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PARABOLIC SYSTEMS: THE $GF(3)$ CASE

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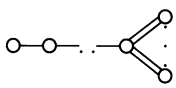
ABSTRACT. Parabolic systems defined over $GF(q)$ have been classified by Timmesfeld for $q \geq 4$ and by Stroth for $q = 2$ (see references). We deal with the case $q = 3$.

Parabolic systems have been classified by Niles, Timmesfeld, Stroth, and Heiss, if the field of definition is $GF(2)$ or has at least four elements. [Ni, Tim1, Tim2, Tim5, Tim7, St1, St2, St3, He]. We treat the $GF(3)$ case, where only partial results by Thiel exist so far [Th]. Our result says that strong parabolic systems in characteristic 3 have spherical diagram, and therefore essentially generate only finite groups of Lie type with the same diagram. This is the content of Theorem A. If we drop the assumption that the parabolic systems have to be strong, some infinite families of systems occur, whose diagrams are



or complete bipartite graphs with only double or triple bonds, and the systems are classified. This is Theorem B. The results of this paper are used in the determination of locally finite classical Tits chamber systems with a transitive group of automorphisms having finite chamber stabilizers. This classification, in turn, could be used in the proof of the theorem of Kantor, Liebler, and Tits that determines all classical affine buildings of rank at least 3 having a discrete chamber-transitive group of automorphisms.

The organization of the paper is as follows. The proof of Theorem A is given in §3, while the proof of Theorem B is contained in §4. Definitions, notation and some preliminaries are given in §1, while in §2 the relevant FF-modules for some Lie-type groups defined over $GF(3)$ are determined.

1. DEFINITIONS, NOTATION, PRELIMINARIES

We are mainly concerned with characteristic 3, hence our notation and the definitions reflect this fact. Let G be a finite group, we set $\widetilde{G} := O^{3'}(G)$, and $\overline{G} = \widetilde{G}/O_3(G)$. If S is a subgroup of G, by S_G we denote the largest normal subgroup of G contained in S. If $\{X_1, i \in I\}$ is a system of subgroups of G, we set X_{ij} for the group generated by X_i and X_j. If X is a finite simple group of Lie type, $\mathrm{PSL}_2(3)$ or ${}^2G_2(3)$, or a direct product of such groups (these

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groups will be denoted either by their symbol or their name as a matrix group, if they are classical), then any finite group G satisfying $G = \widetilde{G}$ and $G/Z(G) \cong X$ is said to be a group of Lie-type X. If X has Lie rank n, then G is said to be a rank n Lie-type group.

A group of order $2^4 \cdot 3$ with $G = \overline{G}$ and $O_2(G)$ elementary abelian is named D (there is only one isomorphism type, and D is a product of two subgroups isomorphic to $\mathrm{PSL}_2(3)$!). A finite group G with $G = \widetilde{G}$ whose Sylow 3-subgroups have three elements and $G/Z(G)$ is isomorphic to D is called of type D.

(1.1) **Definition.** Let G be a group generated by finite subgroups $X_1, \dots, X_n$ satisfying the following conditions:

(i) $\bigcap X_i$ contains a 3-group S such that $S \in \mathrm{Syl}_3(X_{ij})$ for all $i, j \leq n$.
(ii) $\overline{X}_i$ is a rank 1 Lie-type group in characteristic 3 for $i \leq n$.
(iii) $\overline{X}_{ij}$ is a rank 2 Lie-type group in characteristic 3 for $i \neq j$ or is of type D.

Then $X = \{X_1, \dots, X_n\}$ is called a *parabolic system of rank* n *in characteristic* 3 in G. If type D never occurs in (iii), the parabolic system is said to be *strong*.

To a (strong) parabolic system X of rank n in characteristic 3 there belongs a diagram that serves as a "type" of the system. Vertices (nodes) of the diagram are the indices $i \in I$, and no bond (resp. a bond of strength 1, 2, or 3—i.e., a single, double or triple bond) is drawn between the vertices i and j, if the type of $\overline{X_{ij}}$ is a direct product of two groups that are rank 1 groups or D (resp. is $A_2(q)$, resp. is $B_2(q)$, ${}^2A_3(q)$ or ${}^2A_4(q)$, resp. is $G_2(q)$ or ${}^3D_4(q)$) for some power q of the prime 3. The diagram contains exactly the same information as a Coxeter matrix $M = (m(i, j)_{i,j})$, where the entries $m(i, j)$ for $i \neq j$ are equal to 2 (resp. 3, resp. 4, resp. 6) and we will use both ways to describe the diagrams of parabolic systems. Forgetting about the strength of the bonds in the diagram, we get the graph of the diagram and may talk about connected components of the diagram. In the whole paper, we always assume that together with a (strong) parabolic system X we are given the 3-group S occurring in the definition, and the diagram Δ.

The following theorems are listed for easy reference. They were proved by Timmesfeld in arbitrary characteristic; we need only the characteristic 3, so we state them in a somewhat restricted form.

(1.2) **Theorem.** *Let* $X = \{X_1, \dots, X_n\}$, $n \geq 3$, *be a parabolic system in characteristic* 3 *in the group* G *having a connected spherical diagram* Δ. *Assume* $S_G = 1$. *Then* $G_0 = \langle \widetilde{X}_1, \dots, \widetilde{X}_n \rangle$ *is a normal subgroup of* G *and the following holds*:

(a) G_0 *is a finite group of Lie type in characteristic* 3 *with diagram* Δ.
(b) S *is a Sylow* 3*-subgroup of* G_0.
(c) *the groups* X_i *are "essentially" the rank* 1 *parabolic subgroups of* G_0 *containing the Borel subgroup* B *of* G_0 *normalizing* S,

i.e., the groups X_i *are of the form* $B\widetilde{X_i}$.

Proof. [Tim5, (3.2)].

As an immediate consequence we get that a parabolic system in characteristic

3 is automatically strong, if all subdiagrams of type $\circ \quad \circ$ $(A_1 \times A_1)$ of Δ are contained in connected spherical subdiagrams of Δ.

The nonconnected diagrams are treated in the following theorem. See also the beginning of §4.

(1.3) **Theorem.** *Let $\{X_i, i \in I\}$ be a strong parabolic system in characteristic 3 in the group G having diagram Δ. Let Δ_j, $j \in J$, be the connected components of Δ, and let $Y_j = \langle O^3(\widetilde{X_i}), \ i \in \Delta_j \rangle$. Assume $S_G = 1$. Then the subgroups Y_j are normal in G and commute pairwise.*

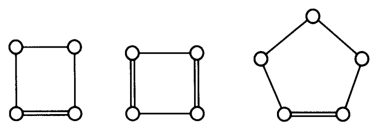
Proof. [Tim2, (4.4)].

In §3, we will need to have a list of all connected nonspherical diagrams all of whose proper subdiagrams are spherical.

(1.4) *Let Δ be a connected nonspherical Coxeter diagram of rank at least 4, whose proper subdiagrams are spherical. Assume Δ contains only single or double bonds ($m(i, j) \leq 4$ for all i, j). Then Δ is one of the following:*

(a) *the extended Dynkin diagram of type $\widetilde{A}_r$, $\widetilde{B}_r$, $\widetilde{C}_r$, $\widetilde{D}_r$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$ $(r \geq 3)$,*

(b)



Proof. Clear.

(1.5) *Let $\{X_1, \dots, X_n\}$ be a parabolic system in characteristic 3 in the group G. Assume $X_i = \widetilde{X_i}$ for $i = 1, \dots, n$ and $S_G = Z(G) = 1$. Let t be an element of order r, r a prime different from 3, in G normalizing S and X_i for $i = 1, \dots, n$. Then for at least one i, S does not contain $[X_i, t]$.*

Proof. Assume the contrary; then $[S, t] = [X_i, t]$ for all $i \leq n$, hence $[S, t]$ is normalized by all X_i, and $[S, t] \leq S_G = 1$. Now $[X_i, t] = 1$ for all $i \leq n$, and $t \in Z(G)$, a contradiction to the hypothesis $Z(G) = 1$.

(1.6) **Corollary.** *Let $X = \{X_1, \dots, X_n\}$ be a (strong) parabolic system in characteristic 3 in the group G with diagram Δ. Suppose $X_i = \widetilde{X_i}$ holds for all i and $S_G = 1$. Let, for $i = 1, \dots, n$, t_i be involutions in X_i normalizing S that commute pairwise.*

(a) *Assume that $m(n, n-1) = m(n-1, n-2) = 3$ and $m(n, i) = m(n-2, i) = 2$ for $i \leq n-3$. Then $t_n t_{n-2} \in Z(G)$.*

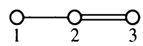
(b) *Assume $\overline{X_n}$ is isomorphic to $\mathrm{SL}_2(3)$ or $\mathrm{SL}_2(9)$, and t_n centralizes $\overline{X_i}$ for all i with $m(i, n) \neq 2$. Then $t_n \in Z(G)$.*

Proof. Clearly, the elements t_i normalize the subgroups $\widetilde{X_j}$ for all j. Consider the case (a). Being involutions, the elements t_n and t_{n-2} centralize $\overline{X_n}$ (resp. $\overline{X_{n-2}}$) and are contained in the commutator subgroup of $\widetilde{X_n}$, $\widetilde{X_{n-2}}$ respectively. Hence they also centralize all X_j where $m(n, j) = m(n-2, j) = 2$. Inspection of the group $\langle X_n, X_{n-1}, X_{n-2} \rangle$ shows that $t := t_n t_{n-2}$ also centralizes $\overline{X_{n-1}}$. The result (a) now follows from (1.5).

Consider case (b). The same argument as above shows $[t_n, \overline{X_i}] = 1$. Again the result follows from (1.5).

The next facts are clear but will be needed in §3.

(1.7) *Let G be a perfect central extension of* $\mathrm{PSp}_6(3)$ *or of* $\Omega_7(3)$. *Let B be a Borel subgroup of G, and let X_1, X_2, X_3 be the three rank 1 parabolic subgroups of G containing B corresponding to the diagram*



Then the following holds:

(i) $\overline{X_{23}} \cong \mathrm{PSp}_4(3)$ *if and only if* $G \cong \Omega_7(3)$.

