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COMPLEX GEODESICS AND ITERATES OF HOLOMORPHIC MAPS ON CONVEX DOMAINS IN $\mathbf{C}^n$

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ABSTRACT. We study complex geodesics $f: \Delta \to \Omega$, where Δ is the unit disk in $\mathbf{C}$ and Ω belongs to a class of bounded convex domains in $\mathbf{C}^n$ with no boundary regularity assumption. Along with continuity up to the boundary, existence of such complex geodesics with two prescribed values $z, w \in \overline{\Omega}$ is established. As a consequence we obtain some new results from iteration theory of holomorphic self maps of bounded convex domains in $\mathbf{C}^n$.

0. INTRODUCTION

Let Ω be a domain in $\mathbf{C}^n$. Ω is convex if for each pair of points $x, y \in \Omega$, the segment $L_{xy} = \{(1-t)x + ty: 0 \le t \le 1\}$ is contained in Ω. Ω is strictly convex if for each pair of points $x, y \in \overline{\Omega}$, the segment $\overset{\circ}{L}_{xy} = \{(1-t)x+ty: 0 < t < 1\}$ is contained in Ω. Ω is strongly convex if it has C^2 boundary and a defining function with positive definite real Hessian. Finally, Ω is strictly linearly convex [L2] if: (i) it has C^2 boundary, (ii) through each boundary point $p \in \partial\Omega$ there passes a complex hyperplane which is disjoint from Ω, and (iii) it retains properties (i) and (ii) under small C^2 perturbations. We have the following implications: strongly convex $\Rightarrow$ strictly convex $\Rightarrow$ convex, and strongly convex $\Rightarrow$ strictly linearly convex. There is no such relationship between the convex domains and the strictly linearly convex domains (for example strictly linearly convex does not, in general, imply convex).

Let Δ denote the unit disk in $\mathbf{C}$. A complex geodesic is a mapping $f \in \mathrm{HOL}(\Delta, \Omega)$ which preserves the Kobayashi distance between each pair of points in Δ (see §1 for precise definitions).

Theorem 0.1. *Let $\Omega \subset \mathbf{C}^n$ be strictly linearly convex. We have*

(i) [L2] *For each pair of points $z, w \in \Omega$ there is a complex geodesic $f \in \mathrm{HOL}(\Delta, \Omega)$ which contains $\{z, w\}$ in its image.*

(ii) [L2] *The mapping f above extends to a continuous mapping $\hat{f}: \overline{\Delta} \to \overline{\Omega}$, with $\hat{f}(\partial\Delta) \subset \partial\Omega$.*

(iii) [CHL] *For each pair of points $z, w \in \overline{\Omega}$ there is a complex geodesic $f \in \mathrm{HOL}(\Delta, \Omega)$ whose continuous extension $\hat{f}$ contains $\{z, w\}$ in its image.*

In this article we study a class of convex domains which we call the m-convex domains (Definition 2.6). This class contains the strongly convex domains but

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requires no boundary regularity. In this case an analogue of Theorem 0.1(i) is immediate, thanks to

Theorem 0.2 [RW]. *Let $\Omega \subset\subset \mathbf{C}^n$ be convex. For each pair of points $z, w \in \Omega$ there is a complex geodesic $f \in \mathrm{HOL}(\Delta, \Omega)$ which contains $\{z, w\}$ in its image.*

After fixing some notation and definitions (§1) we obtain analogues of Theorem 0.1(ii) and (iii) for the class of m-convex domains (§§2 and 3). Finally (in §4) we apply our work to obtain some new results from iteration theory of holomorphic self-maps of convex domains in $\mathbf{C}^n$.

The content of this article will constitute part of the author's doctoral thesis. The author is grateful to Ian Graham, his thesis advisor, for numerous useful discussions and suggestions.

1. Preliminaries

In this section we recall some standard definitions to be used in the sequel. Much of the terminology is taken from [A3]. Ω denotes a domain (= connected open set) and Δ denotes the unit disk in $\mathbf{C}$.

Definition 1.1. Let $\Omega \subset \mathbf{C}^n$. The Kobayashi metric $\kappa_\Omega \colon T(\Omega) \to \mathbf{R}^+$ is given by

$$\kappa_\Omega(z; v) = \inf\{|u| \colon \exists F \in \mathrm{HOL}(\Delta, \Omega) \text{ such that } F(0) = z,\ dF_0(u) = v\}.$$

Definition 1.2. Let $\Omega \subset \mathbf{C}^n$. The Kobayashi distance $k_\Omega \colon \Omega \times \Omega \to \mathbf{R}^+$ is given by

$$k_\Omega(z, w) = \inf\left\{ \int_0^1 \kappa_\Omega(\gamma(t); \dot\gamma(t))\, dt \colon \gamma \colon [0, 1] \to \Omega \text{ is a } C^1 \text{ curve} \right.$$
$$\left. \text{with } \gamma(0) = z,\ \gamma(1) = w \right\}.$$

General properties of κ_Ω and k_Ω may be found, for example, in [K1, K2, or KR]. A result of Lempert (Théorème 1 in [L1]) asserts that if $\Omega \subset \mathbf{C}^n$ is convex then

$$k_\Omega(z, w) = \inf\{\rho_\Delta(0, \lambda) \colon \exists F \in \mathrm{HOL}(\Delta, \Omega) \text{ such that } F(0) = z,\ F(\lambda) = w\},$$

where

$$\rho_\Delta(\mu, \lambda) = \frac{1}{2} \log \frac{|1 - \overline{\mu}\lambda| + |\lambda - \mu|}{|1 - \overline{\mu}\lambda| - |\lambda - \mu|}$$

is the hyperbolic or Poincaré distance on Δ (see [VE] for example). Clearly then $k_\Delta = \rho_\Delta$.

