TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 338, Number 2, August 1993

ASYMPTOTIC HOMOTOPY CYCLES FOR FLOWS AND Π_1 DE RHAM THEORY

DIEGO BENARDETE AND JOHN MITCHELL

In memory of K. T. Chen

ABSTRACT. We define the asymptotic homotopy of trajectories of flows on closed manifolds. These homotopy cycles take values in the 2-step nilpotent Lie group which is associated to the fundamental group by means of Malcev completion. The cycles are an asymptotic limit along the orbit of the product integral of a Lie algebra valued 1-form. Propositions 5.1–5.7 show how the formal properties of our theory parallel the properties of the asymptotic homology cycles of Sol Schwartzman. In particular, asymptotic homotopy is an invariant of topological conjugacy, and, in certain cases, of topological equivalence.

We compute the asymptotic homotopy of those measure-preserving flows on Heisenberg manifolds which lift from the torus T^2 (Theorem 8.1), and then show how this invariant distinguishes up to topological equivalence certain of these flows which are indistinguishable homologically (Theorem 9.1). We also compute the asymptotic homotopy of those geodesic flows for Heisenberg manifolds which come from left invariant metrics on the Heisenberg group (Example 8.1), and then show how this invariant distinguishes up to topological conjugacy certain of these flows which are indistinguishable homologically.

1. INTRODUCTION

Our goal is to investigate the asymptotic topology of orbits of flows on manifolds. The asymptotic *homology* of orbits was studied by Sol Schwartzman in [S]. The (real) homology class of a *closed* orbit in a manifold M may be computed by choosing a set of closed 1-forms representing a basis of real cohomology, and then integrating the forms over the orbit. Schwartzman defined the asymptotic homology class of any (possibly nonclosed) orbit as the asymptotic time average of the integrals of these 1-forms. For measure-preserving flows, existence almost everywhere of these averages is given by Birkhoff's ergodic theorem.

In order to extend this method to the study of asymptotic homotopy, one needs first of all the "Π_1 de Rham theorem" of Chen [C2] and Sullivan [Su1]. We present in §3 a simple proof of a version of the theorem. The theorem states that the "nilpotent torsion free part" of the homotopy class of a closed orbit

Received by the editors August 28, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F25; Secondary 58A12, 58F17, 57R99, 28B10, 22E25.

Key words and phrases. Nonabelian de Rham theory, flows, asymptotic homology, asymptotic homotopy, topological conjugacy, topological equivalence, Heisenberg manifolds, geodesic flows, product integral, nilpotent Lie groups.

may be computed in terms of *iterated integrals* of appropriate 1-forms on M. We consider these 1-forms to be the components of a Lie algebra valued 1-form on M. Then the iterated integrals give the coordinates of a curve in the Lie group, which is an "antiderivative" or "Lie integral" (i.e., product integral) of the curve in the Lie algebra obtained by evaluating the form along the orbit.

We next define the asymptotic homotopy of an orbit in terms of asymptotic averages of such Lie integrals. At present our theory of asymptotic limits applies to the integration along a curve on a manifold of any 1-form taking values in any 2-step nilpotent Lie algebra (this restriction is explained in §5). The appropriateness of our definitions and theory is supported by showing how the formal properties of asymptotic homotopy are parallel to the formal properties of asymptotic homology. (Propositions 5.1–5.7 parallel Propositions 2.1–2.7.) We also introduce a "homotopy foliation" to exploit the global character of asymptotic homotopy, which necessarily arises since $\Pi_1(M, p)$ and $\Pi_1(M, q)$ are only identifiable up to conjugacy.

The generalization of Birkhoff's ergodic theorem to this context (Lie algebra valued functions) has not been studied, so we restrict our applications to cases where we can demonstrate the existence of these limits either by using Birkhoff's ergodic theorem, or by making a direct computation. For example, in §5 we give a proof of the rigidity of nilflows on 2-step nilmanifolds. In Example 8.1, we compute the asymptotic homotopy of geodesic flows for Heisenberg manifolds and then use this computation to distinguish certain geodesic flows up to topological conjugacy. As a further example, we compute the asymptotic homotopy of measure-preserving flows on the Heisenberg manifold which are lifts of toral flows (Theorem 8.1). We can then distinguish up to topological equivalence certain of these flows which are indistinguishable using asymptotic homology (Theorem 9.1, Example 9.1).

David Fried's theory of homology directions [Fr], which can be considered as a "projectivization" of asymptotic homotopy cycles, gives significant information about cross-sections to flows even in cases where a given orbit may have nonunique homology directions. We may therefore expect our theory to be of interest even in cases where unique asymptotic homotopy cycles fail to exist (for *homotopy directions* see the end of §5).

Asymptotic homology has also been applied to study Hamiltonian mechanics, the continuous eigenfunctions of measure-preserving flows (Schwartzman [S], Arnold and Avez [AA, p. 147]), the asymptotic linking of orbits in $\mathbf{R}^3$ (Arnold [A], Michael Freedman and Zheng-Xu He [FH]), and the structure of the group of measure-preserving homeomorphisms (Fathi [Fa]). Also the rotation vectors of maps of the torus which are homotopic to the identity (for recent work see [F, FM, LM, MZ]) can be considered as the asymptotic homology of the suspension flow on T^3. Asymptotic homotopy cycles may be expected to be of use in these matters as well.

In §2, we review asymptotic homology. In §3, we describe the Malcev completion, Lie integrals, and iterated integrals, and state and give a new proof of a version of the Π_1 de Rham Theorem. In §4, we define and give the properties of asymptotic limits and averages in 2-step connected, simply connected, nilpotent Lie groups. In §5, we define and give the formal properties of asymptotic homotopy. As an application we give a new proof of the classification theorem for nilflows on 2-step nilmanifolds. In §6, we discuss the homotopy foliation. In §7, we describe the measure-preserving flows on Heisenberg manifolds, and then we single out those that lift from flows on the torus. In §8, we compute the asymptotic homotopy of these lifted flows, while, in §9, we use asymptotic homotopy to distinguish certain of these flows up to topological equivalence. In §8 we also compute the asymptotic homotopy of geodesic flows for Heisenberg manifolds.

Conventions. Unless otherwise noted, all manifolds and 1-forms are taken to be smooth, and "lim" denotes the limit as $t \to \infty$.

2. Asymptotic homology

In this section we restate without proof some of Sol Schwartzman's results [S] on asymptotic homology cycles in a form which anticipates the generalization to asymptotic homotopy cycles presented in this paper. In particular, we make explicit the sense in which the asymptotic homology cycles are invariants of the flow and we use this invariance to offer a homological proof of the classical classification theorem for flows on the torus induced by one-parameter subgroups. Since we are restricting our attention to smooth flows we can use de Rham cohomology instead of the Bruschlinsky groups (the group of continuous functions $f\colon M \to S^1$ modulo the subgroup of functions which can be lifted to $\mathbf{R}$) which Schwartzman uses in order to deal with continuous flows on compact metric spaces.

Suppose M is a smooth manifold without boundary, $H_1(M)$ is the first homology group with real coefficients, and $H^1(M)$ is the first de Rham cohomology group. $H^1(M)$ is canonically isomorphic to $(H_1(M))^*$ [W, p. 154]. If $\{\sigma_1, \dots, \sigma_m\}$ is a basis of $H_1(M)$ with dual basis $\{[\omega_1], \dots, [\omega_m]\}$ in $H^1(M)$ then the coordinates of the homology class $[\alpha]$ of a closed continuous curve α with respect to the basis $\{\sigma_1, \dots, \sigma_m\}$ are $(\int_\alpha \omega_1, \dots, \int_\alpha \omega_m)$. Note that $\int_\alpha \omega_i$ is defined, since α can be arbitrarily well approximated by smooth curves α_k [BT, p. 213], and $\lim_{k\to\infty} \int_{\alpha_k} \omega_i$ is independent of the approximation chosen since the forms ω_i are closed.

For the rest of this section, we assume that M, $\{\sigma_i\}$, and $\{\omega_i\}$ are given as above, and that if $f\colon M \to M'$ is a homeomorphism, then M' is similarly equipped with $\{\sigma_i'\}$ and $\{\omega_i'\}$. f_* will denote the induced homology isomorphism.

Notation 2.1 (*Restriction of a curve to an initial segment*). If $\alpha\colon \mathbf{R} \to M$ is a curve, then $\alpha_{|t} \equiv \alpha_{|[0,t]}$.

Definition 2.1 (Asymptotic homology). (a) Let $\alpha\colon \mathbf{R} \to M$ be a continuous curve. Then, if the limits indicated below exist, $\mu(\alpha)$, the asymptotic homology of α, is $\mu_1(\alpha)\sigma_1 + \cdots + \mu_m(\alpha)\sigma_m$, where $\mu_i(\alpha) \equiv \lim \frac{1}{t} \int_{\alpha_{|t}} \omega_i$.

(b) If $\phi\colon M \times \mathbf{R} \to M$ is a continuous flow and α is the trajectory of ϕ starting at a point p in M (i.e., $\alpha(0) = p$), then $\mu_i(p) \equiv \mu_i(\alpha)$ and $\mu(p) \equiv \mu(\alpha)$.

We will write μ_i and μ if the curve or point is understood, and we will sometimes consider μ to be the m-tuple of numbers $(\mu_1, \dots, \mu_m)$, where the basis $\{\sigma_1, \dots, \sigma_m\}$ is understood.

The following propositions, especially Propositions 2.1 and 2.2, justify the term asymptotic homology.

Proposition 2.1 (Asymptotic homology agrees with homology on a closed path). *Let* $\alpha\colon \mathbf{R} \to M$ *be a smooth curve for which there exists* $T > 0$ *such that, for all* t, $\alpha(t+T) = \alpha(t)$. *Then* $\mu(\alpha)$ *exists and equals* $\frac{1}{T}([\alpha_{|T}])$.

Definition 2.2 (bounded closings). Suppose $\alpha\colon \mathbf{R} \to M$ is a continuous curve starting at p and $\tilde{p}$ is a lift of p to $\widetilde{M}$, the universal covering of M. A family of closings of α is a collection of continuous paths $\{\beta_t | t \geq 0\}$ such that β_t starts at $\alpha(t)$ and ends at p. The family is bounded if there is some compact set in $\widetilde{M}$ which, for all $t \geq 0$, contains the lift $\tilde{\beta}_t$ of β_t which ends at $\tilde{p}$.

Observe that a family of smooth closings is bounded iff the set of their lengths is bounded, where length is measured in some Riemannian metric on M. Also note that, since M is assumed to be compact, a family of bounded closings always exists.

Proposition 2.2 [S, p. 275] (Asymptotic homology in terms of integrating 1-forms agrees with asymptotic homology in terms of closing paths). *If* $\alpha\colon \mathbf{R} \to M$ *is a smooth curve,* $\{\beta_t\}$ *is a family of bounded closings of* α, $\gamma_t = \alpha_{|t}\beta_t$ *is the resulting loop based at* p, *and* $[\gamma_t]$ *is its class in* $H_1(M)$, *then* $\mu(\alpha)$ *exists iff* $\lim[\gamma_t]/t$ *exists and if they exist they are equal.*

Proposition 2.3 (Asymptotic homology is constant along a path). *If* $\alpha\colon \mathbf{R} \to M$ *is a continuous path and* $\beta\colon \mathbf{R} \to M$ *is given by* $\beta(t) = \alpha(t + t_0)$ *for some* $t_0 \in \mathbf{R}$, *then* $\mu(\alpha)$ *exists iff* $\mu(\beta)$ *exists and if they exist they are equal.*

Definition 2.3 (Topological conjugacy). The flows ϕ and ϕ' on the manifolds M and M' are topologically conjugate if there exists a homeomorphism $f\colon M \to M'$ such that for all $x \in M$ and $t \in R$, $f \circ \phi(x, t) = \phi'(f(x), t)$. That is, a topological conjugacy takes orbits to orbits preserving the parameterization.

Proposition 2.4 (Asymptotic homology is an invariant of parameter-preserving homeomorphisms). (a) *If* $f\colon M \to M'$ *is a homeomorphism taking the curve* α *on* M *to the curve* α' *on* M' *(i.e.,* $\alpha' = f \circ \alpha$*), then* $\mu(\alpha)$ *exists iff* $\mu(\alpha')$ *exists and* $f_*\mu(\alpha) = \mu(\alpha')$, *where* f_* *is the induced map on homology.*

(b) *If* $f\colon M \to M'$ *is a topological conjugacy of the flows* ϕ *and* ϕ', *then* $\mu(p)$ *exists iff* $\mu(f(p))$ *exists and if they exist* $f_*\mu(p) = \mu(f(p))$.

Definition 2.4 (Closed and open rays in $H_1(M)$). Suppose σ is a nonzero element in $H_1(M)$. Then $\{r\sigma | r \geq 0\}$ is a closed ray and $\{r\sigma | r > 0\}$ is an open ray.

Note that since the zero element is in each closed ray, being in the same closed ray is not an equivalence relation.

Definition 2.5 (Reparametrization of a curve). The continuous curve $\alpha\colon \mathbf{R} \to M$ is a reparametrization by s of a continuous curve $\alpha'\colon \mathbf{R} \to M$ if $\alpha(t) = \alpha'(s(t))$, where $s\colon \mathbf{R} \to \mathbf{R}$ is a continuous monotone function such that $s(0) = 0$.

Definition 2.6 (Topological equivalence). The flows ϕ on M and ϕ' on M' are topologically equivalent if there exists a homeomorphism $f\colon M \to M'$ such that, for all $x \in M$, there exists $s\colon \mathbf{R} \to \mathbf{R}$ such that $f \circ \alpha$ is a reparametrization by s of α', where $\alpha(t) = \phi(x, t)$ and $\alpha'(t) = \phi'(f(x), t)$.

Proposition 2.5 (Asymptotic homology and reparametrization). *If $\alpha: \mathbf{R} \to M$ and $\alpha': \mathbf{R} \to M$ are continuous curves such that α is a reparametrization by s of α', then if $\mu(\alpha)$ and $\mu(\alpha')$ both exist they are on the same closed ray, and if they are both nonzero they are on the same open ray. The following statements provide more detail:*

(a) *If $\mu(\alpha')$ and $\mu(\alpha)$ exist and $\mu(\alpha')$ is nonzero, then $r = \lim \frac{s}{t}$ exists and $\mu(\alpha) = r\mu(\alpha')$.*
(b) *If $\mu(\alpha')$ and $r = \lim \frac{s}{t}$ exist, then $\mu(\alpha)$ exists and equals $r\mu(\alpha')$.*

Note that it is possible for $\mu(\alpha')$ to exist, and $\lim \frac{s}{t}$ to be infinite or oscillating making $\mu(\alpha)$ diverge to infinity or oscillate. Also note that if $\mu(\alpha) = \mu(\alpha') =$ the zero element, then no conclusion can be drawn about $\lim \frac{s}{t}$.

Proposition 2.6 (The invariance of asymptotic homology for homeomorphisms not preserving the parameter). (a) *Let $\alpha: \mathbf{R} \to M$ and $\alpha': \mathbf{R} \to M'$ be continuous curves, and let $f: M \to M'$ be a homeomorphism such that $f \circ \alpha$ is a reparametrization by s of α'. If $\mu(\alpha)$ and $\mu(\alpha')$ exist, then $f_*\mu(\alpha)$ and $\mu(\alpha')$ are on the same closed ray, and if both $\mu(\alpha)$ and $\mu(\alpha')$ are nonzero, then $f_*\mu(\alpha)$ and $\mu(\alpha')$ are on the same open ray.*

(b) *If $f: M \to M'$ is a topological equivalence of flows ϕ and ϕ', then $f_*\mu(p)$ and $\mu(f(p))$ are on the same closed ray if they both exist, and they are on the same open ray if in addition $\mu(p)$ and $\mu(f(p))$ are both nonzero.*

Definition 2.7 (Lattice in a Lie group). A lattice in a Lie group G is a discrete subgroup Δ such that G/Δ (and equivalently $\Delta\backslash G$) have finite G-invariant Borel measures which are positive on open sets.

Definition 2.8 (Flows induced by one-parameter subgroups). If $\phi: \mathbf{R} \to G$ is a one-parameter subgroup of a Lie group, and $\Gamma < G$ is a lattice, then

$$\phi^*: (\Gamma\backslash G, \mathbf{R}) \to (\Gamma\backslash G)$$

is the flow given by right translation by ϕ, i.e., $\phi^*_t(\Gamma g) = \Gamma(g\phi(t))$.

Definition 2.9 (Affine map). Suppose Γ and Γ' are lattices in the Lie groups G and G', the element a is in G', and the isomorphism $A: G \to G'$ extends an isomorphism from Γ to Γ'. Then $\overline{aA}: \Gamma\backslash G \to \Gamma'\backslash G'$, $\Gamma x \mapsto \Gamma(Ax)a$ is an affine map.

Proposition 2.7 (Homology proof of classical theorem on toral flows; see [I, p. 36; B, p. 502] for alternate proofs). *Suppose the element $v \in \mathbf{R}^n$ determines the one-parameter subgroup $\phi_v: t \mapsto tv$ of $\mathbf{R}^n$, which induces the flow ϕ^*_v on $T^n = \mathbf{Z}^n\backslash\mathbf{R}^n$ (Definition 2.8). Let $\{\sigma_1, \dots, \sigma_n\}$ be the standard basis of $H_1(T^n)$, i.e., σ_i is the class of the closed curve in T^n which lifts to the line segment in $\mathbf{R}^n$ from $(0, \dots, 0)$ to $(0, \dots, 1, \dots, 0)$, where there is a one in the ith place.*

(a) *For all $p \in T^n$, $\mu(p)$ exists, and, with respect to $\{\sigma_i\}$, $\mu(p) = v$.*
(b) *ϕ^*_v and $\phi^*_{v'}$ are topologically conjugate iff ϕ^*_v and $\phi^*_{v'}$ are affinely conjugate, which holds iff there exists an invertible integral $n \times n$ matrix A such that $\mathrm{Av} = v'$.*
(c) *ϕ^*_v and $\phi^*_{v'}$ are topologically equivalent iff ϕ^*_v and $\phi'^*_{v'}$ are affinely equivalent, which holds iff there exists an invertible integral $n \times n$ matrix A and $r > 0$ such that $\mathrm{Av} = rv'$.*

Proof. (a) Let ω_i be the 1-form on T^n induced by dx_i on $\mathbf{R}^n$. Then $\{[\omega_i]\}$ is dual to $\{\sigma_i\}$. If α is any integral curve of ϕ_v^* and $\tilde{\alpha} = (x_1, \dots, x_n)$ is its lift to $\mathbf{R}^n$, then

$$\left(\int_{\alpha_{|t}} \omega_1, \dots, \int_{\alpha_{|t}} \omega_n\right) = \left(\int_{\tilde{\alpha}_{|t}} dx_1, \dots, \int_{\tilde{\alpha}_{|t}} dx_n\right)$$
$$= (x_1(t) - x_1(0), \dots, x_n(t) - x_n(0)) = tv.$$

So dividing by t and taking limits, we obtain $\mu(\alpha) = v$.

(b) If $f\colon T^n \to T^n$ is a homeomorphism, then, with respect to the basis $\{\sigma_i\}$, f_* is an invertible integral $n \times n$ matrix. Av $= v'$ follows by applying Proposition 2.4(b) on topological conjugacy and part (a) above.

(c) Apply Proposition 2.6(b) on topological equivalence, and part (a) above. Q.E.D.

Proposition 2.8 (Asymptotic homology exists almost everywhere for a measure-preserving flow). *Let ϕ be a measure-preserving flow on M. Then $\mu(p)$ exists for almost all $p \in M$.*

Proof. Apply Birkhoff's ergodic theorem to the functions $\omega_i(V)$, where V is the vector field inducing ϕ. Q.E.D.

We conclude with the following comments.

