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ON COMPLETE MANIFOLDS OF NONNEGATIVE kTH-RICCI CURVATURE

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ABSTRACT. In this paper we establish some vanishing and finiteness theorems for the topological type of complete open riemannian manifolds under certain positivity conditions for curvature. Key tools are comparison techniques and Morse Theory of Busemann and distance functions.

1. INTRODUCTION AND MAIN RESULTS

One of the most important aspects of riemannian geometry deals with the relationship between the curvature properties of a riemannian manifold and its topological structure.

Two classical theorems are Gromoll-Meyer's theorem [GM] and Cheeger-Gromoll's Soul Theorem [CG1]. Gromoll-Meyer's theorem asserts that any complete open riemannian manifold of positive sectional curvature is diffeomorphic to $\mathbf{R}^n$. Cheeger-Gromoll's Soul Theorem says that for any complete open manifold of nonnegative sectional curvature, there is a totally geodesic compact submanifold S, to be called a soul, such that M is diffeomorphic to the normal bundle $\nu(S)$ of S in M (the diffeomorphism does not come from the exponential map of S, in general). Therefore it is quite natural to study complete open manifolds under certain partial positivity for curvature.

For a riemannian n-manifold M, we say the kth-Ricci curvature of M, for some $1 \leq k \leq n-1$, satisfies $\mathrm{Ric}_{(k)} \geq H$, at a point $x \in M$ if for all $(k+1)$-dimensional subspaces $V \subset T_xM$, the curvature tensor $R(x, y)z$ satisfies

$$\sum_{i=1}^{k+1} \langle R(e_i, v)v, e_i \rangle \geq H, \qquad v \in V,$$

where $\{e_1, \ldots, e_{k+1}\}$ is any orthonormal basis for V. By $\mathrm{Ric}_{(k)}(M) \geq H$ we mean $\mathrm{Ric}_{(k)} \geq H$ at all points $x \in M$. In a similar way we can define that $\mathrm{Ric}_{(k)} > H$ at a point x, and $\mathrm{Ric}_{(k)}(M) > H$. Clearly, $\mathrm{Ric}_{(n-1)}(M) \geq H$ if and only if $\mathrm{Ric}(M) \geq H$, and $\mathrm{Ric}_{(1)}(M) \geq H$ if and only if $K_M \geq H$,

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where $\mathrm{Ric}(M)$ and K_M denote the Ricci curvature and the sectional curvature, respectively.

1. We say a complete open n-manifold M is *proper* if the Busemann function b_p at some point p is proper. This property of the Busemann functions is independent of a choice of p (see §2.1 below for definition and basic properties). Any complete open n-manifold M of nonnegative sectional curvature outside a compact subset must be proper. Moreover in this case M has finite topological type (cf. [CG1, GW2]). H. Wu [W2] has proved that any complete open n-manifold M with $\mathrm{Ric}_{(k)}(M) > 0$ for some $1 \le k \le n-1$ and nonnegative sectional curvature outside a compact subset, has the homotopy type of a CW complex with *finitely* many cells each of dimension $\le k-1$. By using the techniques developed in [SH, W2], we will establish the following

Theorem 1. *Let M be a proper open n-manifold with $\mathrm{Ric}_{(k)}(M) > 0$ for some k, $1 \le k \le n-1$. Then M has the homotopy type of a CW complex with (possibly infinitely many) cells each of dimension $\le k-1$. In particular, $H_i(M;\mathbf{Z}) = 0$, for $i \ge k$.*

Theorem 1 should be viewed as a generalized version of the Gromoll-Meyer theorem [GM]. In the case of $k = n-1$, Theorem 1 tells us that any proper open n-manifold of positive Ricci curvature has the homotopy type of a CW complex with cells each of dimension $\le n-2$. Hence $H_{n-1}(M;\mathbf{Z}) = 0$. This is an analogue of a vanishing theorem for closed manifolds which says that any oriented closed n-manifold M of positive Ricci curvature satisfies $H_1(M;\mathbf{R}) = H_{n-1}(M;\mathbf{R}) = 0$ (cf. e.g. [BY]). In [Y], by using a different method, S. T. Yau proves that any complete open n-manifold M of positive Ricci curvature satisfies $H_{n-1}(M,\mathbf{R}) = 0$. His method, however, does not give a vanishing theorem for $H_k(M,\mathbf{R})$ for $k \le n-2$ under the positivity conditions for the kth-Ricci curvature. One notices that M. Anderson also proves a relative result that if M is a complete open n-manifold of nonnegative Ricci curvature, then the first Betti number $b_1(M) := \dim H_1(M;\mathbf{Q}) \le n-1$. For further information see [An].

Recently, Sha-Yang [SY1, 2] have constructed n-dimensional complete open manifolds of infinite topological type for all $n \ge 4$, on which the metrics can be chosen to be proper, have positive Ricci curvature and bounded curvature[1]. Topologically, their examples are obtained by removing infinitely many disjoint balls D_i^{p+1}, $i = 0, 1, \dots, +\infty$, from $\mathbf{R}^{p+1}$ and then gluing $S^{n-p-1} \times (\mathbf{R}^{p+1} \setminus \coprod_{i=0}^{\infty} D_i^{p+1})$ with $D^{n-p} \times \coprod_{i=0}^{\infty} S_i^p$ together by the indentity maps along the corresponding boundaries, where $2 \le p \le n-2$. Let $M_{n,p}$ denote the resulting manifolds. Clearly, the singular homology groups $H_{n-2}(M_{n,n-2};\mathbf{Z})$ are infinitely generated. In this sense, Theorem 1 is sharp. In dimension 3, Schoen-Yau [SHY] prove that all complete open 3-manifolds of positive Ricci curvature is diffeomorphic to R^3. Thus there is no nontrivial examples in this case.

It seems to be difficult to determine whether a complete open riemannian manifold M is proper or not, even if M has nonnegative Ricci curvature. However, we will show that if (with respect to a point) M has small diameter growth of ends, then M is proper. For a subset A of M, denote by $\mathrm{dia}(A)$

[1]This property of the sectional curvature is not stated explicitly in [SY1, 2]

the diameter of A measured in M, i.e. $\mathrm{dia}(A) = \sup_{x,y\in A} d(x, y)$. We will prove that if for some point $p \in M$,

$$\limsup_{r\to+\infty} \frac{\mathrm{dia}(S(p, r))}{r} < 1, \tag{1}$$

where $S(p, r) := \{x \in M; d(p, x) = r\}$ denotes the geodesic sphere of radius r around p, then M is proper (see Corollary 2 below in §2.2). Therefore one has

Corollary 1. *Let M be a complete open n-manifold* $\mathrm{Ric}_{(k)}(M) > 0$ *for some $1 \le k \le n-1$. Suppose that for some $p \in M$, (1) holds. Then M has the homotopy type of a CW complex with cells each of dimension $\le k-1$. In particular, $H_i(M; \mathbf{Z}) = 0$ for $i \ge k$.*

In the case of $k = n-1$, Corollary 1 tells us that if a complete open n-manifold M of positive Ricci curvature satisfies (1), then M has the homotopy type of a CW complex with cells each of dimension $\le n-2$. To my best knowledge, all known examples of positive Ricci curvature satisfy (1).

2. It was proved by M. Gromov [G1] that there is a constant $C(n)$ depending on only n such that for any closed n-manifold M of nonnegative sectional curvature, the total Betti number of M with respect to any field $\mathbf{F}$ satisfies

$$\sum_{k=0}^{n} b_i(M; \mathbf{F}) \le C(n).$$

By the Soul Theorem of Cheeger-Gromoll [CG1], this theorem is also valid for complete open n-manifolds of nonnegative sectional curvature. Examples in [SY1, 2 and AKL], however, show that this theorem does not hold for complete n-manifold of nonnegative Ricci curvature. But one can still obtain some topological obstruction to complete open manifolds with nonnegative Ricci curvature and bounded curvature. Let M be a complete open riemannian n-manifold and let $p \in M$. For $r > 0$, put $B(p, r) = \{x \in M; \ d(p, x) < r\}$. Let $b_i(p, r)$ denote the rank $i_* : H_i(B(p, r); \mathbf{F}) \hookrightarrow H_i(M; \mathbf{F})$, where $\mathbf{F}$ is an arbitrary field. We will prove

Theorem 2. *Let M be a complete open n-manifold with Ricci curvature* $\mathrm{Ric}(M) \ge 0$ *and sectional curvature $K_M \ge -1$. Then there is a constant $C(n)$ depending only on n such that*

$$\sum_{i=0}^{n} b_i(p, r) \le C(n)(1+r)^n, \qquad r > 0.$$

As we see from Sha-Yang's examples, in order to prove a finite topological type theorem for manifolds in Theorem 2, additional conditions are required. Recently, Abresch-Gromoll [AG] have proved that a complete open n-manifold M of nonnegative Ricci curvature has finite topological type if M has essential diameter growth of order $o(r^{1/n})$, provided that the curvature is bounded from below. In §2.2 we will introduce the notion of essential diameter of ends. There are several definitions for the (essential) diameter of ends. It seems to the author that the definition given by Cheeger [C] is the simplest one. Denote by $\mathscr{D}(p, r)$ the one defined in [C], which is called the essential diameter of ends at distance r from p. Roughly speaking, $\mathscr{D}(p, r)$ is the maximum of the diameters of the connected components, Σ, of $S(p, r)$, with $\gamma(r) \in \Sigma$ for some ray emanating from the point p. We will generalize the Abresch-Gromoll's theorem to the case $\mathrm{Ric}_{(k)}(M) \geq 0$.