Let now X_{23} be isomorphic to $\mathrm{Sp}_4(3)$, *and let t be an involution in X_{23} centralizing $\overline{X_{23}}$. Then*

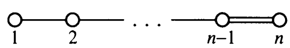
(ii) *If* $t \in \widetilde{X_3}$, *then* $G \cong \mathrm{Spin}_7(3)$ *and* $t \in Z(G)$.

(iii) *If* $t \in \widetilde{X_2}$, *then* $G \cong \mathrm{Sp}_6(3)$ *or* $\mathrm{PSp}_6(3)$.

(iv) $G \cong \mathrm{PSp}_6(3)$ *if and only if* $\overline{X_{13}} \cong \mathrm{Sl}_2(3) * \mathrm{Sl}_2(3)$.

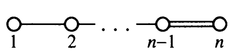
Proof. Easy exercise.

(1.8) *Let G be* $\mathrm{Sp}_{2n}(3)$, *B some Borel subgroup of G and $X_1, \dots, X_n$ the rank 1 parabolic subgroups of G containing B corresponding to the diagram*



Let H be some Cartan subgroup of G contained in B and t_i the involution in $H \cap \widetilde{X_i}$. Then $t := t_1 t_3 ..$ generates the center of G.

(1.9) *Let G be a perfect central extension of* $\Omega_{2n+1}(3)$ *or of* $\Omega^-_{2n+2}(3)$, *let B be some Borel subgroup of G and $X_1, \dots, X_n$ the rank 1 parabolic subgroups of G containing B corresponding to the diagram*



Assume t is an involution in $\widetilde{X_n}$ centralizing $\overline{X_n}$. Then $t \in Z(G)$.

2. Some FF-modules

In this section, we want to collect material that will be helpful to treat some cases in §3. There, the situation is similar to the $GF(2)$-case [St2, Tim7] where Niles' construction of a Tits system does not work. One considers the amalgam of two properly chosen "maximal parabolics" G_1 and G_2 of the parabolic system instead, and tries to get contradictions by comparing the action of both parabolics on their composition factors in the common 3-group S. In this situation, (definitions will be given in §3), one can sometimes assume that one of these composition factors is a so-called FF-module for G_1 resp. G_2. Therefore, it is helpful to have a list of all FF-modules for certain Lie-type groups to work with. But whereas in the characteristic 2 such an enemies' list is available [Co], we have to determine some FF-modules for Lie-type groups defined over $GF(3)$ ourselves.

Let us recall the definition, p is an arbitrary prime here. Let G be a finite group that acts faithfully on the elementary abelian p-group V. Assume there is a nontrivial p-subgroup A of G having the property

(FF) $$|V| \leq |A|\,|C_V(A)|.$$

Assume A is elementary abelian; then A is called an *offending subgroup* of G on V, and V is called a *failure-of-factorization module* (FF-module) *in characteristic* p for G.

In the determination of irreducible FF-modules V for a specific group G, one is almost done as soon as the $GF(p)$-dimension of V is under control. Therefore one wants to get hold of a nice offending subgroup A such that G is generated by few conjugates of A.

(2.1) **Lemma.** *Let V be an FF-module in characteristic p for the finite group G. Let U be a p-subgroup of G containing an offending subgroup. Assume $N_G(U)$ acts irreducibly on $U/\phi(U)$ and $\phi(U)$. Then U or $\phi(U)$ satisfy condition (FF) on V.*

Proof. Set $C = \{X, X \leq U\}$ and apply $[CD]$.

(2.2) **Lemma.** *Let G be a finite simple group of Lie type in characteristic p, let $S \in \mathrm{Syl}_p(G)$ and $B = N_G(S)$ some borel subgroup of G, H some complement to S in B. Let $P = U \cdot L$ be a maximal parabolic subgroup of G containing B. Assume*

(i) *G is of type B_n, C_n, ${}^2A_n$ or ${}^2D_{n+1}$ $(n \geq 2)$, or*
(ii) *G is of type D_n and L of type D_{n-1} $(n \geq 3)$.*

Then there is an element $g \in G$ such that $G = \langle U, U^g \rangle$. Furthermore in case (i), g can be chosen to centralize every involution in H.

Proof. Let (G, B, N, R) be the Tits system with the given B, and with N normalizing H, and let $g \in N$ be an element mapping onto the longest element w_0 in the Weyl group $W = N/H$ (with respect to R). Then in case (i), w_0 acts as -1 on the root system (W is of type C), and hence L, which we may assume to be generated by H and some root subgroups only permuted by g, is normalized by g. Also in case (ii), we may assume L is normalized by g.

But certainly P is not normalized by g, hence $G = \langle U, U^g, L \rangle$. Now, the subgroup $\langle U, U^g \rangle$ of G is normalized by G, and the first result follows. Assume hypothesis (i), and let $P_1 = U_1 \cdot L_1$ be any parabolic subgroup of G containing B, with Levi decomposition adjusted to H. Then again g normalizes L_1, hence $L_1 \cap H$, and hence the section assertion follows from [Ni, (4.1)].

(2.3) **Lemma.** *Let V be an irreducible $GF(3)$-module for the finite Lie-type group G in characteristic 3 of type B_n, C_n, $n \geq 2$, or ${}^2D_n$, $n \geq 3$. Let $P = U \cdot L$ be a maximal parabolic subgroup of G containing the Borel subgroup B, and let $h \in B$ be an involution.*

(a) *P centralizes $C_V(U)$, if and only if P centralizes $V/[V, U]$.*
(b) *Assume h centralizes $C_V(U)$. Then $[V, h] \leq [V, U]$.*

Proof. Take $g \in G$ as in (2.2). Then as in the proof of (2.2), we can see that h is centralized and L is normalized by g. Then h centralizes $C_V(U^g) = C_V(U)^g$, which by [Tim4] complements $[V, U]$. The results (a) and (b) follow.

(2.4) **Lemma.** *Let G be a finite group of Lie type in characteristic 3, V some irreducible FF-module in characteristic 3 for G. Then there is an element in G with minimal polynomial $(X-1)^2$ on V.*

Proof. [Tim3, (2.3)].

Elements with minimal polynomial $(X-1)^2$ on some module V are said to be *quadratic* on V. A group A is said to be quadratic on V, if $[V, A, A] = 0$. If a group G acts faithfully on a module V such that it contains a quadratic element on V, then V is called a quadratic module for G. Quadratic irreducible $K[G]$-modules for finite Lie type-groups in odd characteristic p, K some algebraically closed field in characteristic p, have been determined by Premet and Suprunenko [PS]. We recall the part of [PS] that is needed in §3. (Actually, these quadratic modules have already been determined by Thompson in unpublished parts of his quadratic pairs paper, but we prefer to refer to the easily accessible [PS].)

(2.5) **Theorem.** *Let G be a finite group of Lie type in characteristic 3 with connected diagram; let V be an irreducible $GF(3)$-module for G, and assume there is some quadratic element in G. Then, if G is a Chevalley group, V is a "fundamental module" for G, in particular there is a maximal parabolic subgroup $P = U \cdot L$ of G such that $C_V(U)$ is centralized by P'. More precisely:*

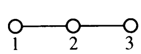
(i) *If G is of type $A_n(3)$, V is an exterior power of the natural module.*
(ii) *If G is of type $B_n(3)$, V is the natural or spin module.*
(iii) *If G is of type $C_n(3)$, V is an exterior power of the natural module.*
(iv) *If G is of type $D_n(3)$, V is the natural or a half spin module.*
(v) *If G is of type ${}^2D_n(3)$, V is the natural $\Omega^-_{2n}(3)$-module or the $GF(9)$-spin module got from the embedding of G into $\Omega^+_{2n}(9)$.*

Proof. For (i) to (iv), see [PS, Theorem 1]. Since $GF(3)$ is a splitting field for G in any case, all modules already exist over $GF(3)$. For (v), see [PS, Theorems 1 and 2]; the (half) spin module for $\Omega^+_{2n}(9)$ cannot be written over $GF(3)$ when it is restricted to $\Omega^-_{2n}(3)$, whereas the natural module can be written over $GF(3)$, if it is restricted to G.

Let us determine some irreducible FF-modules in characteristic 3 for some Lie-type groups defined over $GF(3)$. We assume always that we are given some Sylow 3-subgroup S of our Lie-types groups in characteristic 3, the Borel group $B = N_G(S)$ and the (parabolic) system of rank 1 subgroups X_i of G containing B corresponding to the given diagram. Maximal parabolics G_i (also corresponding to the diagram) will be given in a Levi decomposition $G_i = U_i \cdot L_i$.

We start with rank 3.

(2.6) **Lemma.** *Let G be of type $A_3(3)$ with diagram*



Let V be an irreducible $GF(3)$-module for G.

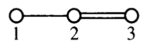
(i) *If V is quadratic for G, then V is a natural, dual, or orthogonal module for G.*

(ii) *If V is FF with $A \leq U_2$ offending, then V is a natural or dual module, in particular $\widetilde{G} \cong \mathrm{SL}_4(3)$.*

(iii) *Let V be an orthogonal module for G and A some offending subgroup. Then A is conjugate to U_1 or U_3 and $[V, A] = C_V(A)$, $|V| = |A|\,|C_V(A)|$.*

Proof. (i) follows from (2.5). In (ii), V is quadratic by (2.4), hence let us assume V is an orthogonal module. By (2.1), also U_2 itself is offending on V, which is certainly not the case. This contradiction proves (ii). Let us still assume V is an orthogonal module for G, and let $A \leq S$ be offending. Claim $|A| = |V/C_V(A)| = |[V, A]| = 3^3$, and A is quadratic. Then the statement follows easily. Let B be an offending subgroup of S with $|B|\,|V/C_V(B)|$ maximal. Then by [Tim3, (2.3)], we may assume B is quadratic, and $|B|\,|V/C_V(B)| = |V|$, $C_V(B) = [V, B]$ of order 3^3 follows, since $[V, B]$ is a singular subspace of V. Since $|B|\,|V/C_V(B)| \leq |V|$ for every quadratic subgroup on V, we may assume the given A contains U_1 or U_3. But since $J(S) = U_2$ is not offending on V, we get the claim.

(2.7) **Lemma.** *Let G be of type $B_3(3)$ with diagram*



Let V be an irreducible $GF(3)$-module for G.