Definition 1.3 [V]. Let $\Omega \subset \mathbf{C}^n$. A complex geodesic is a mapping $f \in \mathrm{HOL}(\Delta, \Omega)$ such that

$$\rho_\Delta(\lambda, \mu) = k_\Omega(f(\lambda), f(\mu)) \quad \forall \lambda, \mu \in \Delta.$$

Corollary 3 of [RW] asserts that if $\Omega \subset\subset \mathbf{C}^n$ is convex and $f \in \mathrm{HOL}(\Delta, \Omega)$ is a complex geodesic then

$$\kappa_\Delta(\lambda; v) = \kappa_\Omega(f(\lambda); f'(\lambda)v) \quad \forall (\lambda, v) \in T(\Delta).$$

Proposition 3.3 of [V] and Theorem 2 of [RW] together imply that if Ω is convex then for $f \in \mathrm{HOL}(\Delta, \Omega)$ to be a complex geodesic it is sufficient that f preserves the Kobayashi distance between just one pair of distinct points in Δ. A complex geodesic composed with an automorphism of Δ is again a complex geodesic (and the image set is unchanged). Such a composition is called a reparameterization.

We fix some further notation. For $z \in \Omega \subset \mathbf{C}^n$, denote by $d_\Omega(z)$ the Euclidean distance from z to $\partial\Omega$. For $\Omega \subset\subset \mathbf{C}^n$ convex, $z \in \Omega$, and $v \in \mathbf{C}^n$, denote by $r_\Omega(z; v)$ the radius of the largest one complex dimensional closed disk, centered at z, tangent to v, and contained in $\overline{\Omega}$. Clearly $r_\Delta(\lambda; v) = 1-|\lambda|$ $\forall \lambda \in \Delta$ and nonzero $v \in \mathbf{C}$.

2. Complex geodesics and continuity at the boundary

In this section we obtain some new boundary estimates for the Kobayashi distance on convex bounded domains in $\mathbf{C}^n$. We define m-convex and use these estimates to obtain the appropriate analogue of Theorem 0.1(ii).

For $\alpha > 1$ denote by Λ_α the image of Δ under the mapping $\lambda \mapsto (\lambda+1)^{1/\alpha}$. For $\alpha = 2$, Λ_α is the interior of one loop of a lemniscate.

Lemma 2.1. *Fix $z_0 \in \Lambda_\alpha$. There is a $C > 0$ such that*

$$k_{\Lambda_\alpha}(z_0, z) \leq C - \tfrac{1}{2}\log d^\alpha_{\Lambda_\alpha}(z) \quad \forall z \in \Lambda_\alpha.$$

Proof. By the triangle inequality we may assume that $z_0 = 1$. The function $f\colon \Delta \to \Lambda_\alpha$ given by $z = f(\lambda) = (\lambda+1)^{1/\alpha}$ defines a biholomorphism, so by the invariance property of the Kobayashi distance we have

$$k_{\Lambda_\alpha}(1, z) = \rho_\Delta(0, z^\alpha - 1) = \frac{1}{2}\log\frac{1+|z^\alpha - 1|}{1-|z^\alpha - 1|} \quad \forall z \in \Lambda_\alpha.$$

Thus it suffices to show that $d^\alpha_{\Lambda_\alpha}(z) \leq 1 - |z^\alpha - 1|$ $\forall z \in \Lambda_\alpha$.

If $\lambda = 0$ then $z = 1$ and $d_{\Lambda_\alpha}(z) = 2^{1/\alpha} - 1$, so the inequality holds. If $\lambda \neq 0$ then $d_\Delta(\lambda) = 1 - |\lambda|$ is the Euclidean length of the segment γ parametrized by $\gamma = t\lambda$, $1 \leq t \leq 1/|\lambda|$. Let $L = f(\gamma)$ and denote by $|L|$ the Euclidean length of L. We have then

$$\begin{aligned} |L| &= \int_\gamma \left|\frac{df}{d\lambda}\right| |d\lambda| = \int_1^{1/|\lambda|} \frac{|\lambda| dt}{\alpha|t\lambda + 1|^{1-1/\alpha}} \\ &\leq \int_1^{1/|\lambda|} \frac{|\lambda|\, dt}{\alpha(1 - t|\lambda|)^{1-1/\alpha}} = (1-|\lambda|)^{1/\alpha} = (1 - |z^\alpha - 1|)^{1/\alpha}. \end{aligned}$$