(1) David Fried [Fr, p. 357] defined homology directions for flows as follows. Let ϕ be a flow on M, m a nonwandering point, and (m_k, t_k) a closing sequence (i.e., $m_k \to m$ and $\phi(m_k, t_k) \to m$). Let α_k be the time t_k trajectory of ϕ through m_k, β_k' a bounded family of paths from m to m_k, and β_k a bounded family of paths from $\phi(m_k, t_k)$ to m. Let γ_k be the resulting loop $\beta_k'\alpha_k\beta_k$ based at m, and $[\gamma_k]$ its homology class in $H_1(M)$. Let D_M be the disjoint union of the space of open rays in $H_1(M)$ and the zero element. Let $p\colon H_1(M) \to D_M$. $\{p(\gamma_k)\}$ has accumulation points in D_M since D_M is compact. Such accumulation points are called homology directions for ϕ. Taking the union of all homology directions for all closing sequences, for all nonwandering points of ϕ we obtain D_ϕ, a compact nonempty set in D_M. Fried then proves several theorems relating D_ϕ to cross-sections of the flow. The strength of the approach to asymptotic homology through homology directions is that it is clearly an invariant of topological equivalence (since $D_\phi = D_{\phi'}$ if ϕ' is any reparametrization of ϕ), and the existence of D_ϕ does not depend on ϕ being measure-preserving. On the other hand, to show the uniqueness almost everywhere of asymptotic homology one needs measure-preserving flows and the ergodic theorem (but see [Rh]), and besides one might be interested in conjugacy questions, where the parametrization is used. The two approaches are complementary, and it would be useful to generalize as far as possible both approaches, in any development of asymptotic homotopy.

(2) if $f\colon M \to M$ is a continuous function, we suspend f and obtain a continuous semiflow on $(\frac{M\times I}{\sim}$, where $(m, 1) \sim (f(m), 0)$. We obtain a flow if f is a homeomorphism. Loosely speaking, the asymptotic homology cycles of Schwartzman or the asymptotic homology directions of Fried are invariants of these semiflows which in the case of $M = S^1$ or $M = T^n$ give us the standard rotation numbers [Fr, p. 364].

3. Π_1 de Rham theory

For Γ any finitely generated group, the Malcev completion is a collection of homomorphisms into a tower of nilpotent Lie groups satisfying certain properties. This notion is an extension of the tensor product $\Gamma \otimes \mathbf{R}$, which is usually defined only for abelian groups. The Π_1 de Rham theorem of Chen and Sullivan states that for a closed manifold M there exist Lie algebra valued 1-forms whose "Lie integrals" on closed paths give the Malcev completion of $\Pi_1(M)$. These integrals can be evaluated using iterated integrals in the sense of Chen.

In what follows, (g, h) is the commutator $ghg^{-1}h^{-1}$ for elements g and h in a group. If H and K are subgroups, then (H, K) is the subgroup generated by commutators (h, k), with $h \in H$ and $k \in K$.

3.1. **Malcev completion.** Recall that the lower central series $\lambda_i(\Gamma)$ of a group Γ is defined by $\lambda_1(\Gamma) \equiv \Gamma$ and $\lambda_{i+1}(\Gamma) = (\lambda_i(\Gamma), \Gamma)$. The root of $\lambda_i(\Gamma)$, $\sqrt{\lambda_i(\Gamma)}$, is $\{g \in \Gamma \mid g^n \in \lambda_i(\Gamma) \text{ for some } n \in \mathbf{Z}\}$. $\sqrt{\lambda_i(\Gamma)}$ is a normal subgroup of Γ, and $\Gamma/\sqrt{\lambda_i(\Gamma)}$ is a torsion free nilpotent group [P, p. 472].

Also recall that if Δ is a lattice in a nilpotent Lie group G (Definition 2.7), then G/Δ must be compact [R].

Definition 3.1 (Malcev completion). Let Γ be a finitely generated group. A Malcev completion of Γ is a tower of nilpotent Lie groups

$$\cdots \to N_n \to N_{n-1} \to \cdots \to N_2 \to N_1 = 1$$

together with a set of homomorphisms $\{\Gamma \to N_i\}$ such that, for all i,

(a) $N_{i+1} \to N_i$ is a surjective homomorphism,

(b)

$$\begin{tikzcd} & N_{i+1} \arrow[d] \\ \Gamma \arrow[ur] \arrow[r] & N_i \end{tikzcd}$$

commutes,

(c) the image of $\Gamma \to N_i$ is a lattice in N_i,

(d) the kernel of $\Gamma \to N_i$ is $\sqrt{\lambda_i(\Gamma)}$.

Malcev showed that any finitely generated, torsion free, nilpotent group is isomorphic to a lattice in a connected, simply connected nilpotent Lie group, and that isomorphisms (epimorphisms) of lattices in such Lie groups extend uniquely to isomorphisms (epimorphisms) of the Lie groups [M]. It follows that any finitely generated group has a Malcev completion which is unique up to isomorphism. Furthermore, isomorphisms (epimorphisms) of such groups induce isomorphisms (epimorphisms) of their Malcev completions in a functorial manner. The homomorphisms from a group to its Malcev completion are natural with respect to those functors.

Note that N_2 is $\Gamma/(\Gamma, \Gamma) \otimes \mathbf{R}$, the abelianization of Γ tensored with $\mathbf{R}$. N_2 is also equal to $N_3/(N_3, N_3)$, the abelianization of N_3.

3.2. **Lie integration of Lie algebra valued 1-forms.** We will now define indefinite and definite integrals of Lie algebra valued functions and 1-forms. We call these integrals "Lie integrals" to distinguish them from the ordinary "additive" integral which treats the Lie algebra as a vector space. Such integrals are also referred to in the literature as "product integrals" [DF].

Definition 3.2 (Lie integrals of Lie algebra valued functions). If $f\colon [a, b] \to L(G)$ is a continuous path in a Lie algebra $L(G)$ of a Lie group G, we define an indefinite left Lie integral ${}_L\int f$ to be a C^1 curve $F(t)$ in G such that $F'(t) = f(t)$, where we identify $F'(t) \in T_{F(t)}G$ with an element of $L(G) = T_eG$ by left translation by $F(t)^{-1}$. The right integral ${}_R\int f$ is defined similarly using the identification of tangent spaces by right translation.

Note that $f(t)$ may be viewed as a time dependent left invariant vector field on G, and so the local existence of $F(t)$ and its uniqueness given a specified initial condition follow from the elementary theory of ordinary differential equations. For global existence (in t) see [KN, p. 69]. (The authors of [KN] identify tangent spaces using right translation.)

Definition 3.3 (Definite Lie integrals of Lie algebra valued functions). If $f\colon [a, b] \to L(G)$ is a continuous path in a Lie algebra $L(G)$, we define its definite left Lie integral ${}_L\int_a^b f$ to be the element of G equal to $\lim \Pi_{i=1}^n \exp(f(t_i^*)\Delta t_i)$, where $\exp\colon L(G) \to G$ is the exponential map and the limit is taken over partitions and choices of t_i^* as in the standard definition of the Riemann integral. The right definition Lie integral ${}_R\int_a^b f$ is $\lim \Pi_{i=n}^1 \exp(f(t_i^*)\Delta t_i)$.

The existence of ${}_L\int_a^b f$ and the "fundamental theorem of calculus," the fact that ${}_L\int_a^b f = F(a)^{-1}F(b)$, where F is an indefinite integral ${}_L\int f$, follow from the proof of the existence and uniqueness of solutions of ordinary differential equations using the method of ε-approximations [CL, pp. 1–20]. Similarly ${}_R\int_a^b f = F(b)F(a)^{-1}$, where $F = {}_R\int f$.

Definition 3.4 (Integration of Lie algebra valued 1-forms). If $\alpha\colon [a, b] \to M$ is a smooth path in a manifold M, and $\omega\colon TM \to L(G)$ is a Lie algebra valued 1-form, then ${}_L\int_\alpha \omega$ is defined to be ${}_L\int_a^b \omega(\alpha'(t))\,dt$. ${}_L\int_\alpha \omega$ is defined for piecewise smooth α by ${}_L\int_\alpha \omega = \Pi\, {}_L\int_{\alpha_i} \omega$, where the α_i are the smooth segments of α. Similarly the right integral is defined by ${}_R\int_\alpha \omega = {}_R\int_a^b \omega(\alpha'(t))\,dt$.

Observe that ${}_L\int_\alpha \omega$ and ${}_R\int_\alpha \omega$ are independent of the parametrization of α. Also note that if α and β are piecewise smooth paths in M such that $\alpha\beta$ is defined, then ${}_L\int_{\alpha\beta} \omega = {}_L\int_\alpha \omega\, {}_L\int_\beta \omega$, while ${}_R\int_{\alpha\beta} \omega = {}_R\int_\beta \omega\, {}_R\int_\alpha \omega$. This explains our preference for the left Lie integrals.

Definition 3.5 (Flat 1-forms). The Lie algebra valued 1-form ω is flat if $d\omega = -[\omega, \omega]$ [Sp, Vol. 2, p. 376].

Note that some authors (e.g., [KN]) use different conventions in defining wedge products and the exterior derivative, and so for them a flat 1-form satisfies $d\omega = -\frac{1}{2}[\omega, \omega]$ [W p. 60; KN, p. 77; Sp, Vol. 1, p. 549].

Proposition 3.1 (Homotopy invariance of ${}_L\int_\alpha \omega$ for flat ω). *If ω is a flat Lie algebra valued* 1*-form, then ${}_L\int_\alpha \omega = {}_L\int_\beta \omega$ when α and β are homotopic rel endpoints.*

Proof. Any $L(G)$ valued 1-form ω on M determines a connection 1-form $\tilde{\omega}$ on the trivial right principal bundle $(M \times G) \to M$ in a standard way such that $\omega = \sigma^* \tilde{\omega}$, where $\sigma: M \to (M \times G)$ is the identity section [KN, Proposition 1.4, p. 66].

Then $\Omega \equiv d\omega + [\omega, \omega]$ is the pullback via σ of $\widetilde{\Omega} = d\tilde{\omega} + [\tilde{\omega}, \tilde{\omega}]$, which is the curvature 2-form of $\tilde{\omega}$. Hence, if Ω vanishes, then $\widetilde{\Omega}$ vanishes on the identity section. Since $\widetilde{\Omega}(X, Y)$ vanishes if X or Y are vertical (this is true by definition for any curvature form [Sp. Vol 2, p. 371]), $\widetilde{\Omega}$ must vanish identically, i.e., the connection $\tilde{\omega}$ is flat. It follows that parallel translation about a contractible loop is trivial [KN, p. 93].

Let γ be a curve in M from p to q, (p, g) an element in the fiber over p, and $\tilde{\gamma}$ the curve in $M \times G$ given by $\tilde{\gamma}(t) = (\gamma(t), g)$. Then the parallel translate of (p, g) along γ is $(q, g({}_R\int_{\tilde{\gamma}} -\tilde{\omega}))$ [KN, p. 69; Sp, Vol. 2, p. 364], which equals $(q, g({}_R\int_\gamma -\omega))$, since by definition of $\tilde{\omega}$, ω is the pullback of $\tilde{\omega}$ via any constant section. Since ${}_R\int_\gamma -\omega = ({}_L\int_\gamma \omega)^{-1}$, it follows that ${}_L\int_\gamma \omega$ is the identity for a contractible loop γ. Letting $\gamma = \alpha\beta^{-1}$ and using the multiplicative property of ${}_L\int \omega$, we obtain the desired result. Q.E.D.

Note that when ω is flat, ${}_L\int_\alpha \omega$ is defined for any continuous curve α, since α can be approximated by smooth curves α_i, and, by Proposition 3.1, $\lim_{i\to\infty} {}_L\int_{\alpha_i} \omega$ is independent of the approximation chosen.

3.3. Π_1 **de Rham theorem.** We next state and prove a version of the Π_1 de Rham theorem of K. T. Chen and D. Sullivan [C2, H, Su1, GM]. A comparison of our approach with those of Chen and Sullivan follows afterward.

Theorem 3.1 (Π_1 de Rham theorem). *If M is a closed manifold, then there exist a tower of connected, simply connected, nilpotent Lie groups*

$$\cdots \to N_n \to N_{n-1} \to \cdots \to N_2 \to N_1 = 1$$

and a sequence of flat Lie algebra valued 1*-forms, $\{\omega_i : TM \to L(N_i)\}$, such that for $p \in M$ the family of homomorphisms $\{\Pi_1(M, p) \to N_i\}$ given by left Lie integrals $\{[\alpha] \mapsto {}_L\int_\alpha \omega_i\}$ is a Malcev completion of $\Pi_1(M, p)$. (We refer to the ω_i as the representing forms.)*

Proof. For a fixed $p \in M$, let $\Pi \equiv \Pi_1(M, p)$. Π has a Malcev completion determining a family of epimorphisms $\{f_i : \Pi \to \Delta_i\}$, where the Δ_i are lattices in the Lie groups N_i (Definition 3.1). Since the N_i are connected, simply connected nilpotent Lie groups, $\exp: L(N_i) \to N_i$ is a diffeomorphism [V, p. 196]. Therefore N_i is contractible, which implies that $\Pi_1 J_i = \Delta_i$ for the space $J_i \equiv \Delta_i \backslash N_i$, and the higher homotopy groups vanish. That is, J_i is a $K(\Delta_i, 1)$ space.

Lemma 3.1. *If K is a connected CW complex and X is a connected space such that $\Pi_i(X) = 0$ for $i > 1$, then the correspondence $f \mapsto \Pi_1 f$ induces a one-to-one correspondence of the homotopy classes of maps from (K, k_0) to (X, x_0) with the homomorphisms from $\Pi_1 K$ to $\Pi_1 X$ (see* [Wh, *p.* 225] *for a proof*).

By the above lemma, it follows that the homomorphisms $\{f_i\colon \Pi \to \Delta_i\}$ determine continuous maps $\{F_i\colon (M, p) \to (J_i, \Delta_i e)\}$ unique up to homotopy, such that $\Pi_1 F_i = f_i$. Since any continuous map between compact manifolds is homotopic to a smooth map [BT, p. 213], the F_i can be chosen to be smooth.

We now show that the F_i can be chosen compatibly, i.e., so that F_{i+1} is a lift of F_i. The epimorphisms pr_i from (N_{i+1}, Δ_{i+1}) to (N_i, Δ_i) given by the Malcev completion (Definition 3.1), induce smooth maps $\overline{pr_i}$ from J_{i+1} onto J_i. Since $\overline{pr_i} \circ F_{i+1}$ and F_i induce the same homomorphism of $\Pi_1(M, p)$ onto $\Pi_1(J_i, \Delta_i e)$, they are homotopic by the lemma above. Since the homotopy can be chosen to be smooth, and can be smoothly lifted to J_{i+1}, it follows that F_{i+1} can be chosen to be a lift of F_i.

Let $\tilde{\omega}_i$ be the canonical $L(N_i)$ valued 1-form on N_i obtained by left translating tangent vectors to the identity.

Since, by definition, a curve δ in N_i is a "left" antiderivative of the curve $\tilde{\omega}_i \circ \delta'$ in $L(N_i)$, it follows that the definite integral ${}_L\int_\delta \tilde{\omega}_i = \delta(0)^{-1}\delta(1)$.

Since $\tilde{\omega}_i$ is left invariant, it induces $\check{\omega}_i$ on J_i which is pulled back by F_i to the desired ω_i on M. Therefore for any curve γ in M, its left Lie integral is obtained by mapping γ into J_i, lifting to N_i, and taking the difference of the endpoints. That is, ${}_L\int_\gamma \omega_i = {}_L\int_{F_i\gamma} \check{\omega}_i = {}_L\int_{\tilde{\gamma}} \tilde{\omega}_i = \tilde{\gamma}(0)^{-1}\tilde{\gamma}(1)$, where $\tilde{\gamma}$ is any lift to N_i of $F_i\gamma$.

It is now easy to see that ω_i has the desired properties. For any loop γ based at p in M, $f_i[\gamma] = \Pi_1 F_i[\gamma]$, which by the standard identification of the fundamental group with the fiber in the universal covering equals $\tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift to N_i starting at e of the loop $F_i \circ \gamma$ based at $\Delta_i e$. By our remark in the previous paragraph, $\int_\gamma \omega_i = \tilde{\gamma}(0)^{-1}\tilde{\gamma}(1) = \tilde{\gamma}(1)$.

Therefore ${}_L\int_\gamma \omega_i = f_i[\gamma]$. Since $\{f_i\}$ is a Malcev completion of $\Pi_1(M, p)$, $\{{}_L\int \omega_i\}$ gives a Malcev completion of $\Pi_1(M, p)$.

We next show that for any $q \in M$, the homomorphisms $[\alpha] \mapsto {}_L\int_\alpha \omega_i$ give a Malcev completion of $\Pi_1(M, q)$. Let β be any path in M from p to q, $\hat{\beta}\colon \Pi_1(M, q) \to \Pi_1(M, p)$ be the induced isomorphism, and $\langle\beta\rangle_i \equiv {}_L\int_\beta \omega_i$. Since

$$(f_i \circ \hat{\beta})[\alpha] = f_i[\beta^{-1}\alpha\beta] = {}_L\int_{\beta^{-1}\alpha\beta} \omega_i = \langle\beta\rangle_i^{-1}\left({}_L\int_\alpha \omega_i\right)\langle\beta\rangle_i,$$

it follows that

$${}_L\int_\alpha \omega_i = \langle\beta\rangle_i((f_i \circ \hat{\beta})[\alpha])\langle\beta\rangle_i^{-1}.$$

Therefore the homomorphism $[\alpha] \to {}_L\int_\alpha \omega_i$ is onto a lattice in N_i conjugate to the original lattice Δ_i. The kernel of the homomorphism is $\sqrt{\lambda_i(\Pi_1(M, q))}$. Q.E.D.

In our proof we start from the Malcev completion of the fundamental group, construct a "nonabelian Jacobian" $F_i\colon M \to J_i$, then produce the 1-forms ω_i by pulling back from J_i. Chen and Sullivan construct the Malcev completion from the algebra of differential forms on M.

Chen starts with n closed forms determining a basis in $H^1(M, \mathbf{R})$. He inductively constructs a noncommutative power series connection, i.e., a flat 1-form taking values in the tensor algebra generated by the vector space $\mathbf{R}^n$ [C2, pp. 184, 197]. By taking appropriate quotients, he obtains flat Lie algebra valued

1-forms determining connections on trivial vector bundles [C2, pp. 191, 192]. The holonomy homomorphisms from $\Pi_1 M$ to the Lie groups are computed using iterated integrals. The kernel of these homomorphisms are shown to be the groups $\{\gamma \in \Pi_1 M | \gamma - 1 \in I^i\}$, where I is the augmentation ideal of the group algebra $\mathbf{R}\Pi_1$ [C2, Corollary 2, pp. 192–196]. For any finitely generated group Γ, $\{\gamma \in \Gamma | \gamma - 1 \in I^i\}$ is sometimes referred to as the ith dimension subgroup $D_i(\Gamma)$, which is isomorphic to the group $\sqrt{\lambda_i \Gamma}$ [P, p. 474].

Sullivan inductively constructs a minimal model of the differential algebra of 1-forms. The minimal model is shown to be a homotopy invariant using a Postnikov tower construction. It is similarly shown that the dual Lie algebra to the minimal model is the Lie algebra corresponding to the Malcev completion of Π_1 [Su1, pp. 41, 48–49; DGMS, p. 259; GM, Chapter 12]. The 1-forms selected by the minimal model can be considered to be the Lie algebra valued 1-form referred to in Theorem 3.1. The Π_1 de Rham theory using minimal models can be proved for triangulable spaces which are not necessarily manifolds.

Both Chen and Sullivan show that if M has a Riemannian metric, then the Hodge decomposition allows the power series connection [C2, p. 187] or minimal model [Su1, p. 42] to be chosen canonically. Presumably this would correspond to the F_i in our proof being harmonic in some sense.

Chen and Sullivan also show that, using similar constructions, $\Pi_n M \otimes \mathbf{R}$ can be determined for $n \geq 2$ [C3, Su2].

3.4. **Iterated integrals.** If G is any Lie subgroup of $\mathrm{Gl}(n, \mathbf{R})$, then the left Lie integral of $L(G)$ valued 1-forms can be computed using iterated integrals in the sense of Chen [C2, p. 185; C1].