Theorem 3. *Let M be a complete open n-manifold of $\mathrm{Ric}_{(k)}(M) \geq 0$ for some $2 \leq k \leq n-1$. Suppose that the sectional curvature $K_M \geq -K$, $K > 0$, and for some point $p \in M$,*

$$\limsup_{r \to +\infty} \frac{\mathscr{D}(p, r)}{r^{1/(k+1)}} < \frac{1}{8} K^{-k/(2(k+1))}.$$

Then M is homeomorphic to the interior of a compact manifold with boundary.

One notices that the diameter growth condition is violated by Sha-Yang's examples. By definition, $\mathscr{D}(p, r) \leq \mathrm{dia}(S(p, r))$ for all r. Thus any growth condition on $\mathrm{dia}(S(p, r))$ implies that on $\mathscr{D}(p, r)$. The advantage of $\mathscr{D}(p, r)$ is that the growth of $\mathscr{D}(p, r)$ can be controlled by the volume growth condition. For a complete open n-manifold M with $\mathrm{Ric}(M) \geq 0$ and volume noncollapse, i.e. $\inf_{x \in M} \mathrm{vol}(B(x, 1)) \geq v > 0$, it is shown in [SW] that for all $r \geq 1$,

$$\mathscr{D}(p, r) \leq C(n) v^{-1} \frac{\mathrm{vol}(B(p, r))}{r}.$$

Thus we have

Theorem 4 [SW]. *Let M be complete with $\mathrm{Ric}_{(k)}(M) \geq 0$ for some $2 \leq k \leq n-1$. Suppose that M has weak bounded geometry, i.e. $K_M \geq -K$, $K > 0$, and $\inf_{x \in M} \mathrm{vol}(B(x, 1)) \geq v > 0$. If the volume growth at a point p satisfies*

$$\lim_{r \to +\infty} \frac{\mathrm{vol}(B(p, r))}{r^{1+1/(k+1)}} = 0,$$

then M is homeomorphic to the interior of a compact manifold with boundary.

In the case of $k = n-1$, Theorem 4, in particular, implies that for a complete open n-manifold M with $\mathrm{Ric}(M) \geq 0$ and bounded geometry, if M has linear volume growth at a point, then M has finite topological type. One should compare this result with Calabi-Yau's theorem ([Y], see also [CGT]) which asserts that any complete open n-manifold of nonnegative Ricci curvature has at least linear volume growth at any point p, more precisely, for all $r \geq 1$,

$$\mathrm{vol}(B(p, r)) \geq \varepsilon(n) \mathrm{vol}(B(p, 1)) r.$$

M. Gromov [G2] proves that a complete manifold M of sectional curvature $-1 \leq K_M < 0$, and $\mathrm{vol}(M) < +\infty$, is diffeomorphic to the interior of a compact manifold with boundary. Furthermore, if in addition the sectional curvature is strictly negative, then M has finitely many ends, E, with $\mathrm{dia}(S(p, r) \cap E) \to 0$ as $r \to +\infty$. We will prove the following relative result.

Theorem 5. *Let M be a complete open manifold with sectional curvature $K_M \geq -K$, $K > 0$. Suppose that M has finitely many ends and for some $p \in M$,*

$$\limsup_{r \to +\infty} \mathscr{D}(p, r) < \ln 2 \cdot K^{-1/2},$$

then M is homeomorphic to the interior of a compact manifold with boundary.

The organization of this paper is as follows. The proof of Theorem 1 is in §3. The proof of Theorem 2 is in §6. The proof of Theorem 3 is in §5. The proof of Theorem 5 is in §4.

Most of the results in this paper were announced in [S1].

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2. Preliminaries

2.1. **Busemann functions.** Let M be a complete open riemannian n-manifold and let $p \in M$. Recall that the *Busemann function* B_γ associated with a ray γ issuing from p is defined as $B_\gamma(x) = \lim_{t \to +\infty} t - d(x, \gamma(t))$, $x \in M$. For arbitrary $r \geq 0$, let

$$R(p, r) = \{\gamma(r);\ \ \gamma \text{ is a ray issuing from } p\},$$

which is a closed subset of the geodesic sphere $S(p, r)$. Set $\rho_p(x) = d(p, x)$. Set $B_p^t(x) = t - d(x, R(p, t))$, $x \in M$. It is clear that $B_p^t(x)$ is increasing in t and $|B_p^t(x)| \leq \rho_p(x)$, $x \in M$. Then the function B_p, defined as $B_p(x) = \lim_{t \to +\infty} B_p^t(x)$, is a Lipschitz function with Lipschitz constant 1. An elementary argument shows that $B_p = \sup_\gamma B_\gamma$, where the supremum is taken over all rays γ issuing from p. For $x \in S(p, r)$, $r \geq 0$, set

$$R_p(x) = d(x, R(p, r)).$$

Since $B_p^t(x)$ is increasing in t, it is easy to see that

$$\rho_p(x) - R_p(x) \leq B_p(x) \leq \rho_p(x). \tag{2}$$

Consider a family of functions $b_p^t : M \mapsto \mathbf{R}$ defined by $b_p^t(x) = t - d(x, S(p, t))$, $t \in [0, +\infty)$. They are Lipschitz continuous (with Lipschitz constant 1) and also satisfy $|b_p^t(x)| \leq \rho_p(x)$ (by the triangle inequality). It is not difficult to show the following

Lemma 1. *Fix any point $x \in M$, $b_p^t(x)$ is decreasing in t, for $t \geq \rho_p(x)$, i.e. for all $t_2 \geq t_1 \geq \rho_p(x)$,*

$$b_p^{t_1}(x) \geq b_p^{t_2}(x). \tag{3}$$

Proof. Let $x' \in S(p, t_2)$ such that $d(x, x') = d(x, S(p, t_2))$. Take a minimal normal geodesic σ from x to x'. For $s_0 := d(x, S(p, t_2)) - t_2 + t_1 \geq 0$, the point $\sigma(s_0)$ satisfies

$$d(p, \sigma(s_0)) \geq t_2 - d(\sigma(s_0), S(p, t_2)) = t_2 - d(x, S(p, t_2)) + s_0 = t_1.$$

Thus $\sigma(s_0) \in M \backslash B(p, t_1)$, that implies

$$d(x, S(p, t_1)) \leq d(x, \sigma(s_0)) = d(x, S(p, t_2)) - t_2 + t_1.$$

This gives (3). Q.E.D.

Thus b_p^t converges to a Lipschitz function b_p, with the convergence being uniform on compact subsets. The function b_p is called the *Busemann function* at point p. Set

$$e_p(x) = \rho_p(x) - b_p(x), \qquad x \in M.$$

e_p is called the *excess function* at point $p \in M$ (compare [GP]). By definition and (2), one has

$$B_p(x) \le b_p(x), \qquad x \in M, \tag{4}$$

$$e_p(x) \le R_p(x), \qquad x \in M. \tag{5}$$

For $r > 0$, set

$$\mathscr{E}(p, r) = \max_{x \in S(p, r)} e_p(x), \tag{6}$$

$$\mathscr{R}(p, r) = \max_{x \in S(p, r)} R_p(x). \tag{7}$$

Then (5) immediately implies that for all r,

$$\mathscr{E}(p, r) \le \mathscr{R}(p, r) \le \mathrm{dia}(S(p, r)). \tag{8}$$

The following lemma is important for further study.

Lemma 2. *Let M be a complete open riemannian n-manifold and let $p \in M$. Then for any point $q \in M$, there is a ray $\sigma_q: [0, +\infty) \to M$ issuing from q such that*

(i) *for all $t > 0$,*

$$b_p^{q,t}(x) := b_p(q) + t - d(x, \sigma_q(t)),$$

supports $b_p(x)$ at q, i.e., $b_p^{q,t}(x) \le b_p(x)$ for all $x \in M$ and $b_p^{q,t}(q) = b_p(q)$.