We observe that $d_{\Lambda_\alpha}(z) \leq |L|$ and the lemma is proved. □

Lemma 2.2. *Let $\Omega \subset \mathbf{C}$ be convex and such that $0 \in \partial\Omega$, $\Lambda_\alpha \subset \Omega$, and $\overline{\Lambda_\alpha} \cap \partial\Omega = \{0\}$. Then for any $\delta > 0$ (small) there is a $\nu > 0$ such that whenever $0 < t < \delta$, we have $\nu d_\Omega(t) \leq d_{\Lambda_\alpha}(t) \leq d_\Omega(t)$.*

Proof. The right-hand inequality is obvious, since $\Lambda_\alpha \subset \Omega$. Set $\Sigma_r = \{\lambda \in \mathbf{C}\colon -\pi/2\alpha < \arg\lambda < \pi/2\alpha, |\lambda| < r\}$. By increasing α (and thus decreasing d_{Λ_α}) if necessary, we may assume that there is an $r_0 > 0$ such that $\Sigma = \Sigma_{r_0} \subset \Omega$. If $t < r_0$ then $t \in \Omega \cap \Sigma \cap \Lambda_\alpha$. Now $d_{\Lambda_\alpha}(t)/d_\Sigma(t) \nearrow 1$ as $t \to 0$, and $d_\Sigma(t) = t\sin(\pi/2\alpha)$ so if t is small enough we can choose a $\nu \in (0, \sin(\pi/2\alpha))$ such that $d_{\Lambda_\alpha}(t) \geq \nu t \geq \nu\, d_\Omega(t)$ for $0 < t < \delta$ and the lemma is proved. □

Now in order to estimate k_Ω for some general domains in $\mathbf{C}^n$, consider the following construction. We can rescale Λ_α by a factor of $r > 0$ and linearly embed the result in $\mathbf{C}^n$ to obtain the set $\Lambda(p, \theta, q, r)$, where p and q are the images of $0, 1, \in \overline{\Lambda}_\alpha$ respectively and $\theta = \pi/\alpha \in (0, \pi)$. We call $\Lambda(p, \theta, q, r)$ a lemniscate with vertex p, aperture θ, centre q and radius r. We call the real segment from p to q the axis of $\Lambda(p, \theta, q, r)$.

Let $\Omega \subset\subset \mathbf{C}^n$ be convex and fix $z_0 \in \Omega$. For each $p \in \partial\Omega$ there is a $\theta \in (0, \pi)$ and an $r > 0$ such that $\Lambda(p, \theta, q, r) \subset \Omega$, where q lies on the real segment from z_0 to p. Now $\Lambda = \Lambda(p, \theta, q, r)$ considered simply as a set of points may be contained in any 2 real dimensional affine subspace containing z_0 and p. In any case, since Ω is bounded we may assume that θ and r are independent of $p \in \partial\Omega$. For our construction, we consider Λ as a complex linear image of Δ, so that Λ necessarily lies in a particular 2 real dimensional (=1 complex dimensional) affine subspace determined by the complex structure of $\mathbf{C}^n$. We say that $\partial\Omega$ is lined with lemniscates when referring to this construction.

Proposition 2.3. *Let $\Omega \subset\subset \mathbf{C}^n$ be convex and fix $z_0 \in \Omega$. There is a $C > 0$ and an $\alpha > 1$ such that*

$$k_\Omega(z_0, z) \leq C - \tfrac{1}{2}\log d_\Omega^\alpha(z) \qquad \forall z \in \Omega.$$

Proof. It suffices to prove the proposition for points $z \in \Omega$ near $\partial\Omega$. We line $\partial\Omega$ with lemniscates and observe that there is a $\rho > 0$, $\rho < r$ such that if $d_\Omega(z) < \rho$ then z lies on the axis of a lemniscate $\Lambda = \Lambda(p, \theta, q, r)$. We may assume that $d_\Omega(q) > \rho$. Set

$$C_1 = \sup\{k_\Omega(x, y) \colon d_\Omega(x) > \rho,\ d_\Omega(y) > \rho\}.$$

By the triangle inequality and the distance decreasing property we have

$$k_\Omega(z_0, z) \leq k_\Omega(z_0, q) + k_\Omega(q, z) \leq C_1 + k_\Lambda(q, z).$$

By Lemma 2.1 there is a $C_2 > 0$ and an $\alpha > 1$ such that

$$k_\Lambda(q, z) \leq C_2 - \tfrac{1}{2}\log d_\Lambda^\alpha(z).$$

Finally, by the lemniscate construction and Lemma 2.2 there is a $C_3 > 0$ such that

$$-\tfrac{1}{2}\log d_\Lambda^\alpha(z) \leq C_3 - \tfrac{1}{2}\log d_\Omega^\alpha(z).$$

Replacing $C_1 + C_2 + C_3$ with C, the proposition is proved. $\square$

Proposition 2.4. *Let $\Omega \subset\subset \mathbf{C}^n$ be convex and fix $z_0 \in \Omega$. There is a $C \in \mathbf{R}$ such that*

$$C - \tfrac{1}{2}\log d_\Omega(z) \leq k_\Omega(z_0, z) \qquad \forall z \in \Omega.$$

