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ON DUBROVIN VALUATION RINGS IN CROSSED PRODUCT ALGEBRAS

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ABSTRACT. Let F be a field and let V be a valuation ring in F. If A is a central simple F-algebra then V can be extended to a Dubrovin valuation ring in A. In this paper we consider the structure of Dubrovin valuation rings with center V in crossed product algebras $(K/F, G, f)$ where K/F is a finite Galois extension with Galois group G unramified over V and f is a normalized two-cocycle. In the case where V is indecomposed in K we introduce a family of orders naturally associated to f, examine their basic properties, and determine which of these orders is Dubrovin. In the case where V is decomposed we determine the structure in the case of certain special discrete, finite rank valuations.

0. INTRODUCTION

Let V be a discrete valuation ring in a field F and let K/F be a finite unramified Galois extension with Galois group G (say). Let W be the integral closure of V in K. In [H] the first author initiated a study of a certain family of orders over V in crossed product algebras $(K/F, G, f)$. If the two-cocycle f takes its values in W then one can form in the obvious way a "crossed product order" $\sum Wx_\sigma \subseteq \sum Kx_\sigma = (K/F, G, f)$. It turns out that these orders have many interesting properties. For example they are primary with an explicitly described radical. Moreover they are a sufficiently large family to include, up to a suitable notion of equivalence, all of the maximal orders over V, in the case where the residue field of V is perfect.

In 1982 Dubrovin in [D_1] defined a notion of valuation ring inside an arbitrary simple Artinian ring. These Dubrovin valuation rings have many properties in common with maximal orders over discrete valuation rings. In particular the Dubrovin valuation rings with center a discrete valuation ring V are precisely the maximal orders over V. In [MW] Morandi and Wadsworth investigated Dubrovin valuation rings over V in $(K/F, G, f)$ where V is an arbitrary valuation ring unramified and indecomposed in K. Among other things they showed that any Dubrovin valuation ring B is integral over V and that its residue ring $B/J(B)$ is a crossed product algebra. This work gave simpler proofs of many results on the structure of division algebras over henselian

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valuation rings. If V is discrete of rank one this case is the same as that considered in §2 of [H], but the viewpoint of the two papers is quite different.

In this paper we consider Dubrovin valuation rings over V in a crossed product algebra $(K/F, G, f)$ where V is any valuation ring of F which is unramified in K. In doing so we combine the viewpoints of [H and MW], and extend the results of both papers. Although even in the case of a perfect residue field the resulting orders do not include all of the Brauer classes as was the case in [H], the class of algebras we consider contains many interesting examples. In fact many constructions in the theory of simple algebras have used such examples.

To describe our results let V be a valuation ring of a field F and let K/F be a Galois extension in which V is unramified. Let W be the integral closure of V in K and let $G = \mathrm{Gal}(K/F)$. Let $f \in Z^2(G, K^\times)$ be a normalized two-cocycle. The first section of the paper contains some necessary preliminaries. In §2 we consider the case where V is indecomposed in K. As in [H] we introduce a special family of orders. Let Y be a finite set of overrings of V, each overring properly contained in F, and assume $V \in Y$. We may write $Y = \{V_1, V_2, \ldots, V_n\}$ where $V = V_n \subsetneq V_{n-1} \subsetneq \cdots \subsetneq V_1 \subsetneq V_0 = F$. Each V_i is indecomposed and unramified in K. Let W_i be the unique extension of V_i to K. Let $H_i = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W_i^\times\}$. We say f is *standard* for Y if for each i, $f(H_i \times H_i) \subseteq W_{i+1}$. It turns out that every cocycle is cohomologous to one that is standard for Y. Now assume f is standard for Y. It turns out that in this case the sets H_i are in fact subgroups of G. For each $\sigma \in G$ we select an overring W_σ in Y as follows: If $\sigma \in H_n$ we let $W_\sigma = W_n = W$. If $\sigma \in H_i - H_{i+1}$ where $i < n$ then we set $W_\sigma = W_{i+1}$. We then set $B_f = \sum W_\sigma x_\sigma$. It turns out that for every choice of Y and every cocycle f standard for Y the set B_f is a ring and in fact a V-order in $(K/F, G, f)$. We call this the standard crossed product order for f (and Y). This family of orders is very well behaved. For example each B_f is a primary ring with an easily described radical. Part of the interest in this family lies in the fact that in the case where V is discrete and finite rank any Dubrovin valuation ring over V in $(K/F, G, f)$ is conjugate to such an order. The main results in §2 are Theorems 2.14 and 2.17 in which we determine which standard crossed product orders are Dubrovin valuation rings. The determination is in terms of conditions on the values of the cocycle and the relations between the subgroups H_i. To give an idea of the sort of conditions that arise consider the special case where Y consists of V alone. If f is standard for Y then there is only one nontrivial subgroup $H = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W^\times\}$. As in [H] we introduce the partial order on the coset space G/H given by $\sigma H \le \tau H$ if $f(\sigma, \sigma^{-1}\tau) \in W^\times$. In Theorem 2.14 we prove that the corresponding order $B_f = \sum W x_\sigma$ is a Dubrovin valuation ring over V if and only if H is a normal subgroup with cyclic quotient and there is a distinguished generator σH of G/H satisfying two conditions: $f(\sigma, \sigma^{-1}) \in J(W) - J(W)^2$ and the partial order is the chain $H \le \sigma H \le \sigma^2 H \le \cdots \le \sigma^{k-1} H$ where $k = |G/H|$. This is a generalization of Theorem 2.3 of [H]. The arguments make heavy use of the notion of a value function as introduced by Morandi in [M].

In the third section we consider the case where V is unramified in K but not necessarily indecomposed. This case turns out to be considerably more complicated and we restrict our attention to those valuations that are discrete of finite rank. Let W be the integral closure of V in K. It turns out once again that a Dubrovin valuation ring in the crossed product algebra $(K/F, G, f) = \sum Kx_\sigma$ is conjugate to an algebra of the form $B = \sum W_\sigma x_\sigma$ where each W_σ is an overring of W contained in K. The aim is to determine the W_σ and to see what one can say about the group G and the values of f. Part of the difficulty arises from the fact that unlike the indecomposed case B is not necessarily integral and so the theory of value functions does not apply. Each W_σ is a Prüfer ring and thus equal to the intersection of the valuation rings that contain it. We first prove that the determination of the rings W_σ can be reduced to the determination of the single ring W_{id}. We say a prime ideal P of W *belongs* to B if W_{id} is contained in the valuation ring W_P. If P is any prime ideal of W we let $D(P)$ denote its decomposition group and $H(P) = \{\sigma \in D(P) | f(\sigma, \sigma^{-1}) \notin P\}$. We prove two basic facts about these groups: If P is a prime of height i belonging to B then $H(P)$ acts transitively on the set of primes of height $i+1$ that belong to B and if $Q \supseteq P$ is a height $i+1$ prime belonging to B then the group $H(Q)$ is normal in $D(Q) \cap H(P)$ with cyclic quotient generated by a coset $\sigma H(Q)$ satisfying two conditions similar to those described above. Again these results generalize §3 of [H]. Along the way we once again give an explicit description of the radical of B. One of the ideas in the proofs is to show that one can find for each prime P a Dubrovin valuation ring "related" to B that satisfies the conditions of §2. This allows us to apply the results obtained there to B. We end with an example in rank 2.

1. Preliminaries

We begin this section with a brief introduction to Dubrovin valuation rings. First recall that a ring B is *primary* if the Jacobson radical $J(B)$ is a maximal ideal of B, that is $B/J(B)$ is simple. The ring B is said to be *Bezout* if every finitely generated one-sided ideal of B is principal. A *Dubrovin valuation ring* is a prime PI ring B which is primary and Bezout. For brevity we will often refer to such a ring simply as a *valuation ring*. Let S be the simple Artinian ring of quotients of B. It is shown in [D_1, D_2] that $BZ(S) = S$, $B \cap Z(S) = V$ is a valuation ring of $Z(S)$, and two-sided ideals of B are linearly ordered by inclusion, as are overrings of B in S. Furthermore if A is an overring of B in S, then A is a valuation ring, $A = BZ(A)$, $J(A) \subseteq B$, and $B/J(A)$ is a valuation ring of $A/J(A)$. If S is a central simple F-algebra and V a valuation ring of F, it is shown in [D_2, §3, Theorem 2] and [BG, Theorem 3.8] that there is a valuation ring B of S with $B \cap F = V$. Also any two valuation rings of S with center V are conjugate [W, Theorem A]. For a fuller introduction to valuation rings see [W].