Definition 3.6 (Iterated integrals). Let $\mathscr{A}$ be an associative algebra over the reals and $\{\eta_i\}_{i=1}^n$ a collection of $\mathscr{A}$ valued 1-forms on a manifold M. If $\alpha: [a, b] \to M$ is a smooth path in M then the iterated integral

$$\int_\alpha \eta_1 \eta_2 \cdots \eta_n \equiv \int_S f_1(t_1) f_2(t_2) \cdots f_n(t_n),$$

where S is the simplex $\{(t_1, \ldots, t_n) \mid a \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq b\}$ and $f_i(t) = \eta_i(\alpha'(t))$.

For example, if η_1 and η_2 are real valued 1-forms, then

$$\int_\alpha \eta_1 \eta_2 = \int_a^b \left(\int_a^t f_1(s)\, ds \right) f_2(t)\, dt,$$

where $f_1(t) = \eta_1(\alpha'(t))$ and $f_2(t) = \eta_2(\alpha'(t))$.

Theorem 3.2 (Chen, Hain [H, §2]; evaluating Lie integrals with iterated integrals). *If G is a Lie subgroup of* $\mathrm{Gl}(n, \mathbf{R})$, *$\omega: TM \to L(G)$ is a Lie algebra valued* 1*-form, and α is a smooth path in M, then the left Lie integral ${}_L\int_\alpha \omega$ is equal to the sum of the infinite series of iterated integrals*

$$I + \int_\alpha \omega + \int_\alpha \omega\omega + \int_\alpha \omega\omega\omega + \cdots.$$

Furthermore, if G is nilpotent and a Lie subgroup of the group of upper triangular matrices (*with* 1 *down the diagonal*), *then the terms of the series become zero after finitely many steps.*

Theorem 3.2 is proved by applying the Picard iteration technique to solve the ordinary differential equation $F'(t) = (L_{F(t)})_* f(t)$ with initial condition $F(0) = I$, where $f(t) = \omega(\alpha'(t))$. Note that in defining the iterated integral $\int_\alpha \omega\omega \cdots \omega$ we are considering ω as taking values in the associative algebra of $n \times n$ matrices over the reals.

Example 3.1. If N is the group of upper triangular 3×3 matrices, and

$$\omega = \begin{pmatrix} 0 & \omega_1 & \omega_3 \\ 0 & 0 & \omega_2 \\ 0 & 0 & 0 \end{pmatrix}$$

is an $L(N)$ valued 1-form on a manifold M, then

$$\begin{aligned} {}_L\int_\alpha \omega &= I + \int_\alpha \omega + \int_\alpha \omega\omega \\ &= I + \begin{pmatrix} 0 & \int_\alpha \omega_1 & \int_\alpha \omega_3 \\ 0 & 0 & \int_\alpha \omega_2 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \int_\alpha \omega_1\omega_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \int_\alpha \omega_1 & \int_\alpha \omega_3 + \int_\alpha \omega_1\omega_2 \\ 0 & 1 & \int_\alpha \omega_2 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where $\int_\alpha \omega_1\omega_2$ is an iterated integral of real valued 1-forms.

The condition that ω is flat, i.e., that $d\omega = -[\omega, \omega]$, is equivalent to $d\omega_3 = -\omega_1 \wedge \omega_2$.

4. Asymptotic limits and averages in nilpotent Lie groups

The average of an infinite sequence in the abelian Lie group $\mathbf{R}^n$ is given by an asymptotic limit of its partial sums. That is $\mathrm{Av}\,\delta_t = \lim \frac{1}{t}\sum_{s=1}^{t} \delta_s$, where $s \in \mathbf{Z}$. For a connected, simply connected, 2-step nilpotent Lie group N, we define an average of an infinite sequence of elements to be the "asymptotic limit" of its partial products. The average, in this sense, of a periodic sequence gives the expected result. We similarly define the average of $\{\delta_t \in L(N) | t \in \mathbf{R}\}$, a family in the Lie algebra $L(N)$, in terms of the asymptotic limit of the family $\{g_t \in N | t \in \mathbf{R},\ g_t = {}_L\int \delta_s\}$. We have not yet determined the appropriate definition of the asymptotic limit for n-step nilpotent Lie groups, where $n > 2$ (this restriction is explained in §5).

4.1. **Preliminaries.** A group Γ is 2-step nilpotent if $\lambda_3(\Gamma) \equiv (\Gamma, (\Gamma, \Gamma))$, the third term in its lower centered series, is trivial. It is clear that for any such group, the commutator subgroup $\lambda_2(\Gamma) \equiv (\Gamma, \Gamma)$ is central in Γ.

For a connected, simply connected, nilpotent Lie group N with Lie algebra $L(N)$, the exponential map $\exp\colon L(N) \to N$ is a diffeomorphism with an inverse denoted by $\log\colon N \to L(N)$ [V, p. 196]. Therefore for all $h \in N$ and $r \in \mathbf{R}$, h^r is defined to be $\exp(r\log(h))$. In particular, elements in N have unique tth roots.

Throughout this section, N is a connected, simply connected, 2-step nilpotent Lie group with Lie algebra $L(N)$, (N, N) is its commutator subgroup, $\overline{N} = N/(N, N)$ is its abelianization, and $\overline{g} \in \overline{N}$ is the image of $g \in N$ under the canonical projection.

Definition 4.1 (Commutator for $\overline{N}$ taking values in (N, N)). If g is in N and $\overline{h}$ and $\overline{k}$ are in $\overline{N}$, then $(g, \overline{h}) \equiv (g, h)$, $(\overline{h}, g) \equiv (h, g)$, and $(\overline{h}, \overline{k}) \equiv (h, k)$.

These commutators are well defined since if $\overline{h}_1 = \overline{h}_2$ and $\overline{k}_1 = \overline{k}_2$, then h_1, and h_2 and k_1, and k_2 differ by elements in (N, N) which is central in N. This commutator is functorial, i.e., $\sigma(g, \overline{h}) = (\sigma g, \overline{\sigma}\overline{h})$, $\sigma(\overline{h}, g) = (\overline{\sigma}\overline{h}, \sigma g)$, and $\sigma(\overline{h}, \overline{k}) = (\overline{\sigma}\overline{h}, \overline{\sigma}k)$, where $\sigma: N \to N$ is a homomorphism inducing $\overline{\sigma}: \overline{N} \to \overline{N}$. Note that $(\ ,\ ): \overline{N} \times \overline{N} \to (N, N)$ is an alternating bilinear form on the vector group (i.e., vector space) $\overline{N}$ taking values in the vector group (i.e., vector space) (N, N).

We make some of our computations for N by considering the isomorphic "pullback group" $\widehat{N}$.

Definition (4.2) ($\widehat{N}$, the pullback group). If N is a connected, simply connected, 2-step nilpotent Lie group, then $\widehat{N}$ is the isomorphic group with its elements the elements of the Lie algebra $L(N)$, and its multiplication $\times$ given by $g \times h = g + h + \frac{1}{2}[g, h]$, where $[\ ,\ ]$ is the Lie bracket in $L(N)$.

By abuse of notation, we consider $\widehat{N}$ to be simultaneously a Lie algebra with its addition, scalar multiplication, and bracket, and a Lie group with multiplication $\times$. $\widehat{N}$ is the group obtained by pulling back the group structure of N to $L(N)$ by means of the exponential map. That is, $g \times h = \log(\exp(g)\exp(h))$, which is evaluated using the Baker-Campbell-Hausdorff formula [V, p. 114].

Lemma 4.1 (Properties of $\widehat{N}$). (a) *If $g \in \widehat{N}$, then $g^r = rg$, where rg is a scalar multiple of g. In particular, $g^{-1} = -g$.*

(b) *If g and h are in $\widehat{N}$, then the commutator (g, h) equals the Lie bracket $[g, h]$.*

(c) *If $g \in \widehat{N}$ and $h \in (\widehat{N}, \widehat{N})$, then $g \times h = g + h$.*

(d) $[g \times h, k] = [g, k] + [h, k]$.

Proof. (a) follow since $L(\widehat{N}) = \widehat{N}$ and $\exp: L(\widehat{N}) \to \widehat{N}$ is the identity. The proofs of (b), (c), and (d) are straightforward computations, where we use the fact that since N is 2-step nilpotent, all commutators of the form $[g, [h, k]]$ vanish in $L(N)$. Q.E.D.

4.2. **Definitions of Alim and average.** The following definition makes precise the notes of a family of elements being asymptotic to a one-parameter subgroup. As is usual in this paper, lim denotes the limit as $t \to \infty$.

Definition 4.3 (Alim and average). If g_t is a sequence $(t \in \mathbf{Z})$ or family $(t \in \mathbf{R})$ in N then:

(a) $\mu \equiv \lim \overline{g}_t/t \in \overline{N}$ is the abelian asymptotic limit.

(b) An element $\rho \in N$ is an asymptotic limit of g_t iff $\lim(\rho^{-t} g_t)^{1/t} = e$, i.e., iff the tth root of the displacement from ρ^t to g_t goes to the identity. We write $\rho = \operatorname{Alim} g_t$.

(c) If δ_t is a sequence in N, then we say $\rho = \operatorname{Av} \delta_t$, an average of δ_t, iff $\rho = \operatorname{Alim} g_t$, where $g_t \equiv \prod_{s=1}^{t} \delta_s$.

(d) If δ_t is a family in $L(N)$, then we say $\rho = \operatorname{Av} \delta_t$, an average of δ_t, iff $\rho = \operatorname{Alim} g_t$, where $g_t \equiv {}_L\int_0^t \delta_s$.

We will show below that $\operatorname{Alim} g_t$ and $\operatorname{Av} \delta_t$ are, in this sense, unique when they exist (Proposition 4.2). However the results below continue to hold for

$$\mathscr{A}\lim g_t \equiv \{\operatorname{Alim} g_{t_k} | g_{t_k} \text{ a subsequence of } g_t\},$$

which is the set of asymptotic limits of subsequences.

Note that we do not assume that our families $\{g_t | t \in \mathbf{R}\}$ are continuous.

4.3. **Properties of Alim.**

Proposition 4.1 (Alim is an isomorphism invariant). *If $\sigma: N \to N'$ is an isomorphism, then $\rho = \operatorname{Alim} g_t$ iff $\sigma\rho = \operatorname{Alim} \sigma g_t$.*

Proof.

$$\sigma \lim(\rho^{-t} g_t)^{1/t} = \lim((\sigma\rho)^{-t}\sigma g_t)^{1/t}.$$

So $\rho = \operatorname{Alim} g_t$ iff $\lim(\rho^{-t} g_t)^{1/t} = e$, iff $\lim((\sigma\rho)^{-t}(\sigma g_t))^{1/t} = e$, iff $\sigma\rho = \operatorname{Alim} \sigma g_t$. Q.E.D.

Note that in the following formula we first compute the abelian asymptotic limit and then use that result to compute Alim.

Proposition 4.2 (Formula for Alim, uniqueness of Alim). *If g_t is a sequence or family in N, then $\rho = \operatorname{Alim} g_t$ iff $\rho = \lim g_t^{1/t}(g_t, \mu)^{1/2}$ (Definition 4.3). Therefore $\operatorname{Alim} g_t$ is unique if it exists.*

Furthermore $\overline{\rho}$, the abelianization of ρ, equals μ, the abelian asymptotic limit.

Proof. If $\rho = \operatorname{Alim} g_t$, then $\lim(\rho^{-t} g_t)^{1/t} = e$. If we abelianize this expression, we obtain $\lim(-t\overline{\rho} + \overline{g}_t)/t = 0$. That is $\overline{\rho} = \lim \overline{g}_t/t \equiv \mu$. (So the existence of an $\operatorname{Alim} g_t$ implies the existence of μ.)

Lemma 4.2 (Alim in $\widehat{N}$). *If g_t is a sequence or family in $\widehat{N}$ (Definition 4.2), then $\rho = \operatorname{Alim} g_t$ iff $\rho = \lim g_t/t + \frac{1}{2}[g_t, \mu]$. (Note that Definition 4.1 and Lemma 4.1(b) allow us to make sense of the expression $[g_t, \mu]$.)*

Proof. In $\widehat{N}$, $\rho = \operatorname{Alim} g_t$ iff

$$\begin{aligned}
&\lim(-t\rho \times g_t)/t = e \\
&\quad \text{iff } \lim\left(-t\rho + g_t + \tfrac{1}{2}[-t\rho, g_t]\right)/t = e \\
&\quad \text{iff } \lim g_t/t + \tfrac{1}{2}[g_t, \rho] = \rho \\
&\quad \text{iff } \lim g_t/t + \tfrac{1}{2}[g_t, \overline{\rho}] = \rho \text{ (Definition 4.1, Lemma 4.1(b))} \\
&\quad \text{iff } \lim g_t/t + \tfrac{1}{2}[g_t, \mu] = \rho. \quad \text{Q.E.D. (Lemma 4.2)}
\end{aligned}$$

Proposition 4.2 follows for a sequence g_t in N by considering the isomorphism $\exp: \widehat{N} \to N$, and applying the isomorphism invariance property of Alim (Proposition 4.1). Q.E.D.

Proposition 4.3 (Alim is unchanged by bounded perturbations on the right). *If $\{h_t\}$ is bounded in N (i.e., is contained in a compact set), then $\operatorname{Alim} g_t$ exists iff $\operatorname{Alim} g_t h_t$ exists and if they exist they are equal.*

Proof. We can assume that g_t, h_t, and $g_t' \equiv g_t \times h_t$ are in $\widehat{N}$ (Definition 4.2).

Suppose $\operatorname{Alim} g_t$ exists. Since $\mu \equiv \lim(\overline{g}_t/t)$ exists, the boundedness of h_t implies that $\mu' \equiv \lim \overline{g}_t'/t = \lim \overline{g}_t/t + \overline{h}_t/t$ exists and equals μ. Suppressing the subscripts, we obtain

$$\begin{aligned}\operatorname{Alim} g' &= \lim \frac{g'}{t} + \frac{1}{2}[g', \mu'] \quad \text{(Lemma 4.2)}\\ &= \lim \frac{g}{t} + \frac{h}{t} + \frac{1}{2t}[g, h] + \frac{1}{2}[g, \mu] + \frac{1}{2}[h, \mu] = \operatorname{Alim} g,\end{aligned}$$

since $\lim h/t = 0$, and $\lim \frac{1}{2t}[g, h] = \lim \frac{1}{2t}[\overline{g}, h] = \frac{1}{2}[\mu, h] = -\frac{1}{2}[h, \mu]$.

The converse follows by setting $g_t = g'_t \times h_t^{-1}$. Q.E.D.

Proposition 4.4 ($\operatorname{Av} \delta_t \equiv \operatorname{Alim} g_t$ is correct for periodic sequences). (a) *If δ_t is a sequence of period T in N and $g_t \equiv \prod_{s=1}^{t} \delta_s$, then* $\operatorname{Av} \delta_t \equiv \operatorname{Alim} g_t = g_T^{1/T} = (\delta_1 \cdots \delta_T)^{1/T}$.

(b) *If δ_t is a family of period T in $L(N)$ and $g_t = {}_L\int_0^t \delta_s$, then* $\operatorname{Av} \delta_t = \operatorname{Alim} g_t = g_T^{1/T} = ({}_L\int_0^T \delta_s)^{1/T}$.

Proof. (a) If t is written as $nT+\varepsilon$, where n and ε are integer valued functions of t with $0 \le \varepsilon < t$, we define $g'_t \equiv \delta_1 \cdots \delta_{nT} = (g_T)^n$. Note that $\lim \frac{n}{t} = \frac{1}{T}$. Therefore the abelianization $\overline{g}'_t = n\overline{g_T}$, so $\mu' \equiv \lim \overline{g'}/t = \overline{g_T}/T$, which is the abelianization of $g_T^{1/T}$. So $(g'_t, \mu') = (g_T^n, g_T^{1/T}) = e$ (Definition 4.1). Therefore (suppressing subscripts),

$$\operatorname{Alim} g' = \lim g'^{(1/t)}(g', \mu')^{1/2} = \lim g'^{(1/t)} = \lim g_T^{1/T}.$$

Since $g'_t = g_t h_t$, where h_t is some bounded sequence, the result follows by Proposition 4.3.

(b) Define $g'_t \equiv_L \int_0^{nT} \delta_s$ and use the same proof as in (a). Q.E.D.

The following example shows the necessity of including the correction term $(g_t, \mu)^{1/2}$ in the definition of Alim.

Example 4.1 (Necessity of correction term). Let $\delta_t = \{a, b, a, b, a, b, a, \dots\}$ be a sequence in M. If $g_t \equiv \prod_{s=1}^{t} \delta_s$, observe that $g_{2t} = (ab)^t$ and $g_{2t+1} = (ab)^t a$. Therefore $\lim g_{2t}^{1/2t} = (ab)^{1/2}$.

For the odd terms, we assume the sequence is in $\widehat{N}$ (Definition 4.2). Then

$$g_{2t+1} = t(a \times b) \times a = t(a \times b) + a + \frac{1}{2}[t(a \times b), a] = t(a \times b) + a + \frac{t}{2}[b, a].$$

So

$$\lim \frac{g_{2t+1}}{2t+1} = \frac{a \times b}{2} + \frac{1}{4}[b, a].$$

Therefore if the sequence is in N,

$$\lim (g_{2t+1})^{1/(2t+1)} = (ab)^{1/2}(b, a)^{1/4}.$$

Therefore, if a and b do not commute, the limit of the tth root of the partial products, $\lim(\prod_{s=1}^{t} \delta_s)^{1/t} = \lim(g_t)^{1/t}$, fails to exist.

Proposition 4.5 ($\operatorname{Alim} hg_t$ for a constant h). *If h is an element in N, then $\operatorname{Alim} g_t$ exists iff $\operatorname{Alim} hg_t$ exists, and then*

$$\operatorname{Alim} hg_t = h(\operatorname{Alim} g_t)h^{-1} = (\operatorname{Alim} g_t)(h, \mu),$$

where $\mu = \lim \overline{g}_t/t$.

Proof. If $\operatorname{Alim} hg_t$ exists, then

$$\begin{aligned}\operatorname{Alim} hg_t &= \operatorname{Alim} hgh^{-1} \quad \text{(by Proposition 4.3)},\\ &= h(\operatorname{Alim} g_t)h^{-1} \quad \text{(by Proposition 4.2)}.\end{aligned}$$

Letting $\rho = \operatorname{Alim} g_t$, we have $h\rho h^{-1} = (h, \rho)\rho = \rho(h, \rho) = \rho(h, \overline{\rho}) = \rho(h, \mu)$.

The converse follows by setting $g_t = h^{-1}(hg_t)$, and applying the same argument. Q.E.D.

Definition 4.4 (Open and closed rays). The elements g and h in N are on the same closed (open) ray if $g = h^r$ for some $r \geq 0$ $(r > 0)$.

By Proposition 4.6 below, if $g_t = g'_{s_t}$ is an order-preserving reparametrization of g'_t, then under certain conditions on g and s, $\rho \equiv \operatorname{Alim} g$ and $\rho' \equiv \operatorname{Alim} g'$ are on the same closed ray. In particular, ρ and ρ' are on the same closed ray if either the correction term (g_t, μ) is bounded, or if $s_t = ct$ for same constant c.

Proposition 4.6 (Alim and reparametrization). *Let* $\{g_t | t \in \mathbf{R}\}$ *and* $\{g'_t | t \in \mathbf{R}\}$ *be families of elements in* N *such that* $g_t = g'_{s_t}$ *is an order-preserving reparametrization of* g'. *Define* $\mu \equiv \lim \overline{g_t}/t$ *and* $\mu' \equiv \lim \overline{g'_t}/t$, *when they exist.* (*In the following statements and proof, consider* s *and* g'_s *to be functions of* t, *and* $\lim$, *as usual, to denote the limit as* $t \to \infty$.)