Proof. We begin by noting that the result is true if Ω is any simply-connected proper subdomain of $\mathbf{C}$ (see Lemma 2.1 in [GP]). Now for $z \in \Omega$, let p be a point of $\partial\Omega$ of minimum distance from z. Since Ω is convex there is a $(2n-1)$ real dimension supporting hyperplane H_p to Ω at p, with $p \in H_p \cap \partial\Omega$. Up to a rotation/translation we may assume that $p = 0$ and $H_0 = \{w = (w^1, \dots, w^n) \in \mathbf{C}^n \colon \operatorname{Re} w^1 = 0\}$. Then (say) $\operatorname{Re} w^1 > 0\ \forall w \in \Omega$ and $z = (x^1, 0, \dots, 0)$ with $x^1 > 0$. Define $f \colon \Omega \to \mathbf{C}$ by $f(w) = w^1$, then

$f(\Omega) \subset \Pi = \{\lambda \in \mathbf{C}:\ \operatorname{Re}\lambda > 0\}$. By the distance decreasing property, the above note, and the choice of p we have then

$$\begin{aligned} k_\Omega(z_0, z) &\geq k_\Pi(z_0^1, z^1) \geq C - \tfrac{1}{2}\log d_\Pi(z^1) \\ &= C - \tfrac{1}{2}\log|z^1| = C - \tfrac{1}{2}\log d_\Omega(z). \quad \square \end{aligned}$$

Proposition 2.5. *Let $\Omega \subset\subset \mathbf{C}^n$ be convex and let $f \in \mathrm{HOL}(\Delta, \Omega)$ be a complex geodesic with $z_0 = f(0) \in \Omega$. There is an $\alpha > 1$ and constants $C_1, C_2 > 0$ such that*

$$C_1(1 - |\lambda|) \leq d_\Omega(f(\lambda)) \leq C_2(1 - |\lambda|)^{1/\alpha} \quad \forall \lambda \in \Delta.$$

Proof. The proposition follows directly from Propositions 2.3 and 2.4, Definition 1.3 and the definition of ρ_Δ. $\square$

Remark 2.6. We can extend the hypothesis of the above proposition slightly by allowing $f(0)$ to be contained in some fixed compact set $V \subset \Omega$. In this case the conclusion remains unchanged except that the constants C_1 and C_2 depend on V.

If $\Omega \subset\subset \mathbf{C}^n$ is convex with C^2 boundary then we can line $\partial\Omega$ with small balls or complex linear images of Δ rather then lemniscates. This construction leads to similar results with $\alpha = 1$. Indeed Proposition 2.3 is then a special case of Proposition 1.2 of [A1], and Proposition 2.5 is a generalization of Proposition 12 of [L1].

Thus the number α is a quantity which indicates the sharpness of the corners of $\partial\Omega$. It is natural then to quantitize the flatness of $\partial\Omega$. This is the content of

Definition 2.7. Let $\Omega \subset\subset \mathbf{C}^n$ be convex. We say that Ω is m-convex if there is a $C > 0$ and an $m \in (0, \infty)$ such that for every $v \in \mathbf{C}^n$ we have

$$r_\Omega(z; v) \leq C d_\Omega^{1/m}(z) \quad \forall z \in \Omega. \tag{1}$$

A ball is 2-convex and thus a strongly convex domain is 2-convex. For an arbitrary m-convex domain in $\mathbf{C}^n$ $(n \geq 2)$ we must have $m \geq 2$. It is possible to have a bounded strictly convex domain in $\mathbf{C}^n$ $(n \geq 2)$ for which (1) is not satisfied for any $C > 0$ or $m \in (0, \infty)$. We say that such a domain is ∞-convex. Conversely, m-convex need not imply strictly convex.

We state a result due to Graham [G] which allows us to relate (1) to the Kobayashi metric.

Theorem 2.8. *Let $\Omega \subset\subset \mathbf{C}^n$ be convex. For each $v \in \mathbf{C}^n$ we have*

$$\frac{|v|}{2r_\Omega(z; v)} \leq \kappa_\Omega(z; v) \leq \frac{|v|}{r_\Omega(z; v)} \quad \forall z \in \Omega. \quad \square$$

We come to the main result of this section.

Proposition 2.9. *Let $\Omega \subset\subset \mathbf{C}^n$ be m-convex and let $f \in \mathrm{HOL}(\Delta, \Omega)$ be a complex geodesic. Then f extends to a continuous mapping $\hat{f}: \overline{\Delta} \to \overline{\Omega}$, with $\hat{f}(\partial\Delta) \subset \partial\Omega$.*

Proof. By Theorem 2.8 we have for any $v \in \mathbf{C}$,

$$\frac{|f'(\lambda)v|}{2r_\Omega(f(\lambda); f'(\lambda)v)} \leq \kappa_\Omega(f(\lambda); f'(\lambda)v) = \kappa_\Delta(\lambda; v) \leq \frac{|v|}{1 - |\lambda|} \quad \forall \lambda \in \Delta.$$