Let V be a valuation ring of a field F and K a finite Galois extension of F with Galois group G. Let W be the integral closure of V in K. In this paper we will only consider the case where V is *unramified* in K, that is for all maximal ideals M of W, the ramification index of W_M over V is one, the residue extension $W_M/J(W_M)$ over $V/J(V)$ is separable and K/F is defectless with respect to V (so $\sum_M [W_M/J(W_M) : V/J(V)] = [K : F]$). It follows that $W_M/J(W_M)$ is Galois over $V/J(V)$ [E, 19.12]. If in addition V is *indecomposed* in K, that is W is a valuation ring, we say K/F is inertial with respect to V. The ring W will be of considerable importance in this paper, so we mention some properties that will be used throughout. Because V is a valuation ring, W is a *Prüfer* ring. Hence any localization of W at a prime ideal is a valuation ring, and the extensions of V to K are precisely the localizations of W at its maximal ideals. Furthermore because K/F is finite, W is semilocal. Every overring of W in K is also Prüfer and is a finite intersection of localizations of W. Proofs of these statements can be found in [E, 11.9, 13.4, 13.7].

Let $f \in Z^2(G, K^\times)$ be a normalized two-cocycle and $\Sigma = (K/F, G, f)$ the corresponding crossed product algebra. Thus we have $\Sigma = \sum_{\sigma\in G} Kx_\sigma$ where multiplication is given by $x_\sigma a = \sigma(a)x_\sigma$ for all $a \in K$ and $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$ for all $\sigma, \tau \in G$. With K as above, if B is a valuation ring of Σ lying over V then we want to show that a suitable conjugate of B can be written in a form compatible with the decomposition $\Sigma = \sum_{\sigma\in G} Kx_\sigma$ and that the precise structure of B can be obtained from K/F and f. The following lemmas give the foundations for determining B.

Lemma 1.1. *If V is unramified in K then for every $\sigma \in G - \{1\}$ and every maximal ideal M of W there is an $x \in W$ such that $\sigma(x) - x \notin M$.*

Proof. Let $D(M) = \{\sigma \in G | \sigma(M) = M\}$, the decomposition group of M. If $\sigma \notin D(M)$ then $\sigma(M) \neq M$, so there is an $x \in M$ such that $\sigma(x) \notin M$ and so $\sigma(x) - x \notin M$. Now suppose $\sigma \in D(M)$. Because K/F is unramified, the inertia group $I(M) = \{\sigma \in D(M) | \overline{\sigma(x)} = \overline{x}$ for all $x \in W_M\}$ is trivial, where W_M is the localization of W at M and $\overline{x} = x + J(W_M)$. Thus if $\sigma \neq 1$, there is an $x \in W$ such that $\overline{\sigma(x)} \neq \overline{x}$, hence $\sigma(x) - x \notin M$. □

It follows from this lemma and [DI, Chapter III, Theorem 1.1] that W/V is a Galois extension of rings.

Lemma 1.2. *The ring W is a finitely generated V-module. There is a valuation ring B of Σ lying over V with $W \subseteq B$.*

Proof. Because K/F is unramified, hence defectless, we have

$$\sum_M [W_M/J(W_M) : V/J(V)] = [K : F].$$

Thus by [E, 18.6] W is a finite V-module.

Let B be a valuation ring of Σ lying over V. Because $BF = \Sigma$ and B is a Bezout ring, the finitely generated B-module WB is principal, say $WB = xB$. Because $1 \in WB$, we have $x \in \Sigma^\times$. Thus $xB = WB = W(WB) \supseteq Wx$, so $W \subseteq xBx^{-1}$, another valuation ring lying over V. □

This lemma was discovered independently by Westmoreland [We].

If B is a valuation ring of Σ containing W then B is a W-W submodule of Σ. The following lemma is the first step towards describing B.

Lemma 1.3. *If T is a W-W submodule of Σ then $T = \sum_{\sigma\in G}(T \cap Kx_\sigma)$. In particular if B is a valuation ring of Σ lying over V and containing W, then $B = \sum_{\sigma\in G} I_\sigma x_\sigma$, where each I_σ is a W-submodule of K.*

Proof. Let $T_\sigma = \{a \in K | ax_\sigma \in T\}$, a W-submodule of K. Clearly $\sum T_\sigma x_\sigma \subseteq T$. To show equality let $\sum_\sigma a_\sigma x_\sigma \in T$. We need to show $a_\sigma x_\sigma \in T$ for all σ. Suppose this is false and that r is minimal with $t = \sum_{i=1}^r a_{\sigma_i} x_{\sigma_i} \in T$, but not all a_{σ_i} in T_{σ_i}. From the minimality it follows that $a_{\sigma_i} \notin T_{\sigma_i}$ for all i. Let $I = \{w \in W | w a_{\sigma_1} x_{\sigma_1} \in T\}$, an ideal of W. Because $I \neq W$ there is a maximal ideal M of W such that $I \subseteq M$. By Lemma 1.1 there is a $u \in W$ such that $\sigma_1(u) - \sigma_2(u) \notin M$. Thus

$$\sigma_2(u)t - tu = (\sigma_2(u) - \sigma_1(u))a_{\sigma_1}x_{\sigma_1} + (\sigma_2(u) - \sigma_3(u))a_{\sigma_3}x_{\sigma_3} + \cdots + (\sigma_2(u) - \sigma_r(u))a_{\sigma_r}x_{\sigma_r} \in T.$$

By the minimality of r we obtain $(\sigma_2(u)-\sigma_1(u))a_{\sigma_1}x_{\sigma_1} \in T$, so $(\sigma_2(u)-\sigma_1(u)) \in I \subseteq M$, a contradiction. □

From these lemmas we see that there is a valuation ring B of Σ that contains W and so decomposes into $B = \sum_{\sigma \in G} I_\sigma x_\sigma$. The task of describing B thus reduces to describing the I_σ. The following simple lemma will be used repeatedly.

Lemma 1.4. *If $\sigma \in G$, then $I_\sigma I^\sigma_{\sigma^{-1}} f(\sigma, \sigma^{-1}) \subseteq I_{\mathrm{id}}$. If all the I_σ are rings, then $f(\sigma, \sigma^{-1}) \in I_{\mathrm{id}}$.*

Proof. Because B is a ring, $(I_\sigma x_\sigma)(I_{\sigma^{-1}} x_{\sigma^{-1}}) \subseteq I_{\mathrm{id}}$. Because $x_\sigma x_{\sigma^{-1}} = f(\sigma, \sigma^{-1})$ we see that $I_\sigma I^\sigma_{\sigma^{-1}} f(\sigma, \sigma^{-1}) \subseteq I_{\mathrm{id}}$. If all the I_σ are rings, then $f(\sigma, \sigma^{-1}) \in I_\sigma I^\sigma_{\sigma^{-1}} f(\sigma, \sigma^{-1}) \subseteq I_{\mathrm{id}}$. □

Notice that this lemma implies that I_{id} is an overring of W and that for all $\sigma \in G$ and all $b \in I^\sigma_{\sigma^{-1}}$ we have $I_\sigma b f(\sigma, \sigma^{-1}) \subseteq I_{\mathrm{id}}$, so in particular I_σ is a fractional ideal over I_{id}. In the case where V is a discrete valuation ring the valuation ring B is necessarily finitely generated as a V-module. It follows that $I_{\mathrm{id}} = W$ and each I_σ is a finitely generated W-submodule of K and so principal over W, because W is a principal ideal domain. If $I_\sigma = c_\sigma x_\sigma$ for all $\sigma \in G$, then by replacing f by the equivalent cocylce g corresponding to replacing x_σ by $c_\sigma x_\sigma$ we can assume $B = \sum_{\sigma \in G} W x_\sigma$. For general valuation rings such a nice decomposition is not always possible (see Example 2.18). However in a number of situations one can considerably simplify the decomposition of B.

Suppose V is a valuation ring of F whose associated value group is $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (n times) ordered antilexiographically. Then the Krull dimension of V is n, and the overrings of V in F are $V = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq F$ where V_i is a valuation ring with value group $\mathbb{Z}^i$ (so V_1 is a discrete valuation ring) and $V_{i+1}/J(V_i)$ is a discrete valuation ring of the field $V_i/J(V_i)$. We will call V a *discrete rank n valuation ring.* For such a V not all V-submodules of F are principal over V, but any such module is necessarily principal over V_i for some i, as the following lemma shows.