(a) *Assume* $\rho = \operatorname{Alim} g_t$ *and* $\rho' = \operatorname{Alim} g'_t$ *exist.*

(i) *If* $\mu' \neq 0$, *then* $r = \lim \frac{s}{t}$ *exists. Furthermore,* ρ *and* ρ' *are on the same closed ray iff* $\rho = \rho'^r$ *iff* $\lim(g_t, \mu')^{(s/t - r)} = e$.

(ii) *If* $\mu = \mu' = 0$ *and* $\rho' \neq e$, *then* $r = \lim \frac{s}{t}$ *exists and* $\rho = \rho'^r$.

(b) *Assume* ρ' *and* r *exist. Then* ρ *exists iff* $\lim(g_t, \mu)^{(s/t - r)}$ *exists.*

Proof. (a)(i) $\mu = \lim \overline{g_t}/t = \lim(\overline{g'_s}/s)(s/t)$. Since $\mu' = \lim \overline{g'_t}/t = \lim \overline{g'_s}/s$ is assumed to be nonzero, $r = \lim \frac{s}{t}$ exists, and $\mu = r\mu'$.

If ρ and ρ' are on the same closed ray, then $\rho = \rho'^a$ for some $a \geq 0$, which implies that $\mu = a\mu'$. Since $\mu = r\mu'$, we obtain $a = r$ and $\rho = \rho'^r$.

Assume g_t and g'_t are in $\widehat{N}$ (Definition 4.2). We have

$$\begin{aligned}\rho = \operatorname{Alim} g_t &= \lim \frac{g_t}{t} + \frac{1}{2}[g_t, \mu] \quad \text{(Lemma 4.2)}\\ &= \lim \frac{g'_s}{s}\frac{s}{t} + \frac{1}{2}[g'_s, r\mu'].\end{aligned}$$

$$r\rho' = r \operatorname{Alim} g' = r\left(\lim \frac{g'_t}{t} + \frac{1}{2}[g'_t, \mu']\right) = r\left(\lim \frac{g'_s}{s} + \frac{1}{2}[g'_s, \mu']\right).$$

Therefore $\rho = r\rho'$ iff $\lim(r - s/t)(g'_s/s) = 0$. Since $\lim g'_s/s + \frac{1}{2}[g'_s, \mu'] = \rho'$, we obtain $\rho = r\rho'$ iff $\lim(r - s/t)[g'_s, \mu'] = 0$, iff $\lim(r - s/t)[g_t, \mu'] = 0$.

Therefore for a sequence in N,

$$\rho = \rho'^r \quad \text{iff} \quad \lim(g_t, \mu')^{(r - s/t)} = e.$$

(a)(ii) Assume g_t and g'_t are in $\widehat{N}$. If $\mu = \mu' = 0$, then $\rho = \lim g_t/t = \lim(g'_s/s)(s/t)$ and $\rho' = \lim g'_t/t = \lim g'_s/s$. So if $\rho' \neq e$, then $r = \lim \frac{s}{t}$ exists, and $\rho = r\rho'$.

(b) Assume g_t and g'_t are in $\widehat{N}$. From the proof of (a)(i) above,

$$g_t/t + \tfrac{1}{2}[g_t, \mu] = r(g'_s/s + \tfrac{1}{2}[g'_s, \mu']) + (s/t - r)(g'_s/s).$$

Therefore, $\rho = \operatorname{Alim} g_t = \lim g_t/t + \frac{1}{2}[g_t, \mu]$ exists iff $\lim(r - s/t)(g'_s/s)$ exists, which holds iff $\lim(r-s/t)[g'_s, \mu']$ exists, iff $\lim(r-s/t)[g_t, \mu]$ exists. Q.E.D.

5. Asymptotic homotopy cycles

The definition and properties of asymptotic homotopy cycles follow directly from the Π_1 de Rham Theorem (Theorem 3.1) and from the properties of asymptotic limits (§4). We restrict ourselves to asymptotic homotopy cycles taking values in 2-step nilpotent Lie groups pending a suitable definition of Alim for n-step nilpotent Lie groups. (Propositions 5.1–5.4, which give the desired properties of asymptotic homotopy cycles, depend on the invariance of Alim under bounded perturbations on the right (Proposition 4.3). Alim, as defined by Definition 4.3, does not have this invariance property for n-step nilpotent Lie groups, $n > 2$.)

5.1. In this section, unless otherwise stated, all curves are assumed to be continuous. As usual, manifolds are assumed to be smooth.

Definition 5.1 (Asymptotic homotopy cycles). If M is a closed manifold with $N \equiv N_3$ the 2-step nilpotent Lie group in its Malcev completion and $\omega: TM \to L(N)$ a representing 1-form given by the Π_1 de Rham Theorem, then an element $\rho = \rho(\alpha) \in N$ is the asymptotic homotopy cycle of a curve $\alpha: \mathbf{R} \to M$ if $\rho(\alpha) = \operatorname{Alim}_L \int_{\alpha_{|t}} \omega$, where $\alpha_{|t}$ is α restricted to $[0, t]$ and ${}_L\int_{\alpha_{|t}} \omega$ is the left Lie integral of $\alpha_{|t}$ (Definitions 3.1, 4.3, 3.4, Theorem 3.1).

If $\overline{N} \equiv N_2$ is the abelianization of N and $\overline{\omega}: TM \to L(\overline{N})$ is the corresponding form, then $\mu(\alpha) \equiv \lim \frac{1}{t} \int_{\alpha_{|t}} \overline{\omega}$ is the asymptotic homology of α.

Note that the above definition of asymptotic homology agrees with that given in §2. Furthermore, when ρ exists, its abelianization $\overline{\rho}$ is μ.

For the rest of this section, we assume that M, N, and ω are given as in Definition 5.1. If $f: M \to M'$ is a homeomorphism, we assume that M' comes similarly equipped with N' and ω'.

Definition 5.2 (Homotopy image of curves and homotopy classes). If $\alpha: I \to M$ is a curve defined on a finite interval, then $\langle\alpha\rangle \equiv {}_L\int_\alpha \omega$ is its homotopy image in N. If $[\alpha]$ is a homotopy class rel endpoints, then $\langle[\alpha]\rangle \equiv \langle\alpha\rangle$ is its homotopy image.

We can then express the asymptotic homotopy cycle $\rho(\alpha)$ as $\operatorname{Alim}\langle\alpha_{|t}\rangle$, the asymptotic limit of its homotopy images.

We can also express $\rho(\alpha) = \operatorname{Alim}_L \int_{\alpha_{|t}} \omega$ as $\operatorname{Av}\omega(\alpha'(t))$, the average of the infinitesimal displacements of the homotopy images of α (Definition 4.3).

Also note that the multiplicative property of left Lie integration (Definition 3.4 and following), can be expressed as $\langle\alpha\beta\rangle = \langle\alpha\rangle\langle\beta\rangle$, for curves α and β on M.

5.2. **Properties of asymptotic homotopy.** The formal properties of asymptotic homotopy are given in Propositions 5.1–5.7. These Propositions parallel Propositions 2.1–2.7 which give the properties of asymptotic homology, and hence show the appropriateness of our definition of asymptotic homotopy.

Proposition 5.1 ($\rho(\alpha)$ gives expected answer for loops). *If $\alpha: \mathbf{R} \to M$ is a curve of period T, then the asymptotic homotopy cycle $\rho(\alpha)$ exists and equals $\langle \alpha_{|T} \rangle^{1/T}$, the Tth root of the homotopy image of the loop $\alpha_{|T}$.*

Proof. Since α and its derivative α' has period T, $\omega \circ \alpha': \mathbf{R} \to L(N)$ has period T. Therefore $\rho(\alpha) = \mathrm{Av}\, \omega(\alpha'(t)) = \mathrm{Alim}\, {}_L\int_0^t \omega(\alpha'(s))$, which equals $({}_L\int_0^T \omega(\alpha'(s)))^{1/T} = \langle \alpha_{|T} \rangle^{1/T}$ by Proposition 4.4(b). Q.E.D.

Proposition 5.2 (Asymptotic homotopy is terms of integrating 1-forms agrees with asymptotic homotopy in terms of closing paths). *If $\alpha: \mathbf{R} \to M$ is a curve in M with $\{\beta_t\}$ a family of bounded closings (Definition 2.2) and $\gamma_t = \alpha_{|t}\beta_t$ the resulting loop based at $\alpha(0)$, then $\rho(\alpha) \equiv \mathrm{Alim}\langle \alpha_{|t} \rangle$ exists iff $\mathrm{Alim}\langle \gamma_t \rangle$ exists, and if they exist they are equal.*

Proof. Since $\{\beta_t\}$ is bounded, $\{\langle \beta_t \rangle\}$ is bounded. Since $\langle \gamma_t \rangle = \langle \alpha_{|t} \rangle \langle \beta_t \rangle$, it follows from Proposition 4.3 that $\rho(\alpha) \equiv \mathrm{Alim}\langle \alpha_{|t} \rangle$ exists iff $\mathrm{Alim}\langle \gamma_t \rangle$ exists. Q.E.D.

We next show that asymptotic homotopy changes in the expected way along a curve. The following characterization of this change was suggested by Bob Williams.

Definition 5.3 (Identification of $\Pi_1 M$ and its Malcev completion for different basepoints). (a) If δ is a curve in M from p to q, then $\hat{\delta}: \Pi_1(M, p) \to \Pi_1(M, q)$ is the standard identification taking a class $[\alpha]$ in $\Pi_1(M, p)$ to the class $[\delta^{-1}\gamma\delta]$ in $\Pi_1(M, q)$.

(b) $\hat{\delta}$ induces $\hat{\delta}_*: N \to N$, since N is the Malcev completion of both $\Pi_1(M, p)$ and $\Pi_1(M, q)$ by means of the integral ${}_L\int \omega$, that is, by the map taking a class $[\alpha]$ to its homotopy image $\langle \alpha \rangle$.

Lemma 5.1. *$\hat{\delta}_*$ is conjugacy by $\langle \delta \rangle^{-1}$.*

Proof. $\Pi_1(M, p)$ and $\Pi_1(M, q)$ map onto lattices Δ_p and Δ_q in N via the homomorphism taking a homotopy class $[\gamma]$ to its homotopy image $\langle [\gamma] \rangle = \langle \gamma \rangle$ (Definition 5.2). $\hat{\delta}: \Pi_1(M, p) \to \Pi_1(M, q)$ induces an isomorphism from Δ_p to Δ_q which extends uniquely to $\delta_*: N \to N$. For $\langle \gamma \rangle \in \Delta_p$, $\hat{\delta}_*\langle \gamma \rangle$ is the homotopy image of $\hat{\delta}[\gamma]$. Since $\hat{\delta}[\gamma] = [\delta^{-1}\gamma\delta]$, we have $\hat{\delta}\langle \gamma \rangle = \langle \delta^{-1} \rangle \langle \gamma \rangle \langle \delta \rangle$. Since $\hat{\delta}_*$ is determined by its action on Δ_p, $\hat{\delta}_*(g) = \langle \delta \rangle^{-1} g \langle \delta \rangle$ for all $g \in N$. Q.E.D.

Proposition 5.3 (Change of asymptotic homotopy along a curve). *If $\alpha: \mathbf{R} \to M$ is a curve in M, let $\beta: \mathbf{R} \to M$ be obtained from α by choosing $\alpha(t_0)$ as an initial point. That is, for some $t_0 \in \mathbf{R}$, $\beta(t) = \alpha(t + t_0)$ for all t. Let δ be the curve along α from $\alpha(0)$ to $\beta(0) = \alpha(t_0)$. Then $\rho(\alpha)$ exists iff $\rho(\beta)$ exists and then $\rho(\beta) = \hat{\delta}_*\rho(\alpha) = \langle \delta \rangle^{-1} \rho(\alpha) \langle \delta \rangle = \rho(\alpha)(\mu, \langle \delta \rangle)$, where μ is the asymptotic homology of both α and β.*

Proof. Since $\alpha_{|t+t_0} = \delta\beta_{|t}$ as curves, $\langle \alpha_{|t+t_0} \rangle = \langle \delta\beta_{|t} \rangle = \langle \delta \rangle \langle \beta_{|t} \rangle$. Therefore, by Proposition 4.5, $\rho(\alpha) \equiv \mathrm{Alim}\langle \alpha_{|t+t_0} \rangle$ exists iff $\rho(\beta) \equiv \mathrm{Alim}\langle \beta_{|t} \rangle$ exists, and in that case $\rho(\alpha) = \langle\delta\rangle\rho(\beta)\langle\delta\rangle^{-1}$. So $\rho(\beta) = \langle\delta\rangle^{-1}\rho(\alpha)\langle\delta\rangle$, which, by Lemma 5.1, is $\hat{\delta}_*\rho(\alpha)$, where $\hat{\delta}_*$ is the isomorphism induced by the curve δ.

Since N is 2-step nilpotent, $\rho(\alpha) = \rho(\beta)(\langle\delta\rangle, \rho(\beta))$, and thus $\rho(\beta) = \rho(\alpha)(\rho(\beta), \langle\delta\rangle)$. Since the abelianization $\overline{\rho(\beta)}$ of $\rho(\beta)$ equals μ, $\rho(\beta) = \rho(\alpha)(\mu, \langle\delta\rangle)$ (Definition 4.1). Q.E.D.

Proposition 5.4 (Asymptotic homotopy is a homeomorphism invariant). *If $\alpha\colon \mathbf{R} \to M$ is a continuous curve and $f\colon M \to M'$ is a homeomorphism, then $\rho(\alpha)$ exists iff $\rho(f\circ\alpha)$ exists, and in that case $f_*\rho(\alpha) = \rho(f\circ\alpha)$, where $f_*\colon N \to N'$ is the isomorphism induced by $\Pi_1 f\colon \Pi_1(M, \alpha(0)) \to \Pi_1(M', f(\alpha(0)))$.*

Proof. Let $\alpha' \equiv f\circ\alpha$. If $\{\beta_t\}$ is a family of bounded closings of α and $\gamma_t \equiv \alpha_{|t}\beta_t$ is the resulting family of loops approximating α, then $\{\beta_t'\} \equiv \{f\beta_t\}$ is a family of bounded closings of α', and $\gamma_t' \equiv \alpha'_{|t}\beta_t' = f(\gamma_t)$ is a family of loops approximating α' (Definition 2.4).

Assume $\rho(\alpha)$ and $\rho(\alpha')$ exists. By Proposition 5.2, $\rho(\alpha) = \operatorname{Alim}\alpha_{|t} = \operatorname{Alim}\gamma_t$ and $\rho(\alpha') = \operatorname{Alim}\langle\alpha'_{|t}\rangle = \operatorname{Alim}\langle\gamma_t'\rangle = \operatorname{Alim}\langle f\gamma_t\rangle$. But $\langle f\gamma_t\rangle = \langle[f\gamma_t]\rangle = \langle\Pi_1 f[\gamma_t]\rangle = f_*\langle[\gamma_t]\rangle = f_*\langle\gamma_t\rangle$, by definition of $\Pi_1 f$ and f_*. So $\operatorname{Alim}\langle f\gamma_t\rangle = \operatorname{Alim} f_*\langle\gamma_t\rangle$, which equals $f_*\operatorname{Alim}\langle\gamma_t\rangle$ by the isomorphism invariance of Alim (Proposition 4.1). Therefore $\rho(\alpha') = f_*\operatorname{Alim}\langle\gamma_t\rangle = f_*\rho(\alpha)$.

The same line of argument shows that $\rho(\alpha')$ exists iff $\rho(\alpha)$ exists. Q.E.D.

Notation 5.1. If ϕ is a flow and α is a trajectory starting at p, then $\rho(p) \equiv \rho(\alpha)$ and $\mu(p) \equiv \mu(\alpha)$.

Corollary 5.1 (Asymptotic homotopy is an invariant of topological conjugacy). *If $f\colon (M, p) \to (M', p')$ is a topological conjugacy of flows ϕ and ϕ' (Definition 2.3), then $\rho(p)$ exists iff $\rho(p')$ exists. In that case $f_*p = p'$, where $f_*\colon N \to N'$ is the isomorphism induced by $\Pi_1 f\colon \Pi_1(M, p) \to \Pi_1(M, p')$.*

Except in special circumstances, the asymptotic homotopy of a curve and its reparametrization are not a rescaling of each other, i.e., not on the same open ray in N. However, they are on the same open ray when the reparametrization is multiplication by a constant, or when the "correction term" $(\langle\alpha_{|t}\rangle, \mu)$ is bounded

Proposition 5.5 (Asymptotic homotopy and reparametrization). *Suppose α is a reparametrization by s of α' (Definition (2.7). If both curves have nontrivial asymptotic homology μ and μ' then $r = \lim\frac{s}{t}$ exists and is nonzero. If furthermore, the asymptotic homotopy ρ and ρ' of α and α' exist, then ρ and ρ' are on the same open ray with $\rho = \rho'^r$ precisely when $\lim(\langle\alpha_{|t}\rangle, \mu)^{s/t-r} = e$. (For the cases not covered in the above statement, apply Proposition 4.6.)*

Proof. Apply Proposition 4.6(a)(i), with $g_t \equiv \langle\alpha_{|t}\rangle$ and $g_t' \equiv \langle\alpha'_{|t}\rangle$. Since we are assuming both μ and μ' are not zero, $\mu' = \frac{\mu}{r}$. Therefore the condition that $\lim(g_t, \mu')^{s/t-r} = e$ is equivalent to the condition $\lim(g_t, \mu)^{s/t-r} = e$. Q.E.D.

Proposition 5.6 (Homeomorphisms not preserving parametrization). *Suppose $f\colon M \to M'$ is a homeomorphism taking a curve α to a reparametrization by s of a curve α' (Definition 2.5), and that $f_*\colon N \to N'$ is the isomorphism induced by $\Pi_1 f\colon \Pi_1(M, \alpha(0)) \to \Pi_1(M, \alpha'(0))$. If both α and α' have nontrivial* *asymptotic homology* μ *and* μ', *then* $r \equiv \lim \frac{s}{t}$ *exists and is nonzero. If furthermore, the asymptotic homotopy* ρ *and* ρ' *of* α *and* α' *exist, then* $f_*\rho$ *and* ρ' *are on the same open ray with* $f_*\rho = \rho'^r$ *precisely when* $\lim(\langle\alpha_{|t}\rangle, \mu)^{s/t-r} = e$.

Proof. The curve $f \circ \alpha$ has asymptotic homotopy $f_*\rho$ by Proposition 5.4. The result follows, since $f \circ \alpha$ is a reparametrization by s of α', and the conditions of Proposition 5.5 are satisfied for $f \circ \alpha$ and α'. Q.E.D.

Corollary 5.2 (Asymptotic homotopy and topological equivalence). *Suppose* f: $(M, p) \to (M', p')$ *is a topological equivalence of flows* ϕ *and* ϕ' *(Definition 2.6),* $f_*: N \to N'$ *is the isomorphism induced by* $\Pi_1 f: \Pi_1(M, p) \to \Pi_1(M', p')$, α *and* α' *are the trajectories starting at* p *and* p', $f(\alpha(t)) = \alpha'(s_t)$, *and the asymptotic homology and homotopy* $\mu \equiv \mu'(p)$, $\mu' \equiv \mu'(p)$, $\rho \equiv \rho(p)$, $\rho' \equiv \rho(p')$ *are all nontrivial. Then* $r = \lim \frac{s}{t}$ *is nonzero, and* $f_*\rho$ *and* ρ' *are on the same open ray precisely when* $\lim(\langle\alpha_{|t}\rangle, \mu)^{s/t-r} = e$.

The following rigidity result was proved in greater generality in [B], using, however, very different techniques.