(i) *If V is quadratic for G, then V is a 7-dimensional natural module or an 8-dimensional spin module for G.*

(ii) *If V is an FF-module for G with $A \leq U_1$ offending, then V is the spin module; in particular $Z(G) \cap \widetilde{X_3} \neq 1$; if $A \leq U_3$ is offending on V, then V is the natural module for G.*

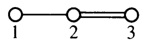
(iii) *If V is the spin module for G and $A \leq U_1$ is offending and quadratic on V, then $C_V(A) = [V, A] = [V, U_1]$.*

(iv) *If V is the natural module for G and A is quadratic and offending on V, then $|A| = |V/C_V(A)| = |[V, A]| = 3^3$ and A is conjugate to $Z(U_3)$; A is not contained in U_2.*

Proof. By (2.4) and (2.5), (i) holds. Hence for (ii), only the natural and spin modules have to be investigated. Using (2.1), we can assume $A = U_1$. But then (ii) follows easily. For (iii), we are done, if A contains elements of rank 4 on V, hence assume all nontrivial elements in A have rank 2 on V, thus A is a singular subspace of the natural module U_1 for L_1. Then $|A| \leq 3^2$, and since G does not possess transvections on V, we have $|A| = 3^2$, and $C_V(A) = C_V(a)$ for all nontrivial $a \in A$. This is clearly impossible.

Finally, assume V is the natural (orthogonal) module for $G = \Omega_7(3)$, and A is quadratic and offending on V. Since A is quadratic, $[V, A]$ is a singular subspace of V, and $C_V(A) = [V, A]^{\perp}$. Now clearly $[V, A]$ is of order 3^3, and also $|A| = 3^3 = |V/C_V(A)|$. Since A centralizes $[V, A]$, A is conjugate to $Z(U_3)$. Assume $A \leq U_2$. Then $[V, A]$ is contained in the 5-space $[V, U_2]$ and contains the singular 2-space $C_V(U_2)$, which is the radical of the space $[V, U_2]$. Now we may assume A equals $Z(U_3)$, since G_2 is transitive on the singular 3-spaces containing $C_V(U_2)$. But $Z(U_3)$ is not contained in U_2, a contradiction. Hence (iv) is proved.

(2.8) **Lemma.** *Let G be of type $C_3(3)$ with diagram*



Let V be an irreducible $GF(3)$-module for G.

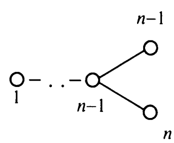
(i) *If V is quadratic for G, then V is the 6-dimensional natural module, the 13-dimensional nontrivial composition factor of the exterior square of the natural module, or the 14-dimensional nontrivial composition factor of the third exterior power of the natural module for G. In particular, if $G \cong \mathrm{PSp}_6(3)$, then* $\dim(V) = 13$.

(ii) *If V is an FF-module for G with $A \le U_1$ or $A \le U_3$ offending, then V is the natural module, in particular $G \cong \mathrm{Sp}_6(3)$.*

Proof. (i) follows from (2.5). Assume $A \le U_1$ is offending on V. By (2.1) we may assume $A = Z(U_1)$, whence G contains transvections on V, and V is certainly the natural module, or U_1 satisfies (FF) on V, whence $\dim(V) \le 10$ by (2.2), and again V is the natural module. If $A \le U_3$, we may assume $A = U_3$ by (2.1), and (2.2) implies $\dim(V) \le 12$. Again the result follows from (i).

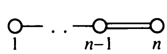
We have to treat also some higher rank cases.

(2.9) **Lemma.** *Let G be of type $D_n(3)$ with diagram*


$(n \ge 3)$

Let V be an irreducible FF-module for G in characteristic 3 with $A \le U_1$ offending. Then V is a spin module for G (for $n = 4$, we may also view the natural module as a spin module!). In particular, unless $n = 4$, we have that $Z(G) \cap (\widehat{X_{n-2}X_n})$ is not reduced to 1.

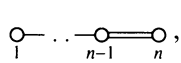
Proof. The case $n = 3$ is just (2.6)(ii), while the case $n = 4$ follows from (2.5). Hence we may assume $n \ge 5$, and assume V is the natural (orthogonal) module for G. But by (2.1) U_1 is offending on V, which is impossible. Hence V is a (half) spin module for G, and now the action of G on V forces $\widehat{X_{n-2,n-1,n}}$ to act as $\mathrm{Sl}_4(3)$, whence the claim by (1.6).

(2.10) **Lemma.** *Let G be of type $B_n(3)$ with diagram* , $n \ge 4$. *Let V be an irreducible $GF(3)$-module for G.*

(i) *If V is quadratic, V is the natural or spin module for G.*
(ii) *If V is an FF-module for G with $A \le U_1$ offending, then V is the spin module, and in particular $Z(G) \cap \widetilde{X}_n$ is not reduced to 1.*

Proof. (i) follows from (2.5), whereas by (2.1) V must be the spin module in (ii). The last assertion follows from (2.7)(ii) by the action of G on its spin module, which behaves somehow "inductive" with respect to maximal parabolic subgroups of type A.

(2.11) **Lemma.** *Let G be of type $C_n(3)$ with diagram*

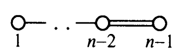
, $n \ge 4$.

Let V be an irreducible FF-module in characteristic 3 for G. Assume A is some offending subgroup with $A \leq U_1$ or $A \leq U_n$. Then V is the natural module for G, in particular $Z(G)$ is nontrivial.

Proof. Assume first $A \leq U_1$. Then by (2.1) either $Z(U_1)$ induces transvections on V, whence obviously the result follows, or U_1 satisfies (FF) on V. Then by (2.2), $\dim(V) \leq 4n-2$. By (2.4) and (2.5), we know that V is a fundamental module for G. Let i be such that $C_V(U_i)$ is 1-dimensional. Then $i = 1$ implies V is the natural module for G, hence assume $i \geq 2$. Certainly $i \neq 2$, since for $i = 2$ we know $\dim(V) = n(2n-1) - 1$ contradicting $n \geq 4$. But for $i > 2$, $C_V(U_1)$ is neither a trivial nor a natural module for L_1, and so we get an easy contradiction to $\dim(V) \leq 4n-2$.

Hence we may assume A is contained in U_n, and so also U_n is offending on V by (2.1). Assume V is not the natural module for G, and choose n minimal with respect to this. Then by (2.8) we may assume inductively that $C_V(U_1)$ is a natural module for L_1, using [Tim3, (2.2)]. Since V is a fundamental module, $C_V(U_2)$ must be 1-dimensional, and $\dim(V) = n(2n-1) - 1$. But from (2.2) we know $\dim(V) \leq n(n+1)$, a contradiction to $n \geq 4$.

(2.12) **Lemma.** *Let G be of type ${}^2D_n(3)$ with diagram*



, $n \geq 4$.

Let V be an irreducible $GF(3)$-module for G.

(i) *If V is quadratic, then V is a natural $\Omega^-_{2n}(3)$-module for G or a (half) spin module (over $GF(9)$) for $\Omega^+_{2n}(9)$ restricted to G.*

(ii) *If V is an FF-module for G with $A \leq U_1$ offending, then V is a spin module, in particular $Z(G) \cap \widetilde{X_{n-1}}$ is not reduced to 1.*

Proof. The first assertion follows from (2.5), and clearly U_1 is not offending on the natural module, hence V is the spin module in (ii) by (2.1). The last assertion again follows from the "inductive" action, hence needs only to be verified for $n = 3$, where it is clear.

In a certain situation in §3, one does not get along with the knowledge of FF-modules, but has to build up a bit more of the 3-group S. The argument needed is due to Timmesfeld. We state what together with (2.2) is sufficient for that situation (in 3.7).

(2.13) **Lemma.** *Let G be a finite group with $G/O_3(G)$ isomorphic to* $\mathrm{PSL}_2(3)$, *$Sl_2(3)$, $PSl_2(9)$ or* $\mathrm{SL}_2(9)$. *Let $t \in G$ be an element satisfying $\langle tO_3(G)\rangle = Z(G/O_3(G))$. Let V be a $GF(3)$-module for G with proper $GF(3)$-subspace W such that the following holds:*

(i) *W is invariant under some Sylow 3-subgroup S of G.*

(ii) *$W = [W, S]C_W(t)$.*

(iii) *$V = \langle W^G \rangle$.*

Then the following holds:

(1) *There is a G-composition factor in V which is a nontrivial* $\mathrm{PSL}_2(3)$ *(resp.* $\mathrm{PSL}_2(9)$*) module for $G/O_3(G)$.*

(2) *There is no quadratic element in $G/O_3(G)$.*

Proof. Clearly $\mathrm{PSL}_2(3^i)$ has no quadratic module in characteristic 3. Hence it is enough to show (1). We may therefore assume t is an involution, $G = O_3(G)C_G(t)$, and $G = S \cdot O^3(C_G(t))$.

By way of contradiction, we assume that every G-composition factor on V is either faithful for $\langle t\rangle$ or trivial for G. But this means that $O^3(C_G(t))$ acts trivially on $C_V(t)$. Consider $U = \langle C_W(t)^G\rangle$. By (ii) and (iii), $U = V$, whereas the above shows $U \leq W$. This contradiction finishes the proof.

3. Strong parabolic systems in characteristic 3

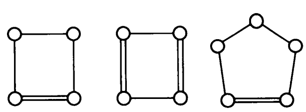
This section is devoted to the proof of Theorem A, as announced in the introduction.

Theorem A. *Let $X = \{X_1, X_2, \dots, X_n\}$ be a strong parabolic system in characteristic 3 in the group G with connected diagram Δ of rank at least 3. Then Δ is spherical and for $G_0 = \langle \widetilde{X_1}, \widetilde{X_2}, \dots, \widetilde{X_n}\rangle$ we have the following: G_0 is a normal subgroup of G, and G_0/S_G is a Lie-type group in characteristic 3 with same diagram Δ.*

Proof. First of all, if Δ is spherical, the rest of the statement is clear by Theorem (1.2). Hence we only have to show that Δ is spherical.