Lemma 1.5. *Let V be a discrete rank n valuation ring of F. If I is a V-submodule of F, then $I = cU$ for some overring U of V.*

Proof. We use induction on $n = \dim(V)$. If $n = 1$ then V is a discrete valuation ring and the result is well known. So suppose $n > 1$. If $I = F$ we are done, so assume $I \neq F$. If $V_1 \supseteq V$ is the rank one overring of V, then V_1 is a discrete valuation ring, so $IV_1 = dV_1$ for some $d \in F^\times$. By replacing I by $d^{-1}I$ we may assume $IV_1 = V_1$. Then by the linear order of V-submodules we see that $J(V_1) \subseteq I$. Thus $I/J(V_1)$ is a $V/J(V_1)$-submodule of $V_1/J(V_1)$. By the induction hypothesis $I/J(V_1) = \bar{c}T$ for some overring T of $V/J(V_1)$.

Let $U = \{x \in V_1 | \overline{x} \in T\}$, an overring of V. If $c \in V_1$ is any preimage of $\overline{c}$, then c is a unit in V_1 and so $J(V_1) \subseteq cU$. Hence $I = cU$. □

Lemma 1.6. *Let V be a discrete rank n valuation ring of F, K/F a finite extension and W the integral closure of V in K. If S is an overring of W in K and I is an S-submodule of K, then $I = cU$ for some overring U of S.*

Proof. By [E, 13.7] S has only finitely many maximal ideals, say $M_1, \ldots, M_r$. Because I is a torsion-free S-module, $I = \bigcap_i IS_{M_i}$. By Lemma 1.5, $IS_{M_i} = c_i U_i$ for some overring U_i of S_{M_i}. By relabeling if necessary we may assume $U_1, \ldots, U_t$ are the minimal rings among the U_i. Then $I = \bigcap_{i=1}^t IU_i$. We want to apply Ribenboim's approximation theorem [R, §E, Theorem 3] to obtain an element $x \in K$ such that $x \equiv c_i \bmod c_i J(U_i)$, $1 \le i \le t$. To do so we need to show $c_i - c_j \in c_i J(U_i) U_{ij} = c_j J(U_j) U_{ij}$ for $i \neq j$, where $U_{ij} = U_i U_j$. Because the U_i are pairwise incomparable, U_i and U_j are proper subrings of U_{ij} and so

$$c_i J(U_i) U_{ij} + c_i U_{ij} = IU_{ij} = c_j U_{ij} = c_j J(U_j) U_{ij}$$

and $c_i - c_j \in IU_{ij}$. Hence such an x exists and so for each i, $x = c_i(1 + m_i)$ for some $m_i \in J(U_i)$. Thus $xU_i = c_i U_i$. Therefore we obtain

$$I = \bigcap_{i=1}^{t} IU_i = \bigcap c_i U_i = \bigcap x U_i = x\left(\bigcap U_i\right) = xU,$$

where U is the overring $\bigcap U_i$. □

We now summarize what we have learned about valuation rings over discrete rank n valuation rings in crossed product algebras.

Proposition 1.7. *Let V be a discrete rank n valuation ring in a field F. Let K/F be a finite Galois extension in which V is unramified and let $G = \mathrm{Gal}(K/F)$. Let W be the integral closure of V in K and let $f \in Z^2(G, K^\times)$ be a normalized two-cocycle. There is a valuation ring B in $(K/F, G, f)$ and a cocycle f' cohomologous to f such that if $(K/F, G, f') = \sum K x_\sigma$ then $B = \sum W_\sigma x_\sigma$ where W_{id} is an overring of W and each W_σ is an overring of W_{id}.*

Proof. We have already seen that there is a valuation ring B in $(K/F, G, f) = \sum K y_\sigma$ such that $B = \sum I_\sigma y_\sigma$ where I_{id} is an overring of W and each I_σ is an I_{id}-submodule of K. By Lemma 1.6 we can write $I_\sigma = c_\sigma W_\sigma$ where $c_\sigma \in K$ and W_σ is an overring of $W_{\mathrm{id}} = I_{\mathrm{id}}$. Hence by replacing f by the cocycle f' corresponding to replacing y_σ by $x_\sigma = c_\sigma y_\sigma$ we obtain the desired form. □

2. Indecomposed case

In this section we consider the case where the valuation ring V is indecomposed and unramified in K. For each cocylce we will construct a natural set of orders which in the discrete, finite rank case contains a valuation ring for that cocycle. We will also show how to determine which of these orders is a valuation ring.

Let Y be a finite set of overrings of V, each properly contained in F, and assume Y contains V. Because the overrings are linearly ordered we may write $Y = \{V_1, V_2, \ldots, V_{n-1}, V_n\}$ where $V = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq V_0 = F$. Because V is indecomposed and unramified in K, it follows that each V_i is also indecomposed and unramified in K. Let W_i be the unique extension of V_i to K (and let $W_0 = K$). Let $f\colon G\times G \to K^\times$ be a (normalized) two-cocycle. For each i, $0 \le i \le n$, let $H_i = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W_i^\times\}$ (note that $H_0 = G$).

Definition. The cocycle f is said to be *standard for* Y if for all i, $0 \le i \le n-1$, $f(H_i \times H_i) \subseteq W_{i+1}$ (and so $\subseteq W_{i+1} \backslash J(W_i)$).

Lemma 2.1. *If f is a standard cocycle for Y, then for all i, $0 \le i \le n$, H_i is a subgroup of G.*

Proof. We proceed by induction on i. If $i = 0$ there is nothing to prove. Now assume $i > 0$. Let $\sigma, \tau \in H_i$. We have the cocycle identities:

$$f^\sigma(\sigma^{-1}, \tau) f(\sigma, \sigma^{-1}\tau) = f(\sigma, \sigma^{-1}),$$
$$f^\tau(\tau^{-1}, \sigma) f(\tau, \tau^{-1}\sigma) = f(\tau, \tau^{-1}),$$
$$f^\sigma(\sigma^{-1}\tau, \tau^{-1}\sigma) = f(\sigma, \sigma^{-1}\tau) f(\tau, \tau^{-1}\sigma).$$

Because $H_i \subseteq H_{i-1}$ and H_{i-1} is a subgroup by induction, all the values in these identities lie in W_i. Because $f(\sigma, \sigma^{-1})$, $f(\tau, \tau^{-1})$ are units in W_i, it follows that $f^\sigma(\sigma^{-1}\tau, \tau^{-1}\sigma)$ is a unit, so $\sigma^{-1}\tau \in H_i$. Hence H_i is a subgroup of G. □

We want to show that every cocycle is equivalent to a standard one. We need the following lemma.

Lemma 2.2. *Let K/F be a finite Galois extension with Galois group G. Let $R_2 \subseteq R_1$ be valuation rings with field of fractions F and assume R_2 is unramified and indecomposed in K. Let S_i be the extension of R_i in K, $i = 1, 2$. Let $f\colon G \times G \to S_1^\times$ be a cocycle. Then there is a cocycle g equivalent to f over $S_1^\times$ such that $g(G \times G) \subseteq S_2$.*

Proof. Because S_2 is a valuation ring the fractional ideal I generated by the $f(\sigma, \tau)$, $\sigma, \tau \in G$, is principal, that is $I = aS_2$ for some $a \in K$. Because $f(G \times G) \subseteq S_1^\times$, we have $IS_1 = S_1$ and so $a \in S_1^\times$. Define a one-cochain α by $\alpha(1) = 1$ and $\alpha(\sigma) = a^{-1}$ for $\sigma \in G - \{1\}$. An easy calculation shows that $g = (\partial\alpha)f$ has the desired property. □

Proposition 2.3. *Every two-cocycle is equivalent to one that is standard for Y.*

Proof. Let $f = f_0$. We will construct a sequence of cocycles $f_1, f_2, \ldots, f_n$ such that for each i, $1 \le i \le n-1$, the following two properties are satisfied.

(1) The cocycle f_i is equivalent to f_{i-1} over $W_{i-1}^\times$, that is there is a one-cochain $\alpha\colon G \to W_{i-1}^\times$ such that $f_i = (\partial\alpha) f_{i-1}$.

(2) We have $f_i(H_{i-1} \times H_{i-1}) \subseteq W_i$, where $H_{i-1} = \{\sigma \in G | f_i(\sigma, \sigma^{-1}) \in W_{i-1}^\times\}$.

It will then follow that f_n is equivalent to f and standard for Y.

To do the first step of the construction, apply Lemma 2.2 to obtain a cocycle f_1 equivalent to f over $K^\times$ with $f_1(G \times G) \subseteq W_1$. Then f_1 is standard for W_1. This finishes the first step of the construction.