Proposition 5.7 (Rigidity of nilflows). *For* N *a 2-step connected, simply connected, nilpotent Lie group and* $w \in N$, *let* $\phi_w: R \to N$ *be the one-parameter subgroup given by* $\phi_w(t) \equiv w^t$. *For* Γ *a lattice in* N, *let* ϕ^*_w *be the induced flow on the nilmanifold* $M = \Gamma\backslash N$ *(Definition 2.8). Let the representing form* $\omega: TM \to L(N)$ *be the canonical form defined by lifting a tangent vector to* TN *and left translating to the identity.*

(a) *For all* $p \in M$, $\rho(p)$ *exists and equals* w.
(b) *The flows* ϕ^*_w *and* $\phi^*_{w'}$ *are topologically conjugate iff they are affinely conjugate (Definition 2.9), which holds iff there exists* $a \in N$ *and an automorphism* A *of* N *extending an automorphism of* Γ *such that* $a(Aw)a^{-1} = w'$.
(c) *The flows* ϕ^*_w *and* $\phi^*_{w'}$ *are topologically equivalent iff they are affinely equivalent, which holds iff there exists* $r > 0$, $a \in N$, *and an automorphism* A *of* N *extending an automorphism of* Γ *such that* $a(Aw)a^{-1} = w'^r$.

Proof. (a) As we showed in our proof of the Π_1 de Rham Theorem (Theorem 3.1), for any curve δ in M, its Lie integral $_L\int_\delta \omega = \tilde{\delta}(0)^{-1}\tilde{\delta}(1)$, where $\tilde{\delta}$ is any lift to N of δ. Since any trajectory α of ϕ^*_w is the projection of a curve $\tilde{\alpha}$ in N of the form $\tilde{\alpha}(t) = g\phi_w(t)$ for some $g \in N$, it follows that $\langle\alpha_{|t}\rangle = {}_L\int_{\alpha_{|t}} \omega = \phi_w(t) \equiv w^t$. Therefore $\rho(\alpha) = \mathrm{Alim}\langle\alpha_{|t}\rangle = w$.

(b), (c) We showed in Lemma 5.1 that for any two points p and q in M, the lattices Δ_p and Δ_q which correspond to $\Pi_1(M, p)$ and $\Pi_1(M, q)$ are conjugate by some element in N. Suppose $p = \Gamma e$. Then $\Delta_p = \Gamma$. If $f_*: N \to N$ is the isomorphism induced by a homeomorphism $f: (M, p) \to (M, q)$ and $\Delta_q = a\Gamma a^{-1}$, then f_* followed by conjugation by a is an automorphism A extending an automorphism of Γ. So $f_*: g \mapsto a(Ag)a^{-1}$ for any $g \in M$.

Since we showed while proving (a) above that $\langle\alpha_{|t}\rangle = w^t$, the correction term $(\langle\alpha_{|t}\rangle, \mu) = (w^t, \rho) = (w^t = w) = e$. Therefore the condition of Proposition 5.6, for invariance of asymptotic homotopy under topological equivalences, is satisfied.

So if $f\colon (M, p) \to (M, q)$ is a topological conjugacy (equivalence) of ϕ^*_w and $\phi^*_{w'}$, then $f_*\rho(p) = \rho(q)$ $(f_*\rho(p) = \rho(q)^r)$ by Corollaries 5.1 and 5.2. Since $\rho(p) = w$ and $\rho(q) = w'$ ((a) above), and $f_* w = aAwa^{-1}$, we have shown $a(Aw)a^{-1} = w'$ $(= w'^r)$.

It is straightforward to show that for any w and w', $a(Aw)a^{-1} = w'$ iff ϕ^*_w and $\phi^*_{w'}$ are conjugate by the affine map $\overline{a^{-1}}\,\overline{A}$ (Definition 2.9). Q.E.D.

We conclude this section with the following comments which parallel our comments at the end of §2.

(1) We define homotopy directions for flows as follows. Assume M, $\{N_i\}$, and $\{\omega_i\}$ give a Malcev completion of $\Pi_1(M)$, as in Theorem 3.1. Let R_i be the space of open rays in N_i disjoint union the identity element (Definition 4.4). R_i is topologically a sphere disjoint union a point. For a flow ϕ on M and a closing sequence (m_k, t_k) based at a nonwandering point m (§2 (end)), let $\{[\gamma_k]\}$ be the resulting sequence in $\Pi_1(M, m)$ and $\{\langle\gamma_k\rangle_i\}$ its homotopy image in N_i. The sequence of rays determined by $\{\langle\gamma_k\rangle_i\}$ has accumulation points in R_i since R_i is compact. We call these accumulation points i-homotopy directions for ϕ at m. Taking the union of all i-homotopy directions over all closing sequences based at M, we obtain $R_i(\phi, m)$, a compact, nonempty set in R_i.

The strength of the approach to asymptotic homotopy through homotopy directions is that it is clearly an invariant of topological equivalence (since $R_i(\phi, m) = R_i(\phi', m)$ if ϕ' is any reparametrization of ϕ), and the existence of $R_i(\phi, m)$ does not depend on ϕ being measure-preserving. However the example discussed in Example 4.1 indicates that $R_i(\phi, m)$ could contain more than one ray even when the trajectory of ϕ through m is periodic.

(2) If the diffeomorphism $f\colon T^2 \to T^2$ is homotopic to a map induced by a matrix which is conjugate to $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ for some $k \in \mathbf{Z}^+$, then the suspension flow is a flow on a Heisenberg manifold $\Gamma_k\backslash H$, where H is the group of 3×3 upper triangular matrices and Γ_k is a lattice depending on k [AGH, p. 47]. The asymptotic homotopy of the suspension flow is then a homotopy rotation element for the diffeomorphism. We thus can extend the notion of rotation vector, which has previously only been defined and studied for maps of the torus which are homotopic to the identity [LM, MZ].

6. Homotopy foliation

Asymptotic homology and asymptotic homotopy are finer invariants when viewed globally; that is, when one considers not just the collection of all asymptotic cycles but rather the function which assigns to a point its asymptotic cycle. This is especially true of asymptotic homotopy. Since $\Pi_1(M, p)$ and $\Pi_1(M, q)$ are only identifiable up to conjugacy, their Malcev completions are also only identifiable up to conjugacy. Therefore it only makes sense to talk of the collection of all *conjugacy classes* of asymptotic homotopy cycles. Though this is adequate in some cases (Example 8.1), in most cases a lot of information would be lost. Therefore we proceed as follows.

Loosely speaking, if $\rho\colon M \to N$ and $\rho'\colon M' \to N'$ are the asymptotic homotopy of two flows ϕ and ϕ' (Notation 5.1), then a topological conjugacy takes graph ρ to graph ρ' while (in certain cases) a topological equivalence takes graph ρ to a rescaling of graph ρ'. It is therefore desirable to determine invariants of graph ρ. We will define a "homotopy foliation" of $M \times N$, the leaves of which are preserved by the map of $M \times N$ induced by a homeomorphism of M. Therefore topological characteristics of the intersection of graph ρ with leaves of the foliation are invariants of the flow ϕ. (For an application of these ideas, see Theorem 9.1.)

6.1. **Graph of asymptotic homotopy.** Assume that a manifold M and a nilpotent Lie group N are given as in Definition 5.1. A function η from a subset of M to N determines in the usual way graph η, which (by abuse of notation) can be thought of as a subset of $M \times N$ or as a function from the given subset of M to $M \times N$.

Note that the asymptotic homotopy function ρ (Definition 5.1, Notation 5.1) is not necessarily defined on all of M, and it is not necessarily continuous.

The following definition is useful in dealing with questions of topological equivalence.

Definition 6.1 (Projective equivalence of graphs). Two graphs in $M \times N$ are projectively equivalent if they have the same domain and if (p, g) being in one of the graphs implies that (p, g^s) is in the other graph for some $s > 0$.

A homeomorphism of M can be lifted algebraically to $M \times N$.

Definition 6.2 (Algebraic lift of a homeomorphism). A homeomorphism $f: M \to M'$ induces the algebraic lift $f_\#: M \times N \to M' \times N'$, which is the homeomorphism given by $f_\#(p, g) = (f(p), f_{p*}(g))$, where $f_{p*}: N \to N'$ denotes the isomorphism induced by $\Pi_1 f: \Pi_1(M, p) \to \Pi(M', f(p))$ via Malcev completion (Definition 3.1).

Clearly $f_\#$ takes graphs to graphs. Also note that if one thinks of $M \times N$ as the trivial bundle over M, then $f_\#$ has the interesting property of being topological on the base and algebraic on the fiber.

The following remark follows directly from Corollaries 5.1 and 5.2.

Remark 6.1 (Invariance of graph ρ). Suppose ϕ and ϕ' are flows on M and M' with asymptotic homotopy ρ and ρ'. If f is a topological conjugacy of ϕ and ϕ', then $f_\#(\text{graph}\,\rho) = \text{graph}\,\rho'$. If f is a topological equivalence of ϕ and ϕ', and if, furthermore, the conditions of Corollary 5.2 hold, then $f_\#(\text{graph}\,\rho)$ is projectively equivalent to graph ρ'.

6.2. **The homotopy foliation.** Recall that a path δ from p to q in M induces an isomorphism $\hat{\delta}_*$ from N to N via Malcev completion (Definition 5.3).

6.3 (Homotopy foliation). (p, g) and (q, h) are on the same leaf of the homotopy foliation $\mathscr{F}$ of $M \times N$ if there exists a path δ from p to q in M such that $\hat{\delta}_*(g) = h$.

Note that (p, g) and (p, g') are on the same leaf iff g and g' are conjugate in N by the homotopy image of some loop based at p (Lemma 5.1). If one uses the leaf topology as opposed to the induced topology, then the restriction to leaves of the projection $(M \times N) \to M$ is a local homeomorphism.

Proposition 6.1 (Algebraic lifts preserve the homotopy foliation). *If $f: M \to M'$ is a homeomorphism, then the algebraic lift $f_\#: M \times N \to M' \times N'$ takes the homotopy foliation $\mathscr{F}$ of $M \times N$ to the homotopy foliation $\mathscr{F}'$ of $M' \times N'$.*

Proof. Let (p, g) and (q, h) be points on the same leaf of $\mathscr{F}$. We need to show that $f_{\#}(p, g) = (f(p), f_{p^*}(g)) \equiv (p', g')$ and $f_{\#}(q, h) = (f(q), f_{q^*}(h)) \equiv (q', h')$ are on the same leaf of $\mathscr{F}'$.

Suppose δ is the path in M from p to q such that $\hat{\delta}_*(g) = h$. We wish to show that if $\delta' = f \circ \delta$, then $\hat{\delta}'_*(g') = h'$.

If $\hat{\delta}$ and $\hat{\delta}'$ are the usual maps on fundamental groups (Definition 5.3), then the following diagram clearly commutes:

$$\begin{tikzcd} \Pi_1(M, p) \arrow[r, "\hat{\delta}"] \arrow[d, "\Pi_1 f"] & \Pi_1(M, q) \arrow[d, "\Pi_1 f"] \\ \Pi_1(M', p') \arrow[r, "\hat{\delta}'"] & \Pi_1(M', q') \end{tikzcd}$$

Since Malcev completion is functorial, then the following diagram also commutes:

$$\begin{tikzcd} N \arrow[r, "\hat{\delta}_*"] \arrow[d, "f_{p*}"] & N \arrow[d, "f_{q*}"] \\ N' \arrow[r, "\hat{\delta}'_*"] & N' \end{tikzcd}$$

Therefore $\hat{\delta}'_*(g') = \hat{\delta}'_*(f_{p*}(g)) = f_{q*}(\hat{\delta}_*(g)) = f_{q*}(h) = h'$. Q.E.D.

Since we have shown that both graph ρ and the homotopy foliation $\mathscr{F}$ are preserved by $f_{\#}$ (Remark 6.1, Proposition 6.1), it follows that topological characteristics of the intersection of graph ρ with leaves of $\mathscr{F}$ are indeed invariants of the flow ϕ.

7. Flows on Heisenberg manifolds

In this section we characterize all measure-preserving flows on Heisenberg manifolds, and then we restrict to those measure-preserving flows which are lifts of flows from the 2-torus. We use standard techniques.

7.1. **Measure-preserving flows on Heisenberg manifolds and tori.** The Heisenberg group is the only three-dimensional connected, simply connected, nonabelian, nilpotent Lie group. It is often represented as the space of 3×3 matrices with zero below the diagonal and one down the diagonal. However, for most of our purposes, we prefer to consider the isomorphic "pullback group" (Definition 4.2, Lemma 4.1) obtained by using the exponential map to transfer to the Lie algebra the group structure of the matrix group. Therefore the Heisenberg group, H, will be the set of elements $\langle x, y, z\rangle$, where x, y, and z are in $\mathbf{R}$, with a topology as in $\mathbf{R}^3$, and with group multiplication given by the rule

$$\langle x, y, z\rangle \cdot \langle x', y', z'\rangle = \left\langle x + x', y + y', z + z' + \frac{xy' - yx'}{2} \right\rangle .$$

Note that $\langle 0, 0, 0\rangle$ is the identity in H. In this representation of the Heisenberg group, since the exponential map is the identity, we can identify the Lie algebra $L(H)$ with H. The Lie bracket in $L(H)$ is given by

$$[\langle a, b, c\rangle, \langle a', b', c'\rangle] = \langle 0, 0, ab' - a'b\rangle.$$

For k a natural number, let $\Gamma_k = \langle l, m, n/k + (lm)/2\rangle \mid l, m, n \in \mathbf{Z}\}$. Γ_k is a lattice in H (Definition 2.7). For $k \neq k'$, Γ_k is not isomorphic to $\Gamma_{k'}$. Every lattice Γ in H is isomorphic to some Γ_k. (See [AGH, p. 46]. Note that we use a different representation of H.)

Definition 7.1 (Heisenberg manifold). If Γ is a lattice in H, then the homogeneous space $\Gamma\backslash H$ is called a Heisenberg manifold. If Γ is one of the lattices Γ_k, then $\Gamma\backslash H$ is called (by us) a standard Heisenberg manifold.

Recall that Malcev [M] showed that isomorphisms of lattices in simply connected nilpotent Lie groups extend to isomorphisms of the Lie group. Thus every Heisenberg manifold is affinely equivalent (Definition 2.9) to a standard Heisenberg manifold. In what follows we state our results for standard Heisenberg manifolds, since, via the affine equivalence, they can easily be translated to any Heisenberg manifold.

For the remainder of this section we will use the following assumptions and notation.

Convention. $M = \Gamma\backslash H$ is a standard Heisenberg manifold. $\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$, $\frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$, and $\frac{\partial}{\partial z}$ are a basis of left invariant vector fields on H, with a dual basis dx, dy, $dz+\frac{y}{2}\,dx-\frac{x}{2}\,dy$ of one-forms. These induce vector fields X_1, X_2, X_3, and one-forms ω_1, ω_2, ω_3 on M which span the tangent and cotangent spaces at each point $p \in M$, and such that $\omega_i X_j(p) = \delta_{ij}$. A vector field V on M can be written $V_1X_1 + V_2X_2 + V_3X_3$ and a one-form η on M can be written $\eta_1\omega_1 + \eta_2\omega_2 + \eta_3\omega_3$, where the V_i and η_i are functions on M. $\omega \equiv \omega_1 \wedge \omega_2 \wedge \omega_3 \equiv \omega_1\omega_2\omega_3$ is a volume form on M which induces the measure on M which we are considering.

Proposition 7.1 (Measure-preserving flows on Heisenberg manifolds). *Let $V = V_1X_1 + V_2X_2 + V_3X_3$ be a vector field on a standard Heisenberg manifold $M = \Gamma\backslash H$. Then the following are equivalent*:

(a) *The flow ϕ induced by V is measure-preserving.*
(b) *The 2-form $i_V\omega$ is closed, where $i_V\omega$ is the interior multiplication of V and ω* [W, *p.* 68].
(c) $X_1V_1 + X_2V_2 + X_3V_3 = 0$, *where the vector fields X_i are considered as differential operators.*
(d) $V_1 = \beta_1 + X_2\eta_3 - X_3\eta_2$, $V_2 = \beta_2 + X_3\eta_1 - X_1\eta_3$, *and* $V_3 = X_1\eta_2 - X_2\eta_1 - \eta_3$, *where η_1, η_2, and η_3 are functions on M, and β_1 and β_2 are real numbers.*

Proof. (a) iff (b) [Ma, p. 35]. V induces a measure-preserving flow on M iff $\mathscr{L}_V\omega = 0$ ($\mathscr{L}_V$ is the Lie derivative), iff $i_V\,d\omega + di_V\omega = 0$ [W, p. 70], iff $di_V\omega = 0$, iff $i_V\omega$ is closed.

(b) iff (c). $i_V\omega = V_1(\omega_2\wedge\omega_3) + V_2(\omega_3\wedge\omega_1) + V_3(\omega_1\wedge\omega_2)$ [W, p. 61]. $di_V\omega = (X_1V_1 + X_2V_2 + X_3V_3)\omega_1\omega_2\omega_3$. So $di_V\omega = 0$ iff $X_1V_1 + X_2V_2 + X_3V_3 = 0$.

(b) iff (d). $[\omega_1]$ and $[\omega_2]$ generate $H^1(M; \mathbf{R})$. By Poincaré duality, $[\omega_2 \wedge \omega_3]$ and $[\omega_3 \wedge \omega_1]$ generate $H^2(M; \mathbf{R})$ [W, p. 226]. Therefore any closed 2-form on M can be written $\beta_1(\omega_2 \wedge \omega_3) + \beta_2(\omega_3 \wedge \omega_1) + d\eta$.

$$d\eta = d(\eta_1\omega_1 + \eta_2\omega_2 + \eta_3\omega_3) = d\eta_1 \wedge \omega_1 + d\eta_2 \wedge \omega_2 + d\eta_3 \wedge \omega_3 - \eta_3(\omega_1 \wedge \omega_2),$$

since $d\omega_1 = d\omega_2 = 0$ and $d\omega_3 = -\omega_1 \wedge \omega_2$. Since $d\eta_i = (X_1\eta_i)\omega_1 + (X_2\eta_i)\omega_2 + (X_3\eta_i)\omega_3$, it follows that

$$d\eta = (X_2\eta_3 - X_3\eta_2)\omega_2 \wedge \omega_3 + (X_3\eta_1 - X_1\eta_3)\omega_3 \wedge \omega_1 + (X_1\eta_2 - X_2\eta_1 - \eta_3)\omega_1 \wedge \omega_2 .$$

Since $i_V\omega = V_1(\omega_2 \wedge \omega_3) + V_2(\omega_3 \wedge \omega_1) + V_3(\omega_1 \wedge \omega_2)$, (b) iff (d) follows by equating the coefficients of $\omega_2 \wedge \omega_3$, $\omega_3 \wedge \omega_1$, and $\omega_1 \wedge \omega_2$. Q.E.D.

Using a similar technique one obtains the following characterization of measure-preserving flows on the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. The measure on T^2 is, as usual, that induced by ordinary Lebesgue measure on $\mathbf{R}^2$.

Proposition 7.2 (Measure-preserving flows on T^2). *A flow on the torus T^2 is measure-preserving iff there exist constants β_1 and β_2 and a doubly periodic function $f: \mathbf{R}^2 \to \mathbf{R}$ such that all trajectories of the lifted flow on $\mathbf{R}^2$ satisfy $x' = \beta_1 + f_y$ and $y' = \beta_2 - f_x$.*

7.2. **Lifted flows.** A standard Heisenberg manifold $M = \Gamma\backslash H$ (Definition 7.1) is a circle bundle over the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ as follows. The homomorphism $\langle x, y, z\rangle \mapsto (x, y)$ from H to $\mathbf{R}^2$ induces a map from M to T^2 such that the following diagram commutes, where the maps $H \to M$ and $\mathbf{R}^2 \to T^2$ are the natural projections:

$$\begin{array}{ccc} H & \longrightarrow & \mathbf{R}^2 \\ \downarrow & & \downarrow \\ M & \longrightarrow & T^2 \end{array}$$

These circle bundles are topologically nontrivial.

Convention. For the rest of this paper we will refer to functions, vector fields, and flows as *lifted* if they are lifted from T^2 to a standard Heisenberg manifold by means of the canonical projection defined above.