Assume the contrary, then we may assume the rank n of X is minimal with respect to being a counterexample, therefore the connected components Δ_j of all proper subdiagrams are spherical, and so the groups $\langle\widetilde{X_i}, i \in \Delta_j\rangle$ are (mod the largest normal subgroup in S) Lie-type groups in characteristic 3 with diagram Δ_j. In our contradiction proof, we surely may assume $X_i = \widetilde{X_i}$ for all $i \in I = \{1, 2, \dots, n\}$, hence $G = G_0$, since G_0 is a normal subgroup of G by the argument in [Ni, (4.4)], and $S_G = Z(G) = 1$.

By (1.4), the diagram Δ is either one of the extended diagrams $\widetilde{A}_r$, $\widetilde{B}_r$, $\widetilde{C}_r$, $\widetilde{D}_r$ $(r = n-1)$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, one of the exceptional diagrams,



or is of rank 3. (If Δ contains a triple bond, the rank n clearly has to be 3.)

We now try and construct a Tits system inside our group G_0 following the method introduced by Niles in [Ni, §4]. In those cases, where the construction is possible, we end up with a Tits system (G_0, B, N, R) of type Δ that has the property that B is finite while $W = N/B\cap N$ is infinite, since the type Δ is nonspherical. This together is impossible by [Tim1, (2.7)]. In the construction, we keep as close to Niles' notation as possible. We already have $X_i = \widetilde{X_i}$, hence also $X_{ij} = \widetilde{X_{ij}}$ for all $i \neq j$. Let B_i denote the normalizer of S in X_i for $i = 1, 2, \dots, n$. Then $\langle B_i, B_j\rangle$ covers the Borel subgroup normalizing S in the Lie-type groups $\overline{X_{ij}}$ for $i \neq j$ by [Ni, (4.1)] and the group $B := \langle B_i, i = 1, 2, \dots, n\rangle$ has the following properties:

B normalizes X_i and X_{ij} for all i, j.

B/S is a finite abelian $3'$-group.

The just-defined group B will be the B of the Tits system to be constructed. Let us now change the strong parabolic system X slightly to avoid notational length. We replace the rank 1 parabolics X_i by $X_i \cdot B$ but call these again X_i. Of course, we get another strong parabolic system with the same diagram Δ and the rank 1 and rank 2 parabolics of the system differ from the old ones only by some abelian $3'$-part at the top. This part can, by the way, only induce diagonal automorphisms on the $\overline{X_{ij}}$, since it induces diagonal automorphisms on the $\overline{X_i}$.

Now pick a complement H to S in B, and define N_i as the normalizer of H in X_i for $i = 1, 2, \ldots, n$. Then Niles' arguments of [Ni, §4] apply directly to our situation and give:

For $N := \langle N_i, i = 1, 2, \ldots, n \rangle$ we have $G_0 = \langle B, N \rangle$.

$B \cap N$ is normal in N and $N_i(B \cap N)/B \cap N$ is of order 2 for all i.

Let r_i denote the nontrivial coset of $B \cap N$ in $N_i(B \cap N)$, and $R := \{r_i, i = 1, 2, \ldots, n\}$; then (G_0, B, N, R) is a Tits system (of type Δ, of course) provided the following conditions are satisfied in our groups X_i and X_{ij}:

$(**)$ The centralizer of H in S/S_{X_i} is trivial.

$(*)$ If X is an H-invariant normal subgroup of S with $X \cdot S_{X_i} = X \cdot S_{X_j} = S$ then also $X \cdot S_{X_{ij}} = S$.

Therefore in our contradiction proof we may assume that at least one X_i does not satisfy $(**)$, or at least one X_{ij} fails to satisfy condition $(*)$. In particular, from Niles' Theorem B (in [Ni]) we know that at least one $\overline{X_i}$ must be of type $A_1(3)$. But we need a bit more detailed information (in our situation!).

(i) X_i does not satisfy $(**)$ if and only if $X_i/O_3(X_i)$ is a central extension of $\mathrm{PSL}_2(3)$.

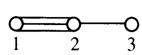
Proof. The if part is trivial, so assume X_i does not satisfy $(**)$ for some i. Since Δ is connected, there is j such that $m(i, j)$ is not 2, hence $\overline{X_i}$ cannot be of type ${}^2G_2(3)$. Therefore [Ni, (3.2)] tells that $\overline{X_i}$ is of type $A_1(3)$. But if $X_1/O_3(X_i)$ has some homomorphic image isomorphic $\mathrm{PGL}_2(3)$, then certainly $(**)$ holds, hence the claim.

(ii) X_{ij} does not satisfy $(*)$ if and only if $\overline{X_{ij}}$ is of type $G_2(3)$ or of type $A_1(3) \times A_1(3)$ *and* X_{ij} has no homorphic image $\mathrm{PSL}_2(3) \times \mathrm{PGL}_2(3)$ or $\mathrm{PGL}_2(3) \times \mathrm{PGL}_2(3)$.

Proof. Again the if part is easy. Hence assume X_{ij} does not satisfy $(*)$ for some $i \neq j$. Summing up the propositions in [Ni, §3], we get that $\overline{X_{ij}}$ is either of type $G_2(3)$ or of type $A_1(3) \times L$, where L is a rank 1 Lie-type group in characteristic 3. Assume $\overline{X_{ij}}$ is of the second type and L is not $A_1(3)$. But then in X_{ij} the two unipotent radicals X_{X_i} and S_{X_j} (mod $O_3(X_{ij})$) are just centralizer and commutator of $H \cap \widetilde{X_{ij}}$ with S, and $(*)$ holds. Hence we assume L is $A_1(3)$ and (ii) follows easily.

As an immediate consequence, we note the following

(3.1) *If* Δ *is*



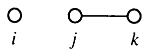
then $\overline{X_{12}}$ *is not of type* ${}^3D_4(3)$.

Proof. Assume the contrary. Then, as can easily be seen in X_{12} and X_{23}, $(**)$ is satisfied in X_i, $i = 1, 2, 3$. And also $(*)$ holds in X_{ij} for all i, j by (ii). Hence Niles' construction works, a contradiction to the above remarks.

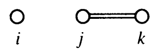
If X_{ij} is of type $A_1(3) \times A_1(3)$, we still have the chance to prove condition (i) for $(**)$ by embedding the subdiagram $\circ \ \circ$ in a suitable rank 3 subdiagram of Δ as follows.

(iii) Assume $\overline{X_{ij}}$ is of type $A_1(3) \times A_1(3)$, then $(*)$ holds in X_{ij} provided there is a vertex k in Δ such that for the subdiagram Δ_{ijk} on $\{i, j, k\}$ one of the following holds:

(a) Δ_{ijk} is



(b) Δ_{ijk} is

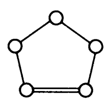


and $(**)$ holds for $X_j \cap \widetilde{X_{jk}}$.

Proof. In case (a) the group $X_i(X_j \cap \widetilde{X_{jk}})$ has certainly a homomorphic image $\mathrm{PSL}_2(3) \times \mathrm{PGL}_2(3)$, whence $(*)$ holds in X_{ij}. In case (b), the same holds under the additional hypothesis.

We are now able to rule out quite a lot of the diagrams left.

(3.2) (a) Δ *is not* $\widetilde{A}_r$ $(r \neq 3)$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$ *or*



(b) *If* Δ *is of type* $\widetilde{C}_r$, $r \geq 3$, *at least one of the groups* $\overline{G_1}$, $\overline{G_n}$ *where* $G_n = \langle X_1, \dots, X_{n-1} \rangle$, $G_1 = \langle X_2, \dots, X_n \rangle$ *is of type* $C_{n-1}(3)$.

Proof. Assume $(**)$ is not satisfied in X_i for some i. Then choose j such that $m(i, j) > 2$. This is possible, since Δ is connected. Now $\overline{X_{ij}}$ must be of type $C_2(3)$, since in all other possible rank 3 groups in characteristic 3 the rank 1 parabolics of type $A_1(3)$ have homomorphic images $\mathrm{PGL}_2(3)$. (a) follows. But it is also easily seen, that in the diagrams in (a) every subdiagram $\circ \ \circ$ can be embedded into a subdiagram $\circ \ \ \circ\!-\!\!-\!\circ$, whence for all X_{ij} $(*)$ holds in view of (iii). Hence (a). Assume Δ is of type $\widetilde{C}_r$, $r \geq 3$, and both maximal parabolics G_1, G_n of type C_{n-1} are not of type $C_{n-1}(3)$. Then as above, $(**)$ holds for all X_i, and by (iii)(b) also $(*)$ for all X_{ij}. Hence (b) follows.

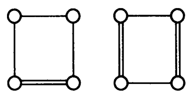
Using work by Timmesfeld [Tim6, Tim7], we can also rule out the rank 3 case.

(3.3) Δ *is not of rank* 3.

Proof. Assume the contrary. Then it follows from [Tim6, (2.3)] and Theorem 2 that Δ has to be a string. In [Tim7, Theorem 1], Timmesfeld also shows that in the cases, where Δ is a string (say $m(1, 3) = 2$), $O_3(X_{12})$ (resp. $O_3(X_{23})$) are centralized by $\widetilde{X_{12}}$ (resp. $\widetilde{X_{23}}$). In fact, he gives a list of all parabolic systems in rank 3, that have a connected diagram. By inspection, a contradiction follows. (Recall that the parabolic system X Ris assumed to be strong!)

It should be remarked, that the situation in (3.3) is highly restricted, so in fact one uses only a very small part of [Tim7].

The discussion above shows that in Δ there are only bonds of strength 1 or 2, leaving us with the following possibilities: Δ is of type $\widetilde{B_r}$, $\widetilde{C_r}$ or $\widetilde{D_r}$ $(r \geq 3)$ or Δ is one of



We have to use a different method now, to get a contradiction, since in the remaining cases (i), (ii), and (iii) do not apply. We choose in our diagram Δ over $I = \{1, \dots, n\}$ two maximal subdiagrams, $\Delta_n = \{1, \dots, n-1\}$ and $\Delta_1 = \{2, \dots, n\}$, say. Then, the groups $G_n = \langle X_1, \dots, X_{n-1} \rangle$ and $G_1 = \langle X_2, \dots, X_n \rangle$ intersect in $G_{1,n} = \langle X_2, \dots, X_{n-1} \rangle$, which maps onto a maximal parabolic subgroup of both Lie-type groups G_1 and G_n (by (1.2), since Δ_1 and Δ_n are spherical; the intersection contains a maximal parabolic subgroup of each group and if this containment was proper, two rank 1 parabolics of X would have to coincide in G, which is certainly not the case).