Now let i be chosen, $1 \le i \le n-1$, and assume we have constructed $f_1, f_2, \ldots, f_i$ satisfying the two properties. We show how to construct f_{i+1}. We have f_i is standard for W_i and so by Lemma 2.1, $H_1, H_2, \ldots, H_i$ are subgroups of G. Let L be the fixed field of H_{i-1}. We apply Lemma 2.2 to the

Galois extension K/L and the valuation rings $W_{i+1} \cap L \subseteq W_i \cap L$. We obtain a cocycle f_{i+1} equivalent to f_i over $W_i^\times$ such that $f_{i+1}(H_i \times H_i) \subseteq W_{i+1}$. □

Let f be a standard cocycle for Y. Let i be an integer, $i \leq n-1$. Let $H = H_i$. The field extension K/K^H is Galois with group H. Moreover at the residue level the field $W_1/J(W_1)$ is a Galois extension of the field $W_1^H/J(W_1)^H$ with Galois group H and $W_n^H/J(W_1)^H$ is a valuation ring in $W_1^H/J(W_1)^H$. The following proposition is clear.

Proposition 2.4. *Let f be standard for Y.*

(a) *If $Y_i = \{V_1, V_2, \dots, V_{i-1}, V_i\}$, then f is standard for Y_i.*
(b) *If $Y' \supseteq Y$ is a finite set of overrings of V, each properly contained in F, then f is standard for Y'.*
(c) *For all i, $i \leq n-1$, letting $H = H_i$, $f_{|H\times H}$ is standard for $\{W_1^H/J(W_1)^H, \dots, W_n^H/J(W_1)^H\}$.* □

Now let f be a cocylce that is standard for $Y = \{V_1, V_2, \dots, V_{n-1}, V_n\}$. Let $\Sigma_f = \sum_{\sigma\in G} Kx_\sigma$ be the central simple crossed product algebra over F corresponding to f. For each $\sigma \in G$, either $\sigma \in H_n$ or there is a unique integer i, $1 \leq i \leq n$, such that $\sigma \in H_{i-1} - H_i$. We will call i the *height* of σ (so we do not define the height for elements of H_n). If $\sigma \in H_n$ we let $W_\sigma = W_n$ and if $\sigma \notin H_n$ we let $W_\sigma = W_i$, where i is the height of σ. Form the subset B_f of Σ_f given by $B_f = \sum_{\sigma\in G} W_\sigma x_\sigma$. Note that B_f depends on Y and not just on f but the notation should not be confusing.

Recall that a V-subalgebra R of Σ is called an *order* over V if $RF = \Sigma$ and R is integral over V.

Proposition 2.5. *Let f be a standard cocycle for Y. The set B_f is an order over V in Σ_f.*

Proof. We have to show B_f is a ring and integral over V (it is then clear that it is an order).

To show that B_f is a ring it suffices to show that if $\sigma, \tau \in g$ then $W_\sigma x_\sigma W_\tau x_\tau \subseteq W_{\sigma\tau} x_{\sigma\tau}$. This in turn reduces to showing that $W_\sigma W_\tau f(\sigma, \tau) \subseteq W_{\sigma\tau}$. So let $\sigma, \tau \in G$. If $\sigma, \tau \in H_n$ the $\sigma\tau \in H_n$ and $W_\sigma = W_\tau = W_{\sigma\tau} = W_n$. Moreover $f(\sigma, \tau) \in W_n$, so the desired inclusion holds. If exactly one of σ, τ is in H_n, say $\sigma \in H_n$, then $\sigma\tau \notin H_n$ and the elements τ and $\sigma\tau$ have the same height. It follows that $W_\sigma \subseteq W_{\sigma\tau}$ and $W_\tau = W_{\sigma\tau}$ and $f(\sigma, \tau) \in W_{\sigma\tau}$, so again the inclusion holds.

So we now may assume neither σ nor τ lies in H_n. Let i be the height of σ, and let j be the height of τ. Hence $W_\sigma = W_i$ and $W_\tau = W_j$. The argument breaks up into cases.

First assume $i > j$. Then $\sigma\tau \in H_{j-1} - H_j$, so $W_{\sigma\tau} = W_j$. Moreover $f(\sigma, \tau) \in W_j$ because $\sigma, \tau \in H_{j-1}$. The inclusion $W_\sigma W_\tau f(\sigma, \tau) \subseteq W_{\sigma\tau}$ is then clear.

The case $j > i$ is handled in the same way.

Now assume $i = j$. If $\sigma\tau \notin H_i$ then $W_\sigma = W_\tau = W_{\sigma\tau} = W_i$ and $f(\sigma, \tau) \in W_i$, so again the inclusion is clear.

Finally assume $i = j$ and $\sigma\tau \in H_i$. In that case (i.e., height$(\sigma\tau) > i$) $W_\sigma = W_\tau = W_i$, but $W_{\sigma\tau} \subseteq W_{i+1}$. However we *claim* that $f(\sigma, \tau) \in J(W_i)$. If so then

$$W_\sigma W_\tau f(\sigma, \tau) = W_i f(\sigma, \tau) \subseteq J(W_i) \subseteq W_n \subseteq W_{\sigma\tau},$$

as desired.

To see the claim we consider the following cocycle identity:

$$f^\sigma(\tau, \tau^{-1}\sigma^{-1}) f(\sigma, \sigma^{-1}) = f(\sigma, \tau) f(\sigma\tau, \tau^{-1}\sigma^{-1}).$$

We know $f^\sigma(\tau, \tau^{-1}\sigma^{-1}) \in W_i$, $f(\sigma, \sigma^{-1}) \in J(W_i)$, and $f(\sigma\tau, \tau^{-1}\sigma^{-1}) \in W_i^\times$. It follows that $f(\sigma, \tau) \in J(W_i)$.

It remains to show that the elements of B_f are integral over V. By [AS, Theorem 2.3] it suffices to show that B_f is generated as a V module by integral elements. Hence it is enough to show that for each $\sigma \in G$, the set $W_\sigma x_\sigma$ consists of integral elements. Let $a \in W_\sigma$. Let k be the order of σ in G. Consider

$$(ax_\sigma)^k = a\sigma(a)\sigma^2(a)\cdots\sigma^{k-1}(a) f(\sigma, \sigma) f(\sigma^2, \sigma)\cdots f(\sigma^{k-1}, \sigma).$$

If $\sigma \in H_{n-1}$ then $W_\sigma = W_n$ and $f(\sigma^m, \sigma) \in W_n$ for all integers m, so $(ax_\sigma)^k \in W_n$. But W_n is integral over V (in fact, finitely generated) and so ax_σ is integral over V.

So assume $\sigma \notin H_{n-1}$ and let i be the height of σ (so $i < n$). Then $W_\sigma = W_i$ and $f(\sigma^m, \sigma) \in W_i$ for all integers m. Moreover,

$$f(\sigma^{k-1}, \sigma) = f(\sigma^{-1}, \sigma) = f^{\sigma^{-1}}(\sigma, \sigma^{-1}) \in J(W_i).$$

Hence $(ax_\sigma)^k \in J(W_i) \subseteq W_n$, so ax_σ is integral over V. □

We will refer to B_f as the crossed product order for f (corresponding to Y). We now want to derive the basic properties of these orders.

Proposition 2.6. *Let f be a standard cocycle for Y.*

(a) *The order B_f is a primary ring with Jacobson radical*

$$J(B_f) = \sum_{\sigma \in H_n} J(W_\sigma) x_\sigma + \sum_{\sigma \notin H_n} W_\sigma x_\sigma.$$

(b) *For each i, $1 \le i \le n-1$, let $B_i = B_f V_i$ (so $B_n = B_f$). Then B_i is the crossed product order over V_i corresponding to the standard cocycle f for $Y_i = \{V_j | j \le i\}$ and we have the inclusions $B_f = B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1$.*

(c) *For each i, $1 \le i \le n-1$, $J(B_i) \subseteq B_f$.*

Proof. (a) We first show that

$$I = \sum_{\sigma \in H_n} J(W_\sigma) x_\sigma + \sum_{\sigma \notin H_n} W_\sigma x_\sigma$$

is an ideal in B_f. Let $I_\sigma = J(W_\sigma)$ for $\sigma \in H_n$ and let $I_\sigma = W_\sigma$ for $\sigma \notin H_n$, so that $I = \sum_{\sigma \in G} I_\sigma x_\sigma$. It suffices to show that for all $\sigma \in G$, $W_\sigma x_\sigma I_\tau x_\tau \subseteq I_{\sigma\tau} x_{\sigma\tau}$ and $I_\tau x_\tau W_\sigma x_\sigma \subseteq I_{\tau\sigma} x_{\tau\sigma}$. This reduces in turn to showing that $W_\sigma I_\tau f(\sigma, \tau) \subseteq I_{\sigma\tau}$ and $I_\tau W_\sigma f(\tau, \sigma) \subseteq I_{\tau\sigma}$, where we are using the fact that for each $\sigma \in G$, the sets I_σ and W_σ are G-stable.

