordinate neighbourhoods such that $\theta_j(U_j)$ is contained in the ball B_1 of radius 1 and center 0 in $\mathbf{R}^{n-1}$. We can also fix a partition of unity $\{\phi_j\}$ subordinate to this covering and denote by z the canonical coordinate of $\mathbf{C}$.

Given $f \in C^0(\mathbf{C} \times S)$ with supp $f \subset K \times S$, $f = \sum_j f_j$, $f_j = \phi_j f$. Let F_j be the expression of f_j using (z, θ_j) as coordinate.

Let χ be a function of $\mathscr{S}(\mathbf{R}^{n+1})$ such that $\hat{\chi} \in C_c^\infty(\mathbf{R}^{n+1})$ (Fourier transform in $\mathbf{R}^{n+1}$) and $\chi \equiv 1$ in a neighbourhood of the origin.

We define

$$S_N F_j = \psi \cdot (\chi_N * F_j) \tag{5.1}$$

where $\chi_N(x) = N^{n+1}\chi(Nx)$, $N \geq 1$, and $\psi \in C_c^\infty(\mathbf{R}^{n+1})$, $\psi \equiv 1$ in a neighbourhood of $K \times B_1$.

Putting $S_N f_j = (S_N F_j) \circ (z, \theta_j)$, then

$$S_N f = \sum_j S_N f_j \tag{5.2}$$

is called the *regularization* of f.

The following estimates are known for $f \in C^\alpha(\mathbf{C} \times S)$ with support in $K \times S$:

$$\|S_N f\|_{\substack{\beta, z \\ \mathbf{C}\times S}} \leq C N^{\beta-\alpha} \|f\|_{\substack{\alpha, z \\ \mathbf{C}\times S}}, \qquad \alpha \leq \beta, \tag{5.3}$$

$$\|(I - S_N) f\|_{\substack{\beta, z \\ \mathbf{C}\times S}} \leq C N^{\beta-\alpha} \|f\|_{\substack{\alpha, z \\ \mathbf{C}\times S}}, \qquad \beta \leq \alpha. \tag{5.4}$$

6. Perturbations of globally integrable structures

Theorem 6.1. *We consider the structure $\mathscr{L}$ given by* (2.7) *where σ and σ_1 are smooth in $\mathbf{C} \times S$. Let $\alpha \in \mathbf{R}$, $0 < \alpha < 1$. Then it is possible to find $\varepsilon_0 > 0$ such that, if*

$$\|\sigma_1\|_{\substack{1+\alpha, z \\ \mathbf{C}\times S}} < \varepsilon_0 \qquad \textit{and} \qquad \|\sigma\|_{\substack{1+\alpha, z \\ \mathbf{C}\times S}} < \varepsilon_0 \tag{6.1}$$

then $\mathscr{L}$ is semiglobally integrable.

In others words, if (6.1) holds for ε_0 suficiently small, then given $R > 0$, there is a function Z_∞ defined in $\Delta_{\frac{R}{4}} \times S$, with nonvanishing differential, such that

$$\delta_0 Z_\infty = 0 \quad \text{in } \Delta_{\frac{R}{4}} \times S. \tag{6.2}$$

Here $\Delta_{\frac{R}{4}}$ is the ball of radius $\frac{R}{4}$ and center $0 \in \mathbf{C}$ and δ_0 is given by (2.8).

Corollary 6.1. *We consider the structure* $\mathscr{L}$ *given by* (2.7). *Let* $\alpha \in \mathbf{R}$, $0 < \alpha < 1$. *Then it is possible to find* $\varepsilon_0 > 0$ *such that, if* (6.1) *holds, then* $\mathscr{L}$ *is globally integrable.*

Theorem 2.1 will also be a corollary of Theorem 6.1.

Proof of the Theorem 6.1. First we take $R = 1$. Using the Nash-Moser procedure, we will construct a sequence $\{z_\nu\}$ of C^∞ functions defined in $\mathbf{C} \times S$ such that $z_\nu \to z_\infty$ when $\nu \to \infty$ in $C^{1+\alpha}(\Delta_{\frac{1}{4}} \times S)$, $0 < \alpha < 1$, and

$$\mathscr{L} = \langle d z_\infty \rangle^\perp \quad \text{in } \Delta_{\frac{1}{4}} \times S. \tag{6.3}$$

We consider the complex (2.8) associated to $\mathscr{L}$. According to the decomposition (2.10), we can write

$$\delta_0 f = (L_1 f d\overline{z}, L_\sigma f) \tag{6.4}$$

where

$$L_1 f = \frac{\partial f}{\partial \overline{z}} + \sigma_1 \frac{\partial f}{\partial z} \quad \text{and} \quad L_\sigma f = d_S f + \frac{\partial f}{\partial z}\sigma \tag{6.5}$$

Here σ_1 is a C^∞ function defined in $\mathbf{C} \times S$, $L_\sigma(f)$ is a smooth 1-form in S and we regard z as a parameter. Using coordinates $(z, \theta_1, \theta_2, \dots, \theta_{n-1})$ and $\sigma = \sum_{j=2}^n \sigma_j d\theta_{j-1}$, (6.5) assumes the form

$$L_j f = 0 \tag{6.6}$$

where L_j are given by (2.5).

If ζ is another global coordinate of $\mathbf{C}$, the new expression of δ_0 in the decomposition (2.8) is

(6.7)

$$\delta_0 f = \left(L_1(\zeta)\frac{\partial f}{\partial \zeta} + L_1(\overline{\zeta})\frac{\partial f}{\partial \overline{\zeta}}, \right.$$
$$\left. d_{s,z}(\zeta)\frac{\partial f}{\partial \zeta} + d_{S,z}(\overline{\zeta})\frac{\partial f}{\partial \overline{\zeta}} + d_{S,\zeta} f + \sigma \wedge \left(\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial \overline{\zeta}}\frac{\partial \overline{\zeta}}{\partial z} \right) \right)$$

where $d_{S,z}$ is the exterior derivative in S, and z is considered as a parameter.

We would like to see the complex (2.8) as a perturbation of a complex associated to a globally integrable structure. For this, we change the generators of $\mathscr{L}$ and define the *operator adapted to* ζ, by setting

$$\delta_0^\zeta f = (L_1(\overline{\zeta})^{-1} L_1 f, \ L_\sigma f - L_1(\overline{\zeta})^{-1} L_1(f) L_\sigma(\overline{\zeta})). \tag{6.8}$$

The reason for this is the following:

First step (definition of z_ν). According to Webster [20] and Treves [17], we will define a sequence of C^∞ functions

$$z_\nu : \mathbf{C} \times S \to \mathbf{C}, \qquad \nu = 1, 2, \dots, \tag{6.9}$$

and a sequence of real numbers

$$\frac{1}{2} < r_{\nu+1} < r_\nu \tag{6.10}$$

such that $z_0 = z$ (the canonical chart of $\mathbf{C}$), $r_0 = 1$ and the following inductive hypotheses will hold:

$$(H_1)_\nu : \qquad (z, \theta) \to (z_\nu, \theta) \text{ is a diffeomorphism from } \mathbf{C} \text{ onto } \mathbf{C},$$

$$(H_2)_\nu : \qquad L_1(\overline{z}_\nu) \neq 0 \quad \text{in } \mathbf{C} \times S,$$

$$(H_3)_\nu : \qquad \overline{\Omega} \subset \Omega_{\nu-1}, \quad \text{where } \Omega_\nu = \{(z, \theta) \in \mathbf{C} \times S : |z_\nu(z, \theta)| < r_\nu\}.$$

Then, taking $\zeta = z_\nu$ in (6.8), and using a system of coordinates $(\theta_1, \theta_2, \dots, \theta_{n-1})$ to S, we will get

$$\delta_0^\nu(f) \stackrel{\text{def}}{=} \delta_0^{z_\nu}(f) = \begin{pmatrix} L_1^v f \\ \vdots \\ L_n^v f \end{pmatrix}$$

$$(6.11) \qquad = \begin{pmatrix} \frac{\partial f}{\partial \overline{z}_v} \\ \frac{\partial f}{\partial \theta_1} \\ \vdots \\ \frac{\partial f}{\partial \theta_{n-1}} \end{pmatrix} + \begin{pmatrix} L_1(z_v) L_1(\overline{z}_v)^{-1} \frac{\partial f}{\partial z_v} \\ [L_2(z_v) - L_2(\overline{z}_v) L_1(\overline{z}_v)^{-1} L_1(z_v)] \frac{\partial f}{\partial z_v} \\ \vdots \\ [L_n(z_v) - L_n(\overline{z}_v) L_1(\overline{z}_v)^{-1} L_1(z_v)] \frac{\partial f}{\partial z_v} \end{pmatrix}$$

$$= \delta_0^{\mathscr{A}_v} f + R_v f.$$

where $\mathscr{A}_\nu$ is the structure given by

$$(6.12) \qquad \mathscr{A}_\nu = \langle d z_\nu \rangle^\perp.$$

We observe here two interesting phenomena: the approximating structure $\mathscr{A}_\nu$ has always the same aspect and the rest R_ν contains terms which permit a quadratic estimate.