(ii) (Wu) *for all $t \ge 0$,*

$$b_p(\sigma_q(t)) = b_p(q) + t. \tag{9}$$

Proof. Take a divergent sequence $t_n \to +\infty$, and a sequence of points $x_n \in S(p, t_n)$ such that $d(q, x_n) = d(q, S(p, t_n))$. Take a normal minimal geodesic σ_n issuing from q to x_n. By passing to a subsequence if necessary, one can assume that $\dot{\sigma}_n(0)$ converges to a unit vector $v \in T_qM$. Set $\sigma_q(s) = \exp_q sv$. It is clear that σ_q is a ray. Notice that for sufficiently large t_n,

$$d(q, S(p, t_n)) = t + d(\sigma_n(t), S(p, t_n)).$$

Thus one obtains

$$\begin{aligned} b_p(x) - b_p^{q,t}(x) &= b_p(x) - b_p(q) - t + d(x, \sigma_q(t)) \\ &= \lim_{t_n \to +\infty} d(q, S(p, t_n)) - d(x, S(p, t_n)) - t + d(x, \sigma_q(t)) \\ &= \lim_{t_n \to +\infty} d(\sigma_n(t), S(p, t_n)) - d(x, S(p, t_n)) + d(x, \sigma_q(t)) \\ &\ge \lim_{t_n \to +\infty} -d(\sigma_n(t), x) + d(x, \sigma_q(t)) \\ &\ge \lim_{t_n \to +\infty} -d(\sigma_n(t), \sigma_q(t)) = 0. \end{aligned}$$

It is obvious that $b_p^{q,t}(q) = b_p(q)$.

The equality (9) was proved by Wu in [W1]. Q.E.D.

An open riemannian manifold M is called *proper*, if M is complete and for some point $p \in M$, the Busemann function b_p is proper, i.e. the subset $b_p^{-1}((-\infty, a])$ is compact for all $a \in \mathbf{R}$. The following lemma shows that b_p is proper for some $p \in M$, then b_q is proper for all $q \in M$.

Lemma 3. *Let M be a complete open riemannian manifold. Suppose that for some point p and some sequence of closed subsets $\mathscr{C} = \{C_n\}_{n=0}^{\infty}$ with $r_n := d(p, C_n) \to +\infty$, $b_p^{C_n} := r_n - d(\cdot, C_n)$ converges to a function $b_p^{\mathscr{C}}$. Then for all q,*

$$b_q(x) \geq -b_p^{\mathscr{C}}(q) + b_p^{\mathscr{C}}(x).$$

In particular,

$$b_q(x) \geq -b_p(q) + b_p(x).$$

Thus b_p is proper if and only if b_q is proper for all points $q \in M$.

Proof. Let $t_n = d(q, C_n)$. Clearly, for $t_n > d(q, x)$, one has $x \in B(q, t_n)$ and $d(x, S(q, t_n)) \leq d(x, C_n)$. Thus

$$\begin{aligned} b_q^{t_n}(x) = t_n - d(x, S(q, t_n)) &\geq d(q, C_n) - d(x, C_n) \\ &= -(r_n - d(q, C_n)) + (r_n - d(x, C_n)). \end{aligned}$$

Letting $r_n \to +\infty$, one obtains $b_q(x) \geq -b_p^{\mathscr{C}}(q) + b_p^{\mathscr{C}}(x)$. Q.E.D.

Remark 1. It was proved in [CG1, GW2] that any complete open manifolds with nonnegative sectional curvature outside a compact subset is proper. In particular [LT], for any point $p \in M$,

$$\lim_{x \to +\infty} \frac{b_p(x)}{\rho_p(x)} = \lim_{x \to +\infty} \frac{B_p(x)}{\rho_p(x)} = 1.$$

The question is when the Busemann function b_p is proper. By (5) one sees that if $R_p(x)$ satisfies

$$\limsup_{r \to +\infty} \frac{R_p(x)}{\rho_p(x)} < 1,$$

then b_p is proper. In the next section we will continue to study the properness of open riemannian manifolds.

2.2. **Diameter of ends.** In this section, we will prove an elementary result for Busemann functions, which tells us how the smallness of the diameter of ends implies the properness of the Busemann functions. In particular, we prove that if a complete open manifold M with finitely many ends has diameter growth of order $o(r)$, then M is proper. No restriction on curvature will be required.

There are several definitions for the (essential) diameter of ends (cf. [AG, S1, C]). Let us first give the most natural one here for manifolds with finitely many ends. The *essential* diameter of ends will be defined in the last of this section. Let M be a complete open riemannian manifold with N ends. Let $R_{\min}$ be the smallest number R such that $M \backslash \overline{B(p, R)}$ has N unbounded connected components. Let $U_1, \ldots, U_N$ be the unbounded connected components of $M \backslash \overline{B(p, R_{\min})}$. Then the diameter of ends at distance $r > R_{\min}$ from p, $\mathscr{W}(p, r)$, is defined as

$$\mathscr{W}(p, r) = \max_{1 \leq i \leq N} \operatorname{dia}(S(p, r) \cap U_i).$$

When M has only one end, $\mathscr{W}(p, r)$ is defined for all $r > 0$, and

$$\mathscr{W}(p, r) = \operatorname{dia}(S(p, r)).$$

Remark 2. By the Cheeger-Gromoll's splitting theorem [CG2], one can conclude that any complete open manifold M of nonnegative Ricci curvature has no more than two ends. In addition, if M has positive Ricci curvature at some point, then M has only one end. Thus in this case $\mathscr{W}(p, r)$ is always defined as the diameter of $S(p, r)$.

Lemma 4. *Suppose M is a complete open riemannian n-manifold with finitely many ends. Then there is R_0 such that for all $r \geq R_0$,*

$$\mathscr{R}(p, r) \leq \mathscr{W}(p, r) \leq \operatorname{dia}(S(p, r)). \tag{10}$$

Proof. Suppose M has N ends. Let $U_1, \dots, U_N$ be the N unbounded connected components of $M \backslash \overline{B(p, R_{\min})}$. Clearly there are only finitely many *bounded* connected components of $M \backslash \overline{B(p, R_{\min})}$, to be denoted by $V_1, \dots, V_K$, with $V_i \cap S(p, 2R_{\min}) \neq \varnothing$. Thus for $R_0 := \max_{1 \leq i \leq K} \sup_{x \in V_i} \rho_p(x) \geq 2R_{\min}$,

$$M \backslash \overline{B(p, R_0)} = \bigcup_{1 \leq i \leq N} U_i \backslash \overline{B(p, R_0)}. \tag{11}$$

By (11), for any $x \in M \backslash \overline{B(p, R_0)}$, there is an unbounded connected component U_{i_0} of $M \backslash \overline{B(p, R_{\min})}$ such that $x \in U_{i_0}$. Take a ray γ issuing from p such that $\gamma|_{(R_{\min}, +\infty)} \subset U_{i_0}$. Let $r_0 = d(p, x)$. Then

$$\begin{aligned} R_p(x) &= d(x, R(p, r_0)) \leq d(x, \gamma(r_0)) \\ &\leq \operatorname{dia}(U_{i_0} \cap S(p, r_0)) \leq \mathscr{W}(p, r_0). \quad \text{Q.E.D.} \end{aligned}$$

(5) and (10) immediately imply the following

Corollary 2. *Let M be a complete open manifold with finitely many ends. Suppose that for some $p \in M$,*

$$\limsup_{r \to \infty} \frac{\mathscr{W}(p, r)}{r} = \zeta < 1,$$

or

$$\limsup_{r \to \infty} \frac{\operatorname{dia}(S(p, r))}{r} = \zeta < 1.$$

Then

$$\liminf_{x \to \infty} \frac{b_p(x)}{\rho_p(x)} \geq 1 - \zeta.$$

In this case, the Busemann function b_p is proper. Thus M is proper.

