TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 338, Number 2, August 1993

CRITICAL LIL BEHAVIOR OF THE TRIGONOMETRIC SYSTEM

I. BERKES

ABSTRACT. It is a classical fact that for rapidly increasing (n_k) the sequence $(\cos n_k x)$ behaves like a sequence of i.i.d. random variables. Actually, this almost i.i.d. behavior holds if (n_k) grows faster than $e^{c\sqrt{k}}$; below this speed we have strong dependence. While there is a large literature dealing with the almost i.i.d. case, practically nothing is known on what happens at the critical speed $n_k \sim e^{c\sqrt{k}}$ (critical behavior) and what is the probabilistic nature of $(\cos n_k x)$ in the strongly dependent domain. In our paper we study the critical LIL behavior of $(\cos n_k x)$ i.e., we investigate how classical fluctuational theorems like the law of the iterated logarithm and the Kolmogorov-Feller test turn to nonclassical laws in the immediate neighborhood of $n_k \sim e^{c\sqrt{k}}$.

1. INTRODUCTION

The purpose of this paper is to study the probabilistic behavior of lacunary trigonometric series. Specifically, we shall give essentially optimal lacunarity conditions under which a subsequence of the trigonometric system satisfies the law of the iterated logarithm and some of its refinements, e.g., the Kolmogorov-Feller upper-lower class test. We shall also investigate critical phenomena related to the LIL, i.e. study the surprising properties of lacunary trigonometric series in the immediate neighborhood of the gap condition where the law of the iterated logarithm and the Kolmogorov-Feller test break down.

It is well known that for rapidly increasing (n_k) the sequences $(\sin n_k x)_{k=1}^\infty$, $(\cos n_k x)_{k=1}^\infty$ behave like sequences of independent random variables.[1] For example, if (n_k) is a sequence of positive integers satisfying

$$n_{k+1}/n_k \geq q > 1 \qquad (k = 1, 2, \dots) \tag{1.1}$$

then by classical results of Salem-Zygmund [14] and Erdős-Gál [6] we have

$$\lim_{N\to\infty} \frac{1}{2\pi}\lambda\left(0 \leq x \leq 2\pi: \sum_{k\leq N} \cos n_k x < t\sqrt{N/2}\right) = \Phi(t) \tag{1.2}$$

Received by the editors February 20, 1991.
1991 *Mathematics Subject Classification*. Primary 42A55, 60F15.
Key words and phrases. Lacunary trigonometric series, weak and strong dependence, law of the iterated logarithm, upper-lower class tests.
Research supported by Hungarian National Foundation for Scientific Research, Grant No. 1905.

[1] Throughout this paper, the probability space for the trigonometric system will be $(0, 2\pi)$, equipped with the Borel σ-field $\mathscr{B}$ and normalized Lebesgue measure $(2\pi)^{-1}\lambda$.

and

$$\overline{\lim_{N\to\infty}}\,(N\log\log N)^{-1/2}\sum_{k\le N}\cos n_k x = 1 \quad \text{a.e.}^2 \tag{1.3}$$

where $\Phi(t) = (2\pi)^{-1/2}\int_{-\infty}^t \exp(-u^2/2)\,du$ is the standard normal distribution function. Actually, much more than (1.2) and (1.3) is true: Philipp and Stout [12] proved that under (1.1) the partial sum process $S(t) = S(t, x) = \sum_{k\le t}\cos n_k x \quad (t \ge 0)$ is nearly Wiener in the sense that without changing its distribution it can be redefined on a suitable probability space together with a Wiener process $\{W(t), t \ge 0\}$ such that

$$S(t) = W(t/2) + O(t^{1/2-\rho}) \quad \text{a.s. as } t \to \infty \tag{1.4}$$

for some constant $\rho > 0$. The approximation (1.4) implies not only the central limit theorem (1.2) and the law of the iterated logarithm (1.3) but it extends a large class of limit theorems of independent r.v.'s to the sequence $(\cos n_k x)$. For example, (1.4) implies easily that $(\cos n_k x)$ obeys the Kolmogorov-Feller test, i.e., for any positive nondecreasing sequence φ_N the inequality

$$\sum_{k\le N}\cos n_k x > \sqrt{N/2}\varphi_N$$

holds a.e. for finitely or infinitely many N according as the series

$$\sum_{N\ge 1}\frac{\varphi_N}{N}\exp\left(-\frac{1}{2}\varphi_N^2\right)$$

converges or diverges. Since the same test is valid for the Wiener process, we see that under (1.1) the partial sum growth of $(\cos n_k x)$ is exactly the same as that of i.i.d.r.v.'s.

All the results formulated above concern the case of the Hadamard gap condition (1.1) and in fact most known probabilistic results for $(\cos n_k x)$ in the literature assume (1.1). (For a survey of the existing results before 1966 see [10]; for modern results see e.g. [11].) Erdős was the first to note that the near independent behavior of $(\cos n_k x)$ remains valid for a large class of sequences (n_k) growing slower than exponential; in fact he proved the following result:

Theorem A (Erdős 1962). *Let (n_k) be a sequence of positive integers satisfying*

$$n_{k+1}/n_k \ge 1 + c_k/\sqrt{k}, \qquad c_k \to \infty. \tag{1.5}$$

Then $(\cos n_k x)$ satisfies the CLT i.e. (1.2) holds. On the other hand, for every $c > 0$ there exists a sequence (n_k) of integers satisfying

$$n_{k+1}/n_k \ge 1 + c/\sqrt{k} \qquad (k \ge k_0)$$

such that the CLT (1.2) is false.

To understand the meaning of (1.5) let us say, given positive numerical sequences (a_N), (b_N), that $a_N \succ b_N$ if $a_{N+1}/a_N \ge b_{N+1}/b_N$ for $N \ge N_0$. Then Erdős' theorem shows that $(\cos n_k x)$ satisfies the CLT if $n_k \succ e^{c_k\sqrt{k}}$ for some $c_k \uparrow \infty$ and for $n_k \succ e^{c\sqrt{k}}$ the result breaks down. An analogous, but slightly less precise result for the LIL was found by Takahashi:

[2] Relations (1.2), (1.3) remain valid also for sine series $\sum \sin n_k x$ and more generally, for $\sum\cos(n_k x + \varphi_k)$ where φ_k are arbitrary real numbers. To simplify the formulas, however, in our paper we shall deal only with pure cosine series.

Theorem B (Takahashi 1972, 1975). *Let (n_k) be a sequence of positive integers satisfying*

$$n_k \succ e^{k^\alpha}, \qquad \alpha > 1/2. \tag{1.6}$$

Then $(\cos n_k x)$ obeys the LIL (1.3). On the other hand, there exists a sequence (n_k) of integers satisfying (1.6) with $\alpha = 1/2$ such that the LIL (1.3) is false. [3]

Theorems A and B show that at the speed $n_k \sim e^{c\sqrt{k}}$ the probabilistic behavior of $(\cos n_k x)$ undergoes a fundamental change: from almost independent the sequence turns to strongly dependent. Due to a series of remarkable papers by Takahashi (see [15–20]) the behavior of $(\cos n_k x)$ on the near independent side of $e^{c\sqrt{k}}$ is fairly well known; on the other hand, practically nothing is known in the strongly dependent domain. In a recent paper [3] we constructed the first class of nongaussian limit distributions of normed sums

$$\frac{1}{a_N}\sum_{k\le N}\cos n_k x - b_N$$

in the strongly dependent case; no complete characterization of the class of limit laws of such sums is known (or seems to be easy). It is not known, either, what asymptotic result replaces the law of the iterated logarithm (1.3) in the strongly dependent domain.

The purpose of this paper is to study the LIL behavior of $(\cos n_k x)$ in the 'critical zone' i.e. in the immediate neighborhood of the critical speed $n_k \sim e^{c\sqrt{k}}$. In view of the strong relation between the central limit theorem and the law of the iterated logarithm, it is natural to expect that the gap condition (1.5) implies also the law of the iterated logarithm (1.3) for $(\cos n_k x)$. Surprisingly, however, this is not the case: in [4] we constructed a sequence (n_k) of integers satisfying (1.5) with a very slowly increasing (c_k) such that (1.3) is false. On the other hand, Theorem B above shows that (1.5) with $c_k \ge k^\varepsilon$ $(\varepsilon > 0)$ implies the LIL (1.3) and in [20] it is proved that (1.5) with $c_k \ge k^\varepsilon$ implies also a version of the strong approximation theorem (1.4). These remarks show that even though the Erdős gap condition (1.5) implies the CLT (1.2) for any $c_k \to \infty$, the partial sum behavior of $(\cos n_k x)$ follows the independent pattern only if c_k has a certain minimal speed and for very slowly increasing c_k (i.e. near the critical speed $e^{c\sqrt{k}}$) the independent behavior of $(\cos n_k x)$ breaks down, e.g. the ordinary LIL (1.3) becomes false. The main result of our paper will show that the change from independent to strongly dependent LIL behavior of $(\cos n_k x)$ takes place at the speed

$$n_k \sim e^{\sqrt{k}(\log\log k)^\alpha}. \tag{1.7}$$

In fact, as α changes, the LIL behavior of $(\cos n_k x)$ goes through a variety of types from "very good" to "very bad". For α large, the LIL behavior of $(\cos n_k x)$ is classical: it satisfies not only the ordinary LIL (1.3) but also the Kolmogorov-Feller test and even a slightly weaker form of the a.s. invariance principle (1.4), namely

$$S(t) = W(t/2) + O(t^{1/2}(\log\log t)^{-\beta}) \quad \text{a.s.} \tag{1.8}$$

[3] The second half of the theorem is implicit in Erdős' example in Theorem A.

where $\beta = \beta(\alpha) \to \infty$ as $\alpha \to \infty$. For α small, $(\cos n_k x)$ fails even the ordinary LIL (1.3) and in fact the cluster set of

$$\left\{ (N \log\log N)^{-1/2} \sum_{k \le N} \cos n_k x\,,\; N \ge 1 \right\}$$

can be asymmetric around zero and contain points with absolute value > 1. For an intermediate range of α's the LIL behavior of $(\cos n_k x)$ is "transitional": it satisfies the ordinary LIL (1.3) but fails the Kolmogorov-Feller test and the upper-lower class behavior of $(\cos n_k x)$ is described by an asymmetric test whose form is different from the Kolmogorov-Feller test and becomes, as α decreases, gradually more and more complicated. At $\alpha = 1/2$ the test blows up and even the ordinary LIL breaks down.

We now formulate our results in detail.

Theorem 1. *Let (n_k) be a sequence of positive integers satisfying*

$$n_k \succ e^{\sqrt{k}(\log\log k)^\alpha} \tag{1.9}$$

for some $\alpha > 5/2$. Then $(\cos n_k x)$ satisfies the Kolmogorov-Feller test i.e. setting $S_N = \sum_{k \le N} \cos n_k x$ we have for any positive nondecreasing function φ_N

$$P\{S_N > \sqrt{N/2}\varphi_N \ \textit{i.o.}\} = 0 \quad \textit{or} \quad 1\,,[4] \tag{1.10}$$

according as

$$\sum_{N \ge 1} \frac{\varphi_N}{N} \exp\left(-\frac{1}{2}\varphi_N^2\right) < +\infty \quad \textit{or} \quad = +\infty. \tag{1.11}$$

Theorem 2. *For any $1/2 \le \alpha < 3/2$ and all sufficiently small $A > 0$ there exists a sequence (n_k) of positive integers satisfying*

$$n_k \succ e^{\sqrt{k/8A}\,(\log\log k)^\alpha} \tag{1.12}$$

such that the Kolmogorov-Feller test (1.10)-(1.11) *fails for the partial sums $S_N = \sum_{k \le N} \cos n_k x$. More precisely, given any positive nondecreasing sequence φ_N satisfying*

$$2^{-1}(\log\log N)^{1/2} \le \varphi_N \le 2(\log\log N)^{1/2} \tag{1.13}$$

the alternative in (1.10) *holds according as*

$$\sum_{N \ge 1} \frac{\varphi_N}{N} \exp\left\{ -\frac{1}{2} \sum_{\nu=0}^{\infty} \frac{a_{N,\nu}}{(\log\log N)^{\alpha\nu}} \varphi_N^{\nu+2} \right\} < +\infty \quad \textit{or} \quad = +\infty. \tag{1.14}$$

Here $a_{N,\nu}$ $(\nu = 0, 1, \dots)$ are explicitly calculable numbers with

$$a_{N,0} = 1\,, \quad a_{N,1} = \frac{\sqrt{A}}{3}\,, \quad a_{N,2} = -\frac{5A}{36}\,, \tag{1.15}$$

$$|a_{N,\nu}| \le \frac{(48\sqrt{A})^\nu}{7}\,, \qquad N \ge 1\,, \ \nu \ge 1. \tag{1.16}$$

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[4] Whenever they concern the trigonometric system, the symbols P and E mean normalized Lebesgue measure and integral in $(0, 2\pi)$, respectively. (Cf. footnote 1.) The symbol "i.o." stands for "infinitely often".

Theorem 3. *Let* (n_k) *be a sequence of positive integers satisfying* (1.9) *for* $\alpha > 1/2$. *Then* $(\cos n_k x)$ *satisfies the upper half of the law of the iterated logarithm, i.e.*

$$\overline{\lim_{N\to\infty}} (N \log\log N)^{-1/2} \left| \sum_{k\le N} \cos n_k x \right| \le 1 \quad \text{a.e.} \tag{1.3a}$$

On the other hand, there is a sequence (n_k) *satisfying* (1.9) *with* $\alpha = 1/2$ *such that* (1.3a) *is false.*

There is a gap between the constants $5/2$ and $3/2$ in Theorems 1 and 2 and thus the upper-lower class behavior of $(\cos n_k x)$ remains open if (1.9) holds with $3/2 \le \alpha \le 5/2$. We believe that the example of Theorem 2 is best possible i.e. Theorem 1 holds actually for $\alpha \ge 3/2$. (See in this respect the remarks at the end of this section.) It seems also likely that Theorem 3 remains valid with ≤ 1 in (1.3a) replaced by $= 1$.

To understand the meaning of (1.14) we mention a few special cases.