Recall now our conventions from the previous subsection. The vector field X_3 is tangent to the fibers of the bundle $M \to T^2$. Therefore a function $g: M \to \mathbf{R}$ is lifted from a function on T^2 iff $X_3 g = 0$. Since X_1 and X_2 lift from T^2, it follows that a vector field $V = V_1X_1 + V_2X_2 + V_3X_3$ on M lifts from T^2 iff X_3V_1 and $X_3V_2 = 0$. It remains, then, to characterize those vector fields on M which both lift and are measure-preserving.

Proposition 7.3 (Lifted measure-preserving flows). *Let $V = V_1X_1 + V_2X_2 + V_3X_3$ be a vector field on a standard Heisenberg manifold $M = \Gamma\backslash H$. Then the following are equivalent:*

(a) *The flow ϕ induced by V is both measure-preserving and lifted from T^2.*
(b) $X_1V_1 + X_2V_2$, X_3V_1, X_3V_2, *and* X_3V_3 *are all* 0, *where the X_i are considered as differential operators on functions.*
(c) $d\theta = 0$ *and* $X_3V_3 = 0$, *where θ is the* 1-*form* $V_1\omega_2 - V_2\omega_1$.
(d) $V_1 = \beta_1 + X_2 f$, $V_2 = \beta_2 - X_1 f$, *and* $V_3 = F$, *where β_1 and β_2 are real numbers and f and F are functions on M which are lifts of functions on T^2.*

Furthermore, if the above conditions hold, then the flow on M is lifted from a flow on T^2 which is itself measure-preserving (with respect to the measure induced by Lebesgue measure on $\mathbf{R}^2$).

Proof. (a) iff (b). By Proposition 7.1, V is measure-preserving iff $X_1V_1+X_2V_2+X_3V_3=0$. Since, as we observed above, $X_3V_1=X_3V_2=0$ for lifted flows, it follows that for V a measure-preserving lift we have

$$0=X_3(X_1V_1+X_2V_2+X_3V_3)=X_1X_3V_1+X_2X_3V_2+X_3X_3V_3=X_3X_3V_3.$$

(X_3 commutes with X_1 and X_2.) However, for all M, g, and X, where M is a compact manifold, g a function on M, and X a vector field on M, if $XXg=0$ then $Xg=0$. ($XXg=0$ implies Xg is constant along trajectories, which implies that g increases or decreases without bound unless Xg is 0.) So $X_3X_3V_3=0$ implies $X_3V_3=0$. Since $X_1V_1+X_2V_2+X_3V_3=0$, condition (b) holds.

Conversely, if (b) holds, $X_1V_1+X_2V_2+X_3V_3=0$, which shows that V is measure-preserving (Proposition 7.1), and $X_3V_1=X_3V_2=0$, which shows that V is a lift.

(b) iff (c). $\theta=V_1\omega_2-V_2\omega_1$.

$$\begin{aligned} d\theta=dV_1\omega_2-dV_2\omega_1&=[(X_1,V_1)\omega_1+(X_2V_2)\omega_2+(X_3V_3)\omega_3]\omega_2\\ &\quad-[(X_1V_2)\omega_1+(X_2V_2)\omega_2+(X_3V_3)\omega_3]\omega_1\\ &=(X_1V_1+X_2V_2)\omega_1\omega_2-(X_3V_1)\omega_2\omega_3-(X_3V_2)\omega_3\omega_1.\end{aligned}$$

Therefore, $d\theta=0$ iff $X_1V_1+X_2V_2=X_3V_1=X_3V_2=0$.

(c) iff (d). Since $H^1(M;\mathbf{R})$ is generated by $[\omega_1]$ and $[\omega_2]$, it follows that any closed 1-form on M can be written in the form $\alpha_1\omega_1+\alpha_2\omega_2+df$, where α_1 and α_2 are real numbers and f is a function on M. Therefore, since θ has no ω_3 component, θ is closed iff

$$\theta=\alpha_1\omega_1+\alpha_2\omega_2+df=(\alpha_1+X_1f)\omega_1+(\alpha_2+X_2f)\omega_2,$$

where f is a function on M such that $X_3f=0$. But $\theta=V_1\omega_2-V_2\omega_1$. Therefore, θ is closed and $X_3V_3=0$ iff $V_1=\beta_1+X_2f$, $V_2=\beta_2-X_1f$, and $V_3=F$, where f and F are functions on M which lift functions on T^2, $\beta_1=\alpha_2$, and $\beta_2=-\alpha_1$.

Finally, it is easy to see that a vector field satisfying condition (d) projects to a vector field on T^2 which satisfies the conditions of Proposition 7.2, and which, therefore, induces a measure-preserving flow on T^2. Q.E.D.

8. Asymptotic homotopy for lifted flows on Heisenberg manifolds

In §8.1 we first obtain an expression for the asymptotic homotopy of a curve on any manifold M which has the Heisenberg group H as the group $N\equiv N_3$ in the Malcev completion (Definition 3.1) of its fundamental group. (An example of such an M which is not homeomorphic to a Heisenberg manifold (Definition 7.1) is the connected sum of two copies of $S^1\times S^2$. The fundamental group of this manifold, which is the free group on two generators, has its $N_3=H$.) We then restrict to the case when M is in fact a Heisenberg manifold. In §8.2 we evaluate the resulting expression in the case of measure-preserving lifted flows. In §8.1 we also compute the asymptotic homotopy of the geodesic flows for Heisenberg manifolds, and we use this computation to distinguish certain of these flows up to topological conjugacy.

8.1. **Asymptotic homotopy on Heisenberg manifolds.**

Proposition 8.1 (Asymptotic homotopy when the Malcev completion is H). *Suppose M is a manifold which has the group $N \equiv N_3$ of its Malcev completion equal to the Heisenberg group H (§7.1), and that $\langle \omega_1, \omega_2, \omega_3 \rangle$ is the representing $L(H)$ valued 1-form given by the Π_1 de Rham Theorem (Theorem 3.1). Then, when the limits exist, the asymptotic homotopy (Definition 5.1) ρ of a curve α on M is given by*

$$\rho = \langle \mu_1, \mu_2, \mu_3 + \mu_{12} \rangle,$$

where, in terms of integrals and integrated integrals (Definition 3.6), we define

$$\mu_i \equiv \lim \frac{1}{t} \int_{\alpha|_t} \omega_i,$$

and

$$\mu_{12} \equiv \lim \left(\frac{1}{2t} \int_{\alpha|_t} (\omega_1\omega_2 - \omega_2\omega_1) + \frac{1}{2} \left(\mu_2 \int_{\alpha|_t} \omega_1 - \mu_1 \int_{\alpha|_t} \omega_2 \right) \right).$$

Proof. By our remarks at the end of §3.2, we can assume that α is smooth. Also note that in our representation of the Heisenberg group the exponential map is the identity, so we can identify $L(H)$ with H.

Recall that the asymptotic homotopy $\rho(\alpha) = \operatorname{Alim} {}_L\!\int_{\alpha|_t} \langle \omega_1, \omega_2, \omega_3 \rangle$ (Definition 5.1).

If we let $f_i(t)$ be ω_i evaluated on the tangent vector $\alpha'(t)$, and let $f \equiv \langle f_1, f_2, f_3 \rangle$ be the resulting $L(H)$ valued function, then ${}_L\!\int_{\alpha|_t} \langle \omega_1, \omega_2, \omega_3 \rangle$ equals ${}_L\!\int_0^t f$ (Definition 3.4), which we can directly compute.

Lemma 8.1. (a) *If $f\colon [a, b] \to L(H)$ is a continuous function given by $f(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$, then ${}_L\!\int_a^b f$ exists and equals*

$$\left\langle \int_a^b f_1, \int_a^b f_2, \int_a^b f_3 + \frac{1}{2} \left(\int_a^b \left(\int_a^t f_1(s)\, ds \right) f_2(t)\, dt \right.\right. \\ \left.\left. - \int_a^b \left(\int_a^t f_2(s)\, ds \right) f_1(t)\, dt \right) \right\rangle.$$

(b) *If $\langle \omega_1, \omega_2, \omega_3 \rangle$ is a $L(H)$ valued 1-form then*

$${}_L\!\int_{\alpha|_t} \langle \omega_1, \omega_2, \omega_3 \rangle = \left\langle \int_{\alpha|_t} \omega_1, \int_{\alpha|_t} \omega_2, \int_{\alpha|_t} \omega_3 + \frac{1}{2} \int_{\alpha|_t} \omega_1\omega_2 - \omega_2\omega_1 \right\rangle.$$

Proof. By using Definition 3.3 and the fact that exp is the identity, we obtain ${}_L\!\int_a^b f(t)\, dt$ is a limit of products of the form

$$\langle f_1(t_i^*)\Delta t_1, f_2(t_1^*)\Delta t_1, f_3(t_1^*)\Delta t_1 \rangle \cdots \langle f_1(t_n^*)\Delta t_n, f_2(t_n^*)\Delta t_n, f_3(t_n^*)\Delta t_n \rangle \\ = \left\langle \sum_{i=1}^n f_1(t_i^*)\Delta t_i, \sum_{i=1}^n f_2(t_i^*)\Delta t_i, \sum_{i=1}^n f_3(t_i^*)\Delta t_i \right. \\ \left. + \frac{1}{2} \sum_{j=2}^n \sum_{i=1}^{j-1} f_1(t_i^*)\Delta t_i f_2(t_j^*)\Delta t_j - f_2(t_i^*)\Delta t_i f_1(t_j^*)\Delta t_j \right\rangle.$$

By using the Riemann integral for the single sums and the Riemann double integral and Fubini theorem for the double sums, we see that the limits exist and have the form indicated.

The result for the 1-form follows by using the definition of iterated integrals (Definition 3.6). (An alternative proof of part (b) of the lemma uses Example 3.1.) Q.E.D. (Lemma 8.1)

We can now compute $\rho(\alpha) = \operatorname{Alim}_L \int_{\alpha|_t} \langle \omega_1, \omega_2, \omega_3 \rangle$ by using the formula for Alim in Lemma 4.2. Note that since the homomorphism $\langle x, y, z \rangle \mapsto (x, y)$ is the abelianization of H, we have $\mu = (\mu_1, \mu_2)$ (Proposition 4.2). Also recall that the Lie bracket in $L(H)$ is given by $[\langle a, b, c \rangle, \langle a', b', c' \rangle] = \langle 0, 0, ab' - a'b \rangle$. Q.E.D.

We now consider the special case where $M = \Gamma \backslash H$ is a Heisenberg manifold (Definition 7.1). The following Corollary shows that when M is a Heisenberg manifold, the asymptotic homotopy of a curve on M can be easily expressed in terms of its lift to H.

Our proof of the Π_1 de Rham Theorem (Theorem 3.1) showed that the representing $L(H)$ valued 1-form on M is given by lifting a tangent vector from TM to TH and then left translating to the identity. Therefore any choice of a basis of left invariant 1-forms on H induces a representing 1-form on M.

Definition 8.1 (Standard representing 1-form). If M is a Heisenberg manifold (Definition 7.1), then the *standard representing* 1-*form* on M is the 1-form $\langle \omega_1, \omega_2, \omega_3 \rangle$, which is induced by the $L(H)$ valued 1-form $\langle dx, dy, dz + \frac{y}{2}\, dx - \frac{x}{2}\, dy \rangle$ on H (§7.1, conventions).

Corollary 8.1 (Asymptotic homotopy on Heisenberg manifolds). *Suppose M is a Heisenberg manifold, $\langle \omega_1, \omega_2, \omega_3 \rangle$ is the standard representing* 1-*form, α is a curve on M, and $\tilde{\alpha}(t) = \langle x(t), y(t), z(t) \rangle$ is the position at time t of a lift of α to H. Then, when the limits exist, the asymptotic homotopy ρ of α is given by*

$$\rho = \langle \mu_1, \mu_2, \mu_3 + \mu_{12} \rangle,$$

where

$$\mu_1 = \lim \frac{x(t) - x(0)}{t}, \qquad \mu_2 = \lim \frac{y(t) - y(0)}{t},$$
$$\mu_3 = \lim \left(\frac{z(t) - z(0)}{t} + \frac{1}{2t} \int_0^t y(s)x'(s) - x(s)y'(s)\, ds \right),$$

and

$$\mu_{12} = \lim \left(\frac{1}{2t} \int_0^t ((x(s) - x(0))y'(s) - (y(s) - y(0))x'(s))\, ds \right.$$
$$\left. + \frac{1}{2}(\mu_2(x(t) - x(0)) - \mu_1(y(s) - y(0))) \right).$$

(*For our purposes, it is not helpful to simplify in the obvious way the $\mu_3 + \mu_{12}$ term.*)

Alternately, observe that

$$
{}_L\int_{\alpha|_t} \langle \omega_1 , \omega_2 , \omega_3 \rangle = \tilde{\alpha}(0)^{-1}\tilde{\alpha}(t) = \langle x(t) - x(0) , y(t) - y(0) , z(t) - z(0) \\ + \tfrac{1}{2}(x(t)y(0) - y(t)x(0))\rangle ,
$$

and $\rho(\alpha)$ *is* Alim *of this expression.*

Example 8.1 (Asymptotic homotopy of geodesic flow on Heisenberg manifolds). In this example, we compute the asymptotic homotopy of the geodesic flow for Heisenberg manifolds, and show that we can thereby distinguish up to topological conjugacy certain of these flows which are indistinguishable by asymptotic homology.

Let us assume that a standard Heisenberg manifold $M = \Gamma\backslash H$ has a metric induced by a left invariant metric on H, and that ϕ is the geodesic flow on the unit tangent bundle $T_1(M)$. Using left translation, we will consider $T_1(M)$ as $M \times U$, where U is the sphere of unit tangent vectors at Γe.

Since $\Pi_1(T_1(M)) = \Pi_1(M)$, the asymptotic homotopy of any curve in $T_1(M)$ is equal to the asymptotic homotopy of its projection to M. Therefore the asymptotic homotopy $\rho(p , v)$ of the trajectory of ϕ starting at the point (p , v) in $M \times U$ is simply the asymptotic homotopy of the geodesic α on M starting at p in the direction v. By Corollary 8.1, $\rho(\alpha) = \operatorname{Alim} g_t$, where g_t is the curve in H obtained by lifting α to H and left translating to the identity. Since the metric is left invariant, g_t is the geodesic starting at the identity with initial direction v. Thus $\rho(p , v) = \rho(\Gamma e , v) = \operatorname{Alim} g_t$.

We have therefore reduced our problem to determining the geodesics in H which start at the identity and then computing their asymptotic limit.

Our basis of left invariant vector fields on H is given by

$$
X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}.
$$

(In §7, X_i denoted the induced vector field on M.) These vector fields evaluated at the identity are a basis of the Lie algebra $L(H)$ which is given by the brackets $[X_1 , X_2] = X_3$ and $[X_1 , X_3] = [X_2 , X_3] = 0$. Since the exponential map from $L(H)$ to H is the identity, the automorphisms of H are the Lie algebra automorphisms of $L(H)$. These automorphisms are given by the matrices

$$
\begin{pmatrix} B & 0 \\ b\ c & \det(B) \end{pmatrix},
$$

where B is in $\mathrm{GL}(2, \mathbf{R})$, and b and c are in $\mathbf{R}$. It is easy to see that for every left invariant metric on H there is an automorphism A of H which is an isometry to a metric which is determined on $L(H)$ by the matrix

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}
$$

[Pa, Introduction]. In computing asymptotic homotopy it suffices to consider such metrics since $\operatorname{Alim} A(g_t) = A(\operatorname{Alim} g_t)$ (Proposition 4.1). We will therefore assume that such a metric, which we will call an ε-metric, is given.

We first determine the Riemannian connection ∇, which for any left invariant vector fields X, Y, and Z satisfies the conditions of being torsion free,

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

and of preserving the metric,

$$\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = 0.$$

By permuting the variables we obtain

$$\langle \nabla_X Y, Z\rangle = \tfrac{1}{2}(\langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle)$$

[Mi, p. 310]. It follows that

$$\nabla_{X_1} X_1 = \nabla_{X_2} X_2 = \nabla_{X_3} X_3 = 0, \qquad \nabla_{X_1} X_2 = -\nabla_{X_2} X_1 = \tfrac{1}{2} X_3,$$
$$\nabla_{X_1} X_3 = \nabla_{X_3} X_1 = -(\varepsilon^2/2) X_2, \qquad \nabla_{X_2} X_3 = \nabla_{X_3} X_2 = (\varepsilon^2/2) X_1.$$

We observe that the sectional curvatures are given by $K(X_1, X_2) = -\frac{3}{4}\varepsilon^2$, and $K(X_1, X_3) = K(X_2, X_3) = \frac{1}{4}\varepsilon^4$ [Mi, p. 311].

We may now write down the geodesic equations. We denote by v the (unit) tangent vector of a geodesic and write $v = v_1 X_1 + v_2 X_2 + v_3 X_3$. We then obtain the following equation by expanding the geodesic equation $\nabla_v v \equiv 0$, using the formulae derived above for the covariant derivatives:

$$\dot v_1 X_1 + \dot v_2 X_2 + \dot v_3 X_2 - v_1 v_3 \varepsilon^2 X_2 + v_2 v_3 \varepsilon^2 X_1 = 0,$$

where the dot denotes differentiation in the parameter t of the geodesic (i.e., $\dot f = \nabla_v f$). Collecting terms, we get the three equations:

$$\dot v_1 + \varepsilon^2 v_2 v_3 = 0, \quad \dot v_2 - \varepsilon^2 v_1 v_3 = 0, \quad \dot v_3 = 0.$$

We assume that our geodesic starts at the identity element $\langle 0, 0, 0\rangle$ in H with its initial tangent vector the unit vector (a, b, c). It follows from the third equation above that $v_3 = c$, and that v_1 and v_2 satisfy the linear system of ordinary differential equations

$$\dot v_1 = -c\varepsilon^2 v_2, \qquad \dot v_2 = c\varepsilon^2 v_1.$$

If $c \neq 0$, the solution is $v_1(t) = \overline{R}\cos(c\varepsilon^2(t+t_*))$ and $v_2(t) = \overline{R}\sin(c\varepsilon^2(t+t_*))$, where $\overline{R}$ and t_* are constants. The vector (v_1, v_2) thus rotates around a circle of radius $\overline{R} = \sqrt{a^2+b^2}$ with angular velocity $c\varepsilon^2$. If $c = 0$, the solution is $v_1 = a$ and $v_2 = b$.

We may now easily pass from the velocity vector $v(t)$ to the coordinates $x(t)$, $y(t)$, and $z(t)$ of the geodesic. By using the definitions of X_1, X_2, and X_3, we obtain $\dot x = v_1$, $\dot y = v_2$, and $\dot z = v_3 + \frac{1}{2}(x v_2 - y v_1)$. If $c \neq 0$, then the point (x, y) rotates with constant speed around a circle of radius $R = \overline{R}/(|c|\varepsilon^2)$. This circle starts at $(0, 0)$ and the period P is $2\pi/(|c|\varepsilon^2)$. Now notice that we can write $\dot z = c + \frac{1}{2}(x\dot y - y\dot x) = c + \dot A$, where $A(t)$ is the signed area swept out by the vector $(x(t), y(t))$ in time t. Therefore, $z(t) = ct + A(t)$. If $c = 0$, we obtain $x = at$, $y = bt$, and $z = 0$.

We now need to determine the asymptotic limit of the geodesic $g_t = \langle x(t), y(t), z(t)\rangle$. We use the formula $\rho = \operatorname{Alim} g_t = \lim g_t/t + \frac{1}{2}[g_t, \mu]$ (Lemma 4.2).

If $c \neq 0$, then $\mu(0, 0)$ since $x(t)$ and $y(t)$ are bounded. Therefore

$$\rho = \lim g_t/t = \lim\langle x/t, y/t, z/t\rangle = \langle 0, 0, c + \lim A(t)/t\rangle .$$

Since $A(t)$ is periodic of period P, when $c > 0$ we obtain

$$\lim A(t)/t = A(P)/P = \pi R^2/P = c\varepsilon^2 R^2/2 = \overline{R}^2/(2c\varepsilon^2) = (1 - \varepsilon^2 c^2)/(2c\varepsilon^2),$$

where we use the fact that $\overline{R}^2 + \varepsilon^2 c^2 = a^2 + b^2 + \varepsilon^2 c^2 = 1$, since (a, b, c) is a vector of unit length. Therefore $\rho = \langle 0, 0, (1 + c^2\varepsilon^2)/(2c\varepsilon^2)\rangle$. When c is negative we obtain the same formula for ρ since, in that case, A is also negative.