Now we consider the coset graph $\Gamma(1, n) = \Gamma(G; G_1, G_n)$. The arguments to follow will give a contradiction independent of the particular group G only using the way G_1 and G_n are amalgamated, i.e., the way their intersection $G_{1,n}$ is embedded in these groups. Hence we may without loss assume in our contradiction proof that G is the amalgamated sum of G_1 and G_n, amalgamated along $G_{1,n}$. (Now we left the strong parabolic system X, since we replaced $X_{1,n}$ by the (infinite) group $X_1 *_B X_n$, but still have $S_G = Z(G) = 1$ and $G = \langle S^G \rangle B$!)

Recall the structure of the groups G_i: we have $G_i = G_i, B$ and G_i is finite Lie-type group in characteristic 3 of type Δ_i; set $M_i := O_3(G_i)$ which is S_{G_i}. The advantage of assuming G is $G_1 *_{G_{1n}} G_n$ lies in the fact that G_i, $i = 1, n$, are now self-normalizing in G, and the graph $\Gamma(1, n)$, on which G acts faithfully by right multiplication, is a tree. The vertices of $\Gamma = \Gamma(1, n)$ are cosets G_{ix}, $i \in \{1, n\}$ and $x \in G$ (two different vertices being adjacent, if their intersection is not empty), and the stabilizer in G of the vertex G_{ix} is just G_i^x.

Vertices of Γ will be denoted by Greek letters, and if α is G_{ix}, then its stabilizer G_α in G will be G_i^x. Moreover, if Z_i is a normal subgroup of G_i, then the normal subgroup $Z_\alpha := Z_i^x$ of G_α is well defined. In particular, we have $M_\alpha = O_3(G_\alpha)$, and certainly M_α fixes all vertices adjacent to α. On the locally finite connected graph Γ, we have a natural distance function $d(\alpha, \beta)$ defined as the minimal length of a path from α to β in Γ, which is even unique since Γ is a tree, and will be denoted by $(\alpha, \alpha+1, \dots, \beta-1, \beta)$.

Assume now that $M_1 \neq 1 \neq M_n$, and Z_i is a G_i-invariant nontrivial elementary abelian subgroup of $Z(M_i)$, $i = 1, n$. Then $M_\alpha \neq 1$ for all α, and we have defined Z_α for all vertices α as above. Clearly Z_α is not contained in all M_β, β in Γ, since G acts faithfully on Γ, hence for α we have $d_\alpha := \min\{d(\alpha, \beta) \colon Z_\alpha \leq M_\beta\}$ and even $d := \min\{d_\alpha \colon \alpha \in \Gamma\}$. A pair (α, δ) of vertices is called *critical*, if $d(\alpha, \delta) = d$ and $Z_\alpha \nleq M_\delta$. Since G is transitive on the two types of vertices (cosets of G_1 and G_n), we obviously have four (different) possibilities for critical pairs, corresponding to the orbits of α and δ. If α is in the orbit of the vertex G_i and δ in the orbit of G_j, $i, j \in \{1, n\}$, then we say the critical pair (α, δ) is of type (i, j).

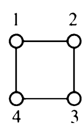
This notation is the setting for the so-called amalgam method and is used in the many papers written on amalgams recently; one of the fundamental properties of critical pairs (α, δ) is the following:

$(*)$ Assume $\overline{G_i}$ acts nontrivially on Z_i for G_i in the orbit of α and in the orbit of δ, then Z_α (or Z_δ) is an FF-module in characteristic 3 for $\overline{G_\alpha}$ (resp. $\overline{G_\delta}$), and Z_δ (resp. Z_α) induces a quadratic offending subgroup on Z_α (resp. Z_δ).

Hence for the choice of Z_i one is led to pick a G_i-invariant subgroup Z_i of $\Omega_1(Z(M_i))$, $i \in \{1, n\}$, with nontrivial $\overline{G_i}$-action, if possible. If $\Omega_1(Z(M_i))$ is a trivial G_i-module, hence lies in the center of S (and $\widetilde{G_i}$), we denote this situation by $G_i \leq N(Z)$. If not, we put $G_i \nleq N(Z)$. Of course, the case $G_1 \leq N(Z)$ *and* $G_n \leq N(Z)$ lead to $M_1 \cap M_n = 1$, since otherwise $\Omega_1(Z(M_1)) \cap \Omega_1(Z(M_n))$ would be a nontrivial normal subgroup of G contained in S. Then $M_1 M_n$ is a direct product and $M_1 M_n / M_1$ is contained in the unipotent radical of $G_{1,n}/M_1$. This bounds the order of M_n, often implies that M_i is trivial under the action of $\widetilde{G_i}$, $i = 1, n$, then $M_1 M_n / M_1$ is trivial for $\widetilde{G_{1,n}}$ and therefore one usually gets a contradiction.

We will treat the remaining cases now one after the other, by choosing the two nodes (denoted $1, n$ above) in the diagram Δ properly, forming the corresponding coset graph Γ of the amalgamated sum G and investigating the action of G on Γ. Since in some situation we have to collect information from more than one amalgam and do not want to change labelling in the diagram Δ, we will be free to pick two nodes i, j and form the tree $\Gamma(i, j)$ in exactly the same manner as $\Gamma(1, n)$. In every case, we have to give the labelling of Δ, the choice of $\Gamma(i, j)$ and to define Z_i and Z_j, if M_i and M_j are not trivial.

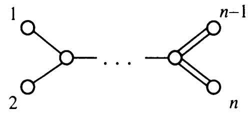
(3.4) Δ *is not*



$(\widetilde{A_3} = \widetilde{D_3})$.

Proof. Since at least one $\overline{X_i}$ is of type $A_1(3)$, we know that all $\overline{G_i}$ are of type $A_3(3)$. Assume $M_1 = 1$. Then all M_i are equal to 1, and $|S| = 3^6$. But then $J(S)$ is normalized by X_{13}, X_{24} hence by G, a contradiction. Hence $M_i \neq 1$ for $i = 1, 2, 3, 4$, and we may choose labelling of the diagram such that $G_1 \nleq N(Z)$ and $G_3 \nleq N(Z)$. Without loss, $\Omega_1(Z(M_1))$ is an FF-module in characteristic 3 for $\overline{G_1}$, with offending subgroup contained in $O_3(X_{24})/M_1$. Now there is also a $\widetilde{G_1}$-composition factor of $\Omega_1(Z(M_1))$ with the same properties, and by (2.6)(ii) $\widetilde{X_{24}}/O_3(X_{24}) \cong \mathrm{SL}_2(3) \times \mathrm{SL}_2(3)$. But now some involution t in H centralizes $\overline{X_i}$ for $i = 1, 2, 3, 4$ by (1.6) and (1.5) gives a contradiction.

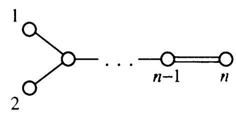
(3.5) Δ *is not*



$(\widetilde{D_{n-1}}\,,\ n \geq 5)$.

Proof. As in the proof of (3.4) we may assume $M_i \neq 1$ and G_i is of type $D_{n-1}(3)$ for $i = 1, 2, n-1, n$ (recall $n \geq 5$). Therefore we may also assume that $G_1 \nleq N(Z)$ and $G_2 \nleq N(Z)$. Hence we may assume by (2.9), that in $\Omega_1(Z(M_1))$ there is a $\overline{G_1}$-spin module involved. For $n \geq 6$, (2.9) implies that some involution in $H \cap \widetilde{X_{n,n-2}}$ centralizes $\overline{X_i}$, $i \geq 2$. But now again (1.6) and (1.5) yield a contradiction. For $n = 5$, the same contradiction follows with some involution in $H \cap \widetilde{L}$, where L is X_{24}, X_{25}, or X_{45}.

(3.6) Δ *is not*

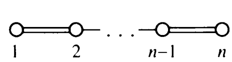


$(\widetilde{B_{n-1}}\,,\ n \geq 4)$.

Proof. Assume the contrary. Then by the structure of $\overline{X_{n-1,n}}$ the groups $\overline{G_1}$ and $\overline{G_2}$ must have the same type (compare (1.7)): $C_{n-1}(3)$, $B_{n-1}(3)$, or ${}^2D_n(3)$. By (1.6) we may assume $\overline{X_{12}}$ is isomorphic to $\mathrm{SL}_2(3) * \mathrm{SL}_2(3)$. Assume $M_1 \cap M_2 = 1$. If now $M_1 \neq 1$, then G_1 is of type $C_{n-1}(3)$ and $|M_1| = |M_2| = 3$. Comparing the orders of $D_{n-1}(3)$ and $C_{n-1}(3)$, one sees that M_n cannot be 1, but must be contained in $O_3(G_{1,n})$, a contradiction to the action of $G_{1,n}$ on $O_3(G_{1,n})$. If $M_1 = M_2 = 1$, this contradiction is got rightaway.

Hence we may assume M_1, M_2, and M_n to be nontrivial, and $G_i \leq N(Z)$ for at most one $i \in \{1, 2, n\}$. Assume $G_1 \leq N(Z)$. Then we consider the graph $\Gamma(2, n)$, taking $Z_i := \Omega_1(Z(M_i))$, $i = 2, \ldots, n$. Since $H \cap \widetilde{G_n}$ centralizes $\Omega_1(Z(S))$ by the structure of $\overline{X_{12}}$, by (2.5) Z_n cannot be a quadratic module for G_n, and any critical pair (α, δ) must be of type (2.2). Considering $G_1 \leq N(Z)$ and using (2.5), some composition factor of Z_2 must be a natural module for $\overline{G_2} = \mathrm{Sp}_{2n-2}(3)$ or $\Omega_{2n-1}(3)$ or $\Omega^-_{2n}(3)$. But again the action of X_{12} on $\Omega_1(Z(S))$ gives a contradiction. Hence we may assume G_1, $G_2 \nleq N(Z)$. Now without loss some composition factor of $\Omega_1(Z(M_1))$, is an irreducible FF-module in characteristic 3 for $\overline{G_1}$ with offending subgroup contained in $O_3(G_{12}/M_1)$. If G_1 has type $C_{n-1}(3)$, we get a contradiction using (2.11), (1.8), and (1.5). If $\overline{G_1}$ has type ${}^2D_n(3)$, the contradiction follows with (2.12), (1.9), and (1.5). If $\overline{G_1}$ is of type $B_{n-1}(3)$, the contradiction follows from (2.10), (1.9), and (1.5).