1. Assume $1 \le \alpha < 3/2$. Then (1.14) reduces to

$$\sum_{N\ge 1} \frac{\varphi_N}{N} \exp\left\{ -\frac{1}{2}\varphi_N^2 - \frac{\sqrt{A}}{6} \frac{\varphi_N^3}{(\log\log N)^\alpha} \right\} < +\infty \quad \text{or} \quad = +\infty.$$

2. Assume $5/6 \le \alpha < 1$. Then (1.14) reduces to

$$\sum_{N\ge 1} \frac{\varphi_N}{N} \exp\left\{ -\frac{1}{2}\varphi_N^2 - \frac{\sqrt{A}}{6} \frac{\varphi_N^3}{(\log\log N)^\alpha} + \frac{5A}{72} \frac{\varphi_N^4}{(\log\log N)^{2\alpha}} \right\} < +\infty \quad \text{or} \quad = +\infty. \tag{1.17}$$

Generally, let $c_k = (k+2)/2k$, $k = 1, 2, \ldots$. Then $c_1 = 3/2$, $c_2 = 1$, $c_3 = 5/6, \ldots, c_1 > c_2 > \cdots$ and $\lim_{k\to\infty} c_k = 1/2$. Now if $\alpha \in [c_{k+1}, c_k)$ then (1.14) reduces to

$$\begin{aligned} \sum_{N\ge 1} \frac{\varphi_N}{N} \exp\Bigg\{ & -\frac{1}{2}\varphi_N^2 - \frac{1}{2} a_{N,1} \frac{\varphi_N^3}{(\log\log N)^\alpha} \\ & - \cdots - \frac{1}{2} a_{N,k} \frac{\varphi_N^{k+2}}{(\log\log N)^{k\alpha}} \Bigg\} < +\infty \quad \text{or} \quad = +\infty. \end{aligned} \tag{1.18}$$

In other words, if α decreases from $3/2$ to $1/2$ then the test (1.10)–(1.14) becomes gradually more and more complicated: passing each value c_k, one new term appears in the exponent in (1.18). For $\alpha > 1/2$ the first term in the exponent in (1.14) dominates (for φ_N satisfying (1.13)) and thus in this case $(\cos n_k x)$ satisfies the ordinary LIL (1.3). For $\alpha = 1/2$, on the other hand, all terms in the exponent in (1.14) have the same order of magnitude as the first term $-\varphi_N^2/2$ and this leads to a change in (1.3). In fact in this case the test (1.10)–(1.14) implies

$$\overline{\lim_{N\to\infty}} (N \log\log N)^{-1/2} \sum_{k\le N} \cos n_k x = 1 - \frac{1}{6}\sqrt{2A} + O(A^{2/3}) \quad \text{a.e.} \tag{1.19}$$

i.e. for sufficiently small A the $\overline{\lim}$ in (1.3) is < 1.

To get further comparisons between the test (1.10)–(1.14) and the Kolmogorov-Feller test (1.10)–(1.11) we recall the well-known fact that if the Kolmogorov-Feller test holds then

$$(1.20)\quad \varphi_N = (2\log_2 N + 3\log_3 N + 2\log_4 N + \cdots + 2\log_{l-1} N + (2+\varepsilon)\log_l N)^{1/2}$$

belongs to the upper or lower class (i.e. the first or second alternative in (1.10) holds) according as $\varepsilon > 0$ or $\varepsilon \le 0$. (Here $\log_l$ denotes the l times iterated logarithm.) Under the test (1.10)–(1.14) this will change, namely we have to insert some new terms between $2\log_2 N$ and $3\log_3 N$ in (1.20) as the following corollary shows. Set

$$(1.21)\qquad \rho_\varepsilon(N) = 3\log_3 N + 2\log_4 N + \cdots + 2\log_{l-1} N + (2+\varepsilon)\log_l N.$$

Then we have

Corollary. *Let* $1/2 < \alpha < 3/2$, *then*

$$(1.22)\qquad \varphi_N = \left(2\log\log N + \sum_{j=0}^{k(\alpha)} b_{N,j}(\log\log N)^{3/2-\alpha-dj} + \rho_\varepsilon(N)\right)^{1/2}$$

belongs to the upper or lower class with respect to $(\cos n_k x)$ *according as* $\varepsilon > 0$ *or* $\varepsilon \le 0$. *Here* $k = k(\alpha)$ *is the integer defined by* $\alpha \in [c_{k+2}, c_{k+1})$, $d = \alpha - 1/2 > 0$ *and the* $b_{N,j}$ *are explicitly calculable numbers (actually polynomials of the* $a_{N,j}$ *in* (1.14)). *In particular* $b_{N,0} = -2\sqrt{2A}/3$, $b_{N,1} = 11A/9$.

Again we mention a few special cases.

1. Let $1/2 < \alpha < 3/2$. Then

$$\varphi_N = (2\log\log N - c(\log\log N)^{3/2-\alpha})^{1/2}$$

belongs to the upper or lower class according as $c < 2\sqrt{2A}/3$ or $c \ge 2\sqrt{2A}/3$.

2. Let $1 \le \alpha < 3/2$. Then

$$\varphi_N = \left(2\log\log N - \frac{2\sqrt{2A}}{3}(\log\log N)^{3/2-\alpha} + \rho_\varepsilon(N)\right)^{1/2}$$

belongs to the upper or lower class according as $\varepsilon > 0$ or $\varepsilon \le 0$.

3. Let $5/6 \le \alpha < 1$. Then

$$\varphi_N = \left(2\log\log N - \frac{2\sqrt{2A}}{3}(\log\log N)^{3/2-\alpha} + \frac{11A}{9}(\log\log N)^{2-2\alpha} + \rho_\varepsilon(N)\right)^{1/2}$$

belongs to the upper or lower class according as $\varepsilon > 0$ or $\varepsilon \le 0$.

A remarkable additional property of the sequence (n_k) constructed in Theorem 2 is that the sequences $(\cos n_k x)$ and $(-\cos n_k x)$ have different upper-lower class behavior. Indeed, the proof of Theorem 2 shows that for any positive nondecreasing sequence φ_N satisfying (1.13) we have

$$P\{-S_N > \sqrt{N/2}\,\varphi_N \text{ i.o.}\} = 0 \quad\text{or}\quad 1$$

according as

$$\sum_{N\geq 1}\frac{\varphi_N}{N}\exp\left\{-\frac{1}{2}\sum_{\nu=0}^{\infty}\frac{b_{N,\nu}}{(\log\log N)^{\alpha\nu}}\varphi_N^{\nu+2}\right\}<+\infty \quad\text{or}\quad =+\infty.$$

Here $b_{N,\nu}$ $(\nu\geq 0)$ are explicitly calculable numbers satisfying the same inequalities

$$|b_{N,\nu}|\leq\frac{(48\sqrt{A})^{\nu}}{7},\qquad N\geq 1,\ \nu\geq 1$$

as we have for the $a_{N,\nu}$ but the sequences $(a_{N,\nu})_{\nu\geq 0}$ and $(b_{N,\nu})_{\nu\geq 0}$ are different. In particular,

$$b_{N,0}=1,\quad b_{N,1}=-\frac{\sqrt{A}}{3},\quad b_{N,2}=-\frac{5A}{36}.$$

An immediate consequence of this asymmetry is that for $\alpha=1/2$ the cluster set of $(N\log\log N)^{-1/2}\sum_{k\leq N}\cos n_k x$ is not symmetric around zero; in fact we have

$$\underline{\lim}_{N\to\infty}(N\log\log N)^{-1/2}\sum_{k\leq N}\cos n_k x=-1-\frac{1}{6}\sqrt{2A}+O(A^{2/3})\quad\text{a.e.}$$

Hence

$$\overline{\lim}_{N\to\infty}(N\log\log N)^{-1/2}\left|\sum_{k\leq N}\cos n_k x\right|>1\quad\text{a.e.}$$

if A is small enough.

There is a remarkable similarity between the change of the upper-lower class behavior of $(\cos n_k x)$ described by Theorem 2 and the change of the upper-lower class behavior of independent sequences (X_n) under the condition

$$\begin{gathered}EX_n=0,\quad EX_n^2<+\infty,\quad s_n^2=\sum_{k\leq n}EX_k^2\to+\infty,\\ |X_n|\leq K\frac{s_n}{(\log\log s_n)^{\beta}}\qquad(\beta\geq 1/2)\end{gathered}\tag{1.23}$$

as β changes. Feller [8] showed that under (1.23) the upper-lower class behavior of (X_n) is described by an integral test whose form is getting more and more complicated as β decreases; as β approaches $1/2$, successively higher and higher moments of the (X_n) enter the test. Formal analogy with Theorem 2 leads to the conjecture that $\alpha=3/2$ is the critical exponent for the Kolmogorov-Feller test for $(\cos n_k x)$ i.e. Theorem 1 holds actually for $\alpha\geq 3/2$. It is also natural to expect that under conditions similar to those of Theorem 1 $(\cos n_k x)$ obeys Chung's test for partial maxima, i.e. setting $S_N^*=\max_{1\leq M\leq N}|\sum_{k\leq M}\cos n_k x|$ we have for any positive increasing sequence ψ_N

$$P\{S_N^*<\sqrt{N/2}\,\psi_N^{-1}\text{ i.o.}\}=0\quad\text{or}\quad 1\tag{1.24}$$

according as

$$\sum_{N\geq 1}\frac{\psi_N^2}{N}\exp\left\{-\frac{8\psi_N^2}{\pi^2}\right\}<+\infty\quad\text{or}\quad=+\infty.\tag{1.25}$$

The method of the proof of Theorem 1 shows (see the remark at the end of §3) that $(\cos n_k x)$ satisfies the test (1.24)–(1.25) if (1.9) holds with $\alpha > 7/2$; whether $\alpha > 5/2$ (or even $\alpha \geq 3/2$) is also sufficient remains open. The proof of Theorem 1 also shows that under (1.9) with $\alpha > 3/2$ $(\cos n_k x)$ satisfies the a.s. invariance principle (1.4) in the slightly weaker form

$$S(t) = W(t/2) + O(t^{1/2}(\log\log t)^{(5-2\alpha+\varepsilon)/4}) \quad \text{a.s.} \tag{1.26}$$

for any $\varepsilon > 0$. It should be noted that the proof of Theorem 1 will *not* proceed via the a.s. approximation (1.26). In fact, while this traditional approach would work, it would mean a loss of precision: (1.26) implies the Kolmogorov-Feller test (1.10)–(1.11) only for $\alpha > 7/2$ i.e. under stronger assumptions than we assumed in Theorem 1. Hence, while in the standard theory of weakly dependent r.v.'s (see e.g. [12]) the a.s. invariance principle, the Kolmogorov-Feller and the Chung tests are obtained simultaneously, under the same conditions, in the very delicate trigonometric situation around the critical speed $e^{c\sqrt{k}}$ the difference between the above limit theorems becomes essential.

It is worth noting that condition (1.13) in Theorem 2 cannot be omitted completely since (1.16) guarantees the convergence of

$$\sum_{\nu=0}^{\infty} a_{N,\nu}(\log\log N)^{-\alpha\nu} x^{\nu+2}$$

only for $|x| < (48\sqrt{A})^{-1}(\log\log N)^{\alpha}$ and thus for too large φ_N the series in the exponent of (1.14) may become divergent. However, a standard argument shows (see [9, Lemma 2] and [8, p. 398]) that for any positive nondecreasing φ_N the probability in (1.10) does not change if we replace φ_N by the nondecreasing sequence ψ_N defined by

$$\psi_N = \begin{cases} 2^{-1}(\log\log N)^{1/2} & \text{if } \varphi_N \leq 2^{-1}(\log\log N)^{1/2}, \\ 2(\log\log N)^{1/2} & \text{if } \varphi_N \geq 2(\log\log N)^{1/2}, \\ \varphi_N & \text{otherwise.} \end{cases}$$

Thus Theorem 2 permits us to decide, for any nondecreasing φ_N, if the probability in (1.10) is 0 or 1.

In conclusion we note that Theorem 2 states only that the LIL behavior of $(\cos n_k x)$ is bad for *some* sequences (n_k) satisfying (1.9), $1/2 \leq \alpha < 3/2$ but not that *all* sequences (n_k) with the same speed have bad LIL behavior. As we shall prove at the end of §4, given any sequence (n_k) satisfying (1.12) for some $A > 0$, $\alpha > 0$, there exists a sequence (m_k) such that $|m_k - n_k| \leq \text{const} \cdot k^3$ (and thus $m_k \sim n_k$) such that $(\cos m_k x)$ satisfies the Kolmogorov-Feller test (1.10)–(1.11). This remark also shows that near the critical speed $n_k \sim e^{c\sqrt{k}}$ the LIL behavior of $(\cos n_k x)$ becomes very unstable: small relative changes in (n_k) lead to essential changes in the behavior of $(\cos n_k x)$.

The proof of Theorems 1 and 2 will be given in §§2–3 and in §4, respectively. As we noted above, for $\alpha = 1/2$ the sequence $(\cos n_k x)$ constructed in Theorem 2 fails the upper LIL (1.3a) (see the remarks on the asymmetric behavior of $(\cos n_k x)$) and thus the second half of Theorem 3 is also contained in the proof of Theorem 2. The proof of the first half of Theorem 3 requires combinatorial tools similar to those used for Theorem 1 but the details are considerably more complicated. Hence to keep our paper at a reasonable length, we will give the proof in a subsequent paper.

2. Some lemmas

The crucial step of the proof of Theorem 1 is Lemma 2.1 below giving a fairly sharp estimate for the number of solutions of a certain diophantine equation.

Lemma 2.1. *Let* $\{n_j\,,\ M+1 \le j \le M+N\}$ *be a finite sequence of positive integers such that*

$$n_{j+1}/n_j \ge 1 + c/\sqrt{j}\,, \qquad M+1 \le j \le M+N-1. \tag{2.1}$$

Let $p \ge 2$ *be an integer and assume that*

$$c\sqrt{Np} \le \sqrt{M+N} \le cN/48. \tag{2.2}$$

Let finally $\varepsilon_1\,, \dots\,, \varepsilon_p$ *be a sequence of* ± 1 *'s and* d *an arbitrary integer. Then the number of solutions of the equation*

$$\begin{aligned} &\varepsilon_1 n_{i_1} + \varepsilon_2 n_{i_2} + \cdots + \varepsilon_p n_{i_p} = d \\ &\qquad\qquad (M+N \ge i_1 \ge i_2 \ge \cdots \ge i_p \ge M+1) \end{aligned} \tag{2.3}$$

is at most

$$2(576 \log p)^p N(\sqrt{M+N}/c)^{p-2}. \tag{2.4}$$

For the proof we need some preparatory lemmas.

Lemma 2.2. *Let* $\{n_j\,, M+1 \le j \le M+N\}$ *satisfy* (2.1) *and assume that* $M+N \ge c^2$. *Then for any* $0 < a < b$ *the interval* $[a\,, b]$ *contains at most*

$$2c^{-1}\sqrt{M+N}\log b/a + 1$$

terms of the sequence $\{n_j\,, M+1 \le j \le M+N\}$.