We now consider the complex associated to $\mathscr{A}_\nu$:

$$(6.13) \quad C^\infty(\mathbf{C} \times S) \xrightarrow{\delta_0^{\mathscr{A}_\nu}} C^\infty\left(\mathbf{C} \times S, \bigwedge^1 \mathscr{A}_\nu^*\right) \xrightarrow{\delta_1^{\mathscr{A}_\nu}} C^\infty\left(\mathbf{C} \times S, \bigwedge^2 \mathscr{A}_\nu^*\right) \xrightarrow{\delta_2^{\mathscr{A}_\nu}} \cdots$$

where

$$\bigwedge^k \mathscr{A}_\nu^* = \frac{\bigwedge^k(\mathbf{C} \times S)}{\langle d z_\nu \rangle}.$$

Using the technique introduced by Kuranishi [8] and Webster [20], we define

$$(6.14) \qquad z_{\nu+1} = z_\nu + \omega_\nu, \quad \omega_\nu = -S_{N_{\nu+1}} \widetilde{K}_1^\nu \delta_0^\nu z_\nu,$$

$$(6.15) \qquad r_{\nu+1} = r_\nu - 2N_{\nu+1}^{-1}, \quad N_{\nu+1} = N_\nu^{\frac{3}{2}}.$$

Here $\delta_0^\nu = \delta_0^{z_\nu}$ is the operator adapted to the chart z_ν (see (6.8)).

$S_{N_{\nu+1}}$ is the smoothing operator defined in $\mathbf{C} \times S$ using the chart z_ν, a covering and a partition of unity of S which we fix from now on, and N_0 is a real number which we will fix in the future. Using (6.15) we can obtain $N_0 > 1$ such that

$$(6.16) \qquad \sum_{j=0}^\infty N_j^{-1} < \frac{1}{4}.$$

$\widetilde{K}_1^\nu$ is the operator obtained from

(6.17)
$$K_1^\nu(F) = K_1^S\omega + \frac{1}{\sigma(S)}\int_S T_{r_\nu}\phi\, d\sigma\,, \quad F = \omega + \phi d\bar{z} \in C^\infty\left(\mathbf{C}\times S\,, \bigwedge^1 \mathscr{A}_\nu^*\right)\,,$$

where

(6.18)
$$T_{r_\nu}\phi(z) = \frac{1}{2\pi i}\int\int_{\zeta\in\Delta_{r_\nu}} \frac{\phi(\zeta)}{z-\zeta} d\zeta \wedge d\bar{\zeta}\,, \qquad z\in\Delta_{r_\nu} = \{z\in\mathbf{C} : |z| < r_\nu\}\,,$$

and

$$K_2^\nu(F) = K_1^S\phi d\bar{z}_\nu + K_2^S\omega\,, \quad F = \omega + \phi d\bar{z}_\nu \in C^\infty\left(\mathbf{C}\times S\,, \bigwedge^2 \mathscr{A}_\nu^*\right)\,, \tag{6.19}$$

putting

$$\widetilde{K}_1^\nu = \chi D_{r_\nu}\varepsilon_1 D_{\frac{1}{r_\nu}} K_1^\nu\,, \quad \widetilde{K}_2^\nu = \chi D_{r_\nu}\varepsilon_2 D_{\frac{1}{r_\nu}} K_1^\nu \tag{6.20}$$

where $\chi\in C_c^\infty(\mathbf{C})$, $\chi\equiv 1$ in a neighbourhood of Δ_1 and $\chi\equiv 0$ out of Δ_2,

$$D_r f = f(rz)\,, \qquad z\in\mathbf{C}\,, \tag{6.21}$$

and ε_i is the extension operator described in (4.13).

We now use the following notation:

$$\|F\|_{\substack{\alpha,\nu\\ \Omega_\nu}} = \|F\|_{\substack{\alpha,z_\nu\\ \Omega_\nu}} = \|F\circ\mathscr{Z}_\nu^{-1}\|_{\substack{\alpha,z\\ \Delta_{r_\nu}\times S}}\,, \quad \mathscr{Z}_\nu(z\,,\theta) = (z_\nu(z\,,\theta)\,,\theta)\,. \tag{6.22}$$

It follows from (4.5), (4.11), (4.12) and (4.13) that

$$\|\widetilde{K}_1^\nu F\|_{\substack{\alpha,\nu\\ \mathbf{C}\times S}} \le C\cdot\|F\|_{\substack{\alpha,\nu\\ \Omega_\nu}}\,, \qquad F\in C^\alpha\left(\mathbf{C}\times S\,, \bigwedge^1 \mathscr{A}_\nu^*\right)\,, \tag{6.23}$$

$$\|\widetilde{K}_2^\nu F\|_{\substack{\alpha,\nu\\ \mathbf{C}\times S}} \le C\cdot\|F\|_{\substack{\alpha,\nu\\ \Omega_\nu}}\,, \qquad F\in C^\alpha\left(\mathbf{C}\times S\,, \bigwedge^2 \mathscr{A}_\nu^*\right)\,. \tag{6.24}$$

It is clear that that homotopy formula

$$\delta_0^{\mathscr{A}_\nu}\widetilde{K}_1^\nu + \widetilde{K}_2^\nu\delta_1^{\mathscr{A}_\nu} = I \tag{6.25}$$

is still valid in $\Delta_{r_\nu}\times S$.

The definition of $z_{\nu+1}$ does not depend on systems of coordinates of S. So, to prove the convergence of z_ν in the space $C^\alpha(\mathbf{C}\times S)$ we can use coordinates but this does not mean a localization in the variables of S.

We take a chart $(\theta_1\,,\theta_2\,,\dots\,,\theta_{n-1})$ from the covering of S fixed before.

Using $(z\,,\theta_1\,,\dots\,,\theta_{n-1})$ as coordinates, the equation (2.14) can be written in the following form:

$$\begin{aligned} L_1 f &= \frac{\partial f}{\partial\bar{z}} + \sigma_1\frac{\partial f}{\partial z} = 0\,,\\ L_j f &= \frac{\partial}{\partial\theta_{j-1}} + \sigma_j\frac{\partial f}{\partial z} = 0\,, \qquad j = 2\,,\dots\,,n\,. \end{aligned} \tag{6.26}$$

and the operator adapted to the chart z_ν has the form (6.11).

Putting

$$\delta_1^\nu f = (L_j^\nu f_i - L_i^\nu f_j)_{1\le i<j\le n}\,, \qquad f\in C^\infty\left(\mathbf{C}\times S\,, \bigwedge^1 \mathscr{A}_\nu^*\right)\,, \tag{6.27}$$

and using the fact that $[L_i, L_j] = 0$, we get

$$\delta_1^\nu \cdot \delta_0^\nu = 0. \tag{6.28}$$

Returning to (6.14), we want to prove that $\{z_\nu\}$ is a Cauchy sequence in $C^{1+\alpha}(\Delta_{\frac{1}{4}} \times S)$, $0 < \alpha < 1$; for this we must estimate

$$\begin{aligned} ||z_{\nu+1} - z_\nu||_{\substack{1+\alpha,\nu \\ C\times S}} &= ||w_v||_{\substack{1+\alpha,v \\ C\times S}} = ||-S_{N_{\nu+1}}\widetilde{K}_1^\nu \delta_0^\nu z_\nu||_{\substack{1+\alpha,\nu \\ C\times S}} \\ &\le CN_{\nu+1}||\widetilde{K}_1^\nu \delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ C\times S}} \le CN_{\nu+1}||K_1^\nu \delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}} \\ &\le CN_{\nu+1}||\delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}}. \end{aligned} \tag{6.29}$$

We are using (6.14), (5.3) and (6.23).

Second step (estimate for $\delta_0^\nu z_{\nu+1}$). Using (6.14), (6.25) and (6.11), we can write

$$\begin{aligned} \delta_0^\nu z_{\nu+1} &= \delta_0^\nu z_\nu + \delta_0^\nu w_\nu = (\delta_0^{\mathscr{A}_\nu} K_1^\nu + K_2^\nu \delta_1^{\mathscr{A}_\nu})(\delta_0^\nu z_\nu) + (\delta_0^{\mathscr{A}_\nu} + R_\nu)(w_\nu) \\ &= \delta_0^{\mathscr{A}_\nu} K_1^\nu \delta_0^\nu z_\nu + K_2^\nu \delta_1^{\mathscr{A}_\nu} \delta_0^\nu z_\nu - (\delta_0^{\mathscr{A}_\nu} + R_\nu)(S_{N_{\nu+1}} K_1^\nu \delta_0^\nu z_\nu) \end{aligned}$$

in Ω_ν.