Now we will introduce a weaker concept of diameter of ends for complete open manifolds (compare [C]). Let M be complete with $p \in M$ fixed. For $r > 0$, the connected components, Σ, of $\partial(M \backslash \overline{B(p, r)})$, are called the *boundary* components of $M \backslash \overline{B(p, r)}$. Set

$$\mathscr{D}(p, r) = \sup_{\Sigma} \operatorname{dia}(\Sigma),$$

where the supremum is taken over all boundary components Σ of $M \backslash \overline{B(p, r)}$ with $\Sigma \cap R(p, r) \neq \varnothing$. We call $\mathscr{D}(p, r)$ the essential diameter of ends at distance r from p. Clearly, one has

$$\mathscr{D}(p, r) \leq \operatorname{dia}(S(p, r)). \tag{12}$$

Remark 3. Without any additional assumption there is no relation between $\mathscr{R}(p, r)$ and $\mathscr{D}(p, r)$. However, one still has that for all boundary components, Σ, of $M \backslash \overline{B(p, r)}$ with $\Sigma \cap R(p, r) \neq \varnothing$,

$$R_p(x) \leq \mathscr{D}(p, r), \qquad x \in \Sigma. \tag{13}$$

3. Open manifolds M with positive kth-Ricci curvature

3.1. **Smoothing theorem.** In [SH, W2], J. Sha and H. Wu independently studied the k-convexity for certain distance functions on a riemannian n-manifold (with boundary). Let (M, g) be a riemannian n-manifold (not necessary to be complete) and let $p \in M$. Let f be a continuous function defined in a neighborhood of $p \in M$. Let $\gamma : (-a, a) \to M$ be a normal geodesic with $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = v \in T_pM$. As in [W2], we define the following extended real number:

$$Cf(p; v) = \liminf_{r \to 0} \frac{1}{r^2}\{f \circ \gamma(r) + f \circ \gamma(-r) - 2f \circ \gamma(0)\}.$$

We say that f belongs to $C(k)$ at $p \in M$ for some k, $1 \leq k \leq n$, if f is Lipschitz continuous in a neighborhood W of p, and there are positive constants ε and η such that if $x \in W$ and $\{v_1, \ldots, v_k\}$ is set in T_xM with $|\langle v_i, v_j \rangle - \delta_{ij}| < \varepsilon$, then

$$\sum_{i=1}^{k} Cf(x; v_i) \geq \eta.$$

We say that f belongs to $C(k)$ on a subset A of M if f is defined on a neighborhood U of A, such that $f \in C(k)$ at every point $p \in U$. Similarly, a function f is said to be C^∞ on a subset A, if f is defined on a neighborhood U of A such that $f \in C^\infty(U)$.

Clearly, a C^2 function $f : M \to \mathbf{R}$ belongs to $C(k)$ on M if and only if

$$\sum_{i=1}^{k} \nabla^2 f(V_i, V_i) > 0,$$

for any set of orthonormal vector fields $\{V_1, \ldots, V_k\}$ locally defined in M. Thus if a smooth Morse function $f : M \to \mathbf{R}$ belongs to $C(k)$ on M, then the index of f at each critical point satisfies that $\operatorname{ind}(f) \leq k - 1$. By Theorem 1.1 in [GW1] Wu proves the following smoothing theorem for $C(k)$.

Theorem 6 [W2]. *Let (M, g) be a riemannian n-manifold. Let $f: M \to \mathbf{R}$ belong to $C(k)$ on M, and $\xi: M \to \mathbf{R}$ be a positive continuous function. Then there exists a C^∞ function $F: M \to \mathbf{R}$ which belongs to $C(k)$ such that $|F - f| < \xi$.*

For the applications below, one needs a refinement of Wu's smoothing theorem for proper functions $f : M \to \mathbf{R}$.

Lemma 5. *Let (M, g) be a riemannian n-manifold. Let $f : M \to \mathbf{R}$ be proper and belong to $C(k)$ on M, and $\xi: M \to \mathbf{R}$ be a positive continuous function. Then there exists a proper Morse function $F: M \to \mathbf{R}$ which belongs to $C(k)$ on M such that $|F - f| < \xi$.*

The proof of the above lemma strongly relies on the argument in §2 of [M1]. The outline of the proof is as follows. By Theorem 6, one can assume f is a smooth proper function belonging to $C(k)$ on M. Take a sequence $a_1 < a_2 < \cdots \to +\infty$, such that all a_i are the regular values of f. Let $W_i = f^{-1}([a_i, a_{i+1}])$. One can find a smooth Morse function $F_i\colon W_i \to [a_i, a_{i+1}]$ such that each F_i has no critical points in a neighborhood of ∂W_i. Moreover, each F_i is sufficiently close to f in C^2-topology so that it belongs to $C(k)$ on W_i. Then by gluing all F_i together, one obtains a desired Morse function. Since the proof is elementary, so the details are omitted here (see [S2]). The following algebraic lemma is also elementary. It will be useful to verify that a locally Lipschitz function f belong to $C(k)$ at a point $p \in M$.

Lemma 6. *Let V be an inner product space of dimension n. Let S be a symmetric bilinear form on V. Suppose that for some k, $1 \le k \le n$, and some positive numbers η and A, S satisfies*

(i) $\sum_{i=1}^k S(e_i, e_i) \ge \eta$ *for any orthonormal set* $\{e_1, \dots, e_k\}$ *in* V,
(ii) $|S(v, v)| \le A|v|^2$ *for* $v \in V$.

Then there is $\varepsilon > 0$ depending only on k, η, and A such that for any set $\{v_1, \dots, v_k\}$ in V with $|\langle v_i, v_j\rangle - \delta_{ij}| < \varepsilon$,

$$\sum_{i=1}^k S(v_i, v_i) \ge \frac{\eta}{2}.$$

3.2. **Construction of proper Morse functions.** In this section we will prove Theorem 1. It suffices to prove the following

Theorem 7. *Let M be a proper open n-manifold. Suppose that for some $1 \le k \le n-1$, the kth-Ricci curvature is nonnegative everywhere and positive curvature outside a compact subset. Then M has the homotopy type of a CW complex with (possibly infinitely many) cells each of dimension $\le k-1$. In particular,*

$$H_i(M; \mathbf{Z}) = 0, \quad \textit{for } i \ge k.$$

We begin with the following elementary result.

Lemma 7. *Let M be a riemannian n-manifold. Suppose at some point $p \in M$ the sectional curvature and the kth-Ricci curvature, for some $1 \le k \le n-1$, satisfy $|K_p| \le K$ and $\mathrm{Ric}_{(k)} \ge H$, respectively. Then for any orthonormal set $\{e_1, \dots, e_k\}$ in T_pM and any unit vector v in T_pM,*

$$\sum_{i=1}^k \langle R(v, e_i)e_i, v\rangle \ge -(k-1)K(\alpha^2 + 4\alpha\beta) + H\beta^2,$$

where $\alpha = \sqrt{\sum_{i=1}^k \langle v, e_i\rangle^2}$ and $\beta = \sqrt{1 - \sum_{i=1}^k \langle v, e_i\rangle^2}$.

Proof. Let $V = \mathrm{span}\{e_1, \dots, e_k\}$ and $v = v_1 + v_2$ such that $v_1 \in V$ and $v_2 \perp V$. Clearly, $|v_1|^2 = \sum_{i=1}^k \langle v, e_i\rangle^2$ and $|v_2|^2 = 1 - \sum_{i=1}^k \langle v, e_i\rangle^2$. Let $\{f_1, \dots, f_k\}$ be another orthonormal basis for V such that $v_1 = |v_1| f_1$. Let f_{k+1} be the unit vector such that $f_{k+1} \perp V$ and $v_2 = |v_2| f_{k+1}$. Consider the following identity for $i = 2, \dots, k$,

$$\begin{aligned} 2\langle R(f_i, f_1)f_{k+1}, f_i\rangle &= \langle R(f_i, (f_1 + f_{k+1}))(f_1 + f_{k+1}), f_i\rangle \\ &\quad - \langle R(f_i, f_1)f_1, f_i\rangle - \langle R(f_i, f_{k+1})f_{k+1}, f_i\rangle. \end{aligned}$$

It turns out that $2|\langle R(f_i, f_1)f_{k+1}, f_i\rangle| \leq 4K$. Thus

$$\begin{aligned}\sum_{i=1}^{k}\langle R(v, e_i)e_i, v\rangle &= \sum_{i=1}^{k}\langle R(f_i, v)v, f_i\rangle \\ &= |v_1|^2\sum_{i=2}^{k}\langle R(f_i, f_1)f_1, f_i\rangle \\ &\quad + 2|v_1|\,|v_2|\sum_{i=2}^{k}\langle R(f_i, f_1)f_{k+1}, f_i\rangle \\ &\quad + |v_2|^2\sum_{i=1}^{k}\langle R(f_i, f_{k+1})f_{k+1}, f_i\rangle \\ &\geq -(k-1)K(|v_1|^2 + 4|v_1|\,|v_2|) + H|v_2|^2. \quad \text{Q.E.D.}\end{aligned}$$

Now we are in position to construct a proper function on M as in Theorem 7 such that it belongs to $C(k)$ on M. Then Theorem 7 follows from the standard Morse theory.