We will show $W_\sigma I_\tau f(\sigma, \tau) \subseteq I_{\sigma\tau}$. The argument for the other inclusion is similar and will be omitted. If $\sigma\tau \notin H_n$ then $I_{\sigma\tau} = W_{\sigma\tau}$ and the result follows from the fact that B_f is a ring. So we may assume $\sigma\tau \in H_n$. In that case if $\sigma, \tau \in H_n$ then $W_\sigma = W_n$, $I_\tau = I_{\sigma\tau} = J(W_n)$, and $f(\sigma, \tau) \in W_n$ so the inclusion is clear. Finally assume neither σ nor τ lies in H_n but $\sigma\tau \in H_n$. It follows that σ and τ must have the same height i (say) and as we saw above in the proof of Proposition 2.5 this implies $f(\sigma, \tau) \in J(W_i)$. Hence

$$W_\sigma I_\tau f(\sigma, \tau) = W_i f(\sigma, \tau) \subseteq J(W_i) \subseteq J(W_n) = I_{\sigma\tau}.$$

We show next that I is maximal. Let $H = H_n$. Consider

$$B_f/I = \sum_{\sigma \in G}(W_\sigma/I_\sigma)x_\sigma = \sum_{\sigma \in H} \overline{W}_n x_\sigma,$$

where $\overline{W}_n$ is the residue field of the valuation ring W_n. As we have seen $\overline{W}_n$ is a Galois extension of $\overline{W}_n^H$ with Galois group H and so it is clear that B_f/I is the crossed product algebra $(\overline{W}_\sigma/\overline{W}_\sigma^H, H, f_{|H\times H})$. In particular B_f/I is simple, so I is maximal.

Finally we need to show I is the unique maximal ideal of B_f. Suppose T is another ideal and suppose T is not contained in I. Then $I + T = B_f$. Because T is a $W_n - W_n$ bimodule, we can apply Lemma 1.3 to write $T = \sum_{\sigma \in G} T_\sigma x_\sigma$, where $T_\sigma x_\sigma = T \cap W_\sigma x_\sigma$. Hence it follows that $T_{\mathrm{id}} + I_{\mathrm{id}} = W_n$. This means $T_{\mathrm{id}} + J(W_n) = W_n$. But then it follows that $T_{\mathrm{id}} = W_n$ and so $T = B_f$.

For (b) and (c), part (b) is clear and part (c) follows from the explicit description of $J(B_i)$ given in (a). □

We now want to investigate the connection between the valuation rings in Σ_f lying over V and the orders we have introduced. Let $f: G \times G \to K^\times$ be a normalized cocycle. Let v be a valuation on F corresponding to V and let Γ be the value group of v. Let $\Delta = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ be the divisible hull of Γ. As in [MW, p. 625], we define the function $w: \Sigma_f \to \Delta$ by $w(\sum_\sigma a_\sigma x_\sigma) = \min_\sigma\{v(a_\sigma) + w(x_\sigma)\}$, where v is used to denote the unique extension to K and for each $\sigma \in G$,

$$w(x_\sigma) = \frac{1}{n}\sum_{i=1}^{n-1} v(f(\sigma^i, \sigma)), \qquad n = |G|.$$

It is shown in [MW, Theorem 2.1] that because K/F is inertial with respect to v the function w is a *value function* on Σ_f and the set $B_w = \{s \in \Sigma_f | w(s) \geq 0\}$ is a valuation ring with Jacobson radical $J(B_w) = \{s \in \Sigma_f | w(s) > 0\}$. It follows that $B_w = \sum_{\sigma \in G} N_\sigma x_\sigma$ where $N_\sigma = \{k \in K | w(kx_\sigma) \geq 0\}$. In particular $N_{\mathrm{id}} = W$. For more details on value functions see [M, §2].

Observe that in the case where V is a discrete rank n valuation ring, it follows from Lemma 1.5 that each N_σ is principal over an overring W_σ of W. Hence by possibly changing to an equivalent cocycle g we may assume $B_w = \sum_{\sigma \in G} W_\sigma x_\sigma$ where each W_σ is an overring of $W_{\mathrm{id}} = W$. This motivates the following proposition.

Proposition 2.7. *Let V be a valuation ring of the field F and let K/F be an inertial Galois extension with Galois group G. Let W be the extension of V to K. Let $f \in Z^2(G, K^\times)$ be a normalized two cocycle and let B be a valuation ring in the crossed product algebra $\Sigma_f = \sum_{\sigma \in G} Kx_\sigma$. If B can be expressed as $B = \sum_{\sigma \in G} W_\sigma x_\sigma$ where each W_σ is an overring of W, then f is standard for $Y = \{W_\sigma \cap F | \sigma \in G\}$ and $B = B_f$.*

Proof. First observe that because B is integral over V [MW, Theorem 2.1] and W_{id} is assumed to be an overring of W, it follows that $W_{\mathrm{id}} = W$. We may label the elements of Y so that $Y = \{V_1, V_2, \dots, V_{n-1}, V_n\}$ where $V = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq F$. Let $W = W_n \subseteq W_{n-1} \subseteq \cdots \subseteq W_1 \subseteq K$ be the extensions of the V_i to K. For each i, $0 \le i \le n$, let $H_i = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W_i^{\times}\}$. We now proceed by induction on n. If $n = 1$ then $W_\sigma = W$ for all σ. The fact that B is a ring then implies that $f(G \times G) \subseteq W$ and B is clearly in standard form. Hence we may assume $n > 1$. Let $B_{n-1} = BV_{n-1} = \sum_{\sigma \in G} W_\sigma V_{n-1} x_\sigma$. Then B_{n-1} is a valuation ring and for each σ, $W_\sigma V_{n-1}$ is an overring of W_{n-1}. By the induction hypothesis we infer that f is standard for $Y' = Y - \{V\}$ and B_{n-1} has the standard form. We *claim* that for all $\sigma \in H_{n-1}$, $W_\sigma = W_n$. If so then the fact that B is a ring will imply that $f(H_{n-1} \times H_{n-1}) \subseteq W_n$ and that will complete the proof.

So let $\sigma \in H_{n-1}$. Because B_{n-1} is in standard form, we know $W_\sigma V_{n-1} = W_{n-1}$. Hence $W_\sigma = W_n$ or W_{n-1}. But $W_\sigma x_\sigma x_{\sigma^{-1}} \subset W_{\mathrm{id}} = W_n$, so $W_n \supseteq W_\sigma f(\sigma, \sigma^{-1})$. If $W_\sigma = W_{n-1}$ then $W_n \supseteq W_{n-1} f(\sigma, \sigma^{-1}) = W_{n-1}$ because $f(\sigma, \sigma^{-1}) \in W_{n-1}^{\times}$. This is a contradiction so $W_\sigma = W_n$ as desired. □

Using the observations made before this proposition we obtain the following consequence.

Corollary 2.8. *Let V, F, K and f be as in the proposition with V a discrete rank n valuation ring. Let $Y = \{U | V \subseteq U \subset F$ and U is a ring$\}$. If B is a valuation ring in Σ_f, then there is a cocycle g equivalent to f such that g is in standard form for Y and B is conjugate to B_g.*

Proof. By Proposition 2.7 and the remarks preceding it there is a cocycle g equivalent to f such that g is in standard form for *some* set of overrings Y' and B is conjugate to B_g. However it then follows from part (b) of Proposition 2.4 that we may take $Y' = Y$. □

Having seen that at least in the discrete rank n case every valuation ring is equivalent to one in standard form it is natural to try to characterize the valuation rings among the standard orders. As before let f be a standard cocycle for $Y = \{V_1, V_2, \dots, V_{n-1}, V_n\}$ and let B_f be the corresponding crossed product order. For each i, $1 \le i \le n-1$, we can (as in [H]) introduce a partial ordering on the set of cosets H_{i-1}/H_i as follows. If $\sigma, \tau \in H_{i-1}$ we define $\sigma H_i \le \tau H_i$ if $f(\sigma, \sigma^{-1}\tau) \in W_i^{\times}$.