Now from (5.4) and (6.23), we conclude

(6.31)

$$||\delta_0^{\mathscr{A}_\nu}(I - S_{N_{\nu+1}})\widetilde{K}_1^\nu \delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}} \le CN_{\nu+1}^{-\lambda}||\widetilde{K}_1^\nu \delta_0^\nu z_\nu||_{\substack{1+\lambda+\alpha,\nu \\ \Omega_\nu}} \le CN_{\nu+1}^{-\lambda}||\delta_0^\nu z_\nu||_{\substack{1+\lambda+\alpha,\nu \\ \Omega_\nu}}$$

where λ is a parameter to be determined.

Using (6.11), (5.3) and (6.23), we get

(6.32)

$$||R_\nu S_{N_{\nu+1}} \widetilde{K}_1^\nu \delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}} \le CN_{\nu+1}||L_1(\overline{z}_\nu)^{-1}||_{\substack{\alpha,\nu \\ \Omega_\nu}}(1 + ||\delta_0 \overline{z}_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}})||\delta_0 z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}}||\delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}}$$

We can use (6.28) and (6.24) to estimate

(6.33)

$$\begin{aligned} ||\widetilde{K}_2^\nu \delta_1^{\mathscr{A}_\nu} \delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}} &= ||\widetilde{K}_2^\nu (\delta_1^{\mathscr{A}_\nu} - \delta_1^\nu)\delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}} \le C||(\delta_1^{\mathscr{A}_\nu} - \delta_1^\nu)\delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}} \\ &\le C||L_1(\overline{z}_\nu)^{-1}||_{\substack{\alpha,\nu \\ \Omega_\nu}}(1 + ||\delta_0 \overline{z}_v||_{\substack{\alpha,v \\ \Omega_v}})||\delta_0 z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}}||\delta_0^\nu z_\nu||_{\substack{1+\alpha,\nu \\ \Omega_\nu}}. \end{aligned}$$

In this estimate we lost one derivative, but putting

$$\delta_0^\nu z_\nu = S_{N_{\nu+1}} \delta_0^\nu z_\nu + (I - S_{N_{\nu+1}})\delta_0^\nu z_\nu, \tag{6.34}$$

from (5.3) and (5.4),

$$||S_{N_{\nu+1}} \delta_0^\nu z_\nu||_{\substack{1+\alpha,\nu \\ \Omega_\nu}} \le CN_{\nu+1}||\delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}}, \tag{6.35}$$

$$||(I - S_{N_{\nu+1}})\delta_0^\nu z_\nu||_{\substack{1+\alpha,\nu \\ \Omega_\nu}} \le CN_{\nu+1}^{-\lambda}||\delta_0^\nu z_\nu||_{\substack{1+\lambda+\alpha,\nu \\ \Omega_\nu}}. \tag{6.36}$$

Collecting (6.31), (6.32), and (6.33), we can finally write

(6.37)

$$\begin{aligned} ||\delta_0^\nu z_{\nu+1}||_{\substack{\alpha,\nu \\ \Omega_\nu}} \le\, & CN_{\nu+1}||L_1(\overline{z}_\nu)^{-1}||_{\substack{\alpha,\nu \\ \Omega_\nu}}(1 + ||\delta_0 \overline{z}_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}})||\delta_0 z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}}||\delta_0^\nu z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}} \\ &+ CN_{\nu+1}^{-\lambda}||L_1(\overline{z}_\nu)^{-1}||_{\substack{\alpha,\nu \\ \Omega_\nu}}(1 + ||\delta_0 \overline{z}_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}})||\delta_0 z_\nu||_{\substack{\alpha,\nu \\ \Omega_\nu}}||\delta_0^\nu z_\nu||_{\substack{1+\lambda+\alpha,\nu \\ \Omega_\nu}} \\ &+ CN_{\nu+1}^{-\lambda}||\delta_0^\nu z_\nu||_{\substack{1+\lambda+\alpha,\nu \\ \Omega_\nu}}. \end{aligned}$$

Third step (preparatory lemmas).

Lemma 6.1. *Let $\mathscr{Z}\colon \mathbf{C}\times S \to \mathbf{C}\times S$, $\mathscr{Z}(p, \theta) = (z(p, \theta), \theta)$ be a global diffeomorphism. Using the fixed covering in S, we can define $\|\ \|_{\substack{1+\alpha, z\\ \mathbf{C}\times S}}$, $0 < \alpha < 1$. Let $(\theta_1, \theta_2, \ldots, \theta_{n-1})$ be a chart of this covering and let $\omega\colon \mathbf{C}\to\mathbf{C}$ a C^∞ function such that*

$$\|\omega\|_{\substack{1+\alpha, z\\ \mathbf{C}\times S}} < \frac{1}{2}.$$

Then

(a) $\mathscr{Z}'\colon \mathbf{C}\times S\to\mathbf{C}\times S$, $\mathscr{Z}'(z, \theta) = (z'(z, \theta), \theta) = (z+\omega(z, \theta), \theta)$ *is also a global diffeomorphism and so* $(z', \theta_1, \ldots, \theta_{n-1})$ *is a system of coordinates on* $\mathbf{C}\times S$.

(b) *The Hölder norms in* $C^{1+\alpha}(\mathbf{C}\times S)$ *(or in* $C^{\alpha}(\mathbf{C}\times S)$*) using* $(z, \theta_1, \ldots, \theta_{n-1})$ *or* $(z', \theta_1, \ldots, \theta_{n-1})$ *as coordinates, are equivalent.*

Proof. (a) By hypothesis

$$\|I - D\mathscr{Z}'\|_{C^\alpha(\mathbf{C}\times S)} < \frac{1}{2} \tag{6.39}$$

where I is the identity matrix and $D\mathscr{Z}'$ is the Jacobian matrix of $\mathscr{Z}'$ in the coordinates $(z, \theta_1, \ldots, \theta_{n-1})$. Then $D\mathscr{Z}'$ is nonsingular in $\mathbf{C}\times S$ and if $\mathscr{Z}'(z, \theta) = \mathscr{Z}'(\tilde z, \tilde\theta)$, we get

$$|z-\tilde z| = |w(z, \theta) - w(\tilde z, \theta)| \le \|w\|_{\substack{1, z\\ \mathbf{C}\times S}} |z-\tilde z| < \frac{1}{2}|z-\tilde z|\,. \tag{6.40}$$

(b) Let f belongs to $C^{1+\alpha}(\mathbf{C}\times S)$ and denote by F the expression of f in the coordinates $(z', \theta_1, \ldots, \theta_{n-1})$. Then

$$\begin{aligned}\|f\|_{\substack{1+\alpha, z\\ \mathbf{C}\times S}} &= \|F\circ\mathscr{Z}'\|_{\substack{1+\alpha, z\\ \mathbf{C}\times S}}\\ &\le C(\|F\|_0 + \|F\|_{C^{1+\alpha}(\mathbf{C}\times S)}\|D\mathscr{Z}'\|_0^{1+\alpha} + \|F\|_{C^1(\mathbf{C}\times S)}\|D\mathscr{Z}'\|_{C^\alpha(\mathbf{C}\times S)})\\ &\le C\|F\|_{C^{1+\alpha}(\mathbf{C}\times S)} = C\|f\|_{\substack{1+\alpha, z'\\ \mathbf{C}\times S}}\end{aligned}$$

because $\|D\mathscr{Z}'\|_{C^\alpha(\mathbf{C}\times S)} < \frac{3}{2}$ (see (6.39)).

For the opposite inequality, we observe that $\mathscr{Z}'^{-1}$ is of the same kind as $\mathscr{Z}'$, that is

$$(\mathscr{Z}')^{-1}(p, \theta) = ((z')^{-1}(p, \theta), \theta)\,, \tag{6.42}$$

where

$$(z')^{-1}(z, \theta) = z' - \omega((\mathscr{Z}')^{-1}(z', \theta)) = z' - \omega'(z', \theta). \tag{6.43}$$

Since $\|D(\mathscr{Z}')^{-1}\|_{C^\alpha(\mathbf{C}\times S)} \le 2$, we get similarly the other inequality.