Proof. Let n_q and n_r be the smallest and largest among the n_j 's $(M+1 \le j \le M+N)$ in the interval $[a\,, b]$. Then $n_r/n_q \le b/a$; on the other hand by (2.1) we have

$$\frac{n_r}{n_q} \ge \prod_{j=q}^{r-1}\left(1 + \frac{c}{\sqrt{j}}\right) \ge \left(1 + \frac{c}{\sqrt{M+N}}\right)^{r-q} \ge \exp\left(\frac{c}{2\sqrt{M+N}}(r-q)\right),$$

using the fact that $1 + x \ge e^{x/2}$ for $0 \le x \le 1$. The two estimates for n_r/n_q imply

$$\exp\left(\frac{c}{2\sqrt{M+N}}(r-q)\right) \le b/a\,,$$

whence $r - q + 1 \le 2c^{-1}\sqrt{M+N}\log b/a + 1$, as stated.

To simplify the writing, in the sequel we shall use the symbol $a \asymp 2^j$ to denote $2^j \le a < 2^{j+1}$.

Lemma 2.3. *Assume* $M+N \ge c^2$ *and consider those solutions of* (2.3) *where* $n_{i_\nu}/n_{i_{\nu+1}} \asymp 2^{j_\nu}$ $(\nu = 1\,, \dots\,, p-1)$ *where* $j_1\,, \dots\,, j_{p-1}$ *are fixed nonnegative integers. Then, given* $n_{i_1}\,, \dots\,, n_{i_{k-1}}$ $(2 \le k \le p-1)$, *the number of choices for* n_{i_k} *is at most*

$$\begin{array}{ll} 48c^{-1}\sqrt{M+N} & \text{if } p2^{-j_k} \ge 1/8\,, \\ 48c^{-1}\sqrt{M+N}\cdot p2^{-j_k} & \text{if } c/(32\sqrt{M+N}) \le p2^{-j_k} < 1/8\,, \\ 1 & \text{if } p2^{-j_k} < c/(32\sqrt{M+N}). \end{array} \tag{2.5}$$

The results remain true also for $k = 1$ *(i.e., for the number of choices of* n_{i_1}*) except that in this case* $48c^{-1}\sqrt{M+N}$ *in the first line of* (2.5) *should be replaced by* N.

Proof. Assume $k \geq 2$; the argument for $k = 1$ is identical (the first alternative in (2.5) is trivial for $k = 1$). By $n_{i_{k-1}}/n_{i_k} \asymp 2^{j_{k-1}}$ we have $n_{i_k} \in [2^{-j_{k-1}-1}n_{i_{k-1}}, 2^{-j_{k-1}}n_{i_{k-1}}]$. Hence using Lemma 2.2 it follows that given $n_{i_1}, \dots, n_{i_{k-1}}$, for n_{i_k} we have at most $2c^{-1}\sqrt{M+N}\log 2 + 1 \leq 3c^{-1}\sqrt{M+N}$ choices, no matter which assumption on $p2^{-j_k}$ in (2.5) holds. Thus the estimate in the first line of (2.5) is proved. Assume now $p2^{-j_k} < 1/8$. Let $n_{i_1}, \dots, n_{i_{k-1}}$ be given and let $\varepsilon_1 n_{i_1} + \dots + \varepsilon_{k-1}n_{i_{k-1}} = A$. By $n_{i_k}/n_{i_{k+1}} \asymp 2^{j_k}$ it follows that the numbers $n_{i_{k+1}}, n_{i_{k+2}}, \dots, n_{i_p}$ are all $\leq n_{i_k}2^{-j_k}$ and thus $|\varepsilon_{k+1}n_{i_{k+1}} + \dots + \varepsilon_p n_{i_p}| \leq pn_{i_k}2^{-j_k}$. Hence (2.3) yields

$$A + \varepsilon_k n_{i_k}(1 + \theta p2^{-j_k}) = d, \qquad |\theta| \leq 1.$$

Thus, setting $B = (d - A)/\varepsilon_k$ and using $p2^{-j_k} < 1/8$ and the fact that for $|x| \leq 1/2$ we have $(1+x)^{-1} = 1 + \lambda x$ with $|\lambda| \leq 2$, we get

$$n_{i_k} = B(1 + \theta p2^{-j_k})^{-1} = B(1 + \theta' p2^{-j_k}), \qquad |\theta'| \leq 2. \tag{2.6}$$

Here $B \neq 0$ since $n_{i_k} \neq 0$. Thus using Lemma 2.2 it follows that there are at most

$$\begin{aligned} 2c^{-1}\sqrt{M+N}\log\frac{1+2p2^{-j_k}}{1-2p2^{-j_k}} + 1 &\leq 2c^{-1}\sqrt{M+N}\log(1 + 6p2^{-j_k}) + 1 \\ &\leq 12c^{-1}\sqrt{M+N}p2^{-j_k} + 1 \end{aligned} \tag{2.7}$$

choices for n_{i_k}. It remains now to observe that the last expression in (2.7) is bounded by $48c^{-1}\sqrt{M+N}p2^{-j_k}$ or $3/2$ according as $p2^{-j_k}$ satisfies the inequality in the second or third line of (2.5).

Remark 2.4. In Lemma 2.3 we estimated the number of choices for n_{i_k} in the diophantine equation (2.3) provided $n_{i_1}, \dots, n_{i_{k-1}}$ are given and provided we consider only those solutions of (2.3) such that $n_{i_\nu}/n_{i_{\nu+1}} \asymp 2^{j_\nu}$ $(1 \leq \nu \leq p-1)$ where $j_1, \dots, j_{p-1}$ are fixed nonnegative integers. Note, however, that for the estimate in the second and third line of (2.5) we used only the fact that j_k is fixed (for the estimate in the first line we need also that j_{k-1} is fixed). Observe also that in the third line of (2.5) the number of choices for n_{i_k} is ≤ 1 not only for any fixed j_k with $p2^{-j_k} < c/(32\sqrt{M+N})$ but for all such j_k's combined. In fact, in the proof we saw that if $n_{i_1}, \dots, n_{i_{k-1}}$ are given and $n_{i_k}/n_{i_{k+1}} \asymp 2^{j_k}$ with $p2^{-j_k} < 1/8$, then $n_{i_k} \in I_k$, where $I_k = [B(1 - 2p2^{-j_k}), B(1 + 2p2^{-j_k})]$ where B is a number uniquely determined by $n_{i_1}, \dots, n_{i_{k-1}}$. We saw also that for $p2^{-j_k} < c/(32\sqrt{M+N})$ the interval I_k contains at most one integer. Clearly, for increasing j_k, the intervals I_k are shrinking and thus the union

$$\bigcup_{\{j_k : p2^{-j_k} < c/(32\sqrt{M+N})\}} I_k$$

contains also at most one integer.

Proof of Lemma 2.1. To simplify the writing, we introduce some terminology. Given a solution $(n_{i_1}, \dots, n_{i_p})$ of (2.3), the ratios $n_{i_k}/n_{i_{k+1}}$ $(1 \leq k \leq p-1)$

will be called the *gaps* in this solution. For any fixed $1 \le k \le p-1$, the gap $n_{i_k}/n_{i_{k+1}}$ will be called *small, medium,* or *large* depending on whether $p2^{-j_k}$ lies in the intervals $[1/8, +\infty)$, $[c/(32\sqrt{M+N}), 1/8)$ or $(0, c/(32\sqrt{M+N}))$, respectively, where $n_{i_k}/n_{i_{k+1}} \asymp 2^{j_k}$. (That is, the gap $n_{i_k}/n_{i_{k+1}}$ is small, medium, or large according as in Lemma 2.3 the inequality in the first, second, or third line of (2.5) is valid.) Remark 2.4 shows that given $n_{i_1}, \dots, n_{i_{k-1}}$ in (2.3) (but without fixing any of $j_1, \dots, j_{p-1}$), there is at most one choice for n_{i_k} such that the gap $n_{i_k}/n_{i_{k+1}}$ is large. We now proceed in steps.

1. Consider first those solutions of (2.3) where $n_{i_\nu}/n_{i_{\nu+1}} \asymp 2^{j_\nu}$ $(1 \le \nu \le p-1)$ and all the gaps are small or medium. We separate 2 cases.

(a) The first gap n_{i_1}/n_{i_2} is small. In such a solution, for n_{i_1} there are N possibilities and given $n_{i_1}, \dots, n_{i_{k-1}}$, $2 \le k \le p-1$, for n_{i_k} there are at most $48c^{-1}\sqrt{M+N}\psi(j_k)$ possibilities where the function $\psi(j)$ $(j \ge 0)$ is defined by

$$\psi(j) = \begin{cases} 1 & \text{if } p2^{-j} \ge 1/8, \\ p2^{-j} & \text{if } p2^{-j} < 1/8. \end{cases}$$

Finally, given $n_{i_1}, \dots, n_{i_{p-1}}$, for n_{i_p} there is at most one possibility. Thus the number of such solutions of (2.3) is at most

$$N(48\sqrt{M+N}/c)^{p-2}\prod_{k=2}^{p-1}\psi(j_k). \tag{2.8}$$

(b) The gap n_{i_1}/n_{i_2} is medium. In this case for n_{i_1} there are at most $48c^{-1}\sqrt{M+N}p2^{-j_1}$ possibilities and the number of choices for the other n_{i_k}'s can be estimated as above. Thus the number of solutions of this type is at most

$$(48\sqrt{M+N}/c)^{p-1}p2^{-j_1}\prod_{k=2}^{p-1}\psi(j_k). \tag{2.9}$$

Adding (2.8) and (2.9) and summing for $j_1, \dots, j_{p-1}$ we get an upper estimate for the number of solutions of (2.3) containing only small and medium gaps. Note that

$$\sum_{j=0}^{\infty}\psi(j) = \sum_{p2^{-j}\ge 1/8} 1 + \sum_{p2^{-j}<1/8} p2^{-j} \le 2\log 8p + 1 + 1/4 \le 12\log p \quad (p \ge 2)$$

since the last sum is a geometric series with ratio $1/2$ and first term $< 1/8$. Now in case (a), the gap n_{i_1}/n_{i_2} is small i.e., $p2^{-j_1} \ge 1/8$ thus $j_1 \le 2\log 8p \le 12\log p$. Hence adding (2.8) for $j_1, \dots, j_{p-1}$ we get at most

$$\begin{aligned} N(48\sqrt{M+N}/c)^{p-2}12\log p\prod_{k=2}^{p-1}\left(\sum_{j_k=0}^{\infty}\psi(j_k)\right) \\ \le N(48\sqrt{M+N}/c)^{p-2}(12\log p)^{p-1}. \end{aligned}$$

In case (b), we have $p2^{-j_1} < 1/8$ i.e., adding (2.9) for $j_1, \dots, j_{p-1}$ we get at most

$$(48\sqrt{M+N}/c)^{p-1} \sum_{p2^{-j_1}<1/8} p2^{-j_1} \prod_{k=2}^{p-1} \left(\sum_{j_k=0}^{\infty} \psi(j_k) \right)$$
$$\leq (48\sqrt{M+N}/c)^{p-1} \frac{1}{4} (12 \log p)^{p-2}$$
$$\leq N(48\sqrt{M+N}/c)^{p-2}(12\log p)^{p-2},$$

where in the last step we used the second inequality of (2.2). Thus we proved that the number of solutions of (2.3) containing no large gaps is

$$\leq N(48\sqrt{M+N}/c)^{p-2}(12\log p)^p \qquad (p \geq 2).$$

2. Let us consider now those solutions of (2.3) where there is exactly one large gap, say $n_{i_s}/n_{i_{s+1}}$. Then the s-tuple $(n_{i_1}, \ldots, n_{i_s})$ and the $(p-s)$-tuple $(n_{i_{s+1}}, \ldots, n_{i_p})$ contain no large gaps and by Remark 2.4 for n_{i_s} there is at most one possibility provided the previous n_{i_ν} 's are given. (The same is trivially true for n_{i_p}.) Hence the previous argument can be applied for both $(n_{i_1}, \ldots, n_{i_s})$ and $(n_{i_{s+1}}, \ldots, n_{i_p})$ and it follows that for $(n_{i_1}, \ldots, n_{i_s})$ we have at most

$$N(48\sqrt{M+N}/c)^{s-2}(12\log p)^s \tag{2.10}$$

choices and given $(n_{i_1}, \ldots, n_{i_s})$ we have for $(n_{i_{s+1}}, \ldots, n_{i_p})$ at most

$$N(48\sqrt{M+N}/c)^{p-s-2}(12\log p)^{p-s} \tag{2.11}$$

choices. Here we assumed $2 \leq s \leq p-2$ but the estimates (2.10) and (2.11) remain valid also for $s = 1$ and $s = p-1$. (Indeed, if e.g. $s = 1$ then the gap n_{i_1}/n_{i_2} is large i.e., by Remark 2.4 there is at most one choice for n_{i_1} while (2.10) gives $(cN/48\sqrt{M+N}) \cdot 12 \log p$ which is greater than 1 by the second inequality of (2.2).) Since the location of the large gap $n_{i_s}/n_{i_{s+1}}$ can be chosen in $p-1$ different ways, it follows that the number of solutions of (2.3) containing exactly one large gap is at most

$$N^2(48\sqrt{M+N}/c)^{p-4}(12\log p)^p(p-1).$$

Similarly, the number of solutions of (2.3) containing exactly l large gaps $(0 \leq l \leq p-1)$ is at most

$$N^{l+1}(48\sqrt{M+N}/c)^{p-2-2l}(12\log p)^p \binom{p-1}{l}.$$

Adding for $l = 0, 1, \ldots, p-1$ and using $\binom{p-1}{l} \leq p^l$ we get that the total number of solutions of (2.3) is at most

$$\sum_{l=0}^{p-1} N^{l+1}(48\sqrt{M+N}/c)^{p-2-2l}(12\log p)^p p^l$$
$$= N(48\sqrt{M+N}/c)^{p-2}(12\log p)^p \sum_{l=0}^{p-1} \left(\frac{Nc^2 p}{48^2(M+N)} \right)^l$$
$$\leq 2N(48\sqrt{M+N}/c)^{p-2}(12\log p)^p,$$

where in the last step we used that $Nc^2 p \leq M + N$ by the first inequality of (2.2). Hence Lemma 2.1 is proved.

From Lemma 2.1 we immediately get the following moment estimate for block sums of $(\cos n_k x)$.