Lemma 8. *Let M be as in Theorem 7 and let $p \in M$ be fixed. Then there exists a function $\chi \in C^2(\mathbf{R})$ such that $\chi \circ b_p$ is a proper function and $\chi \circ b_p$ belongs to $C(k)$ on M.*

Proof. Let $a = \min_{x\in M} b_p(x)$. Let $R_0 \geq a$ be a number such that $\mathrm{Ric}_{(k)} > 0$ in $\{y; b_p(y) \geq R_0\}$. For $r \geq a$, define

$$H(r) = \inf\{\mathrm{Ric}_{(k)}(y); r + R_0 - a \leq b_p(y) \leq r + R_0 - a + 1\} > 0.$$

Let $T(r) = \max\{16kH(r)^{-1}, 2(R_0 - a + 1)\}$. Define

$$K(r) = \sup\{|K_y|; r \leq b_p(y) \leq r + T(r)\}.$$

Let $C\colon [a, +\infty) \to \mathbf{R}^+$ be a positive continuous function to be determined later. Set

$$\chi(t) = \int_a^t \exp\left(\int_a^s C(\tau)\,d\tau\right) ds + a.$$

It is easy to check that χ is of class C^2 and has the following properties:

(i) $\chi'(r) \geq 1$ for all $r \in [a, +\infty)$,
(ii) $\chi''(r) = C(r)\chi'(r)$ for all $r \in [a, +\infty)$.

Clearly, (i) above implies $\chi \circ b_p$ is also proper. By choosing an appropriate $C(t)$, we will prove that $\chi \circ b_p$ belongs to $C(k)$ on M.

Fix any point $q \in M$ with $b_p(q) = r$. From Lemma 2 in §2.1 it follows that there exists a ray $\sigma_q(s)$ issuing from q such that for all $s > 0$,

$$b_p^{q,s}(x) = r + s - d(x, \sigma_q(s))$$

supports $b_p(x)$ at q, and $b_p(\sigma_q(s)) = r + s$.

Define $\theta : T_qM \times [0, T(r)] \to M$ as

$$\theta(v, s) = \exp_{\sigma_q(s)}\left(1 - \frac{s}{T(r)}\right) v(s),$$

where $v(s)$ denotes the parallel vector field along $\sigma_q(s)$ such that $v(0) = v$. Define f locally at q as

$$f \circ \exp_q(v) = r + T(r) - \int_0^{T(r)} \left| \frac{\partial \theta}{\partial s}(v, s) \right| ds.$$

Clearly, f supports b_p at q. Take any orthonormal set $\{e_1, \dots, e_k\}$ in T_qM. Let α and β be nonnegative numbers such that $\alpha^2 + \beta^2 = 1$ and $\alpha^2 = \sum_{j=1}^k \langle e_j, \dot\sigma_q(0)\rangle^2$. Then by the first and second variation formulas and Lemma 7, one obtains

$$\begin{aligned}
&C(r)\sum_{j=1}^k |e_j f|^2 + \sum_{j=1}^k \nabla^2 f(e_j, e_j) \\
&\geq C(r)\alpha^2 - \frac{1}{T(r)}(k - \alpha^2) \\
&\quad - (k-1)\int_0^{T(r)} \left(1 - \frac{s}{T(r)}\right)^2 \max |K_{\sigma_q(s)}|(\alpha^2 + 4\alpha\beta)\, ds \\
&\quad + \int_0^{T(r)} \left(1 - \frac{s}{T(r)}\right)^2 \min \mathrm{Ric}_{(k)}(\sigma_q(s))\beta^2\, ds \\
&\geq C(r)\alpha^2 - \frac{k}{T(r)} - (k-1)K(r)(\alpha^2 + 4\alpha\beta)\int_0^{T(r)} \left(1 - \frac{s}{T(r)}\right)^2 ds \\
&\quad + H(r)\beta^2 \int_{R_0 - a}^{R_0 - a + 1} \left(1 - \frac{s}{T(r)}\right)^2 ds \\
&\geq C(r)\alpha^2 - \frac{k}{T(r)} - \frac{1}{3}(k-1)K(r)T(r)(\alpha^2 + 4\alpha\beta) + \frac{1}{4}H(r)\beta^2.
\end{aligned}$$

Clearly, one can find a positive continuous function $C(r)$ depending only on k, $T(r)$, $K(r)$, and $H(r)$ such that

$$C(r)\sum_{j=1}^k |e_j f|^2 + \sum_{j=1}^k \nabla^2 f(e_j, e_j) \geq \frac{1}{16}H(r).$$

Thus

$$\begin{aligned}
\sum_{j=1}^k \nabla^2(\chi \circ f)(e_j, e_j) &= \left[C(r)\sum_{j=1}^k |e_j f|^2 + \sum_{j=1}^k \nabla^2 f(e_j, e_j) \right] (\chi' \circ f)(q) \\
&\geq \frac{1}{16}H(r)(\chi' \circ b_p)(q) = \frac{1}{16}H(r)\chi'(r).
\end{aligned}$$

One can also check that there is a positive continuous function $A(r)$ on $[a, +\infty)$, such that

$$|\nabla^2(\chi \circ f)(v, v)| \leq A(r)|v|^2, \qquad v \in T_qM.$$

It follows from Lemma 6 that there is a positive continuous function $\varepsilon(r)$ on $[a, +\infty)$ depending only on $H(r)$ and $A(r)$, such that for all vector set $\{v_1, \dots, v_k\}$ in T_qM with $|\langle v_i, v_j\rangle - \delta_{ij}| < \varepsilon(r)$,

$$\sum_{j=1}^k \nabla^2(\chi \circ f)(v_j, v_j) \geq \frac{1}{32}H(r).$$

Since $\chi \circ f$ supports $\chi \circ b_p$ at $q \in M$, one obtains

$$\sum_{j=1}^{k} C(\chi \circ b_p)(q\,;\, v_j) \geq \frac{1}{32} H(r).$$

By definition, $\chi \circ b_p$ belongs to $C(k)$ on M. Q.E.D.

Proof of Theorem 7. It follows from Lemmas 5 and 8 that there is a smooth proper Morse function F which belongs to $C(k)$ on M. Clearly, the index of F at each critical point satisfies that $\operatorname{ind}(F) \leq k-1$. Thus Theorem 7 follows from the standard Morse theory [M2]. Q.E.D.

It is a conjecture that any complete open n-manifold of positive Ricci curvature admits a sequence of compact domains $\Omega_1 \subset \Omega_2 \subset \cdots$ such that $M = \bigcup_{i=1}^{k} \Omega_i$ and each Ω_i has smooth boundary with positive mean curvature. This conjecture is affirmative in case that M is a proper open manifold of positive Ricci curvature. In fact, we will prove the following

Theorem 8. *Let M be a complete proper n-manifold $\mathrm{Ric}_{(k)}(M) > 0$ for some $1 \leq k \leq n-1$. Then M admits a sequence of compact domains with smooth boundary $\Omega_1 \subset \Omega_2 \subset \cdots$ such that $M = \bigcup_{i=1}^{\infty} \Omega_i$ and each Ω_i has k-convex boundary.*

Proof. Since M is proper, it follows from Lemmas 5 and 8 that there is a proper smooth function $f\colon M \to \mathbf{R}$ such that f belongs to $C(k)$ on M. Choose a sequence of numbers $a_1 < a_2 < \cdots \to +\infty$ such that each a_i is a regular value of f. Set

$$\Omega_i = \{x \in M\,;\ f(x) \leq a_i\}.$$

Then each Ω_i has a smooth boundary $\partial\Omega_i$. Let $\{e_1, \ldots, e_k\}$ be any orthonormal set in $T_q\partial\Omega_i$, $q \in \partial\Omega_i$, and $\{\tilde{e}_1, \ldots, \tilde{e}_k\}$ be any extension of $\{e_1, \ldots, e_k\}$ to a set of tangent vector fields to $\partial\Omega_i$ near q. Since f belongs to $C(k)$, one has $\sum_{i=1}^{k} \nabla^2 f(e_i, e_i) > 0$. Notice that

$$\sum_{i=1}^{k} \nabla^2 f(e_i, e_i) = \sum_{i=1}^{k} \{\tilde{e}_i(\tilde{e}_i f) - (\nabla_{\tilde{e}_i}\tilde{e}_i) f\} = -\sum_{i=1}^{k} (\nabla_{\tilde{e}_i}\tilde{e}_i) f.$$

Thus $-\sum_{i=1}^{k} (\nabla_{\tilde{e}_i}\tilde{e}_i) f > 0$. Let ξ be the outward-pointing unit normal at $q \in \partial\Omega_i$. Then the second fundamental form h_ξ at q satisfies

$$\begin{aligned}\sum_{i=1}^{k} h_\xi(e_i, e_i) &= -\sum_{i=1}^{k} \langle \nabla_{\tilde{e}_i}\tilde{e}_i, \xi\rangle = -\frac{1}{\|\operatorname{grad} f\|} \sum_{i=1}^{k} \langle \nabla_{\tilde{e}_i}\tilde{e}_i, \operatorname{grad} f\rangle \\ &= -\frac{1}{\|\operatorname{grad} f\|} \sum_{i=1}^{k} (\nabla_{\tilde{e}_i}\tilde{e}_i) f > 0.\end{aligned}$$

Thus Ω_i has k-convex boundary. Q.E.D.