Remark 6.1. Using the fact that

(6.44)

$$\begin{aligned}\|w'\|_{\substack{1+\alpha, z'\\ \mathbf{C}\times S}} &= C(\|w\|_{\substack{1+\alpha, z\\ \mathbf{C}\times S}}\|D(\mathscr{Z}')^{-1}\|_0^{1+\alpha} + \|w\|_{\substack{1, z\\ \mathbf{C}\times S}}\|D(\mathscr{Z}')^{-1}\|_{C^\alpha(\mathbf{C}\times S)} + \|w\|_0)\\ &\le C\|w\|_{\substack{1+\alpha, z\\ \mathbf{C}\times S}}\end{aligned}$$

we get for a compact subset K of $\mathbf{C}$, a constant $C(K)$ such that

$$\|\mathscr{Z}'\|_{C^1(K\times S)} \le C(\|z\|_{\substack{1, z\\ K\times S}} + \|w_\nu\|_{\substack{1, z\\ K\times S}} + 1) \le C(K)\,, \tag{6.45}$$

$$\|(\mathscr{Z}')^{-1}\|_{C^1(K\times S)} \le C(\|z'\|_{\substack{1,z'\\K\times S}} + \|w'_\nu\|_{\substack{1,z'\\K\times S}} + 1) \le C(K) \tag{6.46}$$

Lemma 6.2 (Moser) [9]. *Let p_σ be a sequence of positive real numbers such that*:

$$p_0 \le \frac{N_1^{-\mu}}{2C_\#} \tag{6.47}$$

and

$$p_\sigma \le C_\#(N_\sigma^{2s}p_{\sigma-1}^2 + N_\sigma^{-\lambda}N_{\sigma-1}^{\lambda+t}), \qquad \sigma = 1, \dots, \nu, \tag{6.48}$$

where $N_\nu = N_{\nu-1}^{\frac{3}{2}}$, $N_0 > 1$, μ, λ, s and t *are parameters determined in the following way*: s *and* t *are positive known numbers*; *we first choose* $\mu \ge 4s$ *and then* $\lambda > 0$ *such that* $t + \frac{9}{4}\mu - \frac{\lambda}{2} < -\frac{3}{2}$.

Then, if $N_1^{-1} \le \frac{1}{(2C_\#)^2}$, *we get*

$$p_\sigma \le \frac{N_{\nu+1}^{-\mu}}{2C_\#} \tag{6.49}$$

Proof. See Kuranishi [8].

Fourth step (induction). We now add to the hypotheses $(H_1)_\nu$, $(H_2)_\nu$ and $(H_3)_\nu$ the following inductive hypotheses:

$$(H_4)_\nu \qquad \|\delta_0^{\nu-1}z_\nu\|_{\substack{\alpha,\nu-1\\\Omega_{\nu-1}}} \le C_4[N_\nu\|\delta_0^{\nu-2}z_{\nu-1}\|^2_{\substack{\alpha,\nu-2\\\Omega_{\nu-2}}} + N_\nu^{-\lambda}N_{\nu-1}^{\lambda+1}],$$

$$(H_5)_\nu \qquad \|\delta_0\overline{z}_\nu\|_{\substack{\alpha,\nu\\\Omega_\nu}} \le C_5,$$

$$(H_6)_\nu \qquad \|L_1(\overline{z}_\nu)^{-1}\|_{\substack{\alpha,\nu\\\Omega_\nu}} \le C_6,$$

$$(H_7)_\nu \qquad \|w_{\nu-1}\|_{\substack{1+\alpha,\nu-1\\C\times S}} \le N_\nu^{-1} \quad \text{and} \quad \|w_{\nu-1}\|_{\substack{1+\alpha,0\\C\times S}} \le N_\nu^{-1},$$

$$(H_8)_\nu \qquad \|z_\nu\|_{\substack{1+\alpha+j,0\\\Omega_0}} \le N_\nu^j, \qquad j = 0, 1, \dots, \lambda+1.$$

where C_4, C_5 and C_6 do not depend on ν; $N_0 > 1$ will be chosen a posteriori. If N_0 is sufficiently large, then $\sum_{\nu=0}^\infty {N_\nu}^{-1}$ can be made arbitrarily small.

When $\nu = 1$, we put $\delta_0^{-1} = \delta_0^0$, $\Omega_{-1} = \Omega_0$ and

$$(H_4)_1 \qquad \|\delta_0^0 z_1\|_{\substack{\alpha,0\\\Omega_0}} \le C_4[N_1\|\delta_0^0 z_0\|^2_{\substack{\alpha,0\\\Omega_0}} + N_1^{-\lambda}N_0^{\lambda+1}].$$

We begin by setting

$$z_0 = z, \quad \Omega_0 = \Delta \times S, \tag{6.50}$$

First we prove that the validity of the hypotheses for step ν implies the same hypotheses in the step $\nu+1$, choosing N_0 large and ε_0 small. Finally we will prove the hypotheses when $\nu = 1$.

By (6.14),

$$z_\nu = z_0 + \sum_{j=0}^{\nu-1}\omega_j. \tag{6.51}$$

Using $(H_7)_\nu$,

$$\sum_{j=0}^{v-1} \|w_j\|_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} \leq \sum_{j=0}^{\infty} N_j^{-1} \tag{6.52}$$

This last sum can be made $\leq \frac{1}{4}$ if we take N_0 sufficiently large (see (6.16)). Lemma 1 implies that the norms $\|\ \|_{\substack{\alpha,j\\ \mathbf{C}\times S}}$, $j = 0, \ldots, \nu$, are equivalent. We can now prove $(H_7)_{\nu+1}$. It follows from (6.29) that

$$\|w_\nu\|_{\substack{1+\alpha,\nu\\ \mathbf{C}\times S}} \leq C N_{\nu+1} \|\delta_0^\nu z_\nu\|_{\substack{\alpha,\nu\\ \Omega_\nu}} \tag{6.53}$$

but according to (6.11) and (2.5),

$$\delta_0^v f = \begin{pmatrix} L_1^v f \\ \vdots \\ L_n^v f \end{pmatrix} = M_v \begin{pmatrix} L_1 f \\ \vdots \\ L_n f \end{pmatrix} = M_\nu \delta_0 f \tag{6.54}$$

where

$$M_v = \begin{pmatrix} L_1(\overline{z}_v)^{-1} & 0 & \cdots & 0 \\ -L_2(\overline{z}_v) L_1(\overline{z}_v)^{-1} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -L_n(\overline{z}_v) L_1(\overline{z}_v)^{-1} & 0 & \cdots & 1 \end{pmatrix} \tag{6.55}$$

(the matrix M_ν is well defined due to $(H_2)_\nu$). So,

$$\delta_0^\nu = M_\nu M_{\nu-1}^{-1} \delta_0^{\nu-1}. \tag{6.56}$$

The elements of the matrix M_ν (and M_ν^{-1}) can be estimated by C_5 or C_6 (due to $(H_5)_\nu$ and $(H_6)_\nu$)and using $(H_3)_\nu$, we can write

$$\begin{aligned} \|w_\nu\|_{\substack{1+\alpha,\nu\\ \mathbf{C}\times S}} &\leq C(5, 6) N_{\nu+1} \|\delta_0^{\nu-1} z_\nu\|_{\substack{\alpha,\nu\\ \Omega_\nu}} \\ &\leq C(5, 6) N_{\nu+1} \|\delta_0^{\nu-1} z_\nu\|_{\substack{\alpha,\nu-1\\ \Omega_{\nu-1}}}. \end{aligned} \tag{6.57}$$

Here $C(5, 6)$ is a constant involving only C_5, C_6 and universal constants.

We now use $(H_4)_\nu$ and the Lemma 6.2 with

$$p_v = \|\delta_0^{v-1} z_v\|_{\substack{\alpha,v-1\\ \Omega_{v-1}}}, \tag{6.58}$$

$$s = \frac{1}{2}, \quad \mu = 2, \quad t = 1 \quad \text{and} \quad \lambda = 14. \tag{6.59}$$

We take N_0 large such that

$$N_1^{-1} \leq \frac{1}{(2C_4)^2}. \tag{6.60}$$

On the other hand,

$$p_0 = \|\delta_0^0 z_0\|_{\substack{\alpha,0\\ \Omega_0}} = \left\| \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \right\|_{\substack{\alpha,0\\ \Omega_0}} < \frac{N_1^{-2}}{2C_4} \tag{6.61}$$

if ε_0 is sufficiently small (this can be achieved due to (6.1)).

Lemma 6.2 implies

$$p_\nu \le \frac{N_{\nu+1}^{-2}}{2C_4} \tag{6.62}$$

Hence, using the equivalent norms $\|\ \|_{\substack{1+\alpha,j\\ \mathbf{C}\times S}}$, $j=0,1,\ldots,\nu$, we get

$$\|\omega_\nu\|_{\substack{1+\alpha,\nu\\ \mathbf{C}\times S}} \le \frac{C(5,6)}{C_4} N_{\nu+1}^{-1} \tag{6.63}$$

and

$$\|\omega_\nu\|_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} \le \frac{\widetilde{C}(5,6)}{C_4} N_{\nu+1}^{-1} \tag{6.64}$$

We take $C_4 > 1$ sufficiently large such that $\widetilde{C}(5,6)/C_4 < 1$ (for, C_4 depends on C_5 and C_6 and it may be necessary to increase N_0 and, hence, decrease ε_0). Therefore $(H_7)_{\nu+1}$ holds.

Once more, by using Lemma 1, we can claim that all the norms $\|\ \|_{\substack{1+\alpha,j\\ \mathbf{C}\times S}}$, $j = 0,1,\ldots,\nu+1$ are equivalent and that $(z,\theta)\mapsto(z_{\nu+1},\theta)$ is a global diffeomorphism (due to (6.16) and Lemma 6.1).