Lemma 2.9. *Let i be an integer, $1 \le i \le n-1$.*

(a) *The relation described above is well defined and gives a partial ordering on the set H_{i-1}/H_i.*
(b) *If $\sigma, \tau, \gamma \in H_{i-1}$ and $\sigma H_i \le \gamma H_i$, then $\sigma H_i \le \tau H_i \le \gamma H_i$ if and only if $\sigma^{-1}\tau H_i \le \sigma^{-1}\gamma H_i$.*

Proof. (a) Let $\sigma, \tau \in H_{i-1}$. To show the relation is well defined it suffices to show that if $f(\sigma, \sigma^{-1}\tau) \in W_i^{\times}$ and $h, k \in H_i$ then $f(\sigma h, h^{-1}\sigma^{-1}\tau k) \in W_i^{\times}$. We first observe that if $h \in H_i$ and $g \in H_{i-1}$, then $f(h, g), f(g, h) \in W_i^{\times}$: this follows from the identities

$$f^{h^{-1}}(h, g) f(h^{-1}, hg) = f(h^{-1}, h) \quad \text{and} \quad f^{g}(h, h^{-1}) = f(g, h) f(gh, h^{-1}).$$

The statement $f(\sigma h, h^{-1}\sigma^{-1}\tau k) \in W_i^\times$ is then a consequence of the following identities:

$$f^\sigma(h, h^{-1}\sigma^{-1}\tau k)f(\sigma, \sigma^{-1}\tau k) = f(\sigma, h)f(\sigma h, h^{-1}\sigma^{-1}\tau k),$$
$$f^\sigma(\sigma^{-1}\tau, k)f(\sigma, \sigma^{-1}\tau k) = f(\sigma, \sigma^{-1}\tau)f(\tau, k).$$

The fact that the relation is a partial ordering and satisfies part (b) is now a consequence of the following:

If $\sigma, \tau, \gamma \in H_{i-1}$ then

$$f^\sigma(\sigma^{-1}\tau, \tau^{-1}\gamma)f(\sigma, \sigma^{-1}\gamma) = f(\sigma, \sigma^{-1}\tau)f(\tau, \tau^{-1}\gamma). \quad \square$$

We begin our characterization of those standard orders which are valuation rings in the case where $|Y| = 1$, that is $f(G \times G) \subseteq W$ and $B_f = \sum_{\sigma\in G} W x_\sigma$. The first result shows that this condition is quite restrictive.

Proposition 2.10. *Let V be a valuation ring in F and let K be an inertial Galois extension with Galois group G. Let W be the unique extension of V to K. Let $f \in Z^2(G, K^\times)$ be a normalized two cocycle and assume f is standard for $Y = \{V\}$. If $B_f = \sum_{\sigma\in G} W x_\sigma$ is a valuation ring then every proper overring of B_f is Azumaya.*

Proof. If C is a proper overring of B_f then $V' = Z(C)$ is a proper overring of V and $W' = WV'$ is a proper overring of W. Moreover $C = B_f V'$ and $C \cap K = W'$. In particular C is the standard order corresponding to f viewed as standard for $Y' = \{V'\}$. Let $H' = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W'^\times\}$. By Proposition 2.6 we know

$$J(C) = \sum_{\sigma\in H'} J(W')x_\sigma + \sum_{\sigma\notin H'} W' x_\sigma.$$

But $J(C) \subseteq B_f$. Because W' is a proper overring of W we infer that $H' = G$ and so C is Azumaya. $\square$

To continue the characterization we first prove a more general result. Let f be standard for $Y = \{V_1, V_2, \dots, V_{n-1}, V_n\}$. Let w denote the value function determined by f and let $B_w = \sum_{\sigma\in G} N_\sigma x_\sigma$ denote the corresponding valuation ring. Recall that $N_\sigma = \{k \in K | w(kx_\sigma) \geq 0\}$. It is shown in the proof of [MW, Theorem 2.1] that $w(x_\sigma s) = w(x_\sigma) + w(s)$ for all $\sigma \in G$ and $s \in \Sigma_f$. Moreover letting $\Gamma_w = w(\Sigma_f)$, the map $\alpha: G \to \Gamma_w/\Gamma$ given by $\alpha(\sigma) = w(x_\sigma) + \Gamma$ is a surjective homomorphism. We let I denote the kernel of α, so $I = \{\sigma \in G | w(x_\sigma) \in \Gamma\}$.

Lemma 2.11. *Let $B_w = \sum_{\sigma\in G} N_\sigma x_\sigma$ be the valuation ring of w and assume each N_σ is a ring. Then*:

(a) *For all $\sigma \in G$, $f(\sigma, \sigma^{-1}) \in W$.*

(b) *We have $I = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W^\times\}$.*

Proof. Let $B = B_w$. First observe that for each σ, if N_σ is a ring then $1 \in N_\sigma$ and so $W = N_{\text{id}} \subseteq N_\sigma$.

(a) Because $1 \in N_\sigma$ for all σ, we have $x_\sigma \in B$ for all σ, and so W contains $x_\sigma x_{\sigma^{-1}} = f(\sigma, \sigma^{-1})$.

(b) Because $x_\tau \in B$ for all $\tau \in G$, $w(x_\tau) \geq 0$. Let $H = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W^\times\}$. If $\sigma \in H$, then

$$0 = v(f(\sigma, \sigma^{-1})) = w(x_\sigma x_{\sigma^{-1}}) = w(x_\sigma) + w(x_{\sigma^{-1}}).$$

It follows that $w(x_\sigma) = 0$, so $\sigma \in I$. Conversely if $\sigma \in I$, then $w(x_\sigma) \in \Gamma$, say $w(x_\sigma) = v(a)$, $a \in K$. Note that $v(a) \geq 0$. Because

$$W_\sigma = \{k \in K | w(kx_\sigma) \geq 0\} = \{k \in K | w(k) \geq -w(x_\sigma)\}$$

it follows that $W_\sigma = a^{-1}W$. But W_σ and W are both rings. Hence a is a unit in W, so $w(x_\sigma) = v(a) = 0$. □

Now assume f is standard for $\{V\}$, so $B_f = \sum_{\sigma \in G} W x_\sigma$.

Lemma 2.12. (a) *We have $B_f \subseteq B_w$. In particular if B_f is a valuation ring, then $B_f = B_w$.*

(b) *There is a cocycle g equivalent to f such that $B_w = B_g$ if and only if B_w is finitely generated as a V-module.*

Proof. (a) The ring B_f is finitely generated as a V-module, because W is a finitely generated V-module. In particular each x_σ is integral over V. If k is the order of σ, then $x_\sigma^k = f(\sigma, \sigma)f(\sigma^2, \sigma)\cdots f(\sigma^{k-1}, \sigma)$ is integral over V and lies in K. Thus $x_\sigma^k \in W$ and so $0 \leq w(x_\sigma^k) = kw(x_\sigma)$. Hence $x_\sigma \in B_w$. This means $1 \in N_\sigma$ and thus $W \subseteq N_\sigma$ because N_σ is a W-module. Therefore $B_f \subseteq B_w$. If B_f is a valuation ring, then $B_f = B_w$ because $B_f \cap F = B_w \cap F$.

(b) If $B_w = B_g$ then in particular B_w is finitely generated over V. Conversely, assume B_w is finitely generated. It follows that for each σ the V-submodule N_σ is finitely generated. Because $N_\sigma N_{\sigma^{-1}}^\sigma f(\sigma, \sigma^{-1}) \subseteq N_{\mathrm{id}} = W$, each N_σ is also a fractional ideal, and so we conclude that N_σ is principal over W. It follows that there is a cocycle g equivalent to f such that $B = B_g$. □

It is shown in [M, Proposition 3.2] that if w is any value function then B_w is a finitely generated V-module if and only if $[\Sigma_f : F] = \varepsilon f$, where $f = [\overline{B}_w : \overline{V}]$ and ε is the number of elements in the set $\Lambda = \{\gamma \in \Gamma_w | 0 \leq \gamma < \delta$ for all $\delta \in \Gamma^+\}$. Moreover in this case $\varepsilon = [\Gamma_w : \Gamma]$ and Γ_w/Γ is a cyclic group generated by $\gamma_0 + \Gamma$, where γ_0 is the least positive element of Γ_w. We want to give another characterization of when B_f is a valuation ring.

Lemma 2.13. *Set $\Lambda = \{\gamma \in \Gamma_w | 0 \leq \gamma < \delta$ for all $\delta \in \Gamma^+\}$. Then B_f is a valuation ring if and only if $w(x_\sigma) \in \Lambda$ for all $\sigma \in G$.*

Proof. Suppose B_f is a valuation ring. We have seen that it follows that $B_f = B_w$. Let a be an element of K such that $0 < v(a) < w(x_\sigma)$ for some σ in G. Then $w(a^{-1}x_\sigma) \geq 0$ so $a^{-1}x_\sigma \in B_w = B_f$. Thus $a^{-1} \in W$, so $v(a) \leq 0$. This is a contradiction, so $w(x_\sigma) \in \Lambda$.