(3.7) Δ *is not*



$(\widetilde{C_{n-1}}\,,\ n \geq 4)$.

Proof. Note first, that $\overline{G_1}$ and $\overline{G_n}$ are of type $B_{n-1}(3)$, $C_{n-1}(3)$, or ${}^2D_n(3)$ and H is an abelian 2-group. By (3.2)(b), one of $\overline{G_1}$ and $\overline{G_n}$ is of type $C_{n-1}(3)$. Assume first $M_1 = 1$. Then S is isomorphic to a Sylow 3-group of $\mathrm{Sp}_{2n-2}(3)$, $\Omega_{2n-1}(3)$, or $\Omega^-_{2n}(3)$, and now clearly also $M_n = 1$ and $\overline{G_1}$ and $\overline{G_n}$ are both of type $C_{n-1}(3)$, since the type is determined by the structure of the Sylow 3-subgroup. But then $Z(S)$ is normalized by G_{12} and $G_{n-1,n}$, hence by G, a contradiction. Hence M_1 and M_n are both nontrivial.

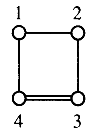
Assume $G_1 \leq N(Z)$ and $G_n \leq N(Z)$. Then the introductory remarks show that $M_1 \cap M_n = 1$ and $\overline{G_{1,n}}$ (viewed as a Levi complement of a maximal parabolic of the group of type $C_{n-1}(3)$) must act trivially on its unipotent radical, of course a contradiction.

Assume $G_1 \not\leq N(Z)$ and $G_n \leq N(Z)$. If now $\Omega_1(Z(S)) \not\leq M_n$, then $\widetilde{G_n} = M_n \cdot C_n$, where C_n is the centralizer of M_n in $\widetilde{G_n}$, and $\overline{G_{1,n}}$ has only central composition factors in M_n, hence the only noncentral $G_{1,n}$-composition factors of $O_3(G_{1,n})$ are contained in $O_3(G_{1,n}/M_n)$. But $G_{1,n}$ has noncentral composition factors in $\Omega_1(Z(M_1))$ and $O_3(G_{1,n}/M_1)$ and we get a contradiction for all possible types of $\overline{G_1}$ and $\overline{G_n}$. Hence $\Omega_1(Z(S)) \leq M_n$, and therefore $\Omega_1(Z(S)) = \Omega_1(Z(M_n))$. Now G_1 cannot have a trivial submodule in $\Omega_1(Z(M_1))$, and $G_{1,n}$ normalizes $\Omega_1(Z(M_1)) \cap Z(S)$.

Consider $\Gamma = \Gamma(1, 2)$ with Z_1 some irreducible G_1-submodule of $\Omega_1(Z(M_1))$, and Z_2 the (unique) irreducible $G_{1,2}$-submodule of Z_1. Clearly, Z_2 is a nontrivial G_2-module. Let (α, δ) be a critical pair in Γ. Clearly, α is of type 1; if now also δ is of type 1, then Z_1 is an FF-module in characteristic 3 for $\overline{G_1}$, and by the action of $G_{1,n}$ cannot be a natural module for $\overline{G_1}$ of type $C_{n-1}(3)$, $B_{n-1}(3)$, or ${}^2D_n(3)$. Hence in this case by (2.10), (2.11), and (2.12), $\overline{G_1}$ must be of type $B_{n-1}(3)$ or ${}^2D_n(3)$ and Z_1 a spin module. In this case, however, (1.6)(b) gives a contradiction. Therefore (α, δ) is of type (1.2). The same argument as above shows $[Z_\alpha, Z_\delta] = 1$. Assume $d = 1$. Then we may take $(\alpha, \delta) = (1, 2)$ and $Z_1/Z_1 \cap M_2$ is centralized by $\widetilde{G_{1,2}}$. But then by (2.3), $G_{1,2}$ fixes $Z_1 \cap Z(S)$, a contradiction. Hence $d \geq 3$, and in particular $V_2 := \langle Z_1^{G_1} \rangle = \langle Z_1^{X_4} \rangle$ is elementary abelian. Since X_1 does not fix Z_1, X_1 acts as (P) $\mathrm{SL}_2(3)$, or (P) $\mathrm{SL}_2(9)$ on V_2. Let $\langle t \rangle = \Omega_1(H \cap \widetilde{X_1})$, then t centralizes $Z_1 \cap Z(S)$, hence $Z_1/[Z_1, S]$ by (2.3), and we may apply (2.13) to $\widetilde{X_1}$, t, V_2, and Z_1. It follows that elements in X_1 acting quadratically on V_2 are contained in $O_3(X_1)$. But this contradicts the quadratic action of $\langle Z_\alpha^{G_{\alpha+1}} \rangle$ on $\langle Z_{\delta-1}^{G_\delta} \rangle$.

Hence we may assume $G_1 \not\leq N(Z)$ and $G_n \leq N(Z)$. Clearly, $Z_1 \cap Z_n \geq \Omega_1(Z(S))$. Consider $\Gamma = \Gamma(1, n)$ with $Z_i = \Omega_1(Z(M_i))$, $i = 1, n$. Without loss, Z_1 is an FF-module for $\overline{G_1}$, and hence by (1.6), (1.9), (2.10), (2.11), and (2.12), all noncentral $\overline{G_1}$-composition factors in Z_1 are natural modules for $\overline{G_1}$, which is of type $C_{n-1}(3)$, $B_{n-1}(3)$ or ${}^2D_n(3)$, and $G_{1,2}$ is the normalizer in G_1 of $\Omega_1(Z(S))$. Now G_n does not have a trivial submodule on Z_n, and hence we may choose Z_n an irreducible nontrivial G_n-submodule and $Z_{n-1} \leq Z_n$ an irreducible nontrivial $\widetilde{G_{n-1}}$-submodule for $\Gamma(n, n-1)$. Now the same arguments as above show that either Z_n is a quadratic module for $\overline{G_n}$, which contradicts the action of $G_{1,2}$, or we get a contradiction to (2.13).

(3.8) Δ *is not*



Proof. Assume the contrary. Since one of the $\overline{X_i}$ is of type $A_1(3)$, by the structure of $\overline{X_{34}}$ we get that (without loss) $\overline{G_1}$ is of type $B_3(3)$ and $\overline{G_2}$ is of type $C_3(3)$. Now clearly $M_1 \neq 1$ and $M_2 \neq 1$, (trivially also $M_4 \neq 1 \neq M_3$) and the structure of $\overline{X_{13}}$ together with (1.6) and (1.7) tells $\overline{G_2} \cong \mathrm{PSp}_6(3)$ and $\overline{G_4} \cong \mathrm{PSL}_4(3)$. This implies easily $\overline{G_1} \cong \mathrm{Spin}_7(3)$ and $\overline{G_3} \cong \mathrm{SL}_4(3)$ by (1.7). And obviously $G_i \leq N(Z)$ can hold for at most one i, and if so, $\Omega_1(Z(M_i)) = \Omega_1(Z(S))$.

Assume first $G_2 \nleq N(Z)$ and $G_4 \nleq N(Z)$. Then the G_i-module $\Omega_1(Z(M_i))$ involves a natural $\mathrm{SP}_6(3)$ (resp. $\mathrm{SL}_4(3)$)-module by (2.6) and (2.8), $i = 2$ (resp. 4), which is certainly a contradiction.

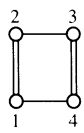
Assume next $G_4 \leq N(Z)$. Consider the graph $\Gamma = \Gamma(1, 2)$ with irreducible nontrivial $\overline{G_i}$-submodules Z_i of $\Omega_1(Z(M_i))$, $i = 1, 2$. Let (α, δ) be a critical pair, and let the order be chosen so that Z_δ is an FF-module for G_δ with $Z_\alpha M_\delta / M_\delta$ offending. Then by (2.8)(ii) the type of (α, δ) is not $(2, 2)$ or $(1, 2)$; and it is not $(2, 1)$ either: by $(2, 7)$ the group $Z_\delta M_\alpha / M_\alpha$ acting quadratically on Z_α would have order at least 3^4, being contained in $O_3(G_{\alpha, \alpha+1}/M_\alpha)$, which is certainly impossible. Hence its type is (1.1), and again Z_1 is an 8-dimensional spin module for G_1. Let $\alpha - 1$ be any vertex of Γ adjacent to α different from $\alpha + 1$. Then, since $(\alpha - 1, \delta - 1)$ is not critical, $Z_{\alpha-1}$ is contained in $M_{\delta-1}$, hence in G_δ and moreover $[Z_\delta, Z_\alpha] = [Z_\delta, O_3(G_{\delta, \delta-1})] = [Z_\delta, Z_\alpha Z_{\alpha-1}]$ by (2.7)(iii). In particular, $[Z_\delta, Z_{\alpha-1}] \leq Z_\alpha$. Since $\alpha - 1$ was chosen arbitrarily, we get $[V_\alpha, Z_\delta] \leq Z_\alpha$ for the G_α-module $V_\alpha = \langle Z_{\alpha-1}^{G} \rangle$. Hence G_α acts trivially on V_α / Z_α, and hence also $\widetilde{G_{12}}$ on $Z_2 / Z_1 \cap Z_2$. Now by (2.3), Z_2 must be contained in Z_1, which is clearly impossible.

Assume finally $G_2 \leq N(Z)$. Then $\Omega_1(Z(S)) = \Omega_1(Z(M_2))$, and we pick irreducible nontrivial G_i-submodules Z_i in $\Omega_1(Z(M_i))$ for $i = 1, 3, 4$. Consider the graph $\Gamma = \Gamma(1, 4)$ first. Let (α, δ) be a critical pair. We want to show that Z_1 is an FF-module for G_1, which must be a natural 7-dimensional module then by (2.7)(ii). Hence assume, there is no critical pair of type $(1, 1)$ and (α, δ) is of type $(1, 4)$ or $(4, 1)$. Then Z_1 is still quadratic and the result follows from (2.7) and the action of G_{12} on Z_1. Hence we may assume (α, δ) is of type $(4, 4)$, Z_4 is an orthogonal module for G_4 by (2.6) and the action of G_{24} on Z_4, and Z_α and Z_δ both offend on each other by (2.6)(iii).