So $(H_1)_{\nu+1}$ is true. We will now prove $(H_3)_{\nu+1}$. From $(H_7)_{\nu+1}$ it follows that

$$\sup_{\mathbf{C}\times S}|z_{\nu+1}-z_\nu| \le N_{\nu+1}^{-1} \tag{6.65}$$

hence

$$|z_\nu| \le |z_{\nu+1}| + N_{\nu+1}^{-1}. \tag{6.66}$$

Since $r_{\nu+1} = r_\nu - 2N_{\nu+1}^{-1}$ and $r_0 = 1$, if $|z_{\nu+1}| < r_{\nu+1}$, we have $|z_\nu| \le r_\nu - N_{\nu+1}^{-1} < r_\nu$. This implies $(H_3)_{\nu+1}$.

Futhermore,

$$r = \inf r_\nu < 1 - 2\sum_{\nu=0}^{\infty} N_{\nu+1}^{-1} \tag{6.67}$$

if we want $r > \frac{1}{2}$ (see (6.10)), we must require

$$\sum_{\nu=0}^{\infty} N_{\nu+1}^{-1} < \frac{1}{4} \tag{6.68}$$

but this follows from (6.16).

Now we will prove $(H_4)_{\nu+1}$.

First, we must note that $\|\sigma_j\|_{\substack{1+\lambda+\alpha,0\\ \Omega_0}}$ are bounded for $j=1,\ldots,n$ (σ_j are smooth in $\mathbf{C}\times S$).

So, from (6.14), (6.4) and (6.5),

$$\|\delta_0 z_\nu\|_{\substack{\alpha,\nu\\ \Omega_\nu}} \le \|\delta_0 z_0\|_{\substack{\alpha,\nu\\ \Omega_\nu}} + C\sum_{j=0}^{\nu-1}\|w_j\|_{\substack{1+\alpha,\nu\\ \Omega_\nu}} \tag{6.69}$$

hence $\|\delta_0 z_\nu\|_{\substack{\alpha,\nu\\ \Omega_\nu}}$ is bounded independently of ν.

Using now (6.37), $(H_5)_\nu$ and $(H_6)_\nu$,

$$\|\delta_0^\nu z_{\nu+1}\|_{\substack{\alpha,\nu\\ \Omega_\nu}} \le C(5,6)(N_{\nu+1}\|\delta_0^\nu z_\nu\|^2_{\substack{\alpha,\nu\\ \Omega_\nu}} + N_{\nu+1}^{-\lambda}\|\delta_0^\nu z_\nu\|_{\substack{1+\alpha+\lambda,\nu\\ \Omega_\nu}}) \tag{6.70}$$

On the other hand, $\overline{\Omega}_\nu \subset \Omega_{\nu-1}$ and due to (6.56),

$$\|\delta_0^\nu z_{\nu+1}\|_{\substack{\alpha,\nu\\ \Omega_\nu}} \le C(5,6)(N_{\nu+1}\|\delta_0^{\nu-1} z_\nu\|^2_{\substack{\alpha,\nu-1\\ \Omega_{\nu-1}}} + N_{\nu+1}^{-\lambda}\|\delta_0^\nu z_\nu\|_{\substack{1+\alpha+\lambda,\nu\\ \Omega_\nu}}) . \tag{6.71}$$

Then, to prove $(H_4)_{\nu+1}$ it is sufficient to show that

$$\|\delta_0^\nu z_\nu\|_{\substack{1+\alpha+\lambda,\nu\\ \Omega_\nu}} \le C(5,6)N_\nu^{\lambda+1} . \tag{6.72}$$

Now (6.11) implies that $\delta_0^\nu z_\nu = R_\nu z_\nu$ and also gives the expression of R_ν in the $(z_0, \theta_1, \ldots, \theta_{n-1})$ coordinates. It is convenient to change $R_\nu z_\nu$ to new coordinates. We then consider

$$\mathscr{Z}_\nu : \mathbf{C}\times S \to \mathbf{C}\times S, \quad (z,\theta)\mapsto (z_\nu(z,\theta),\theta). \tag{6.73}$$

Then

$$\begin{aligned}\|\delta_0^\nu z_\nu\|_{\substack{1+\alpha+\lambda,\nu\\ \Omega_\nu}} \le C(&\|R_\nu z_\nu\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}}\|\mathscr{Z}_\nu^{-1}\|^{1+\alpha+\lambda}_{C^1(\Delta_{r_\nu}\times S)}\\ &+\|R_\nu z_\nu\|_{\substack{1,0\\ \Omega_0}}\|\mathscr{Z}_\nu^{-1}\|_{C^{1+\alpha+\lambda}_{(\Delta_{r_\nu}\times S)}} + |R_\nu z_\nu|_0).\end{aligned} \tag{6.74}$$

We noted in (6.45) and (6.46) that $\|\mathscr{Z}_\nu\|_{C^1(\Delta_{r_\nu}\times S)}$ and $\|\mathscr{Z}_\nu^{-1}\|_{C^1(\Delta_{r_\nu}\times S)}$ are bounded. Then we can use the following lemma:

Lemma 6.3 [6]. *If B_1 and B_2 are two compact convex subsets of $\mathbf{R}^l$; $g\colon B_1\to B_2$, $f\colon B_2\to\mathbf{R}^l$ satisfy*

$$f(g(x)) = x, \qquad x\in B_1,$$

and $\|f\|_{C^1}$ and $\|g\|_{C^1}$ are bounded then $\|g\|_\alpha \le C\|f\|_\alpha$, $\alpha\ge 1$. So,

$$\|\mathscr{Z}_\nu^{-1}\|_{C^{1+\alpha+\lambda}(\Delta_{r_\nu}\times S)} \le C\|\mathscr{Z}_\nu\|_{C^{1+\alpha+\lambda}(\Delta_{r_0}\times S)} . \tag{6.75}$$

and since $\mathscr{Z}_\nu(z,\theta) = (z_\nu(z,\theta),\theta)$,

$$\|\delta_0^\nu z_\nu\|_{\substack{1+\alpha+\lambda,\nu\\ \Omega_\nu}} \le C(\|R_\nu z_\nu\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}} + \|R_\nu z_\nu\|_{\substack{1,0\\ \Omega_0}}\|z_\nu\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}}). \tag{6.76}$$

We now proceed to estimate $R_\nu z_\nu$. The first line of $R_\nu z_\nu$ can be estimated by

$$\begin{aligned}\|L_1(\overline{z}_\nu)^{-1}L_1(z_\nu)\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}} \le &\|L_1(\overline{z}_\nu)^{-1}\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}}\|L_1(z_\nu)\|_0\\ &+\|L_1(\overline{z}_\nu)^{-1}\|_0\|L_1(z_\nu)\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}} .\end{aligned} \tag{6.77}$$

We claim that

$$\|L_1(\overline{z}_\nu)^{-1}\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}} \le C(5,6)\|L_1(\overline{z}_\nu)\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}} . \tag{6.78}$$

This follows by induction over λ. First we note that

$$|L_1(\overline{z}_\nu)| \ge |L_1(\overline{z}_0)| - \sum_{j=0}^{\nu-1}|L_1\overline{\omega}_j| \ge 1 - C\sum_{j=0}^{\nu-1}\|\omega_j\|_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} \ge \frac{1}{2} \tag{6.79}$$

if N_0 is sufficiently large.

So,

(6.80)

$$\|L_1(\overline{z}_v)^{-1}\|_{\substack{\alpha,0\\ \Omega_0}} = \|L_1(\overline{z}_v)^{-1}\|_0 + \sup_{(z,\theta),(z',\theta')\in\Omega_0} \frac{|L_1(\overline{z}_v)^{-1}(z,\theta)-L_1(\overline{z}_v)^{-1}(z',\theta')|}{|(z,\theta)-(z',\theta')|^\alpha}$$
$$= 2 + \sup_{(z,\theta),(z',\theta')\in\Omega_0} \frac{1}{|L_1(\overline{z}_v)(z,\theta)||L_1(\overline{z}_v)(z',\theta')|}\frac{|L_1(\overline{z}_v)(z,\theta)-L_1(\overline{z}_v)(z',\theta')|}{|(z,\theta)-(z',\theta')|^\alpha}$$
$$\leq 2 + 4\|L_1(\overline{z}_v)\|_{\substack{\alpha,0\\ \Omega_0}} \leq C\|L_1(\overline{z}_v)\|_{\substack{\alpha,0\\ \Omega_0}}.$$

To simplify matters, we omit in (6.80) the partition of unity fixed in S. We will make this omission from now on.