Lemma 2.5. *Let* $\{n_j, M+1 \le j \le M+N\}$ *be a finite sequence of positive integers satisfying* (2.1), *further let* $p \ge 2$ *be an even integer and assume* (2.2) *holds. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=M+1}^{M+N}\cos n_k x\right|^p dx \le 2(576p\log p)^p N(\sqrt{M+N}/c)^{p-2}. \tag{2.12}$$

Proof. Using $\cos\alpha\cos\beta = [\cos(\alpha+\beta)+\cos(\alpha-\beta)]/2$ it follows that the integrand in (2.12) equals $2^{-p}\sum\cos(\pm n_{i_1}\pm\cdots\pm n_{i_p})x$, where the sum is extended for all $M+1\le i_1,\ldots,i_p\le M+N$ and all possible choices of the signs ± 1. Since $\int_0^{2\pi}\cos nx\,dx = 0$ or 2π according as the integer n differs from zero or not we get that the left side of (2.12) equals 2^{-p} times the number of solutions of the equation

$$\pm n_{i_1}\pm\cdots\pm n_{i_p} = 0 \qquad (M+1\le i_1,\ldots,i_p\le M+N). \tag{2.13}$$

Fixing the signs in (2.13) and assuming $i_1\ge i_2\ge\cdots\ge i_p$, the number of solutions of (2.13) is bounded by the expression in (2.4). Since there are 2^p possibilities for the choice of the signs ± 1 and $p!\le p^p$ possiblilties for the order of $i_1,\ldots,i_p$, (2.12) follows.

We complement Lemma 2.5 with the following simple estimates for the first and second moments of the trigonometric sums appearing in (2.12).

Lemma 2.6. *Let* $\{n_j, M+1\le j\le M+N\}$ *be a finite sequence of positive numbers satisfying* (2.1) *with some* $c\ge 1$. *Then for any* $0\le a<b\le 2\pi$ *we have*

$$\int_a^b\left(\sum_{j=M+1}^{M+N}\cos n_j x\right)dx = O(N/n_{M+1}), \tag{2.14}$$

$$\int_a^b\left(\sum_{j=M+1}^{M+N}\cos n_j x\right)^2 dx = \frac{1}{2}N(b-a)+O(N^3M/n_{M+1}), \tag{2.15}$$

where the constants implied by the O are absolute.

Proof. (2.14) follows immediately from the fact that

$$\left|\int_a^b\cos\gamma x\,dx\right|\le 2/|\gamma| \qquad (\gamma\ne 0). \tag{2.16}$$

Since the integrand in (2.15) equals

$$\frac{1}{2}N+\frac{1}{2}\sum_{j=M+1}^{M+N}\cos 2n_j x+\sum_{M+1\le\mu<\nu\le M+N}[\cos(n_\nu+n_\mu)x+\cos(n_\nu-n_\mu)x]$$

(2.15) follows from (2.16) and the estimate

$$\sum_{M+1\le\mu<\nu\le M+N}\left(\frac{1}{n_\nu+n_\mu}+\frac{1}{n_\nu-n_\mu}\right)\le\sum_{M+1\le\mu<\nu\le M+N}\frac{2}{n_\nu-n_\mu}$$
$$\le 2N\sum_{\nu=M+1}^{M+N-1}\frac{1}{n_{\nu+1}-n_\nu}\le 2N\sum_{\nu=M+1}^{M+N-1}\frac{\sqrt{\nu}}{n_\nu}\le\frac{2N^2(M+N)}{n_{M+1}}.$$

In the proof of Theorem 1 we shall also make use of a recent upper-lower class result for martingales, due to Einmahl and Mason [5] which we state here, for the purpose of reference, as a lemma.

Lemma 2.7. *Let $\{X_n, \mathcal{F}_n,\ n\ge 1\}$ be a martingale difference sequence with finite second moments such that*

$$s_n^2 := \sum_{j=1}^n E(X_j^2|\mathcal{F}_{j-1})\to+\infty \quad a.s. \tag{2.17}$$

and

$$|X_n|\le Ms_n/(\log\log s_n)^{3/2} \quad a.s. \tag{2.18}$$

for some constant $M>0$. Let $S_n=\sum_{j=1}^n X_j$. Then for any nondecreasing function $\varphi:(0,\infty)\to(0,\infty)$ we have

$$P(S_n>s_n\varphi(s_n^2)\ i.o.)=0 \quad or \quad 1 \tag{2.19}$$

depending on whether

$$\int_1^\infty\frac{\varphi(t)}{t}\exp\left(-\frac{1}{2}\varphi^2(t)\right)dt<+\infty \quad or \quad =+\infty. \tag{2.20}$$

Moreover, if $\varphi:(0,\infty)\to(0,\infty)$ is nondecreasing and satisfies

$$2^{-1}(\log\log t)^{1/2}\le\varphi(t)\le 2(\log\log t)^{1/2} \qquad (t\ge t_0), \tag{2.21}$$

then

$$P(S_n>s_n\varphi(s_n^2)\ i.o.)=P(S_n>\hat{s}_n\varphi(\hat{s}_n^2)\ i.o.) \tag{2.22}$$

for any sequence $\{\hat{s}_n,\ n\ge 1\}$ of r.v.'s such that

$$\hat{s}_n=s_n(1+O((\log\log s_n)^{-1})) \quad a.s. \tag{2.23}$$

Relation (2.22), which is implicit in [5], complements the upper-lower class test (2.19)–(2.20) by stating the stability of the probability in (2.19) at small perturbations of s_n. This fact will be useful in extending Lemma 2.7 to unbounded martingale difference sequences (see Corollary 2.8) and for small perturbations of martingale difference sequences appearing in the proof of Theorem 1.

Corollary 2.8. *The conclusion of Lemma 2.7 remains valid if condition (2.18) of the lemma is replaced by*

$$\sum_{n\ge1}\frac{(\log\log s_n)^3}{s_n^2}E(|X_n|^2 I(|X_n|\ge Ms_n(\log\log s_n)^{-3/2})|\mathcal{F}_{n-1})<+\infty \quad a.s. \tag{2.24}$$

Proof. This follows from Lemma 2.7 by a simple truncation procedure like e.g. in [8, pp. 399–401]. Without loss of generality it suffices to consider functions

$\varphi(t)$ satisfying (2.21) (see [9, Lemma 2]). Assume that $\{X_n, \ n \geq 1\}$ satisfies the conditions of Lemma 2.7 with (2.18) replaced by (2.24) and put

$$b_n = Ms_n/(\log\log s_n)^{3/2},$$

$$X_n^* = X_n I(|X_n| \leq b_n), \qquad X_n^{**} = X_n I(|X_n| > b_n),$$

$$Y_n = X_n^* - E(X_n^*|\mathcal{F}_{n-1}) = X_n^* - a_n,$$

$$a_n = \int_{|x|\leq b_n} x\, dV_n^{\mathcal{F}_{n-1}}(x),$$

here $V_n^{\mathcal{F}_{n-1}}$ is the conditional distribution function of X_n given $\mathcal{F}_{n-1}$. Clearly, $\{Y_n, \mathcal{F}_n, \ n \geq 1\}$ is a martingale difference sequence with $|Y_n| \leq 2b_n$; set $s_n^{*2} = \sum_{j=1}^n E(Y_j^2|\mathcal{F}_{j-1})$. Following the argument in [8, pp. 399–401] with obvious modifications (the only change is that V_n used in [8] should be replaced in our case by $V_n^{\mathcal{F}_{n-1}}$) we get

$$s_n^{*2} = s_n^2(1 + O((\log\log s_n)^{-3})) \quad \text{a.s.} \tag{2.25}$$

and

$$\sum_{i=1}^n X_i = \sum_{i=1}^n Y_i + O(s_n/(\log\log s_n)^{3/2}) \quad \text{a.s.}$$

Now Lemma 2.7 applies for $\{Y_n, \ n \geq 1\}$ and thus for any nondecreasing function $\varphi\colon (0, \infty) \to (0, \infty)$ satisfying (2.21) and any sequence $\{\hat{s}_n, \ n \geq 1\}$ of r.v.'s satisfying (2.23) we get, using the fact the convergence or divergence of the integral in (2.20) is not affected if we replace $\varphi(t)$ by $\varphi(t) \pm C/\varphi(t)$ for any constant $C > 0$,

$$\begin{aligned} P\left(\sum_{i=1}^n X_i > \hat{s}_n\varphi(\hat{s}_n^2) \text{ i.o.}\right) &= P\left(\sum_{i=1}^n Y_i > \hat{s}_n\varphi(\hat{s}_n^2) + O(s_n/(\log\log s_n)^{3/2}) \text{ i.o.}\right) \\ &= P\left(\sum_{i=1}^n Y_i > \hat{s}_n[\varphi(\hat{s}_n^2) + O(\varphi(\hat{s}_n^2)^{-1})] \text{ i.o.}\right) \\ &= P\left(\sum_{i=1}^n Y_i > \hat{s}_n\varphi(\hat{s}_n^2) \text{ i.o.}\right) \\ &= P\left(\sum_{i=1}^n Y_i > s_n^*\varphi(s_n^{*2}) \text{ i.o.}\right), \end{aligned}$$

where in the last step we used the fact that $\hat{s}_n = s_n^*(1 + O((\log\log s_n^*)^{-1}))$ a.s. by (2.23) and (2.25). Hence the proof of Corollary (2.8) is completed.

Remark 2.9. In Philipp-Stout [13] and Einmahl-Mason [5] various further limit theorems for martingales under conditions similar to (2.17), (2.18) are proved. We formulate here two which we shall need in §3 to prove some supplements to Theorem 1. Let $\{X_n, \mathcal{F}_n, \ n \geq 1\}$ be a martingale difference sequence with finite variances satisfying (2.17) and

$$|X_n| \leq Ms_n/(\log\log s_n)^{\beta} \quad \text{a.s.} \tag{2.26}$$

for some constants $M > 0$ and $\beta > 0$. Let $S_n = \sum_{i=1}^n X_i$ and $S(t) = \sum_{\{i\geq 1: s_i^2 \leq t\}} X_i$ for $t \geq 0$. In [5] it is proved that if (2.17) and (2.26) hold for some $\beta \geq 5/2$ then $\{X_n,\ n \geq 1\}$ satisfies the upper-lower class test corresponding to Chung's LIL, i.e., for any nondecreasing $\varphi: (0, \infty) \to (0, \infty)$ we have

$$P\left(\max_{1\leq j\leq n} |S_j| < s_n/\varphi(s_n^2) \text{ i.o.}\right) = 0 \quad \text{or } 1 \tag{2.27}$$

depending on whether

$$\int_1^\infty \frac{\varphi^2(t)}{t} \exp\left(-\frac{8\varphi^2(t)}{\pi^2}\right) dt < +\infty \quad \text{or} \quad = +\infty. \tag{2.28}$$

Whether the original conditions of Lemma 2.7 or of Corollary 2.8 imply the test (2.27)–(2.28) is still an open problem. In Philipp-Stout [13] it is proved that if (2.17) and (2.26) hold for some $\beta > 1/2$ then without changing its distribution the process $\{S(t),\ t \geq 0\}$ can be redefined on a suitable probability space together with a Wiener process $\{W(t),\ t \geq 0\}$ such that

$$S(t) = W(t) + O(t^{1/2}(\log\log t)^{(3-2\beta)/4}) \quad \text{a.s.} \tag{2.29}$$

Using the truncation method in [8, pp. 399–401] it follows that the same conclusion holds if (2.26) is replaced by

$$\sum_{n\geq 1} \frac{(\log\log s_n)^{2\beta}}{s_n^2} E(|X_n|^2 I(|X_n| \geq M s_n (\log\log s_n)^{-\beta}) | \mathscr{F}_{n-1}) < +\infty \quad \text{a.s.}$$

For β large, (2.29) implies both the Kolmogorov-Feller and Chung tests and much more; on the other hand, to get just the tests (2.19)–(2.20) and (2.27)–(2.28), the direct arguments used in [5] require slightly weaker rates. (For example, (2.29) implies the Kolmogorov-Feller test (2.19)–(2.20) for $\beta \geq 5/2$ while in Lemma 2.7, (2.26) is assumed only for $\beta \geq 3/2$.)

To conclude this section we formulate a maximal inequality for partial sums of Fourier series which we shall need for the proof of Theorem 1.

Lemma 2.10. *Let $p \geq 2$ be an even integer and let $f \in L^p(0, 2\pi)$ be an even function with nonnegative Fourier coefficients. Let $s_n(f)$ denote the nth partial sum of the Fourier series of f. Then*

$$\int_0^{2\pi} \left(\sup_{k\geq 1} |s_{2^k}(f)|\right)^p dx \leq A^p \int_0^{2\pi} |f|^p\, dx, \tag{2.30}$$

where $A > 1$ is an absolute constant.

Proof. We first note the well-known fact that for any $f \in L^p$, $p > 1$ we have

$$\int_0^{2\pi} \left(\sup_{k\geq 1} |\sigma_k(f)|\right)^p dx \leq K \int_0^{2\pi} |f|^p\, dx, \tag{2.31}$$

where $\sigma_k(f)$ denotes the kth $(C, 1)$ mean of the Fourier series of f and $K > 1$ is an absolute constant. (See [21, Chapter IV, Theorem 7.8] and the remark after Theorem 7.5 concerning A_r.) Next we observe that if $p \geq 2$ is an even integer and $g = \sum_{k=0}^\infty b_k \cos kx$, $h = \sum_{k=0}^\infty c_k \cos kx$ are L^2-convergent sums with $0 \leq b_k \leq c_k$ $(k = 0, 1, \ldots)$ then $\|g\|_p \leq \|h\|_p$ (where the p norms can also be $+\infty$). Clearly we can assume $\|h\|_p < +\infty$; let g_N and h_N denote the Nth partial sum of the series defining g and h. Expanding $|g_N|^p$ as in the proof of Lemma 2.5 we get

$$\|g_N\|_p^p = 2^{-p} \sum_{\substack{0\le i_1,\dots,i_p\le N\\ \varepsilon_1,\dots,\varepsilon_p=\pm 1}} b_{i_1}\cdots b_{i_p}\psi(\varepsilon_1 i_1+\cdots+\varepsilon_p i_p),$$

where $\psi(x) = 1$ or 0 according as $x = 0$ or $x \neq 0$. A similar expansion holds for $\|h_N\|_p^p$ and thus we get $\|g_N\|_p \le \|h_N\|_p$. By $\|h\|_p < +\infty$ we have $\|h_N\|_p \to \|h\|_p$ as $N \to \infty$ (see [21, Chapter VII, Theorem 6.4]) and thus the partial sums g_N, $N \ge 1$ and consequently also the $(C, 1)$ means of the Fourier series of g remain bounded in L^p norm whence we get $\|g\|_p < +\infty$. (See the remark preceding Theorem 5.12 in [21, Chapter IV]). Thus $\|g_N\|_p \to \|g\|_p$ and our above claim follows.