By hypothesis and Proposition 2.5 there is an $\alpha > 1$ and a $C > 0$ such that

$$(2) \qquad |f'(\lambda)v| \leq \frac{Cd_\Omega^{1/m}(f(\lambda))|v|}{1-|\lambda|} \leq \frac{C(1-|\lambda|)^{1/\alpha m}|v|}{1-|\lambda|} = C(1-|\lambda|)^{1/\alpha m - 1}|v| \quad \forall \lambda \in \Delta.$$

The desired extension $\hat{f}$ exists due to a well-known result of Hardy and Littlewood (Theorems 3 and 4, Chapter IX, §5 of [GO]). Definition 1.3 ensures that $\hat{f}(\partial\Delta) \subset \partial\Omega$. □

Remark 2.10. The above proof actually yields a bit more. The same Hardy-Littlewood Theorem shows that $\hat{f}$ is Hölder continuous on $\overline{\Delta}$ with exponent $1/\alpha m \in (0, \frac{1}{2})$, that is, there is a $C > 0$ (depending only on $f(0)$) such that

$$|\hat{f}(\lambda_1) - \hat{f}(\lambda_2)| \leq C|\lambda_1 - \lambda_2|^{1/\alpha m} \quad \forall \lambda_1, \lambda_2 \in \overline{\Delta}.$$

After obtaining these results, we learned that Dineen and Timoney have a version of Proposition 2.9 in the special case of bounded m-convex circled domains (Theorem 4.4 of [DT]).

3. Complex geodesics with prescribed boundary data

In this section we obtain an analogue of Theorem 0.1(iii) for the class of m-convex domains. With our results in place, the methods of Chang, Hu, and Lee [CHL] do much of the work for us.

For $\Omega \subset \mathbf{C}^n$, set

$$\mathscr{F} = \{\text{complex geodesics } f \in \mathrm{HOL}(\Delta, \Omega) \colon d_\Omega(f(0)) \geq d_\Omega(f(\lambda))\ \forall \lambda \in \Delta\}.$$

Lemma 3.1. *Let $\Omega \subset\subset \mathbf{C}^n$ be convex. There is an $\alpha > 1$ and $C > 0$ such that for each $f \in \mathscr{F}$ we have*

$$d_\Omega(f(\lambda)) \leq C(1-|\lambda|)^{1/2\alpha} \qquad \forall \lambda \in \Delta.$$

Proof. Let $f \in \mathscr{F}$. By Remark 2.6 it suffices to prove the lemma for $f(0)$ near $\partial\Omega$. Fix $z_0 \in \Omega$ and line $\partial\Omega$ with lemniscates. Let $\rho > 0$ and $C_1 > 0$ be as in the proof of Proposition 2.3. If $d_\Omega(f(0)) < \rho$ then $f(0)$ is on the axis of a lemniscate $\Lambda = \Lambda(p, \theta, q, r)$. Fix $\lambda \in \Delta$, then since $f \in \mathscr{F}$, $f(\lambda)$ is on the axis of a lemniscate $\Lambda' = \Lambda(p', \theta, q', r)$. By the definition of ρ_Δ, the triangle inequality, the distance decreasing property, and our choice of C_1 we have

$$(3) \quad \frac{1}{2}\log\frac{1+|\lambda|}{1-|\lambda|} = k_\Omega(f(0), f(\lambda)) \leq k_\Omega(f(0), q) + k_\Omega(q, q') + k_\Omega(f(\lambda), q') \leq k_\Lambda(f(0), q) + C_1 + k_{\Lambda'}(f(\lambda), q').$$

By Lemma 2.2 and Proposition 2.3 there is an $\alpha > 1$ and a $C_2 > 0$ such that (3) is no larger than

$$-\tfrac{1}{2}\log d_\Omega^\alpha(f(0)) + C_2 - \tfrac{1}{2}\log d_\Omega^\alpha(f(\lambda)).$$

Therefore $d_\Omega^\alpha(f(0))\, d_\Omega^\alpha(f(\lambda)) \leq C(1-|\lambda|)$ for some $C > 0$ and finally the definition of $\mathscr{F}$ implies the desired result. □

Lemma 3.2. *Let $\Omega \subset\subset \mathbf{C}^n$ be m-convex. There is an $\alpha > 1$ and a $C > 0$ such that for each $f \in \mathscr{F}$ we have*

$$|f(\lambda_1) - f(\lambda_2)| \leq C|\lambda_1 - \lambda_2|^{1/2\alpha m} \quad \forall \lambda_1, \lambda_2 \in \Delta.$$

Proof. The first inequality in line (2) combined with Lemma 3.1 shows that there is a $C > 0$ and an $\alpha > 1$ such that for each $f \in \mathcal{F}$ and $v \in \mathbf{C}$ we have

$$|f'(\lambda)v| < C(1-|\lambda|)^{1/2\alpha m - 1}|v| \quad \forall \lambda \in \Delta.$$