Corollary 3. *Let M be a complete open n-manifold with $\mathrm{Ric}_{(k)}(M) > 0$ for some $1 \leq k \leq n-1$. Suppose that for some point $p \in M$,* (1) *holds. Then M*

admits a sequence of compact domains $\Omega_1 \subset \Omega_2 \subset \cdots$ *such that* $M = \bigcup_{i=1}^{\infty} \Omega_i$ *and each* Ω_i *has* k*-convex smooth boundary.*

4. Open manifolds with curvature bounded from below

4.1. **Isotopy Lemma.** For an open manifold M, if there is a proper smooth function $f: M \to R$ such that f has no critical points outside a compact subset of M, by Morse theory, M is diffeomorphic to the interior of a compact manifold with (smooth) boundary. For a complete open riemannian manifold M, the natural candidate of such function on M is the distance function $\rho_p(x) := d(p, x)$ from a fixed point $p \in M$. It is well known in riemannian geometry that $\rho_p(x)$ is only Lipschitz continuous. The concept of critical point of $\rho_p(x)$, therefore, cannot be given in a usual way. Grove-Shiohama [GS] have made the fundamental observation that there is a meaningful definition of "critical point" for such distance function, such that in the absence of critical points, the Isotopy Lemma of Morse theory holds.

A point $q(\neq p)$ is called a critical point of ρ_p if for all unit vector v in the tangent space T_qM, there is a minimal geodesic, γ, from q to p, making an angle, $\angle(v, \dot\gamma(0)) \le \pi/2$, with $\dot\gamma(0)$. As was shown in [GS], if q is not a critical point of ρ_p, then there is a small positive number $\varepsilon > 0$, an open neighborhood U, of q, and a smooth unit vector field W on U, such that for any minimal geodesic, γ, from $x \in U$ to p, γ makes an angle, $\angle(W_x, \dot\gamma(0)) \ge \pi/2 + \varepsilon$ with W_x. By using a partition of unity and the first variation formula, one can prove the following

Isotopy Lemma [GS, G1,C]. *If* $r_1 < r_2 \le +\infty$, *and if* C *is a connected component of* $B(p, r_2)\backslash\overline{B(p, r_1)}$ *such that the closure,* $\overline{C}$, *is free of critical points of* ρ_p, *then there is a homeomorphism*

$$\psi : \Sigma \times (r_1, r_2) \to C,$$

where $\Sigma = \partial C \cap S(p, r_1)$ *is a connected boundary component of* C, *such that* $\rho_p(\psi(x, t)) = t$, *for all* $(x, t) \in \Sigma \times (r_1, r_2)$. *Moreover,* Σ *is a topological submanifold* (*without boundary*).

Remark 4. Note that the closure of $B(p, r_2)\backslash\overline{B(p, r_1)}$ is strictly contained in $\overline{B(p, r_2)}\backslash B(p, r_1)$ in certain cases. If $\overline{B(p, r_2)}\backslash B(p, r_1)$ contains no critical point of ρ_p, then this region is homeomorphic to $S(p, r_1) \times [r_1, r_2]$. Moreover, $S(p, r_1)$ is a (not necessarily connected) topological submanifold without boundary (cf. e.g. [C]).

The Isotopy Lemma above, in particular, implies that if a complete open riemannian manifold M does not contain critical points of ρ_p outside a compact subset, then M has finite topological type. For our purpose we need the following

Lemma 9 (cf. e.g. [C]). *Let* M *be complete, and let* $p \in M$ *be fixed. Suppose that there is* $R_0 > 0$ *such that for all* $r \ge R_0$, *all boundary components,* Σ, *of* $M\backslash\overline{B(p, r)}$ *with* $\Sigma \cap R(p, r) \neq \varnothing$, *are free of critical points of* ρ_p. *Then there is a* $R_1 \ge R_0$, *such that* $M\backslash B(p, R_1)$ *is free of critical points of* ρ_p. *In particular,* M *has finite topological type.*

Outline of the proof. Follow [C], let U be any unbounded connected component of $M\backslash\overline{B(p, R_0)}$. Let Σ_{R_0} be a boundary component of ∂U with

$\Sigma_{R_0} \cap R(p, r) \neq \varnothing$. One can take a ray γ emanating from p with $\gamma(R_0) \in \Sigma_{R_0}$. For $r \geq R_0$, let Σ_r denote the boundary component of $M \backslash \overline{B(p, r)}$ with $\gamma(r) \in \Sigma_r$. By assumption, all Σ_r, $r \geq R_0$, are free of points of ρ_p. By the same argument for the Isotopy Lemma, one can show that there is an embedding $\psi : \Sigma_{R_0} \times (R_0, +\infty) \to U$ such that $\psi(\Sigma_{R_0} \times \{r\}) = \Sigma_r$. It is also easy to see ψ is onto. Thus $\psi(\Sigma_{R_0} \times (R_0, +\infty)) = U$. Clearly there are only finitely many bounded connected components, V, of $M \backslash \overline{B(p, R_0)}$ with $V \cap S(p, 2R_0) \neq \varnothing$. Thus there is $R_1 \geq R_0$, such that $M \backslash B(p, R_1)$ is free of critical points of ρ_p. Q.E.D.

4.2. **Small excess and finite topological type.** In order to prove a complete riemannian manifold M has a finite topological type, one needs to show that there is no critical points outside a compact subset with respect to a fixed point $p \in M$. For this purpose we will prove an important lemma, which tells us that the value of excess function e_p at critical points cannot be small. In [AG] Abresch-Gromoll have proved a similar result for the excess of a thin triangle in M. The following lemma gives us a concrete information for the excess function e_p which can be thought of as the excess of the triangle with one vertice at infinity.

Lemma 10. *Let M be a complete open riemannian manifold with sectional curvature $K_M \geq -K$ for some constant $K > 0$, and let $p \in M$ be fixed. Suppose that $q \neq p$ is a critical point of p. Then*

$$e_p(q) \geq \frac{1}{\sqrt{K}} \ln \frac{\exp \sqrt{K} \rho_p(q)}{\cosh \sqrt{K} \rho_p(q)}. \tag{14}$$

Proof. Take an arbitrary sequence $t_n \to +\infty$ so that $b_p^{t_n}(x) = t_n - d(x, S(p, t_n))$ converges to $b_p(x)$ on M. There is $x_n \in S(p, t_n)$ such that $d(q, x_n) = d(q, S(p, t_n))$. Take a minimal geodesic γ issuing from p to x_n, and a minimal geodesic σ issuing from q to x_n. Since q is a critical point of p, there exists a minimal geodesic τ issuing from q to p such that $\dot\sigma(0)$ and $\dot\tau(0)$ make an angle at most $\pi/2$. Apply Toponogov's Theorem [CE] to the triangle formed by γ, σ, and τ, we obtain

$$\cosh \sqrt{K} t_n \leq \cosh \sqrt{K} d(q, x_n) \cosh \sqrt{K} d(p, q). \tag{15}$$

Multiplying (15) by $2 \exp \sqrt{K}(d(p, q) - t_n)$, and letting $t_n \to +\infty$, we obtain

$$\exp \sqrt{K} \rho_p(q) \leq \exp \sqrt{K} e_p(q) \cosh \sqrt{K} \rho_p(q). \tag{16}$$

Then Lemma 10 follows from (16). Q.E.D.

Proof of Theorem 5. By (13), for sufficiently large r, $\mathscr{D}(p, r) < \ln 2 \cdot K^{-1/2}$ implies that $R_p(x) < \ln 2 \cdot K^{-1/2}$ for x in all boundary components, Σ, of $M \backslash \overline{B(p, r)}$ with $\Sigma \cap R(p, r) \neq \varnothing$. Since $e_p(x) \leq R_p(x)$ (5), then by Lemma 10, one concludes that all such Σ are free of critical points of ρ_p. Theorem 5 therefore follows from Lemma 9. Q.E.D.

Next we will give an upper estimate for the Betti numbers of complete open riemannian manifolds with small excess at infinity. Let $H_k(X; \mathbf{F})$ denote the kth singular homology group of a subset X in a riemannian manifold M, where $\mathbf{F}$ is any fixed field. For any two subsets $i: X \subset Y \subset M$, let $b_k(X, Y)$ denote the rank of $i_*: H_k(X; \mathbf{F}) \hookrightarrow H_k(Y; \mathbf{F})$, and $b_k(X) = b_k(X, X) = \dim H_k(X; \mathbf{F})$. Notice that for subsets $X \subset \widetilde{X} \subset \widetilde{Y} \subset Y$ in M, $b_k(X, Y) \leq b_k(\widetilde{X}, \widetilde{Y})$.