Conversely, suppose $w(x_\sigma) \in \Lambda$ for all $\sigma \in G$. If $\sum_{\sigma \in G} a_\sigma x_\sigma \in B_w$ then $v(a_\sigma) + w(x_\sigma) \geq 0$ for all σ. If $v(a_\sigma) < 0$ for some σ then $0 < -v(a_\sigma) \leq w(x_\sigma)$, contradicting $w(x_\sigma) \in \Lambda$. Thus $a_\sigma \in W$ for all σ, so $B_w \subseteq \sum_{\sigma \in G} W x_\sigma = B_f$. But $B_f \cap F = V$, so $B_w = B_f$. □

Let $H = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W^\times\}$. We can now characterize those cocycles f for which $B_f = \sum_{\sigma \in G} W x_\sigma$ is a valuation ring. The characterization generalizes Theorem 2.3 of [H].

Theorem 2.14. *Suppose K/F is Galois and inertial with respect to the valuation rings W/V. Let $f \in Z^2(G, K^\times)$ be a normalized cocycle with $f(G \times G) \subseteq W$. The ring $B_f = \sum_{\sigma \in G} W x_\sigma$ is a valuation ring if and only if the following conditions are satisfied:*

(a) *H is normal in G and the quotient G/H is cyclic.*

(b) *Either* $H = G$, *in which case* B_f *is Azumaya, or there is an element* σ *in* G *such that* σH *generates* G/H *and such that the partial ordering on* G/H *is the simple chain* $H \le \sigma H \le \sigma^2 H \le \sigma^3 H \le \cdots \le \sigma^{k-1} H$ *where* k *is the order of* σ. *Moreover* $f(\sigma, \sigma^{-1}) \in J(W) - J(W)^2$.

Proof. Assume B_f is a valuation ring. Then we know $B_f = B_w$, the valuation ring of the value function w determined by f. By Lemma 2.13, $w(x_\sigma) \in \Lambda$ for all $\sigma \in G$. Also $|\Lambda| = |\Gamma_w/\Gamma|$ and Γ_w/Γ is cyclic. By Lemma 2.11, $H = I = \ker(\alpha)$. Hence H is a normal subgroup of G, and because the map $\alpha: G \to \Gamma_w/\Gamma$ is surjective, $G/H \cong \Gamma_w/\Gamma$ is cyclic. If $H = G$ then $f(G \times G) \subseteq W^\times$ and so B_f is Azumaya. Assume therefore that $H \ne G$. Let γ be the least positive element of Γ_w and let σ be an element of G such that $w(x_\sigma) = \gamma$. Then $\alpha(\sigma) = \gamma + \Gamma$ and $\langle \gamma + \Gamma \rangle = \Gamma_w/\Gamma$, so $\langle \sigma H \rangle = G/H$. Hence the order of σH is $\varepsilon = |\Lambda|$. Because

$$x_\sigma^{\varepsilon-1} = \left(\prod_{i=1}^{\varepsilon-2} f(\sigma^i, \sigma) \right) x_{\sigma^{\varepsilon-1}},$$

we have

$$(\varepsilon - 1)\gamma = v\left(\prod_{i=1}^{\varepsilon-2} f(\sigma^i, \sigma) \right) + w(x_{\sigma^{\varepsilon-1}}).$$

Hence $(\varepsilon - 1)\gamma \ge v(\prod_{i=1}^{\varepsilon-2} f(\sigma^i, \sigma))$. But it is easy to see that $(\varepsilon - 1)\gamma \in \Lambda$. Hence $v(\prod_{i=1}^{\varepsilon-2} f(\sigma^i, \sigma)) = 0$, so $f(\sigma^i, \sigma) \in W^\times$ for $0 \le i \le \varepsilon - 1$. It follows that $\sigma^i H \le \sigma^{i+1} H$ for $0 \le i \le \varepsilon - 1$. Moreover $x_\sigma^\varepsilon = (\prod_{i=1}^{\varepsilon-1} f(\sigma^i, \sigma))$, so $\varepsilon\gamma = w(x_\sigma^\varepsilon) = v(f(\sigma^{\varepsilon-1}, \sigma))$. Because γ is the least positive element of Γ_w it follows that $\varepsilon\gamma$ is the least positive element of Γ. Hence $f(\sigma^{\varepsilon-1}, \sigma) \in J(W) - J(W)^2$. We will have finished this direction once we show that $v(f(\sigma^{\varepsilon-1}, \sigma)) = v(f(\sigma, \sigma^{-1}))$. But $\sigma^{\varepsilon-1} = h\sigma^{-1}$ for some $h \in H$ and $f(H \times G) \subseteq W^\times$. From the cocycle identity $f^h(\sigma^{-1}, \sigma) = f(h, \sigma^{-1}) f(h\sigma^{-1}, \sigma)$ we infer

$$\begin{aligned} v(f(\sigma^{\varepsilon-1}, \sigma)) &= v(f(h\sigma^{-1}, \sigma)) = v(f^h(\sigma^{-1}, \sigma)) \\ &= v(f(\sigma^{-1}, \sigma)) = v(f(\sigma, \sigma^{-1})) \end{aligned}$$

where the last equality follows from $f^\sigma(\sigma^{-1}, \sigma) = f(\sigma, \sigma^{-1})$.

For the converse suppose H is normal in G. If $H = G$ then B_f is Azumaya. Otherwise we have $G/H = \langle \sigma H \rangle$ with $f(\sigma, \sigma^{-1}) \in J(W) - J(W)^2$ and $\sigma^i H < \sigma^{i+1} H$ for $0 \le i < |G/H|$. We will be done by Lemma 2.13 if we show $w(x_\sigma) \in \Lambda$ for all $\sigma \in G$. Let $t = |G/H|$. The relation $\sigma^i H < \sigma^{i+1} H$ gives $f(\sigma^i, \sigma) \in W^\times$ for $0 \le i < t - 1$ so $w(x_{\sigma^i}) = iw(x_\sigma)$ for $i < t$ and $w(x_\sigma^t) = v(f(\sigma^{t-1}, \sigma))$. Let $\gamma = w(x_\sigma)$. If $\tau \in G$, say $\tau = \sigma^i h$, then

$$w(x_\tau) = w(x_{\sigma^i}) + w(x_h) - v(f(\sigma^i, h)) = w(x_{\sigma^i}) = i\gamma$$

because $h \in H = I$ and $f(\sigma^i, h) \in W^\times$. As in the first half of the proof we have $v(f(\sigma^{t-1}, \sigma)) = v(f(\sigma, \sigma^{-1}))$. Because $f(\sigma, \sigma^{-1}) \in J(W) - J(W)^2$, we obtain $v(f(\sigma^{t-1}, \sigma)) = w(x_\sigma^t) = t\gamma$ is the least positive element of Γ. Thus $w(x_\tau) = i\gamma < t\gamma$ for all τ, so $w(x_\tau) \in \Lambda$. Thus $B_f = B_w$. □

Recall that from Lemma 2.11 we know $H = I = \ker \alpha$. In particular H can be described using the value function w. It is worth noting that under the hypotheses of Theorem 2.14 the partial ordering on G/H (which is then a total ordering) can also be described in terms of w. In fact $\sigma H \leq \tau H$ if and only if $w(\sigma) \leq w(\tau)$: If $\sigma H \leq \tau H$ then $f(\sigma, \sigma^{-1}\tau) \in W^\times$, so $w(f(\sigma, \sigma^{-1}\tau)) = 0$. Hence from $x_\sigma x_{\sigma^{-1}\tau} = f(\sigma, \sigma^{-1}\tau)x_\tau$, we infer that $w(x_\sigma) + w(x_{\sigma^{-1}\tau}) = w(x_\tau)$ and so that $w(x_\sigma) \leq w(x_\tau)$. (This direction is true in general, that is even if the ordering on G/H is not total.) For the converse suppose $w(\sigma) \leq w(\tau)$. If $\tau H < \sigma H$ then by the first part $w(\tau) \leq w(\sigma)$ and so $w(\tau) = w(\sigma)$. But then $\sigma^{-1}\tau \in \ker\alpha = H$, a contradiction. Because G/H is totally ordered it follows that $\sigma H \leq \tau H$.

Here is a nice application of the theorem. The result can also be obtained using the exact sequence (5.4) of [JW] along with [W, Theorems B, F].