Assume now that $f = \sum_{k=0}^\infty a_k \cos kx$ and p satisfy the assumptions of the lemma. Set $\Delta_N = \sum_{2^N<k\le 2^{N+1}} a_k \cos kx$, $N = 0, 1, \dots$, and

$$f_1 = a_0 + a_1\cos x + \sum_{N \text{ even}} \Delta_N, \qquad f_2 = \sum_{N \text{ odd}} \Delta_N.$$

Since $f \in L^2$, the series defining f_1 and f_2 converge in L^2 norm and by $f \in L^p$ and the above remark we have $f_1 \in L^p$, $f_2 \in L^p$. Clearly, if N is odd then $s_k(f_1)$ does not change for $2^N \le k \le 2^{N+1}$ and thus

$$\begin{aligned} 2^N s_{2^N}(f_1) &= s_{2^N}(f_1) + s_{2^N+1}(f_1) + \cdots + s_{2^{N+1}-1}(f_1)\\ &= 2^{N+1}\sigma_{2^{N+1}-1}(f_1) - 2^N\sigma_{2^N-1}(f_1), \end{aligned}$$

whence

$$|s_{2^N}(f_1)| = |s_{2^{N+1}}(f_1)| \le 2|\sigma_{2^{N+1}-1}(f_1)| + |\sigma_{2^N-1}(f_1)| \le 3\sup_{k\ge 1}|\sigma_k(f_1)|.$$

A similar remark holds if f_1 is replaced by f_2 and thus using (2.31), $f = f_1 + f_2$, $\|f_1\|_p \le \|f\|_p$, $\|f_2\|_p \le \|f\|_p$ and the Minkowski inequality, we get (2.30) with $A = 6K$.

3. Proof of Theorem 1

Assume that (n_k) satisfies (1.9) with $\alpha > 5/2$. Then we get, using the mean value theorem,

$$\frac{n_{k+1}}{n_k} \ge \exp\left(\frac{1}{2\sqrt{k+1}}(\log\log k)^\alpha\right) \ge 1 + \frac{(\log\log k)^\alpha}{3\sqrt{k}} \qquad (k \ge k_0). \tag{3.1}$$

We now approximate the trigonometric functions $X_k = \cos n_k x$ by step-functions Y_k as follows. Given $k \ge 1$, define $l = l(k)$ and $m = m(k)$ by $2^l \le n_k < 2^{l+1}$, $m = [l + 5\log k]$ where [ ] denotes integral part. Then set $Y_k = E(X_k|\mathscr{F}_{m(k)})$ where $\mathscr{F}_j$ denotes the σ field generated by the intervals $[2\pi i 2^{-j}, 2\pi(i+1)2^{-j})$ $(0 \le i \le 2^j - 1)$ and E, P denote expectation and probability in the probability space $((0, 2\pi), \mathscr{B}, (2\pi)^{-1}\lambda)$. Clearly

$$|X_k - Y_k| \le n_k \cdot 2\pi 2^{-m} \le 4\pi \cdot 2^{-5\log k} \ll k^{-3}, \tag{3.2}$$

where $\ll$ means the same as the O notation. Let us divide the set of positive integers into consecutive blocks $\Delta_1, \Delta_1', \Delta_2, \Delta_2', \ldots$ such that

$$|\Delta_k| = k^4, \qquad |\Delta_k'| = k^3, \tag{3.3}$$

where $|A|$ denotes, for any set $A \subset \mathbf{R}$, the number of integers contained in A. Set

$$T_k = \sum_{\nu\in\Delta_k} X_\nu, \qquad T_k' = \sum_{\nu\in\Delta_k'} X_\nu,$$

$$D_k = \sum_{\nu\in\Delta_k} Y_\nu, \qquad D_k' = \sum_{\nu\in\Delta_k'} Y_\nu, \qquad \overline{D}_k = D_k - E(D_k|\mathscr{G}_{k-1}),$$

where $\mathscr{G}_{k-1}$ denotes the σ-field generated by $D_1, \ldots, D_{k-1}$.

Lemma 3.1. *We have*

$$|E(D_k|\mathscr{G}_{k-1})| = O(k^{-2}), \qquad E(D_k^2|\mathscr{G}_{k-1}) = \tfrac{1}{2}|\Delta_k| + O(k^{-2}),$$

where the constants implied by the O are absolute.

Proof. Let $p = p(k)$ and $q = q(k)$ denote the largest integer of the block Δ_{k-1} and the smallest integer of the block Δ_k, respectively. Let l be the integer defined by $2^l \le n_p < 2^{l+1}$ and set $m = [l + 5\log p]$. Clearly, each Y_ν, $1 \le \nu \le p$, is $\mathscr{F}_m$ measurable and thus $\mathscr{G}_{k-1} \subset \mathscr{F}_m$. Thus to prove Lemma 3.1 it suffices to show that for every atom $[a, b)$ of $\mathscr{F}_m$ we have

$$\begin{aligned} &(b-a)^{-1}\left|\int_a^b D_k\,dx\right| = O(k^{-2}), \\ &(b-a)^{-1}\int_a^b D_k^2\,dx = \frac{1}{2}|\Delta_k| + O(k^{-2}). \end{aligned} \tag{3.4}$$

Now by (3.2), (3.3)

$$|D_k - T_k| \ll \sum_{\nu\in\Delta_k} \nu^{-3} \le \sum_{\nu=(k-1)^4}^{\infty} \nu^{-3} \ll k^{-8} \tag{3.5}$$

and thus by $|T_k| \le k^4$ we get

$$|D_k^2 - T_k^2| \ll k^{-4}. \tag{3.6}$$

Here, and in the rest of the proof of the lemma, the constants implied by $\ll$ are absolute. Clearly $q = \sum_{i\le k-1}(i^4 + i^3) + 1 \le 2k^5$, $q - p = (k-1)^3$ and thus by (3.1)

$$\begin{aligned} \frac{n_q}{n_p} &\ge \prod_{j=p}^{q-1}\left(1 + \frac{(\log\log j)^\alpha}{3\sqrt{j}}\right) \ge \prod_{j=p}^{q-1}\left(1 + \frac{1}{\sqrt{j}}\right) \ge \left(1 + \frac{1}{\sqrt{q}}\right)^{q-p} \\ &\ge \exp\left(\frac{1}{2}(q-p)/\sqrt{q}\right) \ge \exp\left(\frac{1}{8}\sqrt{k}\right) \qquad (k \ge k_0). \end{aligned} \tag{3.7}$$

Hence using Lemma 2.6, relations (3.1), (3.3), (3.7), $b - a = 2\pi 2^{-m}$, and $2^m \le 2^l p^5 \le n_p p^5 \le n_p k^{25}$, we get

$$\left|(b-a)^{-1}\int_a^b T_k\,dx\right| \ll 2^m k^4/n_q \ll (n_p/n_q)k^{30} \ll k^{-2} \tag{3.8}$$

and

$$(3.9)\qquad \left|(b-a)^{-1}\int_a^b T_k^2\,dx - \frac{1}{2}|\Delta_k|\right| \ll 2^m k^{12}k^5/n_q \ll (n_p/n_q)k^{42} \ll k^{-2}.$$

Now (3.4) follows from (3.8), (3.9), (3.5), and (3.6).

Put $U_k = E(D_k|\mathscr{G}_{k-1})$ then

$$E(\overline{D}_k^2|\mathscr{G}_{k-1}) = E((D_k - U_k)^2|\mathscr{G}_{k-1}) = E(D_k^2|\mathscr{G}_{k-1}) - U_k^2,$$

since U_k is $\mathscr{G}_{k-1}$ measurable. Thus Lemma 3.1 and (3.5) imply

$$(3.10)\qquad E(\overline{D}_k^2|\mathscr{G}_{k-1}) = \tfrac{1}{2}|\Delta_k| + O(k^{-2})$$

and

$$(3.11)\qquad |\overline{D}_k - T_k| = O(k^{-2}),$$

where the constants implied by the O are absolute. Now we state our key lemma.

Lemma 3.2. *Let*

$$(3.12)\qquad s_n^2 = \sum_{k=1}^n E(\overline{D}_k^2|\mathscr{G}_{k-1}).$$

Then for any nondecreasing function $\varphi\colon(0,\infty)\to(0,\infty)$ *we have*

$$(3.13)\qquad P\left(\sum_{k=1}^n \overline{D}_k > s_n\varphi(s_n^2)\ \textit{i.o.}\right) = 0 \quad \textit{or} \quad 1$$

depending on whether the integral in (2.20) *converges or diverges. Moreover, for any nondecreasing function* $\varphi\colon(0,\infty)\to(0,\infty)$ *satisfying* (2.21) *we have*

$$(3.14)\qquad P\left(\sum_{k=1}^n \overline{D}_k > s_n\varphi(s_n^2)\ \textit{i.o.}\right) = P\left(\sum_{k=1}^n \overline{D}_k + \tau_n > \hat{s}_n\varphi(\hat{s}_n^2)\ \textit{i.o.}\right)$$

for any sequences $\{\tau_n,\ n\ge 1\}$ *and* $\{\hat{s}_n,\ n\ge 1\}$ *of r.v.'s such that*

$$(3.15)\qquad \tau_n = O(s_n/(\log\log s_n)^{1/2})\quad \textit{a.s.},$$

$$(3.16)\qquad \hat{s}_n = s_n(1 + O((\log\log s_n)^{-1}))\quad \textit{a.s.}$$

Remark. Relation (3.14) expresses the fact that the probability

$$P\left(\sum_{k=1}^n \overline{D}_k > s_n\varphi(s_n^2)\ \text{i.o.}\right)$$

does not change at small perturbations of $\sum_{k=1}^n \overline{D}_k$ and s_n. Actually, it suffices to prove the statement for $\tau_n = 0$ since the convergence or divergence of the integral in (2.20) is not changed if φ is replaced by $\varphi \pm C/\varphi$ for any constant $C > 0$ and by (3.15), (3.16), and (2.21) we have

$$P\left(\sum_{k=1}^n \overline{D}_k + \tau_n > \hat{s}_n\varphi(\hat{s}_n^2)\ \text{i.o.}\right) = P\left(\sum_{k=1}^n \overline{D}_k > \hat{s}_n(\varphi(\hat{s}_n^2) + O(\varphi(\hat{s}_n^2)^{-1}))\ \text{i.o.}\right).$$

Proof of Lemma 3.2. Clearly, $\{\overline{D}_k, \mathcal{G}_k, \ k \geq 1\}$ is a martingale difference sequence with finite variances and thus by Corollary 2.8 it suffices to prove that

$$\sum_{k\geq 1} \frac{(\log\log s_k)^3}{s_k^2} E(\overline{D}_k^2 I(|\overline{D}_k| \geq s_k(\log\log s_k)^{-3/2})|\mathcal{G}_{k-1}) < +\infty \quad \text{a.s.} \tag{3.17}$$

To verify (3.17) we first observe that by (3.12), (3.10), and (3.3) we have

$$s_n^2 = \frac{1}{2}\sum_{k=1}^{n} |\Delta_k| + O(1) = \frac{1}{10}n^5 + O(n^4). \tag{3.18}$$

Next we note that setting $\rho_k = s_k(\log\log s_k)^{-3/2}$ we have, for any integer $p_k \geq 2$,

$$E(\overline{D}_k^2 I(|\overline{D}_k| \geq \rho_k)|\mathcal{G}_{k-1}) \leq \rho_k^{-(p_k-2)} E(|\overline{D}_k|^{p_k}|\mathcal{G}_{k-1})$$

since ρ_k is $\mathcal{G}_{k-1}$ measurable. Hence a sufficient condition for (3.17) is

$$\sum_{k\geq 1} \frac{(\log\log s_k)^{3p_k/2}}{s_k^{p_k}} E(|\overline{D}_k|^{p_k}|\mathcal{G}_{k-1}) < +\infty \quad \text{a.s.},$$

which, in view of (3.18) and the Beppo Levi theorem, will be proved if we show that

$$\sum_{k\geq 1} \frac{(8\log\log k)^{3p_k/2}}{k^{5p_k/2}} E(|\overline{D}_k|^{p_k}) < +\infty. \tag{3.19}$$

We now choose $p_k = 2[\log\log k]$ and use Lemma 2.5 to get

$$E(|T_k|^{p_k}) \leq 2(576 p_k \log p_k)^{p_k} k^4 \frac{(k^{5/2})^{(p_k-2)}}{c_k^{p_k-2}}, \tag{3.20}$$

where $c_k = (\log\log k)^\alpha/4$. (To verify the assumptions of Lemma 2.5 in the present case note that by (3.1) we have $n_{j+1}/n_j \geq 1 + c/\sqrt{j}$ for $j \in \Delta_k$ with $c = c_k$ and also

$$c_k(k^4 p_k)^{1/2} \leq \left(\sum_{i=1}^{k} i^4 + \sum_{i=1}^{k-1} i^3\right)^{1/2} \leq c_k k^4/48$$

for $k \geq k_0$.) (3.11) and the Minkowski inequality show that (3.20) remains valid, with an extra factor 2^{p_k} on the right-hand side, if T_k on the left-hand side is replaced by $\overline{D}_k$. Thus the kth term of the series in (3.19) is

$$\begin{aligned}
&\leq \frac{(8\log\log k)^{3p_k/2}}{k^{5p_k/2}} 2^{2p_k}(576 p_k \log p_k)^{p_k} k^4 \frac{(k^{5/2})^{(p_k-2)}}{c_k^{p_k-2}} \\
&\leq \frac{1}{k}(8\log\log k)^{3p_k/2} \frac{(576 p_k^{1+\varepsilon})^{p_k} 16^{p_k}}{(\log\log k)^{\alpha(p_k-2)}} \\
&\leq \frac{1}{k}\frac{(2^{37})^{\log\log k}}{(\log\log k)^{(2\alpha-5-2\varepsilon)\log\log k - 2\alpha}} \leq \frac{1}{k}\frac{1}{(\log\log k)^{(2\alpha-5-3\varepsilon)\log\log k}} \\
&\leq \frac{1}{k(\log k)^2} \qquad (k \geq k_0)
\end{aligned} \tag{3.21}$$

for any sufficiently small $\varepsilon > 0$ since $\alpha > 5/2$. Thus (3.19) is verified and the proof of Lemma 3.2 is completed.