Theorem 9. *Given n, $K > 0$, $D > 0$, $\theta < \ln 2$. There is a constant $C = C(n, K, D, \theta) > 0$ such that if a complete open n-manifold M satisfies the bounds*:

(1) $K_M \geq -K$,

(2) *for some $p \in M$ and all $r \geq D$,*

$$\mathscr{E}(p, r) < \frac{1}{\sqrt{K}}\theta,$$

or

$$\mathscr{R}(p, r) < \frac{1}{\sqrt{K}}\theta,$$

or

$$\operatorname{dia}(S(p, r)) < \frac{1}{\sqrt{K}}\theta.$$

Then M has finite topological type and its total Betti number satisfies

$$\sum_{i=0}^{n} b_i(M) \leq C.$$

Proof. By Lemma 10, there is $D_0 = D_0(K, D, \theta) > 0$ such that $M \backslash B(p, D_0)$ is free of critical points of ρ_p. Thus by the Isotopy Lemma and its remark, M and $B(p, D_0 + 1)$ are homeomorphic to $B(p, D_0)$, respectively. Thus by Theorem 11 below

$$\begin{aligned}\sum_{k=0}^{n} b_k(M) &= \sum_{k=0}^{n} b_k(B(p, D_0)) = \sum_{k=0}^{n} b_k(B(p, D_0), B(p, D_0 + 1)) \\ &\leq \sum_{k=0}^{n} b_k(B(p, D_0), T_1 B(p, D_0)) \leq C(n, K, D, \theta). \quad \text{Q.E.D.}\end{aligned}$$

5. Open manifolds with nonnegative kth-Ricci curvature

5.1. **Better estimates of excess functions.** Recall that we always have the estimate $e_p(x) \leq R_p(x)$, $x \in M$ (5). No conditions on curvature are required in this case. The basic idea of the proof of Theorem 3 is to find a better estimate of $e_p(x)$ in terms of $R_p(x)$ if $\operatorname{Ric}_{(k)}(M) \geq 0$. We start with the following

Lemma 11. *Let M be a complete n-manifold with $\operatorname{Ric}_{(k)}(M) \geq 0$ for some k, $1 \leq k \leq n-1$. Let C_p be the cut locus of p. Then the distance function ρ_p is smooth at any point $x \in \Omega_p := M \backslash C_p \cup \{p\}$ and for any orthonormal set $\{e_1, \ldots, e_{k+1}\}$ in $T_x M$ with* $\operatorname{grad} \rho_p(x) \in \operatorname{span}\{e_1, \ldots, e_{k+1}\}$,

$$\sum_{j=1}^{k+1} \nabla^2 \rho_p(e_j, e_j) \leq \frac{k}{\rho_p(x)}.$$

Proof. The proof is quite standard. Let γ be the minimal normal geodesic issuing from p with $\gamma(r) = x$, $r = \rho_p(x)$. For each $u \in T_x M$, let $u(t)$ be the parallel vector field along γ with $u(r) = u$. Then define $\theta\colon T_xM \times [0, r] \to M$ as

$$\theta(u, t) = \exp_{\gamma(t)} \frac{t}{r} u(t).$$

Set

$$f(u) = \int_0^r \left| \frac{\partial \theta}{\partial t}(u, t) \right| dt.$$

Clearly, $\rho_p \circ \exp_x^{-1}$ supports f at $u = 0$, i.e., $\rho_p \circ \exp_x^{-1} u \le f(u)$, for all u close to 0, and the equality holds at $u = 0$. Hence by the second variation formula [CE] one obtains

$$\begin{aligned} \nabla^2 \rho_p(u, u) \le \frac{d^2}{ds^2} f(su)|_{s=0} &= \frac{1}{r}(1 - \langle u, \operatorname{grad} \rho\rangle^2) \\ &\quad - \int_0^r \left(\frac{s}{r}\right)^2 \langle R(u(t), \dot\gamma(t))\dot\gamma(t), u(t)\rangle\, dt. \end{aligned}$$

Thus

$$\sum_{j=1}^{k+1} \nabla^2 \rho_p(e_j, e_j) \le \frac{1}{r}\left(k + 1 - \sum_{i=1}^{k+1} \langle e_j, \operatorname{grad} \rho_p\rangle^2\right) = \frac{k}{r}. \quad \text{Q.E.D.}$$

By using a modification of the argument given in [AG], we can prove the following

Lemma 12. *Let M be a complete open n-manifold with $\operatorname{Ric}_{(k)}(M) \ge 0$ for some $1 \le k \le n - 1$. Then for all $x \in M$,*

$$e_p(x) \le 4R_p(x) \left(\frac{R_p(x)}{\rho_p(x)}\right)^{1/k}.$$

Proof. In case of $k = 1$, M has nonnegative sectional curvature. By Toponogove Theorem (cf. [CE]), one can easily obtain

$$e_p(x) \le \frac{1}{2} \frac{R_p(x)^2}{\rho_p(x)}.$$

For $2 \le k \le n - 1$ and $r > 0$, set

$$\varphi_r(t) = \frac{1}{(k-1)(k+1)}(t^{1-k} - r^{1-k})r^{k+1} + \frac{1}{2(k+1)}(t^2 - r^2).$$

It is easy to check that

(a) $\varphi_r''(t) + (k/t)\varphi'(t) = 1$,

(b) $\varphi_r'(t) < 0$ for $0 < t < r$,

(c) $\varphi_r(r) = 0$.

Now fix a point $x \in M$. Take $C = 2k\rho_p(x)^{-1}$ and $l = R_p(x)$.

First we assume that $R_p(x) < \frac{1}{2}\rho_p(x)$. Take any r with $R_p(x) < r < \frac{1}{2}\rho_p(x)$. Define $f\colon \overline{B(x, r)} \to \mathbf{R}$ as

$$f(y) = C\varphi_r(d(x, y)) - e_p(y), \qquad y \in \overline{B(x, r)},$$

where $e_p(y) = d(p, y) - b_p(y)$ is the excess function at p. We claim that f has no locally maximal point in $B(x, r)\backslash\{x\}$. We will prove it by contradiction. Suppose f has a locally maximal value at some point $x_0 \in B(x, r)\backslash\{x\}$. Take a normal minimal geodesic γ issuing from p to x_0 and a normal minimal geodesic τ issuing from x to x_0. By triangle inequality one can prove that $-\varepsilon - d(\cdot, \gamma(\varepsilon))$ supports $-d(\cdot, p)$ at x_0 and $-\varepsilon - d(\cdot, \tau(\varepsilon))$ supports $-d(\cdot, x)$ at x_0, respectively. By Lemma 2, there is a ray σ_{x_0} issuing from x_0 such that $b_p^{x_0, t}(y) := b_p(x_0) + t - d(y, \sigma_{x_0}(t))$ supports $b_p(y)$ at x_0. Therefore for small $\varepsilon > 0$,

$$f_\varepsilon(y) := C\varphi_r(\varepsilon + d(y, \tau(\varepsilon))) + b_p(x_0) + \frac{1}{\varepsilon} - d\left(y, \sigma_{x_0}\left(\frac{1}{\varepsilon}\right)\right) - \varepsilon - d(y, \gamma(\varepsilon))$$

is smooth near x_0 and supports $f(y)$ at x_0. Thus f_ε is locally maximal at x_0 and for all $v \in T_{x_0}M$,

$$\nabla^2 f_\varepsilon(v, v) \leq 0.$$

Let $\{e_1, \dots, e_{k+1}\}$ be and arbitrarily orthonormal set in $T_{x_0}M$ such that $\dot{\gamma}(d(p, x_0))$, $\dot{\tau}(d(x, x_0))$, and $\dot{\sigma}_{x_0}(0)$ is in $\operatorname{span}\{e_1, \dots, e_{k+1}\}$. By Lemma 11, one has

$$\begin{aligned} 0 \geq \sum_{i=1}^{k+1} \nabla^2 f_\varepsilon(e_i, e_i) &\geq C\left[1 + \varphi_r'(d(x, x_0))\left(\frac{k}{d(x_0, \tau(\varepsilon))} - \frac{k}{d(x, x_0)}\right)\right] \\ &\quad - k\varepsilon - \frac{k}{d(x_0, \gamma(\varepsilon))}. \end{aligned} \tag{17}$$