Now the same proof as in the case $G_4 \leq N(Z)$ implies G_4 centralizes $\langle Z_1^{G_n} \rangle / Z_4$. But this contradicts the action of G_{14} on $Z_1 / Z_1 \cap Z_4$. Hence indeed we have Z_1 a natural module for $\overline{G_1}$.

Consider now the graph $\Gamma = \Gamma(1, 2)$ with the same Z_1, and Z_2 the centralizer of S on Z_1. Then pick a critical pair (α, δ), it must have type $(1, 2)$ or $(1, 1)$. The second case, however, contradicts (2.7), hence the type is $(1, 2)$. Assume $d = 1$. Then $[[Z_1, M_2], Z_1] = 1$ and all noncentral G_2-composition factors of M_2 are quadratic, hence 13-dimensional by (2.8). If, however, $d \geq 3$, then $V_2 = \langle Z_1^{G} \rangle$ is abelian, and $V_{\alpha+1}$ and V_δ act quadratically on each other. Again, (2.8) tells that all noncentral $\overline{G_2}$-composition factors on V_2 are 13-dimensional. Let V be a minimal nontrivial G_{12}-submodule of V_2. Then by what we just said and the action of G_{12} on the 13-dimensional fundamental module the noncentral G_{12}-composition factor of V is 4-dimensional. But inside Z_1, we see a 6-dimensional submodule with 5-dimensional noncentral composition factor. This contradiction finishes the proof.

(3.9) Δ *is not*



Proof. Assume the contrary; clearly $(**)$ holds for all X_i, hence at least one rank 2 group of type $A_1(3) \times A_1(3)$ is involved in the parabolic system, and we know immediately the types of $\overline{G_i}$, $i = 1, \dots, 4$. Without loss $\overline{G_1}$ is of type $B_3(3)$ and $\overline{G_2}$ is of type $C_3(3)$ by the structure of $\overline{X_{34}}$ and (1.7). Also by (1.7) and the structure of $\overline{X_{13}}$, and $\overline{X_{24}}$, we get $\overline{G_1} \cong \overline{G_3} \cong \mathrm{Spin}_7(3)$ and $\overline{G_2} \cong \overline{G_4} \cong \mathrm{PSp}_6(3)$. Clearly $M_i \neq 1$ for $i = 1, 2, 3, 4$ since Sylow 3-subgroups of $B_3(3)$ and $C_3(3)$ are not isomorphic. Also $\Omega_1(Z(S)) \leq M_i$ for all i, and finally $G_i \leq N(Z)$ can be true for at most one i.

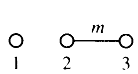
Assume $G_2 \not\leq N(Z)$ and $G_4 \not\leq N(Z)$. Then consider the graph $\Gamma(2, 4)$ with $Z_i = \Omega_1(Z(M_i))$. We immediately get a (quadratic) FF-module for $\overline{G_2}$ or $\overline{G_4}$ and hence by (2.8), noncentral composition factors of, say, Z_2 are isomorphic to the 13-dimensional fundamental module V for G_2, moreover an offending subgroup is contained in the unipotent radical U of a line stabilizer P. By (2.1), and since no transvections are reduced on by G_2, also U satisfies (FF) on V. This is impossible, since P fixes a point on V.

Hence assume $G_2 \leq N(Z)$. Then consider the graph $\Gamma(1, 4)$. Let Z_i be irreducible $\overline{G_i}$-submodules of $\Omega_1(Z(M_i))$, $i = 1, 4$, and let (α, δ) be a critical pair. By (2.8), the type is not $(4, 4)$. By (2.7), Z_1 is a spin module for $\overline{G_1}$, and hence again by (2.7), and the structure of $O_3(G_{14}/M_4)$, we get that the type of (α, δ) is $(1, 1)$. Therefore again $[Z_\alpha Z_{\alpha-1}, Z_\delta] \leq Z_\alpha$ follows, and hence G_α acts trivially on $\langle Z_{\alpha-1}^G \rangle Z_\alpha / Z_\alpha$, a contradiction to the action of G_{14} on $Z_4/Z_1 \cap Z_4$.

4. Parabolic systems that are not strong

In this section, we consider parabolic systems in characteristic 3 that do not have to be strong any more. That means, $\overline{X_{ij}}$ of type D is allowed. The following lemma will be used later in the proof of Theorem B, but also indicates why we may restrict our interest to the case of connected diagrams.

(4.1) **Lemma.** *Let* $\{X_1, X_2, X_3\}$ *be a parabolic system in characteristic* 3 *in* G *with diagram*



i.e., $m(1,2) = m(1,3) = 2$ and $m(2,3) > 2$. Then the parabolic system is strong.

Proof. Assume by way of contradiction that $\overline{X_{12}}$ is of type D. Of course, we may assume $X_i = \widetilde{X_i}$ for $i = 1, 2, 3$ and $S_G = Z(G) = 1$. Then X_1 is not contained in X_{23} and clearly X_{23} has index 4 in G. Let K be the largest normal subgroup of G contained in X_{23}, then $S \not\leq K$, and even $\mathrm{PSL}_2(3)$ is a subgroup of X_1K/K. Since $O_3(X_{23})$ is not contained in K, there is $x \in O_3(X_{23}) \leq O_3(X_2) = O_3(X_1)$ not contained in K. This contradicts the structure of X_1.

If X is a parabolic system in characteristic 3 in a group G having a diagram Δ that is not connected, then we may apply a version of Theorem (1.3) to get a decomposition of $O^3(\overline{G})$ corresponding to the decomposition of Δ.

Consider now a parabolic system X in characteristic 3 in the group G that is not strong, i.e. there are i, j in the diagram Δ such that $\overline{X_{ij}}$ is of type D, and assume Δ is connected. Then it is interesting, how the vertices i, j are "embedded" in Δ.

(4.2) **Lemma.** *Let $\{X_1, X_2, X_3\}$ be a parabolic system in characteristic 3 in G, with connected diagram Δ and $\overline{X_{13}}$ of type D. Assume $S_G = Z(G) = 1$. Then $\widetilde{X_{12}} \cong \widetilde{X_{23}}$ and one of the following holds:*

(a) *$m(1,2) = 4$ and $\widetilde{X_{12}}$ is isomorphic to* $\mathrm{PSp}_4(3)$, $Z_3 \times \mathrm{PSp}_4(3)$, *or* $U_4(3)$.

(b) *$m(1,2) = 6$ and $\widetilde{X_{12}}$ is isomorphic to $G_2(3)$ or ${}^3D_4(3)$.*

Proof. As already used in the proof of (3.3), work by Timmesfeld [Tim7] shows that $\widetilde{X_{12}}$ and $\widetilde{X_{23}}$ act trivially on $O_3(X_{12})$ (resp. $O_3(X_{23})$). Now inspection of the outcome of [Tim7, Theorem 1] gives the desired result.

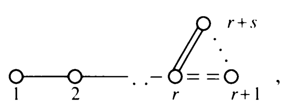
It should be mentioned that unless $\overline{X_{12}}$ is of type $G_2(3)$ or $\mathrm{PSp}_4(3)$, the types of the X_i (i.e., the labelling) is uniquely determined in (4.2).

We come now to the proof of Theorem B.

Theorem B. *Let $X = \{X_1, \ldots, X_n\}$ be a parabolic system in characteristic 3 in G, with connected diagram Δ, n at least 3. Then either the system is strong (and Δ is spherical by Theorem A) or Δ is one of the following:*

(i) *a complete bipartite graph with only triple or only double bonds*

(ii)

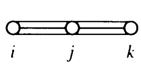


, $r, s \geq 2$,

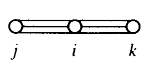
hence of type $Y(r,s)$ in the notation of [St].

Proof. We may assume X is not strong. Then assume first that there is a triple bond contained in Δ, say $m(i,j) = 6$ for some $i, j \in \Delta$. Let k be an arbitrary vertex in Δ different from i and j.

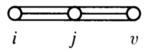
Claim the subdiagram on $\{i, j, k\}$ is either



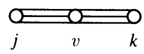
or



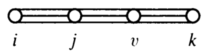
The claim follows from (4.2), if k is connected to i or j in Δ, since the subsystem $\{X_i, X_j, X_k\}$ is not strong by Theorem A. Hence, by way of contradiction, we may assume k is at distance 2 from $\{i, j\}$ in Δ. Thus, we have a vertex $v \in \Delta$, connected to i or j and to k, while k is connected to neither i nor j. Clearly, by Theorem A, the system $\{X_i, X_j, X_k, X_v\}$ is not strong, and also the system $\{X_i, X_j, X_v\}$ is not strong. Without loss, the diagram on $\{i, j, v\}$ is



and hence also the system $\{X_v, X_j, X_k\}$ is not strong and its diagram is also



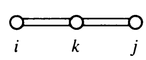
Hence the diagram on $\{i, j, v, k\}$ is



and we get a contradiction to (4.1).

This contradiction proves the claim and it follows that all bonds in Δ are triple bonds, every vertex being adjacent to either i or j, and by (4.2) case (i) follows. So we may assume there are no triple bonds contained in Δ. If there are no single bonds either contained in Δ, then the same argument as above gives case (i) again.

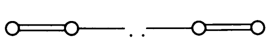
So we may assume there are single bonds contained in Δ. Let i, j in Δ such that the X_{ij} is of type D. Then i and j are connected in Δ by a path $(i, k, \dots, v, j)$, and by (4.1) and (4.2) we have the subdiagram



Consider the graph $\widetilde{\Delta}$ got from Δ by first removing all single bonds, then all isolated vertices.

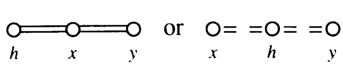
Claim: $\widetilde{\Delta}$ is a star with central vertex k.

It is clear that $\widetilde{\Delta}$ is connected, since Δ does not contain subdiagrams of type



or circuits. The corresponding subsystem would have to be strong, contradicting Theorem A.