The Hardy-Littlewood Theorem completes the proof. □

Lemma 3.3. *Let $\Omega \subset\subset \mathbf{C}^n$ be m-convex. Let $\mathcal{G}$ be a family of complex geodesics with the property that there is an $a > 0$ such that* $\operatorname{diam}(f(\Delta)) \geq a$ *for each $f \in \mathcal{G}$. Then there is a compact set $V \subset \Omega$ such that $f(\Delta) \cap V \neq \varnothing$ for each $f \in \mathcal{G}$.*

Proof. By reparameterizing each element of $\mathcal{G}$ (if necessary) we may assume that $\mathcal{G} \subset \mathcal{F}$. Write $\Omega = \bigcup V_j$ with $V_1 \subset V_2 \subset \cdots$ each compact. If the lemma is not true then for each j there is an $f_j \in \mathcal{G}$ such that $f_j(\Delta) \cap V_j = \varnothing$. By Lemma 3.2 we may assume that $f_j \to f$ uniformly on Δ. We have then $f(\Delta) \cap V_j = \varnothing$ for each j and hence $f(\Delta) \subset \partial\Omega$. We must also have $\operatorname{diam}(f(\Delta)) \geq a$. Since Ω is m-convex this is impossible (the appropriate maximum principle is provided by Lemma 1 of [G], which originally appears in [TW]), and we have the desired contradiction. □

We come to the main result of this section.

Proposition 3.4. *Let $\Omega \subset\subset \mathbf{C}^n$ be m-convex. Let $z \neq w \in \overline{\Omega}$. There is a complex geodesic $f \in \mathrm{HOL}(\Delta, \Omega)$ whose continuous extension* (*see Proposition* 2.9) *contains $\{z, w\}$ in its image.*

Proof. Let $\{z_j\}, \{w_j\} \subset \Omega$ with $z_j \to z$ and $w_j \to w$. By Theorem 0.2 there are complex geodesics $f_j \in \mathrm{HOL}(\Delta, \Omega)$ with $\{z_j, w_j\} \subset f_j(\Delta)$. Now $\operatorname{diam}(f_j(\Delta)) \geq |z_j - w_j|$ and $z \neq w$ so we may assume that $\operatorname{diam}(f_j(\Delta)) \geq a$ for some $a > 0$. By Lemma 3.3 we can reparameterize each f_j (if necessary) to ensure that there is a compact set $V \subset \Omega$ such that $\{f_j(0)\} \subset V$. By Remark 2.10 we have $f_j \to f$ uniformly on Δ. f has the desired property. □

4. An application

In this section we apply some of the work of the previous sections to obtain a new generalization of the Denjoy-Wolff Theorem (Theorem 4.1 below). First we do some groundwork.

Let $\Omega \subset C^n$ and $z, w \in \Omega$. A C^1 function $\gamma: [0, 1] \to \Omega$ with $\gamma(0) = z$ and $\gamma(1) = w$ which attains the infimum in Definition 1.2 is called a real geodesic between z and w. If $\Omega = \Delta$, real geodesics are arcs of circles which intersect $\partial\Delta$ at right angles (see Chapter 2, §7 of [VE]). Theorem 0.2 implies that if Ω is convex then for any pair of points $z, w \in \Omega$, there is a real geodesic between z and w. If $u \in \Omega$ is a point of the image of such a real geodesic then $k_\Omega(z, w) = k_\Omega(z, u) + k_\Omega(u, w)$. When complex geodesics extend continuously to the boundary of Δ it is clear that real geodesics extend along with them.

Let $\Omega \subset \mathbf{C}^n$ and $F \in \mathrm{HOL}(\Omega, \Omega)$. We consider convergence (uniformly on compact subsets) of the sequence defined by $F^1 = F$, $F^{j+1} = F \circ F^j$, $j = 1, 2, \ldots$. The structure of the set $\operatorname{Fix}(F) = \{z \in \Omega: F(z) = z\}$ plays an important part. If $\Omega = \Delta$, then (assuming F is not the identity mapping) the Schwarz Lemma implies that either $\operatorname{Fix}(F) = \varnothing$ or $\operatorname{Fix}(F)$ is a single point. With regard to the former case, the following result was proved in 1926.

Theorem 4.1 (Denjoy [D], Wolff [W2]). *Let* $f \in \mathrm{HOL}(\Delta, \Delta)$ *with* $\mathrm{Fix}(f) = \varnothing$. *There is an* $x \in \partial\Delta$ *such that* $f^j \to x$ *uniformly on compact subsets of* Δ *(that is,* $f^j \to f$ *where* $f(\lambda) = x \ \forall \lambda \in \Delta$*).* □

The main tool used in the proof of this theorem is the following generalization of the Schwarz Lemma.

Theorem 4.2 (Wolff [W1]). *Let* $f \in \mathrm{HOL}(\Delta, \Delta)$ *with* $\mathrm{Fix}(f) = \varnothing$. *There is an* $x \in \partial\Delta$ *such that every disk* D_x *in* Δ, *tangent to* $\partial\Delta$ *at* x *has the property that*

$$f^j(D_x) \subset D_x \quad \forall j. \quad \square$$

Of course the x's appearing in the above two theorems are the same. For a nice account of iteration theory on Δ see [B].