If $c = 0$, then $\mu = (a, b)$ and so $[g_t, \mu] = \mu_2 x(t) - \mu_1 y(t) = 0$. Therefore $\rho = \lim g_t/t = \langle a, b, 0\rangle$.

Let us summarize. Suppose we have an ε-metric on M, i.e., a metric induced by a left invariant metric on H for which the vector fields X_1, X_2 and X_3 are orthogonal, with X_1 and X_2 of unit length and X_3 of length $\varepsilon > 0$. If $v = (a, b, c)$ is a unit vector in U, then $\rho(p, v)$, the asymptotic homotopy of a trajectory of the geodesic flow starting at (p, v), is $\langle 0, 0, (1 + c^2\varepsilon^2)/(2c\varepsilon^2)\rangle$ when $c \neq 0$, and is $\langle a, b, 0\rangle$ when $c = 0$. The asymptotic homology $\mu(v, p)$ is $(0, 0)$ when $c \neq 0$, and is (a, b) when $c = 0$. Therefore asymptotic homology does not depend on ε. Also note that both asymptotic homology and asymptotic homotopy are discontinuous at $c = 0$.

We now show that if $\varepsilon \neq \varepsilon'$, then the induced geodesic flows ϕ_ε and $\phi_{\varepsilon'}$ are not topologically conjugate.

Recall that if p and q are points in some manifold M, then the Malcev completions of $\Pi_1(M, p)$ and $\Pi_1(M, q)$ are conjugate. That is, if Γ_p and Γ_q are the lattices in N corresponding to p and q, then there is a conjugacy of N taking Γ_p to Γ_q (Lemma 5.1). From this it follows that if $f\colon M \to M'$ is a topological conjugacy of flows ϕ and ϕ', then for *any* two points $p \in M$ and $p' \in M'$ there is a lattice-preserving automorphism $A\colon (N, \Gamma_p) \to (N', \Gamma_{p'})$ which preserves central asymptotic homotopy cycles. That is, A induces a bijection from $\{\rho(q) | q \in M,\ \rho(q) \text{ central in } N\}$ to $\{\rho(q') | q' \in M',\ \rho(q') \text{ central in } N'\}$.

In the present case, the two manifolds are $M \times U$ and $M \times U'$, the groups are $N = N' = H$, and the lattices, $\Gamma_{(p, v)}$ and $\Gamma_{(p', v')}$ depend only on p and p'. If we take $p = p' = \Gamma e$, we obtain $\Gamma_{(p, v)} = \Gamma_{(p', v')} = \Gamma$, where Γ is one of the standard lattices Γ_k (§7.1). The center of H consists of elements of the form $\langle 0, 0, z\rangle$. The characterization given above of the automorphisms of H shows that, for an automorphism $A\colon (H, \Gamma) \to (H, \Gamma)$, the associated B is in $\mathrm{GL}(2, \mathbf{Z})$, and thus the determinant of B is 1 or -1. Therefore the characterization of automorphisms shows that A either fixes the center or takes central elements to their inverses. But if $\varepsilon \neq \varepsilon'$, such an action cannot take the set of central elements ρ for ϕ_ε to the set of central elements ρ' for $\phi_{\varepsilon'}$. The reason is that, for a given ε, the minimum of the set of nonzero $|z|$, such that $\langle x, y, z\rangle$ is a central asymptotic homotopy element, is $1/\varepsilon$ which is achieved when $a = b = 0$ and $c^2\varepsilon^2 = 1$.

8.2. **Asymptotic homotopy for lifted flows.** Asymptotic homotopy for lifted flows can be expressed in terms of "ergodic averages" of functions.

Definition 8.2 (Ergodic average). If ϕ is a flow on a manifold M, then $\operatorname{Av}(f)_p$, the ergodic average of f at a point p, is equal to $\lim \frac{1}{t}\int_0^t f(\alpha(s))\,ds$, where α is the trajectory starting at p.

The terms μ_i which appear in the expression for asymptotic homotopy (Proposition 8.1) are just the ergodic averages of the functions $\omega_i(V)$, where V is the vector field inducing a flow ϕ. Consequently, by Birkhoff's ergodic theorem, if ϕ is measure-preserving, then μ_i exist almost everywhere. Unfortunately, for general measure-preserving flows, it is unclear whether and where the μ_{12} term exists.

The situation improves when we restrict to measure-preserving flows on a Heisenberg manifold which are lifts of toral flows. In that case, as Corollary 8.1 indicates, μ_{12} depends entirely on the toral flow, which is itself measure-preserving (Proposition 7.3). Furthermore the trajectories of a measure-preserving flow on the 2-torus are quite explicitly computable. We are therefore able, in Theorem 8.1 below, to prove the existence almost everywhere of asymptotic homotopy, and to express it in terms of ergodic averages along the toral flow.

Recall that a lifted measure-preserving flow is determined by constants β_1 and β_2 and by functions f and F which we can think of as defined on the torus (Proposition 7.3). The partial derivatives f_x and f_y are also functions on the torus.

Theorem 8.1. (Asymptotic homotopy exists for measure-preserving lifts). *Suppose M is a standard Heisenberg manifold with the standard representing $L(H)$ valued* 1*-form, and that the measure-preserving lifted flow ϕ is determined by the constants β_1 and β_2, and by the functions f and F defined on T^2 (Proposition* 7.3). *Let $\overline{p}$ denote the projection to T^2 of a point p in M. Then the following statements hold when the indicated ergodic averages along the toral flow exist*:

(a) $\mu_1(p) = \operatorname{Av}(\beta_1 + f_y)_{\overline{p}}$, $\mu_2(p) = \operatorname{Av}(\beta_2 - f_x)_{\overline{p}}$, *and* $\mu_3(p) = \operatorname{Av}(F)_{\overline{p}}$.

(b) *If* $\beta_1 \neq 0$, *then* $\mu_{12}(p) = \operatorname{Av}(f)_{\overline{p}} - \operatorname{Av}(ff_y)_{\overline{p}}/\beta_1 - \mu_1 f(\overline{p})/\beta_1$.

(c) *If* $\beta_2 \neq 0$, *then* $\mu_{12}(p) = \operatorname{Av}(f)_{\overline{p}} - \operatorname{Av}(ff_x)_{\overline{p}}/\beta_2 - \mu_2 f(\overline{p})/\beta_2$.

(d) *If* $\beta_1 = \beta_2 = 0$, *then, for almost every p, $\overline{p}$ is fixed or periodic for the toral flow. If $\overline{p}$ is fixed, then $\mu_{12}(p) = 0$. If $\overline{p}$ is periodic of period T, then*

$$\mu_{12}(p) = \frac{1}{2T}\int_0^T (x(t) - x(0))y'(t) - (y(t) - y(0))x'(t)\,dt,$$

where $(x(t), y(t))$ is a lift to $\mathbf{R}^2$ of the loop based at $\overline{p}$.

Consequently the asymptotic homotopy $\rho = \langle \mu_1, \mu_2, \mu_3 + \mu_{12}\rangle$ exists almost everywhere on M. Furthermore, wherever ρ is defined by (a)–(d) *above, it satisfies the condition to be an invariant of topological equivalence* (*Corollary* 5.2).

Proof. (a) By Proposition 8.1, $\mu_i = \operatorname{Av}(\omega_i(V))_p = \operatorname{Av}(V_i)_p$, where V is the vector field which has components V_i and which determines the flow ϕ. Since the flow is lifted, the components V_i also lift from functions on T^2. But for lifted flows, if $\tilde{g}$ is a function on M which lifts from a function g on T^2, then $\operatorname{Av}(\tilde{g})_p$ equals $\operatorname{Av}(g)_{\overline{p}}$. Since V_1, V_2, and V_3 lift from $\beta_1 + f_y$, $\beta_2 - f_x$, and F respectively (Proposition 7.3), part (a) of the theorem follows.

(b) We need the following lemma which characterizes trajectories of measure-preserving flows on T^2.

Lemma 8.2 (Measure-preserving flows on T^2). *Suppose a measure-preserving flow on T^2 is determined by the equations $x' = \beta_1 + f_y$ and $y' = \beta_2 - f_x$ (Proposition 7.2).*

(a) *If f' is the derivative of f along the trajectory $(x(t), y(t))$ of the lifted flow on $\mathbf{R}^2$, then*

$$-\beta_2 x'(t) + \beta_1 y'(t) + f'(x(t), y(t)) = 0$$

and

$$-\beta_2(x(t) - x(0)) + \beta_1(y(t) - y(0)) + f(x(t), y(t)) = f(x(0), y(0)).$$

(b) *If μ_1 and μ_2 are the ergodic averages of the components of the vector field, then they exist almost everywhere, and where they are defined we have $-\beta_2\mu_1 + \beta_1\mu_2 = 0$.*

Proof. (a) If we multiply the equation for x' by y' and the equation for y' by x' and then subtract, we obtain $-\beta_2 x' + \beta_1 y' + f_y y' + f_x x' = 0$. Since $f_y y' + f_x x' = f'$, we obtain the first expression. Then we integrate this expression from 0 to t to get the second expression.

(b) Since $\mu_1 = \lim(x(t) - x(0))/t$ and $\mu_2 = \lim(y(t) - y(0))/t$, if we use part (a) of this lemma and divide

$$-\beta_2(x(t) - x(0)) + \beta_1(y(t) - y(0)) + f(x(t), y(t)) = f(x(0), y(0))$$

by t, and then take the limit, we obtain $-\beta_2\mu_1 + \beta_1\mu_2 = 0$. We use the fact that the function f is bounded since it lifts from a continuous function on the torus. μ_1 and μ_2 exist almost everywhere by the ergodic theorem. Q.E.D. (Lemma 8.2)

Now fix a point p in M for which $\mu_1(p)$ and $\mu(p)$ are defined. The trajectory of the flow ϕ through p lifts to a curve on H whose position at time t is $\langle x(t), y(t), z(t)\rangle$. By Corollary 8.1

$$\mu_{12} = \lim\left(\frac{1}{2t}\int_0^t ((x(s) - x(0))y'(s) - (y(s) - y(0))x'(s))\,ds \right.$$
$$\left. + \frac{1}{2}(\mu_2(x(t) - x(0)) - \mu_1(y(s) - y(0)))\right).$$

Since the flow ϕ is measure-preserving and lifted, $x(t)$ and $y(t)$ determine a trajectory of a measure-preserving flow on T^2 and therefore satisfy Lemma 8.2. Now think of f, f', f_x, and f_y as doubly periodic functions on $\mathbf{R}^2$, and let $f(t)$, $f'(t)$, $f_x(t)$, and $f_y(t)$ denote their value at the point $(x(t), y(t))$. Since we are assuming that $\beta_1 \neq 0$, we can divide by β_1 in Lemma 8.2(a), and obtain

$$y'(s) = \frac{-f'(s) + \beta_2 x'(s)}{\beta_1} \quad \text{and} \quad y(s) - y(0) = \frac{f(0) - f(s) + \beta_2(x(s) - x(0))}{\beta_1}.$$

Also, if we let $k = \mu_1/\beta_1$, we obtain by Lemma 8.2

$$\begin{aligned}\mu_2(x(t) - x(0)) - \mu_1(y(t) - y(0)) &= k(\beta_2(x(t) - x(0)) - \beta_1(y(t) - y(0))) \\ &= k(f(t) - f(0)).\end{aligned}$$

We can now substitute into the expression for μ_{12} and obtain

$$\begin{aligned}\mu_{12} = \lim\Bigg(&\frac{1}{2\beta_1 t}\int_0^t (x(s)-x(0))(-f'(s)+\beta_2 x') \\ &- (f(0)-f(s)+\beta_2(x(s)-x(0)))x'(s)\,ds + \frac{k}{2}(f(t)-f(0))\Bigg) \\ = \lim\Bigg(&\frac{1}{2\beta_1 t}\int_0^t (f(s)-f(0))x'(s)\,ds \\ &- \frac{1}{2\beta_1 t}\int_0^t (x(s)-x(0))f'(s)\,ds + \frac{k}{2}(f(t)-f(0))\Bigg).\end{aligned}$$

We then integrate by parts and obtain

$$\begin{aligned}\mu_{12} &= \lim\Bigg(\frac{1}{2\beta_1 t}\int_0^t (f(s)-f(0))x'(s)\,ds - \frac{1}{2\beta_1 t}(x(t)-x(0))f(t) \\ &\qquad + \frac{1}{2\beta_1 t}\int_0^t x'(s)f(s)\,ds + \frac{k}{2}(f(t)-f(0))\Bigg) \\ &= \lim\Bigg(\frac{1}{\beta_1 t}\int_0^t x'(s)f(s)\,ds - \frac{f(0)}{2\beta_1 t}(x(t)-x(0)) \\ &\qquad - \frac{1}{2\beta_1 t}(x(t)-x(0))f(t) + \frac{k}{2}f(t) - \frac{k}{2}f(0)\Bigg) \\ &= \lim\Bigg(\frac{1}{\beta_1 t}\int_0^t (\beta_1 + f_y(s))f(s)\,ds - f(0)\left(\frac{k}{2} + \frac{x(t)-x(0)}{2\beta_1 t}\right) \\ &\qquad + f(t)\left(\frac{k}{2} - \frac{x(t)-x(0)}{2\beta_1 t}\right)\Bigg) \\ &= \frac{1}{\beta_1 t}\mathrm{Av}((\beta_1 + f_y)f)_{\overline{p}} - kf(0) + 0 \\ &= \mathrm{Av}(f)_{\overline{p}} - \frac{1}{\beta_1}\mathrm{Av}(ff_y)_{\overline{p}} - \frac{\mu_1}{\beta_1}f(\overline{p}).\end{aligned}$$

So part (b) of theorem is proved.

(c) The proof of part (c) is similar to the proof of (b).

(d) T^2 is the disjoint union of the sets A, B, and C, where, for the function f, A is the set of critical points, B is the set of regular points whose images are regular values, and C is the set of regular points whose images are critical values. ($\overline{p}$ is a critical point iff both $f_x(\overline{p})$ and $f_y(\overline{p})$ are 0 [Hi, p. 22].)

Lemma 8.3 (Regular points with critical images are measure 0). *The set C has measure* 0.

Proof. Let $D = B \cup C$ be the set of regular points of f. D is open. By the implicit function theorem [Hi], f on D looks locally like the projection π from $\mathbf{R}^2$ to R. That is, for each $\overline{p} \in D$, there exists an open set U containing $\overline{p}$ and a diffeomorphism $h\colon U \to \Omega \subset \mathbf{R}^2$ such that $f_{|U} = \pi \circ h$, where $\pi(x, y) = x$. By Sard's Theorem [Hi, p. 69], the set S of critical values of the function f has measure 0 in $\mathbf{R}$. Therefore $\pi^{-1}(S)$ has measure 0 in $\mathbf{R}^2$, and $f_{|U}^{-1}(S) = h^{-1}(\pi^{-1}(S))$ has measure 0 in T^2. Note that $f_{|U}^{-1}(S) = U \cap C$. By the Lindelöf theorem [D, p. 174], every open cover of D has a countable subcover.

Therefore we can find a countable subcover $\{U_i\}$ of D such that $U_i \cap C$ has measure 0. Therefore C has measure 0. (This proof applies to any smooth $f\colon M \to \mathbf{R}$, where M is a second countable manifold.) Q.E.D. (Lemma 8.3)

Since we are assuming that β_1 and β_2 are 0, it follows that the equations $x' = f_y$ and $y' = -f_x$ determine the flow on T^2. Note that $\overline{p}$ is a critical point (i.e., $\overline{p} \in A$) iff it is a fixed point of the flow on T^2. Therefore if $\overline{p} \in A$, then $\mu_{12}(\overline{p}) = 0$ (Corollary 8.1).

If $\overline{p}$ is in B, then $f(\overline{p}) = c$ is a regular value, and so $f^{-1}(c)$ is a compact embedded 1-manifold in T^2 [Hi, p. 22]. That is, $f^{-1}(c)$ is a disjoint union of circles. Furthermore, the vector field is tangent to and nonvanishing on these circles. Therefore each of these circles is a periodic orbit of the flow.

So suppose p is a point in M whose image $\overline{p}$ in T^2 is a periodic point for the flow on T^2. Proposition 8.1 shows that if α is the trajectory through p, then

$$\operatorname{Alim} L \int_{\alpha_{|t}} \langle \omega_1, \omega_2, \omega_3 \rangle = \langle \mu_1, \mu_2, \mu_3 + \mu_{12} \rangle .$$

The same proof shows that

$$\operatorname{Alim} L \int_{\alpha_{|t}} \langle \omega_1, \omega_2, 0 \rangle = \langle \mu_1, \mu_2, \mu_{12} \rangle .$$

But since $\langle \omega_1, \omega_2, 0 \rangle$ when evaluated on α is a periodic function into $L(H)$, we can apply Proposition 4.4(b) and conclude that

$$\operatorname{Alim} L \int_{\alpha_{|t}} \langle \omega_1, \omega_2, 0 \rangle = \left(L \int_{\alpha_{|T}} \langle \omega_1, \omega_2, 0 \rangle \right)^{1/T} ,$$

which, by Lemma 8.1(b),

$$= \left\langle \frac{1}{T} \int_{\alpha_{|T}} \omega_1 , \frac{1}{T} \int_{\alpha_{|T}} \omega_2 , \frac{1}{2T} \int_{\alpha_{|T}} \omega_1\omega_2 - \omega_2\omega_1 \right\rangle .$$

Therefore

$$\mu_{12} = \frac{1}{2T} \int_{\alpha_{|T}} \omega_1\omega_2 - \omega_2\omega_1 = \frac{1}{2T} \int_0^T (x(t) - x(0))y'(t) - (y(t) - y(0))x'(t)\, dt .$$

So part (d) of the theorem is proved.

ρ exists almost everywhere since the ergodic averages in parts (a), (b), and (c) above exist almost everywhere by the ergodic theorem, while case (d) is itself a statement of existence almost everywhere. It remains to show that ρ is an invariant of topological equivalence by verifying that the condition of Corollary 5.2 is satisfied.

First assume that μ_1 and μ_2 are not both 0. It is sufficient to show the boundedness of the "correction term" $(\langle \alpha_{|t} \rangle, \mu)$ (Definition 4.1), where $\mu = (\mu_1, \mu_2)$ (Definition 5.1). But

$$\begin{aligned} (\langle \alpha_{|t} \rangle, \mu) = [\langle \alpha_{|t} \rangle, \mu] &= \mu_2 \int_{\alpha_{|t}} \omega_1 - \mu_1 \int_{\alpha_{|t}} \omega_2 \\ &= \mu_2(x(t) - x(0)) - \mu_1(y(t) - y(0)) \end{aligned}$$

(Lemma 4.1(b), Definition 5.2, Lemma 8.1(b)). If β_1 and β_2 are not both 0, then

$$\begin{aligned}\mu_2(x(t)-x(0))-\mu_1(y(t)-y(0)) &= k(\beta_2(x(t)-x(0))-\beta_1(y(t)-y(0))) \\ &= k(f(t)-f(0)),\end{aligned}$$

which is bounded (Lemma 8.2(b), proof of Theorem 8.1(b)). If $\beta_1=\beta_2=0$, then almost every orbit on T^2 is stationary or periodic. In either case it is easy to see from the definitions of μ_1 and μ_2 that $\mu_2(x(t)-x(0))-\mu_1(y(t)-y(0))$ is bounded.

If $\mu_1 = \mu_2 = 0$, then we use Proposition 4.6(a)(ii) to see that ρ is an invariant of topological equivalence. Q.E.D.

9. Asymptotic homotopy distinguishes regular hypercircular flows

In this section we discuss "regular hypercircular" flows on standard Heisenberg manifolds. These flows are a class of lifted measure-preserving flows for which asymptotic homology and homotopy are defined everywhere. They all have the same asymptotic homology, but differences in asymptotic homotopy allow some of them to be distinguished up to topological equivalence.