Observe now that by (3.3) the sequence $\{\sqrt{2}k^{-3/2}T_k'\}$ is an orthonormal system and thus the Rademacher-Mensov convergence theorem (see e.g. [21, Chapter XIII, Theorem 10.21]) implies that $\sum_{k\geq 1} k^{-2}(\log k)^{-2}T_k'$ is a.s. convergent. Hence by the Kronecker lemma and (3.18) we get

$$\sum_{k=1}^{n} T_k' = O(n^2 \log^2 n) = O(s_n/(\log\log s_n)^{1/2}) \quad \text{a.s.}$$

The last relation, together with (3.11), yields

$$\sum_{k=1}^{n}(T_k + T_k') = \sum_{k=1}^{n} \overline{D}_k + O(s_n/(\log\log s_n)^{1/2}) \quad \text{a.s.} \tag{3.22}$$

Moreover, setting $N_k = \sum_{i=1}^{k}(i^4 + i^3)$, (3.3) and (3.18) yield

$$\sqrt{N_k/2} = s_k(1 + O((\log\log s_k)^{-1})) \quad \text{a.s.} \tag{3.23}$$

From (3.22), (3.23), and Lemma 3.2 (cf. also the perturbational statement (3.14)) it follows that for any nondecreasing function $\varphi\colon (0, \infty) \to (0, \infty)$ satisfying (2.21) and any sequence $\{\tau_k,\ k \geq 1\}$ of r.v.'s satisfying (3.15) we have

$$P\left(\sum_{j=1}^{k}(T_j + T_j') + \tau_k > \sqrt{N_k/2}\,\varphi(N_k/2) \text{ i.o.}\right) = 0 \quad \text{or} \quad 1$$

depending on whether $I(\varphi)$ converges or diverges. Here, for brevity, we introduced the symbol $I(\varphi)$ for the integral in (2.20). Since $I(\varphi) < +\infty$ iff $I(\varphi_1) < +\infty$, where $\varphi_1(t) = \varphi(t/2)$, we have proved the following lemma.

Lemma 3.3. *Let* $S_N = \sum_{j=1}^{N} \cos n_j x$. *Then for any nondecreasing function* $\varphi\colon (0, \infty) \to (0, \infty)$ *satisfying* (2.21) *we have*

$$P(S_{N_k} > \sqrt{N_k/2}\varphi(N_k) \textit{ i.o.}) = 0 \quad \textit{or} \quad 1 \tag{3.24}$$

depending on whether $I(\varphi)$ *converges or diverges. Moreover, the probability in* (3.24) *does not change if we replace* S_{N_k} *by* $S_{N_k} + \tau_k$, *where* $\{\tau_k,\ k \geq 1\}$ *is any sequence of r.v.'s satisfying* $\tau_k = O((N_k/\log\log N_k)^{1/2})$ *a.s.*

To complete the proof of Theorem 1 it remains to show that for any nondecreasing function $\varphi\colon (0, \infty) \to (0, \infty)$ satisfying (2.21) we have

$$P(S_N > \sqrt{N/2}\varphi(N) \text{ i.o.}) = P(S_{N_k} > \sqrt{N_k/2}\varphi(N_k) \text{ i.o.}). \tag{3.25}$$

This statement will be an easy consequence of the following lemma.

Lemma 3.4. *Let* $M_k = \max_{N_k \leq j \leq N_{k+1}} |S_j - S_{N_k}|$. *Then*

$$M_k = O((N_k/\log\log N_k)^{1/2}) \quad \textit{a.s.} \tag{3.26}$$

To deduce (3.25) from Lemma 3.4 note that if $I(\varphi) = +\infty$ then the right-hand side of (3.25) and thus also the left-hand side is 1 and hence we may assume $I(\varphi) < +\infty$. Then the right-hand side of (3.25) is 0 and by (3.26) and the last statement of Lemma 3.3 we have $S_{N_k} + M_k \leq \sqrt{N_k/2}\varphi(N_k)$ a.s. for

$k > k_0$. But then for any $N_k \le N \le N_{k+1}$, $k \ge k_0$ we have $S_N \le S_{N_k} + M_k \le \sqrt{N_k/2}\varphi(N_k) \le \sqrt{N/2}\varphi(N)$ i.e., the left-hand side of (3.23) is also 0.

Proof of Lemma 3.4. Set

$$Z_k = \sum_{j=N_k+1}^{N_{k+1}} \cos n_j x, \quad p(k) = \max\{i: n_i \le 2^k\}, \quad H = \{p(1), p(2), \dots\}.$$

Clearly

$$M_k \le \max_{\substack{N_k \le j \le N_{k+1} \\ j \in H}} |S_j - S_{N_k}| + \max_{\{i: p(i) \le N_{k+1}\}} |p(i+1) - p(i)| := J_1 + J_2. \tag{3.27}$$

By (3.1) we have for $k \ge k_0$

$$2 \ge n_{p(k+1)}/n_{p(k)+1} \ge \prod_{m=p(k)+1}^{p(k+1)-1} \left(1 + \frac{(\log\log m)^\alpha}{3\sqrt{m}}\right)$$

$$\ge 1 + \sum_{m=p(k)+1}^{p(k+1)-1} \frac{(\log\log m)^\alpha}{3\sqrt{m}} \ge 1 + (p(k+1) - p(k) - 1)\frac{(\log\log p(k+1))^\alpha}{3\sqrt{p(k+1)}},$$

whence $p(k+1)/p(k) \to 1$ and

$$p(k+1) - p(k) \ll p(k)^{1/2}(\log\log p(k))^{-\alpha}.$$

Thus for J_2 in (3.27) we get, using $N_{k+1}/N_k \to 1$ and $\alpha \ge 1/2$,

$$|J_2| \ll (N_k/\log\log N_k)^{1/2} \quad \text{a.s.} \tag{3.28}$$

On the other hand, applying Lemma 2.10 for $f = Z_k$ we get for any even integer $p \ge 2$

$$E|J_1|^p \le A^p E|Z_k|^p, \tag{3.29}$$

where $A > 1$ is an absolute constant. Hence choosing $p = p_k = 2[\log\log k]$ and using (3.29), Lemma 2.5, $N_k \sim k^5/5$, $N_{k+1} - N_k \sim k^4$, and the Markov inequality, we get, setting $c_k = (\log\log k)^\alpha/4$,

$$\begin{aligned} P(|J_1| \ge \sqrt{N_k/\log\log N_k}) &\le \frac{(\log\log N_k)^{p_k/2}}{N_k^{p_k/2}} A^{p_k} E|Z_k|^{p_k} \\ &\le \frac{(12\log\log k)^{p_k/2}}{k^{5p_k/2}} 4A^{p_k}(576 p_k \log p_k)^{p_k} k^4 \frac{(k^{5/2})^{(p_k-2)}}{c_k^{p_k-2}}. \end{aligned} \tag{3.30}$$

Observe that for $k \ge k_0$ the last expression in (3.30) is smaller than the first expression in (3.21) and thus it cannot exceed $k^{-1}(\log k)^{-2}$. Hence (3.30) and the Borel-Cantelli lemma imply

$$|J_1| \ll (N_k/\log\log N_k)^{1/2} \quad \text{a.s.} \tag{3.31}$$

Now Lemma 3.4 follows from (3.27), (3.30), and (3.31).

The just completed proof of Theorem 1 yields, with trivial modifications, various related limit theorems for $(\cos n_k x)$ under condition (1.9). For example, replacing Lemma 2.7 in the proof of Theorem 1 with the martingale version of the Chung test formulated in Remark 2.9 we get that $(\cos n_k x)$ satisfies the

Chung test (1.24)–(1.25) if (1.9) holds with $\alpha > 7/2$. Similarly, using the a.s. invariance principle mentioned in Remark 2.9 we get, without any difficulty, the a.s. invariance principle (1.26) for $(\cos n_k x)$.

4. Proof of Theorem 2

Lemma 4.1. *We have*

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{j=1}^{n}\cos jx\right)^3 dx = \frac{3}{8}N^2 + O(N), \tag{4.1}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{j=1}^{n}\cos jx\right)^4 dx = \frac{1}{3}N^3 + O(N^2). \tag{4.2}$$

Proof. We prove (4.2). As in the proof of Lemma 2.4, the left side of (4.2) equals $1/16$ times the number of solutions of

$$\pm j_1 \pm j_2 \pm j_3 \pm j_4 = 0, \qquad 1 \le j_1, j_2, j_3, j_4 \le N. \tag{4.3}$$

Simple calculations show that the equation

$$\pm j_1 \pm j_2 = \nu, \qquad 1 \le j_1, j_2 \le N$$

has $2N$, $2N-1-|\nu|$, $2N+1-|\nu|$, or 0 solutions according as $\nu = 0$, $1 \le |\nu| \le N$, $N+1 \le |\nu| \le 2N$, or $|\nu| > 2N$. Thus the number of solutions of (4.3) equals

$$(2N)^2 + \sum_{1\le|\nu|\le N}(2N-1-|\nu|)^2 + \sum_{N+1\le|\nu|\le 2N}(2N+1-|\nu|)^2 = \frac{16N^3}{3} + O(N^2)$$

and (4.2) is proved.

Let $a_k = 2^{k^2}$, $m_k = [Ak/(\log\log k)^{2\alpha}]$ $(k \ge 3)$; $m_1 = m_2 = 0$ and $M_k = \sum_{i=1}^k m_i$ where $\alpha \ge 1/2$ and A is an absolute constant with $0 < A \le 10^{-6}$. Using the easily verifiable formula

$$\sum_{k=3}^{n}\frac{k^\beta}{(\log\log k)^\gamma} = \frac{1}{\beta+1}\frac{n^{\beta+1}}{(\log\log n)^\gamma}(1+O((\log\log n)^{-1})) \quad (\beta \ne -1) \tag{4.4}$$

we get

$$M_k = \frac{A}{2}\frac{k^2}{(\log\log k)^{2\alpha}}(1+O((\log\log k)^{-1})). \tag{4.5}$$

Let $I_k = \{a_k, 2a_k, \dots, m_k a_k\}$; clearly the sets I_k, $k = 1, 2, \dots$, are disjoint. Define the sequence (n_k) by $(n_k) = \bigcup_{j=1}^\infty I_j$. We prove that $(\cos n_k x)$ satisfies the requirements of Theorem 2. As a first step we show that

$$n_k \succ \exp\left\{\frac{1}{4\sqrt{A}}\sqrt{k}(\log\log k)^\alpha\right\}. \tag{4.6}$$

Indeed, if $M_{k-1} < j < M_k$ then setting $i = j - M_{k-1}$ and using (4.5), $M_k/M_{k-1} \to 1$ and the mean value theorem we get

$$\begin{aligned}\frac{n_{j+1}}{n_j} &= 1+\frac{1}{i} \ge 1+\frac{1}{m_k} \ge 1+\frac{(\log\log k)^{2\alpha}}{Ak}\\ &\ge 1+\frac{(\log\log M_k)^\alpha}{2\sqrt{A}\sqrt{M_{k-1}}} \ge 1+\frac{(\log\log j)^\alpha}{2\sqrt{A}\sqrt{j}} \ge \exp\left(\frac{(\log\log j)^\alpha}{4\sqrt{A}\sqrt{j}}\right)\\ &\ge \exp\left(\frac{1}{4\sqrt{A}}\left\{\sqrt{j+1}(\log\log(j+1))^\alpha - \sqrt{j}(\log\log j)^\alpha\right\}\right)\end{aligned}$$

for $j \ge j_0$. On the other hand, if $j = M_k$ then $n_{j+1}/n_j = a_{k+1}/(m_k a_k) \ge 2 \ge 1 + 1/m_k$ i.e., we get the same lower bound for n_{j+1}/n_j as above. Thus we proved (4.6). Set now $q_k = a_{k+1}/a_k$ and

$$X_k = \sum_{j=M_{k-1}+1}^{M_k} \cos n_j x, \qquad f_k(x) = \sum_{j=1}^{m_k} \cos jx.$$

Further let $\rho_k(x)$ $(0 \le x \le 2\pi)$ be the function which equals $2\pi j/q_k$ provided $2\pi j/q_k \le (a_k x)_{2\pi} < 2\pi(j+1)/q_k$ for some integer $0 \le j \le q_k - 1$; here $(t)_{2\pi}$ denotes the residue of t mod 2π. Clearly, $\rho_k(x)$ is constant on each interval $[2\pi j/a_{k+1}, 2\pi(j+1)/a_{k+1})$ $(0 \le j \le a_{k+1} - 1)$ and is periodic with period $2\pi/a_k$. Thus the functions $\rho_k(x)$, $k = 1, 2, \ldots,$ are independent r.v.'s over the probability space $((0, 2\pi), \mathscr{B}, (2\pi)^{-1}\lambda)$. Further,

$$|\rho_k(x) - (a_k x)_{2\pi}| \le 2\pi/q_k \qquad (0 \le x \le 2\pi). \tag{4.7}$$

Now let $Y_k = f_k(\rho_k(x))$, then using $X_k = f_k(a_k x) = f_k((a_k x)_{2\pi})$, $|f_k'| \le m_k^2$, (4.7), and the mean value theorem we get $|X_k - Y_k| \le 2\pi m_k^2/q_k \ll 2^{-k}$ which, in view of $EX_k = 0$, implies

$$|X_k - Z_k| \ll 2^{-k}, \tag{4.8}$$

where $Z_k = Y_k - EY_k$. Moreover, the Z_k are independent r.v.'s over the probability space $((0, 2\pi), \mathscr{B}, (2\pi)^{-1}\lambda)$. Since $|X_k| \le m_k$, the last relation and the mean value theorem imply

$$|X_k^2 - Z_k^2| \ll 2^{-k} m_k \ll 2^{-k/2} \tag{4.9}$$

and similarly

$$|X_k^3 - Z_k^3| \ll 2^{-k/2}, \qquad |X_k^4 - Z_k^4| \ll 2^{-k/2}. \tag{4.10}$$

Now using (4.5) and (4.9) we get

$$\sigma_k^2 := EZ_k^2 = EX_k^2 + O(2^{-k/2}) = \frac{1}{2}m_k + O(2^{-k/2}), \tag{4.11}$$

$$s_n^2 := \sum_{k=1}^n EZ_k^2 = \frac{1}{2}M_n + O(1), \tag{4.12}$$

whence we get by (4.5)

$$s_n = \frac{\sqrt{A}}{2}\frac{n}{(\log\log n)^\alpha}(1 + O((\log\log n)^{-1})). \tag{4.13}$$