Similarly, we can show that

$$\|L_1(\overline{z}_\nu)^{-j}\|_{\substack{\alpha,0\\ \Omega_0}} \leq C_j\|L_1(\overline{z}_\nu)^j\|_{\substack{\alpha,0\\ \Omega_0}}, \qquad j=1,2,\dots, \tag{6.81}$$

and using $(H_5)_\nu$

$$\|L_1(\overline{z}_\nu)^{-j}\|_{\substack{\alpha,0\\ \Omega_0}} \leq C_j(5,6)\|L_1(\overline{z}_\nu)\|_{\substack{\alpha,0\\ \Omega_0}}, \qquad j=1,2,\dots. \tag{6.82}$$

A straightforward calculation yields

$$\begin{aligned}\|L_1(\overline{z}_\nu^{-1})\|_{\substack{1+\alpha,0\\ \Omega_0}} &\leq C\left(\left\|\frac{-1}{L_1(\overline{z})^2}\right\|_{\substack{\alpha,0\\ \Omega_0}}\|L_1(\overline{z}_v)\|_{\substack{1,0\\ \Omega_0}}\right.\\ &\qquad\left.+\left|\frac{-1}{L_1(\overline{z}_\nu)^2}\right|_0\|L_1(\overline{z}_\nu)\|_{\substack{1+\alpha,0\\ \Omega_0}}+|L_1(\overline{z}_\nu)^{-1}|_0\right)\\ &\leq C(5,6)\|L_1(\overline{z}_\nu)\|_{\substack{1+\alpha,0\\ \Omega_0}}\end{aligned} \tag{6.83}$$

This prove (6.78) for $\lambda = 0$. The general case follows by induction using the fact that the kth-derivative of $L_1(\overline{z}_\nu)^{-1}$ is equal to a finite number of derivatives of order $\leq k$ of the function $z \mapsto \frac{1}{z}$ evaluated at $L_1(\overline{z}_\nu)$ with derivatives of $L_1(\overline{z}_\nu)$ of orders $k_1, k_2, \dots,$ with $k_1+k_2+\cdots \leq k$ which can be estimated using (6.81).

Returning to (6.77) and using (6.78),

$$\begin{aligned}&\|L_1(\overline{z}_\nu)^{-1}L_1(z_\nu)\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}}\\ &\leq C(5,6)(\|L_1(\overline{z}_\nu)^{-1}\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}}+\|L_1(z_\nu)\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}})\\ &\leq C(5,6)\|z_v\|_{\substack{2+\alpha+\lambda,0\\ \Omega_0}}\end{aligned} \tag{6.84}$$

Analogously, we can estimate all the others terms of $R_\nu z_\nu$. Going back to (6.76),

$$\|\delta_0^\nu z_\nu\|_{\substack{1+\alpha+\lambda,\nu\\ \Omega_\nu}} \leq C(5,6)(\|z_\nu\|_{\substack{2+\alpha+\lambda,0\\ \Omega_0}}+\|z_\nu\|_{\substack{2,0\\ \Omega_0}}\|z_\nu\|_{\substack{1+\alpha+\lambda,0\\ \Omega_0}}). \tag{6.85}$$

Hence $(H_8)_\nu$ implies

$$\|\delta_0^\nu z_\nu\|_{\substack{1+\alpha+\lambda,\nu\\ \Omega_\nu}} \leq C(5,6)(N_\nu^{\lambda+1}+N_\nu N_\nu^\lambda). \tag{6.86}$$

So, (6.71) yields

$$\|\delta_0^\nu z_\nu + 1\|_{\substack{\alpha,\nu\\ \Omega_\nu}} \leq C(5,6)(N_{\nu+1}\|\delta_0^{\nu-1}z_\nu\|^2_{\substack{\alpha,\nu-1\\ \Omega_{\nu-1}}}+N_{\nu+1}^{-\lambda}N_\nu^{\lambda+1}). \tag{6.87}$$

To get $(H_4)_\nu$, we choose C_4 larger than this last constant C(5,6) (N_0^{-1} and ε_0 can be shrunk).

We will now prove $(H_8)_{\nu+1}$.

Let $j \in \mathbf{Z}$, $0 \leq j \leq \lambda + 1$.

Due to $(H_8)_\nu$,

$$\|z_{\nu+1}\|_{\substack{\alpha+1+j,0 \\ \Omega_0}} \leq N_\nu^j + \|w_\nu\|_{\substack{\alpha+1+j,0 \\ \Omega_0}}. \tag{6.88}$$

Changing coordinates

$$\|w_\nu\|_{\substack{1+\alpha+j,0 \\ \Omega_0}} \leq C(\|w_\nu\|_{\substack{1+\alpha+j,\nu \\ \mathbf{C}\times S}} \|\mathscr{Z}_\nu\|^{1+\alpha+j}_{C^1(\Delta_{r_\nu}\times S)} + \|w_\nu\|_{\substack{1,\nu \\ \mathbf{C}\times S}} \|\mathscr{Z}_\nu\|_{C^{1+\alpha+j}_{(\Delta_{r_\nu}\times S)}} + \|w_\nu\|_0) \tag{6.89}$$

where $\mathscr{Z}_\nu$ is given by (6.73). Hence

$$\begin{aligned} \|w_\nu\|_{\substack{1+\alpha+j,0 \\ \Omega_0}} \leq C(&\|w_\nu\|_{\substack{1+\alpha+j,\nu \\ \mathbf{C}\times S}} (1 + \|z_\nu\|_{\substack{1,0 \\ \Omega_0}})^{1+\alpha+j} \\ &+ \|w_\nu\|_{\substack{1,\nu \\ \mathbf{C}\times S}} (1 + \|z_\nu\|_{\substack{1+\alpha+j,0 \\ \Omega_0}}) + \|w_\nu\|_0). \end{aligned} \tag{6.90}$$

By using $(H_8)_\nu$ once more, we obtain

$$\|w_\nu\|_{\substack{1+\alpha+j,0 \\ \Omega_0}} \leq C(\|w_\nu\|_{\substack{1+\alpha+j,\nu \\ \mathbf{C}\times S}} + N_\nu^j) \tag{6.91}$$

because $\|z_\nu\|_{\substack{1,0 \\ \Omega_0}}$ and $\|w_\nu\|_{\substack{1,\nu \\ \mathbf{C}\times S}}$ are bounded.

But now, (5.3), (6.54), (6.56), (6.23), $(H_5)_\nu$ and $(H_6)_\nu$ yield

$$\begin{aligned} \|w_\nu\|_{\substack{\alpha+1+j,\nu \\ \mathbf{C}\times S}} = \|S_{N_{\nu+1}} \widetilde{K}_0^\nu \delta_0^\nu z_\nu\|_{\substack{\alpha+1+j,\nu \\ \mathbf{C}\times S}} &\leq C N_{\nu+1}^{j+1} \|\delta_0^\nu z_\nu\|_{\substack{\alpha,\nu \\ \Omega_\nu}} \\ &\leq C(5,6) N_{\nu+1}^{j+1} \|\delta_0^{\nu-1} z_\nu\|_{\substack{\alpha,\nu-1 \\ \Omega_{\nu-1}}}. \end{aligned} \tag{6.92}$$

Hence from (6.58) and (6.62),

$$\|w_\nu\|_{\substack{\alpha+1+j,\nu \\ \mathbf{C}\times S}} \leq C(5,6) N_{\nu+1}^{j+1} \frac{N_{\nu+1}^{-2}}{2C_4} \tag{6.93}$$

and from (6.88), (6.15), $C_4 > 1$,

$$\begin{aligned} \|z_{\nu+1}\|_{\substack{1+\alpha+j,0 \\ \Omega_0}} &\leq C(5,6)(N_{\nu+1}^{j-1} + N_\nu^j) \\ &= C(5,6)\left(N_{\nu+1}^{-1} + \left(\frac{N_\nu}{N_{\nu+1}}\right)^j\right) N_{\nu+1}^j < N_{\nu+1}^j \end{aligned} \tag{6.94}$$

since N_0 is sufficiently large.

We now prove $(H_5)_{\nu+1}$. It follows from (6.14) that

$$\|\delta_0 \overline{z}_{\nu+1}\|_{\substack{\alpha,\nu+1 \\ \Omega_{\nu+1}}} \leq C\|\delta_0 \overline{z}_{\nu+1}\|_{\substack{\alpha,0 \\ \Omega_0}} \leq C\left(\|\delta_0 \overline{z}_0\|_{\substack{\alpha,0 \\ \Omega_0}} + \sum_{j=0}^{\nu} \|w_j\|_{\substack{1+\alpha,0 \\ \mathbf{C}\times S}}\right). \tag{6.95}$$

Since

$$\delta_0 \overline{z}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and $\sum_{j=0}^{\nu} ||w_j||_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} \leq \sum_{j=0}^{\infty} N_j < \frac{1}{4}$ (see (6.4), (6.5) and (6.16)), it is sufficient to take $C_5 > \frac{3}{2}C$, where C is the constant in the right hand of (6.95).