We have previously used the fact that a Heisenberg manifold is a circle bundle over the 2-torus (§7.2). It is also a torus bundle over the circle $S^1=\mathbf{R}/\mathbf{Z}$ as follows. The homomorphism $\langle x, y, z\rangle \mapsto y$ from H to $\mathbf{R}$ induces a map from a standard Heisenberg manifold M to S^1 such that the following diagram commutes (Definition 7.1), where the maps $H \to M$ and $\mathbf{R} \to S^1$ are the natural projections:

$$\begin{array}{ccc} H & \longrightarrow & \mathbf{R} \\ \downarrow & & \downarrow \\ M & \longrightarrow & S^1 \end{array}$$

These torus bundles are topologically nontrivial.

Notation 9.1 (The fiber of the torus bundle). The fiber over the point $y+\mathbf{Z}$ will be denoted by T_y.

The flows defined below have the property that the tori T_y are invariant sets, that the flows on T_y are conjugate to the usual linear flows on tori (Definition 2.9), and that under the standard map $M \to T^2$ (§7.2) the flows on T_y project to unit speed flows on a circle.

Definition 9.1 (Regular hypercircular flows). A lifted measure-preserving flow on a standard Heisenberg manifold is a regular hypercircular flow determined by a function of period one, $\tau\colon \mathbf{R}\to\mathbf{R}$, if it is determined by the vector field having components (Proposition 7.3) $V_1=1$, $V_2=0$, and $V_3=F$, where $F(\Gamma\langle x, y, z\rangle)\equiv\tau(y)$, and where the function given by $\tau(y)-y$ has only a finite number of critical points on the unit interval, and all those critical points are nondegenerate $(\tau''\neq 0)$.

Proposition 9.1 (Asymptotic homotopy of regular hypercircular flows). *The asymptotic homotopy of a regular hypercircular flow determined by* τ *exists at every point on a standard Heisenberg manifold* M, *and is given by the formula* $\rho(\Gamma\langle x, y, z\rangle)=\langle 1, 0, \tau(y)\rangle$. *At every point in* M, *the asymptotic homology*

$\mu = (\mu_1, \mu_2) = (1, 0)$. *The asymptotic homotopy is an invariant of topological equivalence in the sense of Corollary* 5.2.

Proof. We apply the formulas in Theorem 8.1, using $\beta_1 = 1$, $\beta_2 = 0$, $f = 0$, and F as given in Definition 9.1. Since $f = 0$, at all $p \in M$ we immediately obtain $\mu_1 = 1$, $\mu_2 = 0$, and $\mu_{12} = 0$. Since $V_2 = 0$, y remains constant along the lift to H of every trajectory (§7.1, Conventions). Since F depends only on y, it remains constant along every trajectory. Therefore at all $p = \Gamma\langle x, y, z\rangle$, we have $\mu_3 = \mathrm{Av}(F)_p = F(p) = \tau(y)$. By Theorem 8.1, $\rho = \langle \mu_1, \mu_2, \mu_3 + \mu_{12}\rangle$ satisfies the condition required to be an invariant of topological equivalence. $\mu = (1, 0)$ is the asymptotic homology since it is the abelianization of ρ (Definition 5.1). Q.E.D.

We will distinguish regular hypercircular flows by using the homotopy foliation (§6).

Theorem 9.1 (Distinguishing regular hypercircular flows). *If the regular hypercircular flows determined by* τ_1 *and* τ_2 *are topologically equivalent* (*Definition* 2.6), *then the functions given by* $\tau_1(y) - y$ *and* $\tau_2(y) - y$ *have the same number of critical points on the unit interval* $[0, 1)$.

Proof. We first note that in this setting the homotopy foliation (Definition 6.3) can be computed more explicitly.

Lemma 9.1 (Homotopy foliation of standard Heisenberg manifolds). *Suppose* $p = \Gamma b$ *and* $q = \Gamma c$ *are two points on a standard Heisenberg manifold* $M = \Gamma\backslash H$ (*Definition* 7.1). *Then* (p, g) *and* (q, h) *are on the same leaf of the homotopy foliation* $\mathscr{F}$ *of* $M \times H$ *iff there is some* $a \in H$ *such that* a *translates* p *to* q *and* a^{-1} *conjugates* g *to* h (*i.e.,* $ba = c$ *and* $a^{-1}ga = h$).

Proof. (p, g) and (q, h) are on the same leaf iff there exists a path δ from p to q such that $\langle\delta\rangle^{-1} g\langle\delta\rangle = h$, where $\langle\delta\rangle = {}_L\int_\delta \langle\omega_1, \omega_2, \omega_3\rangle$ for the standard representing 1-form $\langle\omega_1, \omega_2, \omega_3\rangle$ (Definition 6.3, Lemma 5.1, Definition 5.2, Definition 8.1). But one can compute (or see) that if a curve $\delta\colon [0, 1] \to M$ lifts to a curve $\tilde{\delta}$ on H then ${}_L\int_\delta \langle\omega_1, \omega_2, \omega_3\rangle = \tilde{\delta}(0)^{-1}\tilde{\delta}(1)$ (Lemma 8.1(b), Proposition 5.7 (proof)). So for a curve δ from p to q in M which lifts to a curve in H from b to c, we have $\langle\delta\rangle = b^{-1}c$. The lemma follows by letting $a = b^{-1}c$. Q.E.D. (Lemma 9.1)

We now define two rather ad hoc invariants of the intersection of graph ρ with the homotopy foliation $\mathscr{F}$ which are suited to this problem.

Definition 9.2 (Disconnecting set and FTI property). (a) Suppose ρ is the asymptotic homotopy of a flow ϕ on a manifold M. Then a point $p \in M$ is in the disconnecting set S if for any neighborhood U of p there exists some leaf $\mathscr{L}$ of $\mathscr{F}$ such that the projection to U of the intersection $\mathscr{L} \cap \operatorname{graph}\rho \cap (U \times H)$ is disconnected topologically.

(b) A flow on a standard Heisenberg manifold M has the *finite torus intersection* (FTI) property if for every leaf $\mathscr{L}$ the projection to M of the intersection $\mathscr{L} \cap \operatorname{graph}\rho$ is equal to the disjoint union of a finite number of tori in M.

Remark 9.1 (Disconnecting sets and the FTI property are conjugacy invariants). If two flows are topologically conjugate by the homeomorphism f, then f restricts to a homeomorphism of their disconnecting sets, and they both have or both lack the FTI property (Remark 6.1, Proposition 6.1).

Lemma 9.2 (FTI property and disconnecting set for regular hypercircular flows). (a) *Regular hypercircular flows have the FTI property. Moreover the projection of* $\mathscr{L} \cap \operatorname{graph} \rho$ *is the disjoint union of tori of the form* T_y (*Notation* (9.1).

(b) *The disconnecting set* S *of a regular hypercircular flow determined by* τ (*Definition* 9.1) *is the union of the tori* T_y *for all* y *which are critical points of the function given by* $\tau(y) - y$.

Proof. (a) Let $\mathscr{L}_{g_0}$ be the leaf containing the point $(\Gamma e, g_0)$ in $M \times H$. (Every leaf $\mathscr{L}$ can be represented (nonuniquely) as an $\mathscr{L}_{g_0}$.) Then (p, g) is on $\mathscr{L}_{g_0}$ iff there exists an a such that $\Gamma a = p$ and $a^{-1} g_0 a = g$ (Lemma 9.1). In particular, $\mathscr{L}_{g_0}$ intersects graph ρ at $(p, \rho(p))$ iff there exists an a such that $\Gamma a = p$ and $a^{-1} g_0 a = \rho(p)$. If we let $g_0 = \langle x_0, y_0, z_0 \rangle$ and $a = \langle x, y, z \rangle$, this last condition becomes $\langle x_0, y_0, z_0 - (x y_0 - y x_0) \rangle = \langle 1, 0, \tau(y) \rangle$, which is equivalent to $x_0 = 1$, $y_0 = 0$, and $z_0 = \tau(y) - y$ (§7.1, Proposition 9.1). Therefore $(p, \rho(p))$ is on the leaf $\mathscr{L}_{g_0}$ iff there exists a real number y such that p is in the torus T_y, and $x_0 = 1$, $y_0 = 0$, and $\tau(y) - y = z_0$. This implies that whenever a leaf $\mathscr{L}$ intersects graph ρ over a point p in M, then that leaf intersects graph ρ over the torus T_y which contains p. The FTI property follows since, for any z_0, the equation $\tau(y) - y = z_0$ has only finitely many solutions (Definition 9.1).

(b) Now note that $\hat{y}$ is a critical point of $\tau(y) - y$ iff for all neighborhoods of $\hat{y}$ there exist numbers z_0 for which the equation $\tau(y) - y = z_0$ has more than one solution (Definition 9.1). Therefore if p is in $T_{\hat{y}}$ and $\hat{y}$ is a critical point of $\tau(y) - y$, then for all neighborhoods U of p there exists a leaf $\mathscr{L}$ such that the projection to U of $\mathscr{L} \cap \operatorname{graph} \rho \cap (U \times H)$ has nontrivial intersection with finitely many, but more than one, tori of the form T_y. Therefore such a point p is in the disconnecting set. If p is in $T_{\hat{y}}$ and $\hat{y}$ is not a critical point of $\tau(y) - y$, then there exists an interval I containing $\hat{y}$ in which, for all z_0, the equation of $\tau(y) - y = z_0$ has at most one solution. Let $U = \bigcup T_y$, where the union is over $y \in I$, and let $\mathscr{L} = \mathscr{L}_{g_0}$ for some $g_0 = \langle 1, 0, z_0 \rangle$. By part (a) above, the projection of $\mathscr{L} \cap \operatorname{graph} \rho \cap (U \times H)$ to U consists of finitely many tori of the form T_y, and furthermore any such y must be contained in some integer translate of I. Suppose that T_{y_1} and T_{y_2} are both in the projection but that $y_1 \neq y_2$. Then both $\tau(y_1) - y_1$ and $\tau(y_2) - y_2$ equal z_0. This is impossible if y_1 and y_2 are in the same translate, and if we pick I small enough it is impossible even if they are in different translates (Definition 9.1). So the projection is empty or it consists of a single T_y, which is a connected set.

Therefore p is in the disconnecting set S, iff p is in T_y for y a critical point of $\tau(y) - y$. Q.E.D. (Lemma 9.2)

Observe that as a direct consequence of Remark 9.1 and Lemma 9.2 we obtain a weaker version of Theorem 9.1 in which "topological equivalence" is replaced by "topological conjugacy." To deal with topological equivalences we need to understand the effect of reparametrization on the disconnecting set.

Lemma 9.3 (Rescaling and the disconnecting set). *Let* ρ *and* ρ' *be the asymptotic homotopy with disconnecting sets* S *and* S' *of the flows* ϕ *and* ϕ' *on a standard Heisenberg manifold* M, *where* ϕ *is regular hypercircular while* ϕ' *has the FTI property. Furthermore suppose that there is a positive rescaling function*

$r: M \to \mathbf{R}^+$ such that $\rho'(p) = r(p)\rho(p)$ for all $p \in M$ (Lemma 4.1(i)). Then S' is a subset of S.

Proof. If τ is the determining function of ϕ (Definition 9.1), then $\rho(p) = \langle 1, 0, \tau(y)\rangle$ and $\rho'(p) = \langle r(p), 0, r(p)\tau(y)\rangle$, where $p = \Gamma\langle x, y, z\rangle$ (Proposition 9.1). As in the proof of Lemma 9.2(a), a leaf $\mathscr{L}_{g_0}$ determined by $g_0 = \langle x_0, y_0, z_0\rangle$ intersects graph ρ' at $(p, \rho'(p))$ iff there exists $\langle x, y, z\rangle$ such that $p = \Gamma\langle x, y, z\rangle$ (i.e., $p \in T_y$) and $x_0 = r(p)$, $y_0 = 0$, and $z_0/x_0 = \tau(y) - y$. Since $\tau(y) - y$ assumes any value z_0/x_0 at most finitely many times, a leaf $\mathscr{L}$ intersects graph ρ' over some *subset* of finitely many disjoint tori of the form T_y. However, by the FTI property hypothesis on ϕ', the intersection must project *onto* some finite number of tori, which we now see must be of the form T_y. Therefore on *any* torus T_y the function $r(p)$ must be constant.

Now assume $p \in S'$. By our remarks in the previous paragraph, this can only occur if, for any neighborhood U of p, there is some leaf $\mathscr{L}_{g_0}$ such that the projection to U of $\mathscr{L}_{g_0} \cap \operatorname{graph}\rho' \cap (U \times H)$ contains points in two or more tori of the form T_y. Let p be in $T_{\hat{y}}$ for some $\hat{y}$. Then for any open interval I about $\hat{y}$, there exists a real number z_0/x_0 such that the function $\tau(y) - y$ assumes the value z_0/x_0 two or more times. Let $h_0 = \langle 1, 0, z_0/x_0\rangle$. Then, by our argument in the proof of Lemma 9.2(a), the projections to U of $\mathscr{L}_{h_0} \cap \operatorname{graph}\rho \cap (U \times H)$ contains points in two or more tori of the form T_y. This implies that $p \in S$. Q.E.D. (Lemma 9.3)

We can now prove Theorem 9.1 Suppose ϕ_1 and ϕ_2 are equivalent regular hypercircular flows determined by τ_1 and τ_2, and that the equivalence f is a conjugacy of ϕ_1 with a continuous flow ϕ_3 which is a reparametrization of ϕ_2. Let n_1 and n_2 be the number of critical points on the unit interval of $\tau_1(y) - y$ and $\tau_2(y) - y$. By Lemma 9.2, ϕ_1 has the FTI property and the disconnecting set S_1 of ϕ_1 consists of n_1 tori of the form T_y. By Remark 9.1, ϕ_3 has the FTI property and the disconnecting set S_3 of ϕ_3 consists of n_1 tori. By Corollary 5.1 on topological conjugacy, $f_*\rho_1(p) = \rho_3(f(p))$. By Proposition 9.1, the asymptotic homotopy and homology of both ϕ_1 and ϕ_2 are everywhere nontrivial, and the condition for applying Corollary 5.2 is satisfied. By Corollary 5.2 on topological equivalence, there exists $r: M \to \mathbf{R}^+$ such that $\rho_3(p) = r(p)\rho_2(p)$. By Lemma 9.3, the disconnecting set S_3 is contained in the disconnecting set S_2 of the flow ϕ_2. By Lemma 9.2, S_2 consists of n_2 tori. Therefore $n_1 \le n_2$. Since topological equivalence is a symmetric relation, we can similarly show $n_2 \le n_1$. Therefore $n_1 = n_2$. Q.E.D.

Example 9.1 (Flows distinguished by asymptotic homotopy). On any standard Heisenberg manifold, no two of the regular hypercircular flows determined by the functions $\tau_n(y) = \cos(2n\pi y)$ are topologically equivalent, where $n = 0, 1, 2, \ldots$ (Theorem 9.1).

Acknowledgments

This work was greatly helped by conversations with Michael Shub, Dennis Sullivan, David Fried, K. T. Chen, Felix Albrecht, George Leger, Bob Williams, and many others at City University of New York, The University of Illinois at Urbana, and Tufts University.

REFERENCES

[A] V. I. Arnold, *The asymptotic Hopf invariant and its applications*, Selecta Math. Soviet. **5** (1986), 327–345.

[AA] V. I. Arnold and A. Avez, *Ergodic problems of classical mechanics*, Benjamin, New York, 1968.

[AGH] L. Auslander, L. Green, and F. Hahn, *Flows on homogeneous spaces*, Princeton Univ. Press, Princeton, N. J., 1963.

[B] D. Benardete, *Topological equivalence of flows on homogeneous spaces, and divergence of one-parameter subgroups of Lie groups*, Trans. Amer. Math. Soc. **306** (1988), 499–527.

[BT] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Springer-Verlag, New York, 1982.

[C1] K. T. Chen, *Algebras of iterated path integrals and fundamental groups*, Trans. Amer. Math. Soc. **156** (1971), 359–379.

[C2] ——, *Extension of C^∞ function algebra by integrals and Malcev completion of Π_1*, Adv. in Math. **23** (1977), 181–210.

[C3] ——, *Iterated path integrals*, Bull. Amer. Math. Soc. **83** (1977), 831–878.

[CL] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, Krieger, Malabar, Fla., 1984.

[D] J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

[DF] J. D. Dollard and C. N. Friedman, *Product integration with applications to differential equations*, Addison-Wesley, Reading, Mass., 1979.

[DGMS] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.

[Fa] A. Fathi, *Structure of the group of homeomorphisms preserving a good measure on a compact manifold*, Ann. Sci. École Norm. Sup. (4) **13** (1980), 45–93.

[F] J. Franks, *Realizing rotation vectors for torus homeomorphisms*, preprint.

[FM] J. Franks and M. Misiurewicz, *Rotation sets of toral flows*, preprint.

[Fr] D. Fried, *The geometry of cross sections to flows*, Topology **21** (1982), 353–371.

[FH] M. H. Freedman and Zheng-Xu He, *Incompressible flows: energy and asymptotic crossing number*, preprint.

[GM] P. A. Griffith and J. W. Morgan, *Rational homotopy theory and differential forms*, Birkhäuser, Boston, Mass., 1981.

[H] R. M. Hain, *The geometry of the mixed Hodge structure on the fundamental group*, Algebraic Geometry (Bowdoin 1985) (Spencer Bloch, ed.), Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, R. I., 1987.

[He] S. Helgason, *Differential geometry, Lie groups, symmetric spaces*, Academic Press, New York, 1978.

[Hi] M. W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1976.

[Hr] M. Herman, *Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension* 2, Comment. Math. Helv. **58** (1983), 453–502.

[I] M. C. Irwin, *Smooth dynamical systems*, Academic Press, New York, 1980.

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. 1, Interscience-Wiley, New York, 1963.

[LM] J. Llibre and R. S. MacKay, *Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity*, Ergodic Theory Dynamical Systems **11** (1991), 115–128.

[M] A. I. Malcev, *On a class of homogeneous spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 9–32; English transl., Amer. Math. Soc. Transl. (1) **9** (1962), 276–307.

[Ma] R. Mañé, *Ergodic theory and differentiable dynamics*, Springer-Verlag, New York, 1987.

[Mi] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 293–329.

[MZ] M. Misiurewicz and K. Ziemian, *Rotation sets for maps of tori*, J. London Math. Soc. (2) **40** (1989), 490–506.

[Pa] P. Pansu, *Geometrie du groupe d'Heisenberg*, Doctoral Thesis (3ème cycle), University of Paris VII, 1982.

[P] D. S. Passman, *The algebraic structure of group rings*, Wiley, New York, 1977.

[R] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972.

[Rh] F. Rhodes, *Asymptotic cycles for continuous curves on geodesic spaces*, J. London Math. Soc. (2) **6** (1973), 247–255.

[S] S. Schwartzman, *Asymptotic cycles*, Ann. of Math. (2) **66** (1957), 270–284.

[Sp] M. Spivak, *A comprehensive introduction to differential geometry*, 2nd ed., Publish or Perish, Houston, 1979.

[Su1] D. Sullivan, *Differential forms and the topology of manifolds*, Proc. Internat. Conf. on Manifolds and Related Topics in Topology (Tokyo, 1973) (A. Hattori, ed.), Univ. of Tokyo Press, 1975, pp. 37–49.

[Su2] ——, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331.

[V] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Springer-Verlag, New York, 1984.

[W] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, New York, 1983.

[Wh] G. W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, New York, 1978.

Department of Mathematics, Tufts University, Medford, Massachusetts 02155

Current address: Department of Mathematics, University of Hartford, West Hartford, Connecticut 06117

E-mail address: dbenarde@uhasun.hartford.edu

Prime Computer, Bedford, Massachusetts 01730

Current address: Computervision Corporation, 9805 Scranton Road, San Diego, California 92121

E-mail address: jwm@calmasd.prime.com