Further using Lemma 4.1, $X_k = f_k(a_k x)$, the periodicity of f_k, (4.10), and (4.4) we get

$$\begin{aligned}\sum_{k=1}^n EZ_k^3 &= \sum_{k=1}^n EX_k^3 + O(1) = \sum_{k=1}^n \frac{1}{2\pi}\int_0^{2\pi} f_k^3(a_k x)\,dx + O(1)\\ &= \sum_{k=1}^n \frac{1}{2\pi}\int_0^{2\pi} f_k^3(x)\,dx + O(1) = \sum_{k=1}^n \frac{3}{8}m_k^2 + O\left(\sum_{k=1}^n m_k\right)\\ &= \frac{A^2}{8}\frac{n^3}{(\log\log n)^{4\alpha}}(1+O((\log\log n)^{-1})).\end{aligned} \tag{4.14}$$

Similarly

$$\sum_{k=1}^n EZ_k^4 = \frac{A^3}{12}\frac{n^4}{(\log\log n)^{6\alpha}}(1+O((\log\log n)^{-1})), \tag{4.15}$$

$$\sum_{k=1}^n (EZ_k^2)^2 = \frac{A^2}{12}\frac{n^3}{(\log\log n)^{4\alpha}}(1+O((\log\log n)^{-1})). \tag{4.16}$$

Set

$$\lambda_n = 4\sqrt{A}(\log\log n)^{-\alpha}. \tag{4.17}$$

Then by (4.8), $|X_k| \le m_k$, and (4.13) we have

$$|Z_n| \le \lambda_n s_n \qquad (n \ge n_0).$$

Also by (4.13), $\alpha \ge 1/2$, and $A \le 10^{-6}$ we have

$$\lambda_n \le \frac{1}{200}(\log\log s_n)^{-1/2} \qquad (n \ge n_0).$$

Hence applying Feller's general upper-lower class criterion (see [8, Theorems 1 and 11]) for the independent sequence $\{Z_n,\ n \ge 1\}$ we get that for any positive nondecreasing sequence ψ_n we have

$$P\left(\sum_{k=1}^n Z_k > s_n\psi_n \text{ i.o.}\right) = 0 \quad \text{or} \quad 1 \tag{4.18}$$

depending on whether

$$\sum_{n\ge 1}\frac{\sigma_n^2}{s_n^2}\psi_n \exp\left\{-\frac{1}{2}\psi_n^2(1+Q_n(\psi_n))\right\} < +\infty \quad \text{or} \quad = +\infty. \tag{4.19}$$

Here $Q_n(x) = \sum_{\nu=1}^\infty q_{n,\nu}x^\nu$ is a function analytic for $|x| < 1/(12\lambda_n)$ whose coefficients can be explicitly calculated from the moments of the Z_i's. Actually $q_{n,\nu}$ depends on the first $\nu+2$ moments of $Z_1, \dots, Z_n$, e.g.

$$\begin{aligned} q_{n,1} &= \frac{1}{3s_n^3}\sum_{k=1}^n EZ_k^3,\\ q_{n,2} &= \frac{1}{12s_n^4}\sum_{k=1}^n EZ_k^4 - \frac{1}{4s_n^4}\sum_{k=1}^n (EZ_k^2)^2 - \frac{1}{4s_n^6}\left(\sum_{k=1}^n EZ_k^3\right)^2.\end{aligned} \tag{4.20}$$

Further

$$|q_{n,\nu}| \le \frac{1}{7}(12\lambda_n)^\nu \qquad (n \ge 1,\ \nu \ge 1). \tag{4.21}$$

Using (4.13)–(4.16), (4.20), and (4.21) we get

$$q_{n,1} = \frac{\sqrt{A}}{3}(\log\log n)^{-\alpha}(1 + O((\log\log n)^{-1})), \tag{4.22}$$

$$q_{n,2} = -\frac{5A}{36}(\log\log n)^{-2\alpha}(1 + O((\log\log n)^{-1})), \tag{4.23}$$

and

$$|q_{n,\nu}| \le \frac{(48\sqrt{A})^\nu}{7}(\log\log n)^{-\nu\alpha} \qquad (n \ge 1,\ \nu \ge 1). \tag{4.24}$$

From (4.21) we easily get

$$|Q_n(x)| \le 1/4, \qquad |Q_n'(x)| \le 12\lambda_n \quad \text{for } |\lambda_n x| \le 1/24. \tag{4.25}$$

We also note the fact that if $4^{-1}(\log\log n)^{1/2} \le \psi_n \le 4(\log\log n)^{1/2}$ (actually, it suffices to assume $\lambda_n\psi_n \le 1/48$) then replacing ψ_n by $\psi_n \pm C/\psi_n$ ($C > 0$ is an arbitrary constant), the convergence or divergence of the series (4.19) is not affected. (This is observed in [8] and easily verified since (4.25) and the mean value theorem show that replacing ψ_n by $\psi_n \pm C/\psi_n$ the exponent in (4.19) changes by $O(1)$ if $\lambda_n\psi_n \le 1/48$.) This remark implies that the probability in (4.18) does not change if $\sum_{k=1}^n Z_k$ is replaced by $\sum_{k=1}^n Z_k + \tau_n$ where $\{\tau_n,\ n \ge 1\}$ is any sequence of r.v.'s such that $\tau_n = O(s_n/(\log\log s_n)^{1/2})$. Now letting $S_N = \sum_{j\le N} \cos n_j x$, we have by (4.8) and (4.13)

$$S_{M_k} = \sum_{i=1}^k X_i = \sum_{i=1}^k Z_i + O(1) = \sum_{i=1}^k Z_i + O(s_k/(\log\log s_k)^{1/2})$$

and thus we proved the following result.

Lemma 4.2. *Assume φ_n is a nondecreasing sequence satisfying*

$$2^{-1}(\log\log n)^{1/2} \le \varphi_n \le 2(\log\log n)^{1/2}. \tag{4.26}$$

Then

$$P(S_{M_k} > s_k\varphi_{M_k}\ i.o.) = 0 \quad or \quad 1 \tag{4.27}$$

depending on whether

$$\sum_{k\ge1} \frac{\sigma_k^2}{s_k^2}\varphi_{M_k}\exp\left\{-\frac{1}{2}\varphi_{M_k}^2(1 + Q_k(\varphi_{M_k}))\right\} < +\infty \quad or \quad = +\infty. \tag{4.28}$$

Here

$$Q_k(x) = \sum_{\nu=1}^\infty \frac{c_{k,\nu}}{(\log\log k)^{\nu\alpha}}x^\nu \tag{4.29}$$

is a power series converging for $|x| \le 10(\log\log n)^\alpha$ whose coefficients $c_{k,\nu}$ are explicitly calculable numbers with

$$c_{k,1} = \frac{\sqrt{A}}{3}(1 + O((\log\log k)^{-1})), \quad c_{k,2} = \frac{-5A}{36}(1 + O((\log\log k)^{-1})), \tag{4.30}$$

and

$$|c_{k,\nu}| \leq \frac{(48\sqrt{A})^\nu}{7} \qquad (k \geq 1, \ \nu \geq 1). \tag{4.31}$$

In what follows we shall prove that

$$P(S_N > \sqrt{N/2}\varphi_N \text{ i.o.}) = P(S_{M_k} > s_k \varphi_{M_k} \text{ i.o.})$$

and we shall also write the sum (4.28) in a simpler form, not containing quantities depending explicitly on the sequence M_k. We break the argument into steps.

Lemma 4.3. *Let the nondecreasing function* φ_n *satisfy* (4.26) *and assume that* $\varphi_{M_k}^2 - \varphi_{M_{k-1}}^2 \leq 1$. *Then*

$$|\varphi_i^2(1 + Q_k(\varphi_i)) - \varphi_j^2(1 + Q_k(\varphi_j))| \leq 2 \quad \textit{for } M_{k-1} \leq i \leq j \leq M_k. \tag{4.32}$$

Proof. Clearly the left side of (4.32) is bounded by

$$(\varphi_j^2 - \varphi_i^2)(1 + |Q_k(\varphi_i)|) + \varphi_j^2|Q_k(\varphi_i) - Q_k(\varphi_j)|. \tag{4.33}$$

By $\alpha \geq 1/2$, $A \leq 10^{-6}$, (4.5), (4.17), and the second inequality of (4.26) we have $\lambda_k \varphi_i \leq 1/24$, $\lambda_k \varphi_j \leq 1/24$, and thus (4.25) implies $|Q_k(\varphi_i)| \leq 1/4$. Hence the first term in (4.33) is $\leq (\varphi_{M_k}^2 - \varphi_{M_{k-1}}^2) \cdot 3/2 \leq 3/2$. On the other hand, using (4.25) and the mean value theorem it follows that the second term in (4.33) is

$$\leq \varphi_j^2 \cdot 12\lambda_k(\varphi_j - \varphi_i) \leq \varphi_j \cdot 12\lambda_k(\varphi_j^2 - \varphi_i^2) \leq 12\lambda_k\varphi_j \leq 1/2.$$

Hence (4.32) is proved.

Lemma 4.4. *Assume that* φ_n *is nondecreasing and satisfies* (4.26). *Then the series in* (4.28) *is equiconvergent with*

$$\sum_{n \geq 1} \frac{\varphi_n}{n} \exp\left\{-\frac{1}{2}\varphi_n^2(1 + \widehat{Q}_n(\varphi_n))\right\}, \tag{4.34}$$

where $\widehat{Q}_n = Q_k$ *for* $M_{k-1} < n \leq M_k$.

Proof. Let $\mathscr{H}$ denote the set of those integers $k \geq 1$ such that $\varphi_{M_k}^2 - \varphi_{M_{k-1}}^2 \leq 1$. Using the monotonicity of φ_n, (4.11), (4.12), Lemma 4.3, $M_k/M_{k-1} \to 1$, and the fact that by (4.26) and (4.5) $\varphi_{M_k}/\varphi_{M_{k-1}}$ is bounded, we get for $k \in \mathscr{H}$,

$$\begin{aligned} &\sum_{M_{k-1} < n \leq M_k} \frac{\varphi_n}{n} \exp\left\{-\frac{1}{2}\varphi_n^2(1 + \widehat{Q}_n(\varphi_n))\right\} \\ &\quad \asymp \sum_{M_{k-1} < n \leq M_k} \frac{\varphi_{M_k}}{M_k} \exp\left\{-\frac{1}{2}\varphi_{M_k}^2(1 + Q_k(\varphi_{M_k})) + O(1)\right\} \\ &\quad \asymp \frac{M_k - M_{k-1}}{M_k}\varphi_{M_k} \exp\left\{-\frac{1}{2}\varphi_{M_k}^2(1 + Q_k(\varphi_{M_k}))\right\} \\ &\quad \asymp \frac{\sigma_k^2}{s_k^2}\varphi_{M_k} \exp\left\{-\frac{1}{2}\varphi_{M_k}^2(1 + Q_k(\varphi_{M_k}))\right\}, \end{aligned} \tag{4.35}$$

where $c_k \asymp d_k$ means that c_k/d_k lies between positive constants independent of k. On the other hand, in the proof of Lemma 4.3 we saw that $|Q_k(\varphi_n)| \leq 1/2$

for any $M_{k-1} \le n \le M_k$, $k \ge 1$ (regardless whether $k \in \mathscr{H}$) and thus for any $k \notin \mathscr{H}$ the first sum in (4.35) is bounded by

$$\sum_{M_{k-1}<n\le M_k} \frac{\varphi_{M_k}}{M_{k-1}} \exp\left\{-\frac{1}{4}\varphi^2_{M_{k-1}}\right\} \ll \varphi_{M_k} \exp\{-C\varphi^2_{M_k}\} \ll \exp\left\{-\frac{C}{2}\varphi^2_{M_k}\right\}$$

for some constant $C > 0$; here again we used the boundedness of $\varphi_{M_k}/\varphi_{M_{k-1}}$. Since $\varphi^2_{M_k} - \varphi^2_{M_{k-1}} > 1$ for $k \notin \mathscr{H}$, the terms of the sum

$$\sum_{k\notin\mathscr{H}} \exp\left\{-\frac{C}{2}\varphi^2_{M_k}\right\}$$

decrease at least exponentially and thus the sum of converges. Since $\sigma_k^2/s_k^2 \le 1$, the same argument shows that adding the terms of (4.28) for $k \notin \mathscr{H}$ we get a convergent series. From these facts, the equiconvergence of (4.28) and (4.34) follows immediately.

By (4.29) and the definition of $\widehat{Q}_n$ we have

$$\widehat{Q}_n(x) = \sum_{\nu=1}^{\infty} \frac{c_{\overline{n},\nu}}{(\log\log\overline{n})^{\alpha\nu}} x^\nu, \tag{4.36}$$

where $\overline{n}$ is defined by $M_{\overline{n}-1} < n \le M_{\overline{n}}$. The sum (4.36) is similar to (4.29) but the coefficients on the right-hand side contain the powers of $(\log\log\overline{n})^\alpha$ instead of $(\log\log n)^\alpha$. As the following lemma shows, replacing $\log\log\overline{n}$ by $\log\log n$ in (4.36) will not affect the convergence or divergence of (4.34).

Lemma 4.5. *Let φ_n be a nondecreasing sequence satisfying* (4.26). *Then the series* (4.34) *is equiconvergent with*

$$\sum_{n\ge1} \frac{\varphi_n}{n} \exp\left\{-\frac{1}{2}\varphi_n^2(1 + Q_n^*(\varphi_n))\right\}, \tag{4.37}$$

where

$$Q_n^*(x) = \sum_{\nu=1}^{\infty} \frac{c_{\overline{n},\nu}}{(\log\log n)^{\alpha\nu}} x^\nu. \tag{4.38}$$

Proof. By (4.5) we have $\sqrt{n} \le \overline{n} \le n$ and thus $\log\log n - \log\log\overline{n} \le 1$ for large enough n. Hence by the mean value theorem we get

$$|(\log\log n)^{-\alpha\nu} - (\log\log\overline{n})^{-\alpha\nu}| \le \alpha\nu 2^{\alpha\nu+1}(\log\log n)^{-\alpha\nu-1} \qquad (n \ge n_0).$$

Thus using (4.31), (4.26), $\alpha \ge 1/2$, and $A \le 10^{-6}$ we get

$$\begin{aligned}\varphi_n^2|\widehat{Q}_n(\varphi_n) - Q_n^*(\varphi_n)| &\le \varphi_n^2 \sum_{\nu=1}^{\infty} |c_{\overline{n},\nu}|\, 2\alpha\nu 2^{\alpha\nu}(\log\log n)^{-\alpha\nu-1}\varphi_n^\nu \\ &\le \frac{2\alpha}{7}\frac{\varphi_n^2}{\log\log n}\sum_{\nu=1}^{\infty}\nu\left(\frac{2^\alpha\cdot 48\sqrt{A}\varphi_n}{(\log\log n)^\alpha}\right)^\nu \\ &\ll O(1)\sum_{\nu=1}^{\infty}\nu 2^{-\nu} = O(1)\end{aligned}$$

and Lemma 4.5 follows.