Since $L_1(\overline{z}_0) = 1$, if N_0 is sufficiently large, we get

$$\begin{aligned} |1 - L_1\overline{z}_{\nu+1}| &= |L_1(\overline{z}_0) - L_1(\overline{z}_{\nu+1})| \leq |\delta_0(\overline{z}_0 - \overline{z}_{\nu+1})| \\ &\leq C\sum_{j=0}^{\nu} ||w_j||_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} \leq C\sum_{j=0}^{\infty} N_j^{-1} < \frac{1}{2}. \end{aligned} \tag{6.96}$$

then $L_1(\overline{z}_{\nu+1}) \neq 0$ in $\mathbf{C}\times S$. Hence $(H_2)_{\nu+1}$ is valid.

Similarly

$$\left|1 - \frac{\partial}{\partial x}\overline{z}_{\nu+1}\right| < \frac{1}{2} \quad \text{in } \mathbf{C}\times S. \tag{6.97}$$

It remains to prove $(H_6)_{\nu+1}$.

Since

$$|L_1(\overline{z}_{v+1})| \geq |L_1(\overline{z}_0)| - \left|\sum_{j=0}^{v} L_1 w_j\right| > \frac{1}{2}$$

then

$$||L_1(\overline{z}_{\nu+1})^{-1}||_{\substack{\alpha,\nu+1\\ \Omega_{\nu+1}}} \leq 2 + 4||L_1(\overline{z}_{\nu+1})||_{\substack{\alpha,\nu+1\\ \Omega_{\nu+1}}} \leq 2 + 4C_5 < C_6 \tag{6.98}$$

if C_6 is sufficiently large.

We need to prove the hypotheses when $\nu = 1$.

From (6.29), we have

$$(H_7)_1: \qquad ||w_0||_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} \leq CN_1||\delta_0^0 z_0||_{\substack{\alpha,0\\ \mathbf{C}\times S}}$$

and using (6.1),

$$||w_0||_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} \leq CN_1\varepsilon_0 \leq N_1^{-1} \tag{6.99}$$

since $\varepsilon_0 \leq N_1^{-2}/C$.

Lemma 1 implies $(H_1)_1$ because $N_1^{-1} < \frac{1}{2}$. It is also clear that $\overline{\Omega}_1 \subset \Omega_0$ because

$$|z_0| \leq |z_1| + |w_0| < 1 - N_1^{-1} < 1. \tag{6.100}$$

Now, $(H_4)_1$ can be expressed by

$$||\delta_0^0 z_1||_{\substack{\alpha,0\\ \Omega_0}} \leq C_4[N_1||\delta_0^0 z_0||^2_{\substack{\alpha,0\\ \Omega_0}} + N_1^{-\lambda}N_0^{\lambda+1}]. \tag{6.101}$$

To prove this we use (6.37)

$$||\delta_0^0 z_1||_{\substack{\alpha,0\\ \Omega_0}} \leq CN_1||\delta_0 z_0||_{\substack{\alpha,0\\ \Omega_0}}||\delta_0^0 z_0||_{\substack{\alpha,0\\ \Omega_0}} + CN_1^{-\lambda}||\delta_0^0 z_0||_{\substack{1+\alpha+\lambda,0\\ \Omega_0}}. \tag{6.102}$$

We need show that

$$||\delta_0 z_0||_{\substack{1+\alpha+\lambda,0\\ \Omega_0}} \leq N_0^{\lambda+1} \tag{6.103}$$

but this is obvious because $\delta_0 z_0$ is bounded in $C^{1+\alpha+\lambda}(\Omega_0)$ and N_0 can be taken large.

Now $|L_1(\overline{z}_1)-L_1(\overline{z}_0)| \leq C||\overline{w}_0||_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} < CN_1^{-1} < \frac{1}{2}$ then $L_1(\overline{z}_1) \neq 0$ in $\mathbf{C}\times S$ since $L_1(\overline{z}_0) = 1$. So $(H_2)_\nu$ holds.

It follows from (6.14) that

$$||\delta_0 \overline{z}_1||_{\substack{\alpha,1\\ \Omega_1}} \leq C(||w_0||_{\substack{1+\alpha,0\\ \Omega_0}} + ||\delta_0 \overline{z}_0||_{\substack{\alpha,0\\ \Omega_0}}) \leq C_5 \tag{6.104}$$

where C_5 is chosen after (6.95). Since $|L_1(\overline{z})| \geq |L_1(\overline{z}_0)| - |L_1\overline{w}_0| > \frac{1}{2}$, we get

$$||L_1(\overline{z}_1)^{-1}||_{\substack{\alpha,1\\ \Omega_1}} \leq 2 + 4||L_1(\overline{z}_1)||_{\substack{\alpha,1\\ \Omega_1}} \leq 2 + 4C_5 \leq C_6 \tag{6.105}$$

where C_6 is chosen in (6.98).

Hence $(H_5)_1$ and $(H_6)_1$ hold.

$(H_8)_1$ is a consequence of

$$||w_0||_{\substack{1+\alpha+j,0\\ \Omega_\nu}} \leq C\ N_1^{j+1}||\delta_0^0 z_0||_{\substack{\alpha,0\\ \Omega_0}} \tag{6.106}$$

(see (6.29), the proof is similar), shrinking ε_0 once again if necessary.

So, if N_0 is sufficiently large and ε_0 is sufficiently small, the inductive hypotheses are true for $\nu = 1, 2, \dots$.

Finally we note that

$$\Delta_{\frac{1}{4}} \times S \subset \Omega_\nu, \qquad \nu = 1, 2, \dots. \tag{6.107}$$

Indeed, if $|z_0| \leq \frac{1}{4}$, (6.65), (6.16) and (6.10) imply that

$$|z_\nu| \leq |z_0| + \sum_{j=0}^{\nu-1} N_j^{-1} \leq |z_0| + \frac{1}{4} \leq \frac{1}{2} < \inf r_\nu. \tag{6.108}$$

Now, $(H_7)_{\nu+1}$ yields

$$||z_{\nu+1} - z_\nu||_{\substack{1+\alpha,0\\ \mathbf{C}\times S}} \leq N_{\nu+1}^{-1} \tag{6.109}$$

since $\sum_{j=1}^{\infty} N_j^{-1} < \frac{1}{4}$, $\{z_\nu\}$ is a Cauchy sequence in the space $C^{1+\alpha}(\Delta_{\frac{1}{4}} \times S)$ and hence it converges to a function z_∞ of this space.

It follows from $(H_4)_\nu$ and (6.62) that

$$||\delta_0 z_\nu||_{\substack{\alpha,0\\ \Delta_{\frac{1}{4}}\times S}} \leq C||\delta_0 z_\nu||_{\substack{\alpha,\nu-1\\ \Omega_{\nu-1}}} \leq C(5,6)||\delta_0^{\nu-1} z_\nu||_{\substack{\alpha,\nu-1\\ \Omega_{\nu-1}}} \leq \frac{C(5,6)}{2C_4} \cdot N_{\nu+1}^{-2} \tag{6.110}$$

Since $N_{\nu+1} \to \infty$ when $\nu \to \infty$, we get

$$\delta_0(z_\infty) = 0 \quad \text{in } \Delta_{\frac{1}{4}} \times S. \tag{6.111}$$

It is clear that

$$dz_\infty \neq 0 \quad \text{in } \Delta_{\frac{1}{4}} \times S \tag{6.112}$$

this is a consequence of (6.97). So, z_∞ is a solution of the system (2.5) in $\Delta_{\frac{1}{4}} \times S$ whose differential does not vanish in $\Delta_{\frac{1}{4}} \times S$. But (2.5) is an elliptic system and therefore

$$z_\infty \in C^\infty(\Delta_{\frac{1}{4}} \times S) \tag{6.113}$$

(see [2]).

This proves the semiglobal integrability of the structure $\mathscr{L}$ given by (2.7), because we can proceed analogously from $\Delta_R \times S$, $R > 0$ and $z_0 = z$ (the canonical chart of $\mathbf{C}$), to get a C^∞ solution $z_{R,\infty}$ of $\mathscr{L}$ with $dz_{R,\infty} \neq 0$ in $\Delta_{\frac{R}{4}} \times S$.

Proof of the Corollary 6.1. We must show that $\mathscr{L}$ given by (2.7) is globally integrable. Let $z_{R,\infty} \in C^\infty(\Delta_{\frac{R}{4}} \times S)$ the solution of

$$\delta_0 z_{R,\infty} = 0 \quad \text{in } \Delta_{\frac{R}{4}} \times S \tag{6.114}$$

where δ_0 is the operator (2.8).

If $\alpha' < \alpha$, we know that the embedding $C^{1+\alpha}(\Delta_{\frac{R}{4}} \times S) \hookrightarrow C^{1+\alpha'}(\Delta_{\frac{R}{4}} \times S)$ is compact.