For the same reason, for v, w, in Δ that are no adjacent in $\widetilde{\Delta}$, we also have $m(v, w) = 2$ (they are not adjacent in Δ), and hence the above argument shows that $\widetilde{\Delta}$ is a complete bipartite graph with only double bonds. Let t be a vertex in Δ-$\widetilde{\Delta}$ that is adjacent to some h in $\widetilde{\Delta}$. Certainly, h is contained in a subdiagram of type



in $\widetilde{\Delta}$. In the first case, by (4.1), $m(t, x) = 3$ or $m(t, y) = 3$, both contradicting Theorem A. Hence $\widetilde{\Delta}$ contains a vertex h, that is not at distance 2 from any other vertex of $\widetilde{\Delta}$, hence $\widetilde{\Delta}$ must be a star, and clearly k must be the central vertex, hence the claim follows.

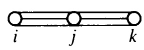
Moreover, k must be equal to h. This implies that any vertex in Δ–$\widetilde{\Delta}$ that is adjacent to some vertex in $\widetilde{\Delta}$, is adjacent to k and to no other vertex. But since Δ does not contain triangles nor subdiagrams of type B_3 by Theorem A, we also get that t was unique. But now Δ–$\widetilde{\Delta}$ is a connected subdiagram of Δ containing only single bonds, and hence is spherical by (4.1) and Theorem A. If it is of Type A_r, we get conclusion (ii).

Hence assume it is of type D_1 or E_1, then however Δ contains a subdiagram of type B_f, a final contradiction.

Let still $X = \{X_1, \ldots, X_n\}$ be a parabolic system in characteristic 3 in G, that is not strong, but has a connected diagram. We want to say a bit more on these systems.

(4.3) *Assume some* $\overline{X_{ij}}$ *is isomorphic to* ${}^3D_4(3)$. *Then* Δ *is a star. If* j *is the central vertex of* Δ, X_j *is of type* $L_2(27)$ *and* $S_G = S_{X_{jk}}$ *for all* $j \neq k$.

Proof. Any connected subdiagram on $\{i, j, k\}$, say, looks like



by Theorem B, and is clearly not strong, hence $\overline{X_j}$ is of type $L_2(27)$. Obviously, Δ is a star with central vertex j. By (4.2), $S_{X_{ij}} = S_{X_{jk}}$ for all $k \neq j$, hence this group is normal in G.

(4.4) *Assume some* X_{ij} *is of type* $G_2(3)$. *Then* $S_G = S_{X_{ij}}$ *for all* $i \neq j$ *adjacent in* Δ.

Proof. Easy application of (4.2).

(4.5) *Assume* Δ *is a complete bipartite graph with only double bonds involved. Assume some* $\overline{X_{ij}}$ *is of type* ${}^2A_3(3)$. *Then* Δ *is a star, with central vertex* j, *say, and* $\overline{X_j}$ *is of type* $A_1(9)$, *and* $S_G = S_{X_{jk}}$ *for all* $k \neq j$.

Proof. Same as (4.3).

Let us fix some notation for the rest of the paper. Recall that the parabolic system X is defined on the index set $I = \{1, 2, \ldots, n\}$. For any nonempty subset J of I we set $X_J = \langle X_i, i \in J\rangle$, and $Q_J = S_{X_J}$. If J consists of all vertices of I but i, we set (as in §3) $G_i := X_J$ and $M_i = Q_J$.

(4.6) *Let* Δ *be of type* $Y(r, s)$, $r, s \geq 2$. *Let* $J = \{1, \ldots, r\}$ *and* $J_i = J \cup \{r+i\}$ *for all* $i \leq s$. *Then all groups* $\overline{X_J}$ *are of the same type* ($B_{r+1}(3)$, $C_{r+1}(3)$ *or* ${}^2A_{2r+1}(3)$), *and* $S_G = S_{X_{J_i}}$ *for* $i = 1, 2, \ldots, s$.

Proof. We may assume $S_G = Z(G) = 1$. Since Δ is not spherical, the system is not strong, and hence obviously $\overline{X_{r+i,r+j}}$ is of type D for all $1 \leq i \neq j \leq s$. The type of $\overline{X_{J_i}}$ is clearly determined by the system $\{X_r, X_{r+1}\}$ (for the case B/C see (1.7)), hence by (4.2) independent of i. For the last claim, assume $S_G = 1$. We may assume $s = 2$. Consider first the case $r = 2$.

We have to show $M_3 = M_4 = 1$. Assume $M_3M_4 > M_4$. Then M_3M_4 is a normal 3-subgroup of X_{12}, hence is contained in Q_{12} and equals Q_{12}, unless $\overline{G_3}$ and $\overline{G_4}$ are of type $B_3(3)$. But $M_3 \leq Q_4 = Q_3$, since the system is not strong, and we get a contradiction, if $M_3M_4 = Q_{12}$, because in G_4 we see that Q_{12}/M_4 is not contained in Q_3/M_4.

Hence we may assume $\overline{G}_3$ and $\overline{G_4}$ are of type $B_3(3)$ and M_3M_4 is the unique X_{12}-invariant subgroup of Q_{12} with $M_4 < M_3M_4 < Q_{12}$.

Assume $M_3 \le M_1$. Then M_3M_4 is contained in $M_1M_4 = Q_{23}$ and Q_{12}, a contradiction to the above.

Therefore, we may assume $M_3 \not\le M_1$, hence M_3M_1/M_1 has order 3 by (4.2). Since $M_3M_4/M_3 \cong M_4/M_3 \cap M_4$ is a nontrivial X_{12}-module, we have nontrivial action of G_i on M_i for $i = 3, 4$, and of course also for $i = 1$. We again apply the amalgam method to get a contradiction.

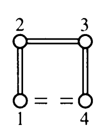
Assume first $G_1 \le N(Z)$. Then by the discussion above $\Omega_1(Z(S)) = \Omega_1(Z(M_1))$ and we consider the graph $\Gamma(3, 4)$ with irreducible nontrivial submodules Z_i of $\Omega_1(Z(M_i))$, $i = 3, 4$. Without loss, Z_3 is an FF-module for $\overline{G_3}$, and hence a natural module by (2.7).

Consider $\Gamma(3, 2)$ with the natural module Z_3 and Z_2 contained in Z_3. Then a critical pair (α, δ) is of type $(3, 3)$ or $(3, 2)$ by the choice of Z_2. But type $(3, 2)$ is impossible, since $O_3(G_{3,2}) = M_2$, while type $(3, 3)$ contradicts (2.7)(iv). Hence we may finally assume $G_1 \not\le N(Z)$ and also $G_4 \not\le N(Z)$. Consider $\Gamma(1, 4)$ with $Z_i = \Omega_1(Z(M_i))$, $i = 1, 4$. Let (α, δ) be a critical pair. Then it is not of type $(4, 4)$ by (2.7), (1.9), and (1.6). But if it is of type $(1, 4)$ or $(4, 1)$, then transvections are induced on the quadratic module Z_4 and we get a contradiction, by (2.7).

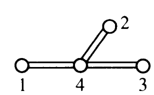
Hence it is of type $(1, 1)$, and the usual argument yields that V_4/Z_4 is trivial for G_4, contradicting the action of X_{23} on $Z_1/Z_1 \cap Z_4$. This finishes the case $r = 2$, and we use induction on r, still $s = 2$. But now clearly $M_{r+2} \le Q_{2,\dots,r,r+1} = M_1$ and also $M_{r+1} \le M_1$, and we get $M_{r+1} = M_{r+2} = 1$ as in the case $r = 2$.

(4.7) *Let Δ be a complete bipartite graph with only double bonds involved. Assume $\overline{X_{ij}}$ is of type $C_2(3)$ for some i, j. Then $|S_{X_{ij}} : S_G| \le 3$.*

Proof. We may assume $S_G = Z(G) = 1$. For $n = 3$, this is contained in (4.2), so consider the case $n = 4$ next. Assume first Δ is



Then we may assume without loss $|Q_{23} : M_1| = 3$. Also $|Q_{34} : M_1| = 3$ and $|S : M_1| = 3^5$. Now, $\overline{X_3}$ is isomorphic to $\mathrm{SL}_2(3)$, and $X_{12} \cong X_{23} \cong X_{34} \cong X_{41}$, hence $M_4 = Q_{12} = Q_{23}$ and $M_2 = Q_{34} = Q_{14}$. Therefore, M_1 is contained in M_2 and M_4, and also $M_3 \le M_2 \cap M_4$. So, since $|M_2M_4 : M_1|$, $|M_2M_4 : M_3|$ are at most 3^2, we have either $M_1 = M_3$, or $M_1M_3 = M_2 \cap M_4$. In both cases, $M_1M_3 = S_G = 1$. The claim follows. Assume now that Δ is



and assume $M_1 \le M_2$. Then clearly $M_1M_2 = Q_{34}$, and if $M_2 \le M_1$, then $|S\colon M_1| = 3^4$, and also $M_1 = Q_{24}$, whence $M_3 \le M_1$. Now $M_2 = M_3$ or

$M_2M_3 = M_1$, so in any case $M_2M_3 = S_G = 1$ and again the claim follows. Hence assume finally in rank 4 that for all pairs (i, j) with i, j different elements of $\{1, 2, 3\}$ we have $M_i \leq M_j$ and $M_j \leq M_i$. Then, however, $M_i \cap M_j \leq M_k$ for $\{i, j, k\} = \{1, 2, 3\}$, and therefore $M_i \cap M_j \cap M_k = M_i \cap M_j$ for all choices of i, j. Hence $M_1 \cap M_2 \cap M_3$ is normal in G, and $M_i \cap M_j = S_G = 1$. But now the action of G_4 on M_4 gives a contradiction. Thus we are lead to n at least 5.

For given i, j we pick three more vertices $1, 2, 3$ in Δ, by the above $|Q_{ij} : M_k| \leq 3$ for $k = 1, 2, 3$. If now M_1 is contained properly in M_2, then also M_3 is contained properly in M_2, and we get the claim $M_2 = M_1M_3 = S_G = 1$. But if no M_k is properly contained in M_h for any different k, h in $\{1, 2, 3\}$, then we get $Q_{ij} = M_1M_2 = M_1M_3 = M_2M_3 = S_G = 1$, again the result.

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