The proofs of these two theorems rely on purely elementary means such as the Schwarz Lemma, Montel's Theorem and a geometrical description of the set D_x. In particular we have

Definition 4.3. The horocycle at $x \in \partial\Delta$ with radius R is given by

$$D_x(R) = \left\{\lambda \in \Delta \colon \frac{|1 - \lambda\overline{x}|^2}{1 - |\lambda|^2} < R\right\}. \tag{4}$$

A useful result of Yang [Y] is the following:

$$\lim_{\mu \to x}[\rho_\Delta(\lambda, \mu) - \rho_\Delta(0, \mu)] = \frac{1}{2}\log\frac{|1 - \lambda\overline{x}|^2}{1 - |\lambda|^2}, \tag{5}$$

which is a direct result of the transitivity of the automorphism group of Δ. The importance of (5) is that although the right-hand side relies heavily on the structure of Δ (compare with (4)), the left-hand side can be generalized to quite general domains in $\mathbf{C}^n$ using the Kobayashi distance. Thus we have the following generalization of horocycle, originally due to Abate [A2].

Definition 4.4. Let $\Omega \subset\subset \mathbf{C}^n$ and fix $z_0 \in \Omega$. We define the small and big horospheres at $x \in \partial\Omega$ with radius R respectively by

$$E_x(z_0, R) = \left\{z \in \Omega \colon \limsup_{w \to x}[k_\Omega(z, w) - k_\Omega(z_0, w)] < \tfrac{1}{2}\log R\right\}$$

and

$$F_x(z_0, R) = \left\{z \in \Omega \colon \liminf_{w \to x}[k_\Omega(z, w) - k_\Omega(z_0, w)] < \tfrac{1}{2}\log R\right\}.$$

Clearly $E_x(z_0, R) \subset F_x(z_0, R)$. Abate [A4] has shown that if Ω is strictly convex with C^3 boundary then $E_x(z_0, R) = F_x(z_0, R)$. In general this is not the case (see [A5]). We remark that if $\Omega = \Delta$ then $D_x(R) = E_x(0, R) = F_x(0, R)$.

These definitions lead one to hope that an analogue of Theorem 4.1 holds for general domains in $\mathbf{C}^n$. Indeed in 1988 Abate [A2] proved a perfect analogue in case $\Omega \subset\subset \mathbf{C}^n$ is strongly convex. In this section we obtain the desired analogue in case $\Omega \subset\subset \mathbf{C}^n$ is m-convex.

The main problem in using small and big horospheres is making sure that they behave roughly the same way that horocycles do near the boundary of the domain in question. To be more precise we have

Definition 4.5 [A6]. A domain $\Omega \subset\subset \mathbf{C}^n$ is F-convex if for each $x \in \partial\Omega$ we have

$$\overline{F}_x(z_0, R) \cap \partial\Omega = \{x\} \quad \forall z_0 \in \Omega, \ R > 0.$$

Certain types of domains are known to be F-convex: strictly pseudoconvex domains with C^2 boundary, domains of strict finite type, etc. (see [A6] and the references given there). We prove

Theorem 4.6. *Let $\Omega \subset\subset \mathbf{C}^n$ be m-convex. Then Ω is F-convex.*

Proof. Fix $x \in \partial\Omega$, $z_0 \in \Omega$, $R > 0$. We begin by showing that $x \in \overline{F}_x(z_0, R) \cap \partial\Omega$. By Proposition 3.4 there is a complex geodesic $f \in \mathrm{HOL}(\Delta, \Omega)$ whose continuous extension (which we also denote by f) contains $\{z_0, x\}$ in its image. Up to reparameterization we may assume that $f(0) = z_0$ and $f(1) = x$. Let $z \in f(D_1(R))$ (see (4)), so that $z = f(\lambda)$ for some $\lambda \in D_1(R)$, and choose a sequence $\{r_j\} \subset \Delta$ with $r_j \nearrow 1$. We have then

$$\begin{aligned}\liminf_{w \to x}[k_\Omega(z, w) - k_\Omega(z_0, w)] &\le \lim_{j\to\infty}[k_\Omega(z, f(r_j)) - k_\Omega(z_0, f(r_j))] \\ &= \lim_{j\to\infty}[k_\Omega(f(\lambda), f(r_j)) - k_\Omega(f(0), f(r_j))] \\ &= \lim_{j\to\infty}[\rho_\Delta(\lambda, r_j) - \rho_\Delta(0, r_j)] < \tfrac{1}{2}\log R.\end{aligned}$$

Hence $f(D_1(R)) \subset F_x(z_0, R)$ and in particular $x \in \overline{F}_x(z_0, R) \cap \partial\Omega$.