The coefficients $c_{\overline{n},\nu}$ in (4.38) depend on n, ν in a rather complicated way. However, as we shall not explicitly compute them (with the exception of the first two), it is worth changing the notation and to write

$$Q_n^*(x) = \sum_{\nu=1}^{\infty} \frac{a_{n,\nu}}{(\log\log n)^{\alpha\nu}} x^\nu, \tag{4.39}$$

where, in view of (4.30), (4.31) and $\log\log \overline{n} \sim \log\log n$ we have

$$\begin{aligned} a_{n,1} &= \frac{\sqrt{A}}{3}(1 + O((\log\log n)^{-1})), \\ a_{n,2} &= -\frac{5A}{36}(1 + O((\log\log n)^{-1})), \end{aligned} \tag{4.40}$$

$$|a_{n,\nu}| \le \frac{(48\sqrt{A})^\nu}{7} \qquad (n \ge 1,\ \nu \ge 1). \tag{4.41}$$

Moreover, the convergence or divergence of the series (4.37) is not affected if we change the coefficients $a_{n,1}$ and $a_{n,2}$ in (4.39) by deleting the error terms $O((\log\log n)^{-1})$ in (4.40). (Indeed, the error made by these deletions in $\varphi_n^2 Q_n^*(\varphi_n)$ is $\ll \varphi_n^3(\log\log n)^{-\alpha-1} + \varphi_n^4(\log\log n)^{-2\alpha-1}$ which is $O(1)$ by (4.26) and $\alpha \ge 1/2$.) Thus instead of (4.40) we can write

$$a_{n,1} = \frac{\sqrt{A}}{3}, \qquad a_{n,2} = -\frac{5A}{36}. \tag{4.42}$$

Lemma 4.6. *Let* φ_n *be a nondecreasing sequence satisfying* (4.26). *Then*

$$P(S_N > \sqrt{N/2}\varphi_N \ i.o.) = P(S_{M_k} > s_k \varphi_{M_k} \ i.o.). \tag{4.43}$$

Proof. By Lemma 4.2 the right-hand side of (4.43) is 0 or 1 according as the series (4.28) converges or diverges. Moreover, the remark following (4.25) shows that the right-hand side of (4.43) does not change if we replace φ_{M_k} by $\varphi_{M_k} \pm 10/\varphi_{M_k}$. Further, by (4.12) we have $s_n = \sqrt{M_n/2} + O(M_n^{-1/2})$.

Assume that the right-hand side of (4.43) is 1. Using the above facts and (4.26) it follows that with probability one we have for infinitely many k

$$\begin{aligned} S_{M_k} &> s_k(\varphi_{M_k} + 10/\varphi_{M_k}) \\ &\ge (\sqrt{M_k/2} - O(M_k^{-1/2}))(\varphi_{M_k} + (\log\log k)^{-1/2}) \ge \sqrt{M_k/2}\varphi_{M_k} \end{aligned}$$

and thus the left-hand side of (4.43) is also 1. Assume now that the right-hand side of (4.43) is 0. Then it remains 0 if we replace φ_{M_k} by $\varphi_{M_k} - 10/\varphi_{M_k}$ and thus we have almost surely for large enough k

$$\begin{aligned} S_{M_k} &\le s_k(\varphi_{M_k} - 4(\log\log k)^{-1/2}) \\ &\le (\sqrt{M_k/2} + O(M_k^{-1/2}))(\varphi_{M_k} - 4(\log\log k)^{-1/2}) \\ &\le \sqrt{M_k/2}(\varphi_{M_k} - 2(\log\log k)^{-1/2}). \end{aligned} \tag{4.44}$$

Now if $M_k \le N < M_{k+1}$ then by (4.5), $\alpha \ge 1/2$, and $A \le 10^{-6}$ we have

$$|S_N - S_{M_k}| \le m_{k+1} \le \sqrt{M_k}(\log\log k)^{-1/2} \qquad (k \ge k_0),$$

and thus by (4.44) we have

$$\begin{aligned} S_N &\le \sqrt{M_k/2}(\varphi_{M_k} - 2(\log\log k)^{-1/2}) + \sqrt{M_k}(\log\log k)^{-1/2} \\ &\le \sqrt{M_k/2}\varphi_{M_k} \le \sqrt{N/2}\varphi_N \qquad (N \ge N_0). \end{aligned}$$

Thus the left-hand side of (4.43) is 0 and the proof of Lemma 4.6 is completed.

Now Theorem 2 follows immediately from Lemmas 4.2–4.6 and the remarks made after the proof of Lemma 4.5.

Remark. Note that we obtained the validity of the test (1.10)–(1.14) for all $\alpha \geq 1/2$, regardless whether $\alpha < 3/2$. For $\alpha \geq 3/2$, however, the series (1.14) is equiconvergent with (1.11) (see the computations below) and thus the test (1.10)–(1.14) of Theorem 2 reduces to the Kolmogorov-Feller test.

In conclusion we prove the remarks we made in the Introduction concerning the test (1.10)–(1.14). First we note that if φ_N satisfies (1.13) then using (1.16) we get for $\alpha > 1/2$, $0 < A \leq 10^{-6}$, and any $k \geq 1$

$$(4.45)\quad \left|\sum_{\nu=k+1}^{\infty} \frac{a_{N,\nu}}{(\log\log N)^{\alpha\nu}}\varphi_N^{\nu+2}\right| \leq \frac{4}{7}\log\log N \sum_{\nu=k+1}^{\infty} (96\sqrt{A}\,(\log\log N)^{1/2-\alpha})^{\nu}$$
$$\leq (\log\log N)^{1-(k+1)(\alpha-1/2)} \qquad (N \geq N_0).$$

Thus if $\alpha \in [c_{k+1}, c_k)$ then the total contribution of all terms $\nu \geq k+1$ of the sum in the exponential in (1.14) is $O(1)$ and thus for such α, (1.14) is equivalent to (1.18). (If $\alpha \geq 3/2$ then the last expression in (4.45) is $O(1)$ for $k = 0$ and thus, in this case, the sum in the exponential in (1.14) is $a_{N,0}\varphi_N^2 + O(1) = \varphi_N^2 + O(1)$ i.e., the test (1.10)–(1.14) reduces to the Kolmogorov-Feller test.) (4.45) also shows that if $\alpha > 1/2$ and (1.13) holds then the contribution of all terms $\nu \geq 1$ in the sum in (1.14) is $\leq (\log\log N)^{1-(\alpha-1/2)} = o(\varphi_N^2)$ whence it follows immediately that $(\cos n_k x)$ obeys the ordinary LIL (1.3). Next we prove that for $\alpha = 1/2$ the test (1.10)–(1.14) implies (1.19). Observe to this end that for $\alpha = 1/2$ and $\varphi_N = c(2\log\log N)^{1/2}$, $0 \leq c \leq 2$ the exponent in (1.14) becomes $\exp(-f_N(c)\log\log N)$ where $f_N(c) = \sum_{\nu=0}^{\infty} 2^{\nu/2} a_{N,\nu} c^{\nu+2}$. By (1.16) the total contribution of all terms $\nu \geq 2$ in the last sum is $O(A)$ and thus by (1.15) we get

$$(4.46)\qquad f_N(c) = c^2 + \tfrac{1}{3}\sqrt{2A}\,c^3 + O(A),$$

where the constants implied by the O's are absolute. Let

$$c_1 = 1 - \tfrac{1}{6}\sqrt{2A} + A^{2/3}, \qquad c_2 = 1 - \tfrac{1}{6}\sqrt{2A} - A^{2/3}.$$

Substituting into (4.46) we get by a simple calculation

$$f_N(c_1) = 1 + 2A^{2/3} + O(A), \qquad f_N(c_2) = 1 - 2A^{2/3} + O(A),$$

and thus $f_N(c_1) > 1$, $f_N(c_2) < 1$ if A is small enough. Hence (1.14) converges for $\varphi_N = c_1(2\log\log N)^{1/2}$, diverges for $\varphi_N = c_2(2\log\log N)^{1/2}$, and thus the $\overline{\lim}$ in (1.19) lies between c_1 and c_2, completing the proof.

Next we prove the Corollary concerning the upper-lower class behavior of the function φ_N in (1.22). For simplicity, we shall give the proof for the case $k(\alpha) = 1$ (i.e., for $5/6 \leq \alpha < 1$) when the sum in (1.22) contains two terms. (This is exactly the third special case listed after the Corollary.) The proof in the general case is the same.

Assume $5/6 \leq \alpha < 1$; then, as we already observed, (1.14) is equivalent to (1.17). Set

$$(4.47)\quad \varphi_N = (2\log\log N + c(\log\log N)^{3/2-\alpha} + c^*(\log\log N)^{2-2\alpha} + \rho_\varepsilon(N))^{1/2},$$

where c and c^* are arbitrary constants and $\rho_\varepsilon(N)$ is defined by (1.21). Writing φ_N as

$$\varphi_N = (2\log\log N)^{1/2}\left(1+\frac{c}{2}(\log\log N)^{1/2-\alpha}+O((\log\log N)^{1-2\alpha})\right)^{1/2}$$

and using the power series of $(1+x)^{3/2}$ we get

(4.48)

$$\begin{aligned}&(\log\log N)^{-\alpha}\varphi_N^3\\ &\quad=\sqrt{8}(\log\log N)^{3/2-\alpha}\left(1+\frac{3c}{4}(\log\log N)^{1/2-\alpha}+O((\log\log N)^{1-2\alpha})\right)\\ &\quad=\sqrt{8}(\log\log N)^{3/2-\alpha}+\frac{3c\sqrt{8}}{4}(\log\log N)^{2-2\alpha}+O(1).\end{aligned}$$

Similarly

$$\begin{aligned}(\log\log N)^{-2\alpha}\varphi_N^4&=4(\log\log N)^{2-2\alpha}(1+O((\log\log N)^{1/2-\alpha}))\\ &=4(\log\log N)^{2-2\alpha}+O(1).\end{aligned}\tag{4.49}$$

Substituting (4.47), (4.48), and (4.49) for the terms of the exponent in (1.17), the exponent becomes

$$\begin{aligned}&-\log\log N+\left(-\frac{c}{2}-\frac{\sqrt{2A}}{3}\right)(\log\log N)^{3/2-\alpha}\\ &\quad+\left(-\frac{c^*}{2}-\frac{c\sqrt{2A}}{4}+\frac{5A}{18}\right)(\log\log N)^{2-2\alpha}-\frac{\rho_\varepsilon(N)}{2}+O(1).\end{aligned}\tag{4.50}$$

Choosing $c=-2\sqrt{2A}/3$ and $c^*=11A/9$ the coefficients of $(\log\log N)^{3/2-\alpha}$ and $(\log\log N)^{2-2\alpha}$ in (4.50) become 0 and thus the series (1.17) reduces to

$$\sum_{N\geq 1}(1+o(1))\frac{(2\log\log N)^{1/2}}{N}\exp\left\{-\log\log N-\frac{\rho_\varepsilon(N)}{2}+O(1)\right\},$$

which is clearly convergent if $\varepsilon>0$ and divergent if $\varepsilon\leq 0$.

Regarding the first of the special cases listed after the Corollary, its validity clearly follows from the Corollary for $c<2\sqrt{2A}/3$ and $c>2\sqrt{2A}/3$. In the case $c=2\sqrt{2A}/3$, φ_N belongs to the lower class for arbitrary $1/2<\alpha<3/2$ since the coefficient $b_{N,1}=11A/9$ in the sum in (1.22) is positive. (Hence to decide for arbitrary $1/2<\alpha<3/2$ if

$$\varphi_N=\left(2\log\log N-\frac{2\sqrt{2A}}{3}(\log\log N)^{3/2-\alpha}+\frac{11A}{9}(\log\log N)^{2-2\alpha}\right)^{1/2}$$

belongs to the upper or lower class (for $1\leq\alpha<3/2$ it is upper class and for $5/6\leq\alpha<1$ it is lower class by special cases 2 and 3 listed after the Corollary) one has to compute coefficient $b_{N,2}$ in (1.22).)

In conclusion we prove the remark made at the end of the Introduction i.e., we show that if (n_k) is the sequence in Theorem 2 or more generally (n_k) is any sequence satisfying (1.12) for some $A>0$, $\alpha>0$ then there exists a sequence (m_k) such that $|m_k-n_k|\ll k^3$ and $(\cos n_k x)$ satisfies the Kolmogorov-Feller test (1.10)–(1.11). In fact, let $I_k=[n_k-4k^3,\, n_k+4k^3]$ $(k\geq 1)$; clearly the intervals I_k are disjoint for $k \geq k_0$. Now the desired (m_k) can be constructed by induction as follows. Let $m_k = n_k$ for $1 \leq k \leq k_0$ and if for some $k \geq k_0$, $m_1, \ldots, m_k$ are already constructed, choose $m_{k+1} \in I_{k+1}$ so that it is different from all numbers of the form $\pm n_{i_1} \pm n_{i_2} \pm n_{i_3}$, $1 \leq i_1, i_2, i_3 \leq k$. Since the number of such sums is $\leq 8k^3$ and I_{k+1} contains more than $8k^3$ integers, this choice is possible. Clearly, the so constructed sequence (m_k) has the property that for large enough ν the equation

$$\nu = m_k \pm m_l \qquad (k > l \geq 1)$$

has at most one solution. Also, $|m_k - n_k| \leq 4k^3$ $(k \geq 1)$ and since (1.12) implies $n_k \succ e^{\sqrt{k}}$, we get

$$\begin{aligned} \frac{m_{k+1}}{m_k} &\geq \frac{n_{k+1} - 4(k+1)^3}{n_k + 4k^3} = \frac{n_{k+1}}{n_k}(1 + O(k^3 e^{-\sqrt{k}})) \\ &\geq \exp\left(\frac{1}{2\sqrt{k+1}}\right)(1 + O(k^{-1})) \geq \left(1 + \frac{1}{4\sqrt{k}}\right)\left(1 - \frac{1}{k}\right) \\ &\geq 1 + \frac{1}{5\sqrt{k}} \qquad (k \geq k_0). \end{aligned}$$

Hence by Theorem 3 of [2] $(\cos n_k x)$ satisfies the a.s. invariance principle (1.4) and consequently the Kolmogorov-Feller test (1.10)–(1.11).

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Mathematical Institute of the Hungarian Academy of Sciences, H-1053 Budapest, Reáltanoda u. 13-15, Hungary

E-mail address: h1127ber@ella.hu