The functions $z_{1,\infty}, z_{2,\infty}, \dots$ belong to $C^{1+\alpha}(\Delta_{\frac{1}{4}} \times S)$ and from (6.109),

$$\|z_{k,\infty} - z\|_{\substack{1+\alpha,z\\ \mathbf{C}\times S}} \leq \sum_{j=0}^{\infty} N_j^{-1} < \frac{1}{4}. \tag{6.115}$$

So $\{z_{k,\infty}\}$ is bounded in $C^{1+\alpha}(\Delta_{\frac{1}{4}} \times S)$ and therefore it has a subsequence $\{z_{k_j,\infty}\}$ converging to a function $z_1^\#$ in $C^{1+\alpha}(\Delta_{\frac{1}{4}} \times S)$.

The functions $z_{k_j,\infty}$, $j = 1, 2, \dots$, belong to $C^{1+\alpha}(\Delta_{\frac{2}{4}} \times S)$ except for a finite number of them. Hence $\{z_{k_j,\infty}\}$ has a new subsequence $\{z_{k_{j_l},\infty}\}$ converging to $z_2^\#$ in $C^{1+\alpha'}(\Delta_{\frac{2}{4}} \times S)$. But then $z_1^\# = z_2^\#$ in $\Delta_{\frac{1}{4}} \times S$.

By diagonalization we get a subsequence $\{\tilde{z}_k\}$ of $\{z_{k,\infty}\}$ which converges to a function $z_\infty^\# \in C^{1+\alpha'}(\Delta_{\frac{k}{4}} \times S)$, $\forall k \in \mathbf{N}$. But

$$\delta_0 \tilde{z}_k = 0 \quad \text{in } \Delta_{\frac{k}{4}} \times S \tag{6.116}$$

and so

$$\delta_0(z_\infty^\#) = 0 \quad \text{in } \mathbf{C} \times S. \tag{6.117}$$

It follows that $z_\infty^\# \in C^\infty(\mathbf{C} \times S)$ because δ_0 is an elliptic operator. Futhermore, since

$$\|\tilde{z}_k - z\|_{\substack{1+\alpha,z\\ \mathbf{C}\times S}} < \frac{1}{4}, \qquad k = 1, 2, \dots, \tag{6.118}$$

we get

$$\|z_\infty^\# - z\|_{\substack{1+\alpha,z\\ \mathbf{C}\times S}} \leq \frac{1}{4} \tag{6.119}$$

hence $dz_\infty^\# \neq 0$ in $\mathbf{C} \times S$.

So, $\mathscr{L}$ is globally integrable.

7. Proof of the Theorem 2.1

We consider the vector fields given by (2.5), with $S = S^{n-1}$ and where σ_j, $j = 1, 2, \dots, n$, are flat at Im z=0. Hence, given $N \in \mathbf{N}$, there are $C = C(N)$ and $R = R(N) < 1$ such that

$$\left\| \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} \right\|_{\substack{1+\alpha,0\\ \Delta_r \times S^{n-1}}} < C r^N, \qquad r \leq R. \tag{7.1}$$

Theorem 6.1 can be used to obtain a solution Z_∞ of the system

$$
\left\{\begin{aligned}
&\left(\frac{\partial}{\partial \bar{z}}+\chi\sigma_1\frac{\partial}{\partial z}\right)Z_\infty=0,\\
&\left(\frac{\partial}{\partial \theta_{j-1}}+\chi\sigma_j\frac{\partial}{\partial z}\right)Z_\infty=0,\qquad j=2,\dots,n.
\end{aligned}\right. \tag{7.2}
$$

with differential $dZ_\infty \neq 0$. Here $\chi(z)=\psi(\frac{z}{R})$, $\psi\in C_c^\infty(\mathbf{C})$, $\psi\equiv 1$ in $\Delta_{\frac{R}{2}}$ and $\psi\equiv 0$ out of Δ_R.

Then

$$
\left\|\chi\begin{pmatrix}\sigma_1\\ \vdots\\ \sigma_n\end{pmatrix}\right\|_{\substack{1+\alpha,0\\ \mathbf{C}\times S^{n-1}}}=\left\|\chi\begin{pmatrix}\sigma_1\\ \vdots\\ \sigma_n\end{pmatrix}\right\|_{\substack{1+\alpha,0\\ \Delta_R\times S^{n-1}}}\le\|\chi\|_{C^{1+\alpha}(\Delta_R)}\left\|\begin{pmatrix}\sigma_1\\ \vdots\\ \sigma_n\end{pmatrix}\right\|_{\substack{1+\alpha,0\\ \Delta_R\times S^{n-1}}} \tag{7.3}
$$
$$
\le CR^{-(1+\alpha)}R^N<\varepsilon_0
$$

if $N\ge 2$ and R is chosen sufficiently small. So (6.1) is true. Since $\chi\equiv 1$ in $\Delta_{\frac{R}{4}}\times S$, we get a solution Z_∞ of the system (2.5) and therefore Z_∞ is a null solution of the system (2.4) with differential $\neq 0$, globally defined for $\theta\in S^{n-1}$.

We consider now

$$
Z(x,t)=Z_\infty\left(x,\frac{|t|^2}{2},\theta(t)\right) \tag{7.4}
$$

where $\theta(t)=t/|t|\in S^{n-1}$, $t\neq 0$.

Z is a C^∞ local solution of the Mizohata system given by (2.1) with differential $\neq 0$. Indeed, if $t\neq 0$, $Z(x,t)$ is a C^∞ function. Z is also continuous at $t=0$ because Z_∞ satisfies the system (2.5) and $\sigma_j\equiv 0$ when $s\le 0$. Hence

$$
\frac{\partial Z_\infty}{\partial\theta_j}+\rho_j\frac{\partial Z_\infty}{\partial z}=\frac{\partial Z_\infty}{\partial\theta_j}\equiv 0\quad\text{if } s\le 0. \tag{7.5}
$$

So $Z_\infty(x,0,\theta)$ does not depend on θ and $Z(x,t)$ can be defined continuously in a neighbourhood of the origin in $\mathbf{R}^{n+1}$.

Now

$$
\frac{\partial}{\partial x}Z(x,t)=\frac{\partial}{\partial x}Z_\infty\left(x,\frac{|t|^2}{2},\theta(t)\right) \tag{7.6}
$$

can be defined at $t=0$ because

$$
\frac{\partial}{\partial\theta}\frac{\partial}{\partial x}Z_\infty(x,s,\theta)=\frac{\partial}{\partial x}\frac{\partial}{\partial\theta}Z_\infty(x,s,\theta)\equiv 0\quad\text{for } s\le 0. \tag{7.7}
$$

The derivatives with respect to t_j are

$$
\frac{\partial Z}{\partial t_j}=t_j\frac{\partial Z_\infty}{\partial s}+\sum_{j=1}^{n-1}\frac{\partial Z_\infty}{\partial\theta_k}\frac{\partial\theta_k}{\partial t_j},\qquad j=1,\dots,n. \tag{7.8}
$$

Since

$$
\frac{\partial}{\partial\theta}\frac{\partial Z_\infty}{\partial s}\equiv 0,\qquad s\le 0, \tag{7.9}
$$

it is only necessary to prove that

$$\frac{\partial z_\infty}{\partial \theta_k}\frac{\partial \theta_k}{\partial t_j} \tag{7.10}$$

can be defined continuously at the origin. We know that

$$\frac{\partial z_\infty}{\partial \theta_k} \tag{7.11}$$

is flat at $s = |t|^2/2 = 0$. So, given $N \in \mathbf{N}$, there is a neighbourhood of the origin in $\mathbf{R}^{n+1}$ and a constant C_N such that

$$\left|\frac{\partial Z_\infty}{\partial \theta_k}\right| \le C_N|s|^N = \frac{C_N}{2}|t|^{2N} \tag{7.12}$$

A simple calculation yields constants $C > 0$ and $M \in \mathbf{N}$ such that

$$\left|\frac{\partial \theta_k}{\partial t_j}\right| \le C|t|^{-M} \tag{7.13}$$

in a neighbourhood of the origin in $\mathbf{R}^{n+1}$.

Taking $N > M/2$, we see that

$$\frac{\partial z}{\partial t_j} \tag{7.14}$$

can be defined continuously at the origin. Hence Z is a C^1 function. The same reasoning shows that Z is a C^∞ function, because the derivatives with respect to t are always multiplied by a flat term at $s = 0$.

The chain rule implies

$$M_j Z = 0, \qquad j = 1, 2, \dots, n, \tag{7.15}$$

where M_j are given by (2.1). From (6.112), it follows that $dZ \neq 0$ in a neighbourhood of the origin.

Finally, we conclude that a strongly pseudoconvex Mizohata structure over $\mathbf{R}^{n+1}$, $n > 2$, is locally integrable since in a convenient system of coordinates it has the form (2.1).

Remark 7.1. The proof of Theorem 6.1 does not fully use that the coefficients of the system (2.1) are smooth. We can get information about the regularity of $Z(x, t)$ from the choice of parameters given by (6.59). If the coefficients of the system (2.1) belong to $C^{15+\alpha}(\mathbf{C} \times S)$ that is enough to construct a solution of class $C^{1+\alpha}$.

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