We now show that x is the only element of $\partial\Omega$ which belongs to $\overline{F}_x(z_0, R)$. To this end, assume that $y \in \overline{F}_x(z_0, R) \cap \partial\Omega$, with $y \ne x$. Then there is a sequence $\{z_j\} \subset F_x(z_0, R)$ with $z_j \to y$. That is, for each j we have

$$\liminf_{w\to x}[k_\Omega(z_j, w) - k_\Omega(z_0, w)] < \tfrac{1}{2}\log R.$$

So for each j there is a w_j (close to x) such that

$$[k_\Omega(z_j, w_j) - k_\Omega(z_0, w_j)] \le \tfrac{1}{2}\log R. \tag{6}$$

We have then $z_j \to y$ and $w_j \to x$. By the proof of Proposition 3.4 and the opening remarks of this section there is a sequence $\{u_j\} \subset \Omega$ contained in a compact set $V \subset \Omega$ such that $k_\Omega(z_j, w_j) = k_\Omega(z_j, u_j) + k_\Omega(u_j, w_j)$. We may assume that $u_j \to u \in \Omega$. From (6) and the triangle inequality we have then

$$\begin{aligned}\tfrac{1}{2}\log R &\ge k_\Omega(z_j, u_j) + k_\Omega(u_j, w_j) - k_\Omega(z_0, u_j) - k_\Omega(u_j, w_j) \\ &= k_\Omega(z_j, u_j) - k_\Omega(z_0, u_j).\end{aligned}$$

Upon letting $j \to \infty$ we obtain the desired contradiction. □

Now to obtain the main result of this section we need only refer to the work of Abate. First we have

Definition 4.7. Let $\Omega_1 \subset \mathbf{C}^n$, $\Omega_2 \subset \mathbf{C}^m$. A sequence of mappings $\{F_j\} \subset \mathrm{HOL}(\Omega_1, \Omega_2)$ is compactly divergent if for each pair of compact sets $V_1 \subset \Omega_1$, $V_2 \subset \Omega_2$ there is a j_0 such that $F_j(V_1) \cap V_2 = \varnothing \ \forall j \ge j_0$.

Lemma 4.8 (Theorem 2.4.20 of [A5]). *Let $\Omega \subset\subset \mathbf{C}^n$ be convex and let $F \in \mathrm{HOL}(\Omega, \Omega)$ be such that $\mathrm{Fix}(F) = \varnothing$. Then $\{F_j\}$ is compactly divergent.* □

The following is the appropriate generalization of Theorem 4.2.

Lemma 4.9 (Theorem 2.3 of [A2]). *Let* $\Omega \subset\subset \mathbf{C}^n$ *be convex and let* $F \in \mathrm{HOL}(\Omega, \Omega)$ *be such that* $\mathrm{Fix}(F) = \varnothing$. *Then there is an* $x \in \partial\Omega$ *such that for any* $z_0 \in \Omega$, $R > 0$ *we have*

$$F^j(E_x(z_0, R)) \subset F_x(z_0, R) \quad \forall j. \quad \square$$

Finally we come to our version of Theorem 4.1.

Proposition 4.10. *Let* $\Omega \subset\subset \mathbf{C}^n$ *be m-convex. Let* $F \in \mathrm{HOL}(\Omega, \Omega)$ *be such that* $\mathrm{Fix}(F) = \varnothing$. *Then* $\{F^j\}$ *converges uniformly on compact subsets of* Ω *to a constant* $x \in \partial\Omega$.

Proof. The proof is similar to that of Theorem 3.5 in [A6]. Let $x \in \partial\Omega$ be given by Lemma 4.9. Let G be a limit point of $\{F^j\}$. By Lemma 4.8 and the boundedness of Ω we must have $G(\Omega) \subset \partial\Omega$. By Lemma 4.9 and Proposition 4.6 we have for any $z_0 \in \Omega$, $R > 0$,

$$G(E_x(z_0, R)) \subset \overline{F}_x(z_0, R) \cap \partial\Omega = \{x\}.$$

Clearly then $G \equiv x$ and the proposition is proved. $\square$

Remark 4.11. An example in [A2] shows that Proposition 4.10 does not hold if Ω is assumed to be merely convex and bounded. As we noted earlier, a strongly convex domain is 2-convex. Thus Proposition 4.10 is currently the most general version of the Denjoy-Wolff Theorem in the sense that the (bounded) m-convex domains is the largest subclass of the bounded convex domains for which such a theorem is known to hold. The method of using complex geodesics for such problems seems to be new.

We note that the Denjoy-Wolff Theorem for the unit ball in $\mathbf{C}^n$ was proved in 1983 by MacCluer in [M]. The methods there are more elementary than those used here or in Abate's work. In particular the Kobayashi distance does not appear. Similar results for the unit ball in $\mathbf{C}^n$ appear in [KU and C].

Finally we point out that this section was devoted exclusively to the situation where $F \in \mathrm{HOL}(\Omega, \Omega)$ is such that $\mathrm{Fix}(F) = \varnothing$. For some very interesting results for the case where $\mathrm{Fix}(F) \neq \varnothing$, one may consult [A2], and especially [A6].

Added in proof. For smoothly bounded (pseudo-)convex domains of finite type, X. Huang (*A boundary rigidity problem of holomorphic mappings on some weakly pseudoconvex domains*, Preprint, 1992) has independently obtained results analogous to those results of §§2 and 3 which involve complex geodesics.

M. Suzuki (*Iterates of holomorphic self-maps on a convex domain*, Kobe J. Math. **6** (1989), 229–232) obtained a result similar to Proposition 4.10 using an approach to iteration theory which does not involve horospheres.

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