Since $kd(x_0, p)^{-1} \leq k(\rho_p(x) - r)^{-1} < 2k\rho_p(x)^{-1} = C$, the right side of (17) is positive for sufficiently small $\varepsilon > 0$. It is a contradiction. Therefore one concludes that f has no locally maximal point in $B(x, r)$. Take $z \in R(p, r_0)$ with $d(x, z) = R_p(x)$ (hence $e_p(z) = 0$), where $r_0 = \rho_p(x)$. Clearly, f does not achieve the maximum on $S(p, r)$. Then for any ρ, $0 < \rho < l = R_p(x)$,

$$0 < f(z) \leq \max_{y \in S(x, \rho)} f(y) = C\varphi_r(\rho) - \min_{y \in S(x, \rho)} e_p(y),$$

which implies

$$e_p(x) \leq \min_{y \in S(x, \rho)} e_p(y) + 2\rho \leq 2\rho + C\varphi_r(\rho).$$

Letting $r \to l = R_p(x)$, one obtains

$$e_p(x) \leq \min_{0<\rho<l} (2\rho + C\varphi_l(\rho)).$$

Notice that $h(\rho) := 2\rho + C\varphi_l(\rho)$ satisfies that

$$\lim_{\rho \to 0^+} h(\rho) = +\infty \quad \text{and} \quad \lim_{\rho \to +\infty} h(\rho) = +\infty.$$

Then $h(\rho)$ has a minimal point $\rho_0 \in (0, +\infty)$.

$$h'(\rho_0) = 2 + \frac{C}{k+1}(\rho_0 - \rho_0^{-k} l^{k+1}) = 0. \tag{18}$$

It follows from (18) that

$$\rho_0 < \left(\frac{C}{2(k+1)} l^{k+1}\right)^{1/k}, \tag{19}$$

and

$$\rho_0 < l. \tag{20}$$

By (18), (19) and (20), one obtains

$$\begin{aligned} e_p(x) &\le \frac{2k}{k-1}\rho_0 + \frac{C}{2(k-1)}(\rho_0^2 - l^2) \le \frac{2k}{k-1}\rho_0 \\ &\le \frac{2k}{k-1}\left(\frac{C}{2(k+1)} l^{k+1}\right)^{1/k} \\ &= \frac{2k}{k-1}\left[\frac{2k}{2(k+1)} \cdot \frac{R_p(x)^{k+1}}{\rho_p(x)}\right]^{1/k} \\ &\le 4R_p(x)\left(\frac{R_p(x)}{\rho_p(x)}\right)^{1/k}. \end{aligned}$$

Second, if $R_p(x) \ge \frac{1}{2}\rho_p(x)$, then by (5), one obtains

$$e_p(x) \le R_p(x) \le 4R_p(x)\left(\frac{R_p(x)}{\rho_p(x)}\right)^{1/k}. \quad \text{Q.E.D.}$$

Remark 5. Let M be as in Lemma 12. Let $e_{p,q}(x) := d(p, x) + d(q, x) - d(p, q)$ denote the excess of a triangle formed by p, q, x in M, by the same argument as above, one can show that

$$e_{p,q}(x) \le 8h(x)\left(\frac{h(x)}{s(x)}\right)^{1/k},$$

where $s(x) = \min\{d(p, x), d(q, x)\}$ and $h(x)$ equals the distance between x and a minimal geodesic joining p and q.

5.2. **Small diameter growth and finite topological type.**

Proof of Theorem 3. By (13), for sufficiently large r, the assumption of Theorem 3 implies that

$$\frac{R_p(x)}{\rho_p(x)^{1/(k+1)}} < \frac{1}{8}K^{k/2(k+1)},$$

for x in any boundary component, Σ, of $M\backslash\overline{B(p, r)}$ with $\Sigma \cap R(p, r) \ne \varnothing$. It follows from Lemmas 10 and 12 that all such components Σ are free of critical points of ρ_p. Then Theorem 3 follows from Lemma 9. Q.E.D.

The same argument as in Theorem 9 also gives the following

Theorem 10. *Given n, $K>0$ and $D>0$. There is a constant $C=C(n, K, D)>0$, such that if a complete open n-manifold M satisfies the bounds*:

(1) $K_M \ge -K$,
(2) $\mathrm{Ric}_{(k)}(M) \ge 0$ *for some* $2 \le k \le n-1$,
(3) *for some* $p \in M$,

$$\frac{\mathscr{R}(p, r)}{r^{1/(k+1)}} < \frac{1}{8}K^{k/2(k+1)}, \qquad r \ge D,$$

or

$$\frac{\mathrm{dia}(S(p, r))}{r^{1/(k+1)}} < \frac{1}{8}K^{k/2(k+1)}, \qquad r \ge D.$$

Then M has finite topological type, and its total Betti number satisfies

$$\sum_{k=0}^{n} b_k(M) \leq C.$$

6. Betti numbers and nonnegative Ricci curvature

In this section, we will study the "topological growth" of the geodesic balls in complete open manifolds of nonnegative Ricci curvature. First let us recall Gromov's theorem [G1]. One can also refer to [A] for the details.

Theorem 11 (Gromov [G1]). *Let M be an n-dimensional complete riemannian manifold with sectional curvature $K_M \geq -1$. Then there is a constant $C(n) > 1$ depending only on n such that for any $0 < \varepsilon \leq 1$ and any bounded subset $X \subset M$,*

$$\sum_{k=0}^{n} b_k(X, T_\varepsilon X) \leq (1 + \mathrm{dia}(X)\varepsilon^{-1})^n C(n)^{1+\mathrm{dia}(X)},$$

$T_\varepsilon X$ denotes the ε-neighborhood of X in M.

This theorem, in particular, tells us that for a complete n-manifold M with sectional curvature $K_M \geq -1$, the total Betti number of the geodesic balls in M has at most exponential growth, more precisely, for all $r > 0$,

$$\sum_{k=0}^{n} b_k(p, r) \leq C(n)^{1+r},$$

where $b_k(p, r) = b_k(B(p, r), M)$, the rank of the natural inclusion

$$i_*: H_k(B(p, r), \mathbf{F}) \to H_k(M, \mathbf{F}).$$

What we will show in this section is, if in addition, M has $\mathrm{Ric}(M) \geq 0$, then the total Betti number of M has polynomial growth of degree n.

Proof of Theorem 2. Let B be any ball in M with radius r and let $\rho > 1$. Denote by ρB the concentric ball of B with radius ρr. By Theorem 11, there is a constant $C_1(n)$ depending only on n such that for all balls B with radius $r \leq 1$ in M,

$$\sum_{i=0}^{n} b_i(B, 5B) \leq C_1(n). \tag{21}$$

The rest of the proof will rely on the following topological lemma which was proved by Gromov [G1].

Lemma 13 [G1, A]. *Let M be a complete riemannian n-manifold and let $p \in M$. For any fixed numbers $r > 0$ and $r_0 \leq 7^{-n-1}$, let $B_j^0 = B(p_j, r_0)$, $j = 1, \dots, N$, be a ball covering of $B(p, r)$ with $p_j \in B(p, r)$. Let $B_j^k = 7^k B_j^0$, $k = 0, \dots, n+1$. Then*

$$\sum_{i=0}^{n} b_i(B(p, r), B(p, r+1))$$
$$\leq (e-1)Nt^n \sup\left\{\sum_{i=0}^{n} b_i(B_j^k, 5B_j^k);\ 0 \leq k \leq n,\ 1 \leq j \leq N\right\},$$

where t is the smallest number such that each ball B_j^n intersects at most t other balls $B_{j'}^n$.

Take $r_0 = 7^{-n-1}$, and let $B(p_j, \frac{1}{2}r_0)$, $j = 1, \ldots, N$, be a maximal set of disjoint balls with $p_j \in B(p, r)$. Then $B_j^0 := B(p_j, r_0)$, $j = 1, \ldots, N$, is a covering of $B(p, r)$. By Bishop-Gromov's volume comparison theorem, one obtains

$$N \leq \left(1 + 4\frac{r}{r_0}\right)^n \leq 4^n 7^{n^2+n}(1+r)^n.$$

Let $B_j^k = 7^k B_j^0$, $k = 0, \ldots, n+1$. Assume that B_j^n intersects t other balls $B_{j'}^n$. By the same volume comparison argument, one obtains $t \leq 5^n$. Since each ball B_j^k has radius ≤ 1, it follows from (21) and Lemma 13 above that

$$\sum_{i=0}^n b_i(B(p, r), M) \leq \sum_{i=0}^n b_i(B(p, r),\ B(p, r+1)) \leq C(n)(1+r)^n. \quad \text{Q.E.D.}$$

This theorem gives a topological obstruction to complete open manifolds M with nonnegative Ricci curvature and sectional curvature bounded from below.

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