Corollary 2.15. *Assume the hypotheses of the proposition and assume that the value group of V is equal to its own divisible hull. If $f \in Z^2(G, K^\times)$ is a normalized two cocycle and B is a valuation ring over V in Σ_f then B is Azumaya.*

Proof. Let B be a valuation ring over V in Σ_f. The condition on the value group implies that the ramification index of B/V is one. Moreover by [MW, Theorem 2.1] B/V is defectless. It then follows from [M, Proposition 3.2] that B is finitely generated as a V-module. By Lemma 2.12 it follows we may assume f is standard for $\{V\}$ and $B = B_f = \sum_{\sigma\in G} Wx_\sigma$. Let $H = \{\sigma \in G | f(\sigma, \sigma^{-1}) \in W^\times\}$. If H is a proper subgroup of G then by the theorem there is an element $\sigma \in G - H$ such that $f(\sigma, \sigma^{-1}) \in J(W) - J(W)^2$. But because the valuation is not discrete $J(W) = J(W)^2$, so this is impossible. Hence $H = G$, so B is Azumaya. □

We now proceed to the general case. We begin with a generalization of Lemma 2.12.

Lemma 2.16. *Let $Y = \{V_1, V_2, \dots, V_{n-1}, V_n\}$ where $V = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq V_0 = F$. There is a cocycle g equivalent to f such that g is standard for Y and $B_w = B_g$ if and only if for all i, $1 \leq i \leq n-1$, $B_w V_i / J(B_w V_{i-1})$ is finitely generated over $V_i/J(V_{i-1})$.*

Proof. If $B_w = B_g$ then we can apply Proposition 2.6 to see that for all i, $1 \leq i \leq n-1$,

$$B_w V_i / J(B_w V_{i-1}) = \sum_{\sigma\in H_i} W_i/J(W_i)x_\sigma$$

which is finitely generated over $V_i/J(V_{i-1})$.

For the converse we proceed by induction on n. Let $B_w = \sum_{\sigma\in G} N_\sigma x_\sigma$ as usual. If $n = 1$ then this is the second part of Lemma 2.12. Assume $n > 1$. The induction hypothesis applied to $Y' = Y - \{V\}$ shows that there is a cocycle g' equivalent to f such that the valuation ring $B_w V_{n-1}$ equals $B_{g'}$. Let $B_{g'} = \sum_{\sigma\in G} W_\sigma y_\sigma$. By Proposition 2.6 we know

$$J(B_{g'}) = \sum_{\sigma\in H_{n-1}} J(W_{n-1})y_\sigma + \sum_{\sigma\notin H_{n-1}} W_\sigma y_\sigma.$$

Because $J(B_{g'}) \subseteq B_w$ we can write

$$B_w = \sum_{\sigma\in H_{n-1}} N_\sigma y_\sigma + \sum_{\sigma\notin H_{n-1}} W_\sigma y_\sigma$$

where $N_{\mathrm{id}} = W$ and $N_\sigma \subseteq W_{n-1}$ with $N_\sigma W_{n-1} = W_{n-1}$ for all $\sigma \in H_{n-1}$. By hypothesis

$$B_w/J(B_{g'}) = \sum_{\sigma \in H_{n-1}} N_\sigma/J(W_{n-1})y_\sigma$$

is a finitely generated $V/J(V_{n-1})$-module and clearly contains $W/J(W_{n-1})$. As in the proof of part (b) of Lemma 2.12 it follows that for each $\sigma \in H_{n-1}$ there is an element $a_\sigma \in W_{n-1}$ such that $N_\sigma/J(W_{n-1})$ is the principal ideal $W/J(W_{n-1})(a_\sigma + J(W_{n-1}))$. Hence $Wa_\sigma + J(W_{n-1}) = N_\sigma$. But then $Wa_\sigma W_{n-1} + J(W_{n-1}) = N_\sigma W_{n-1} = W_{n-1}$ so a_σ is a unit in W_{n-1}. It follows that $J(W_{n-1}) \subseteq Wa_\sigma$ and so $Wa_\sigma = N_\sigma$. It is now easy to see that one can alter g' to obtain a cocycle g equivalent to f such that g is standard for Y and $B_w = B_g$. □

Theorem 2.17. *Let f be a standard cocycle for $Y = \{V_1, V_2, \dots, V_{n-1}, V_n\}$. The crossed product order B_f is a valuation ring if and only if for each i, $1 \le i \le n$, the following conditions are satisfied:*

(a) *The subgroup H_i is normal in H_{i-1}, and the quotient H_{i-1}/H_i is cyclic.*
(b) *Either $H_{i-1} = H_i$ or there is an element σ_{i-1} in H_{i-1} such that $\sigma_{i-1}H_i$ generates H_{i-1}/H_i and such that the partial ordering on H_{i-1} is the simple chain*

$$\sigma_{i-1}H_i \le \sigma_{i-1}^2H_i \le \sigma_{i-1}^3H_i \le \cdots \le \sigma_{i-1}^{k-1}H_i$$

where k is the order of σ_{i-1}. Moreover $f(\sigma_{i-1}, \sigma_{i-1}^{-1}) \in J(W_i) - J(W_i)^2$.

Proof. Let $B = B_f$ and assume B is a valuation ring. We proceed by induction on n. We need to show that for each i, f satisfies properties (a) and (b). If $n = 1$ then this is Theorem 2.14. Hence we may assume $n > 1$. The ring $B_{n-1} = BV_{n-1}$ is also a valuation ring. Moreover we have seen that B_{n-1} is the crossed product over V_{n-1} corresponding to the cocycle f, which is standard for $Y' = Y - \{V\}$. Hence by induction properties (a) and (b) hold for all i, $1 \le i \le n-1$, and we are left with verifying the properties for $i = n$. Let $H = H_{n-1}$. We have $J(B_{n-1}) \subseteq B$ by Proposition 2.6 and $B/J(B_{n-1})$ is a valuation ring in the simple algebra $B_{n-1}/J(B_{n-1})$. Also

$$B_{n-1}/J(B_{n-1}) = \sum_{\sigma \in H} W_{n-1}/J(W_{n-1})x_\sigma,$$

a crossed product algebra for the cocycle $f_{|H\times H}$. Moreover $f_{|H\times H}$ is standard for the valuation ring $W_n/J(W_{n-1})$ and

$$B/J(B_{n-1}) = \sum_{\sigma \in H} W_n/J(W_{n-1})x_\sigma$$

is the crossed product order for $f_{|H\times H}$. Hence the result for $i = n$ follows from the $n = 1$ case.

Conversely assume $B = B_f$ and f has properties (a) and (b). We need to show B is a valuation ring. Again we argue by induction on n, the $n = 1$ case being Theorem 2.14. Thus assume $n > 1$ and let $B_{n-1} = BV_{n-1}$. Then B_{n-1} is the crossed product order for the standard cocycle f (for Y'). Because f satisfies (a) and (b) for $i \le n-1$ we infer by induction that B_{n-1} is a valuation ring. Moreover $J(B_{n-1}) \subseteq B$ and as we have seen $B/J(B_{n-1})$ is the crossed product order for the cocycle $f_{|H\times H}$ inside the crossed product algebra $B_{n-1}/J(B_{n-1})$ (where $H = H_{n-1}$). Because $f_{|H\times H}$ satisfies (a) and (b) for $H_n \subseteq H$ we obtain $B/J(B_{n-1})$ is a valuation ring by the $n=1$ case. But by [D_2, §1, Proposition 2] it then follows that B is a valuation ring, as desired. □

Example 2.18. Here is an example of a valuation ring which is not equal to B_f for any f and Y:

Let F be a field with valuation v whose value group is $\Gamma = \mathbb{Z} + \pi\mathbb{Z}$ with the archimedean ordering induced from the inclusion $\Gamma \subseteq \mathbb{R}$. Let V be the valuation ring of v and suppose $\operatorname{char}(\overline{V}) \neq 2$ and that there is an $a \in V$ with $\overline{a}$ not a square in $\overline{V}$. (For example we could take $F = \mathbb{Q}(\Gamma)$ and $a = 2$.) Let $b \in V$ with $v(b) = \pi$. Set $K = F(\sqrt{a})$, an inertial Galois extension of F with respect to V and Σ the cyclic algebra $(K/F, \sigma, b)$. Then Σ is the quaternion algebra $(a, b)_F$ with generators i, j satisfying $i^2 = a$, $j^2 = b$, $ij = -ji$. Also $K = F(i)$ and $\Sigma = K \oplus Kj$. By [JW, Example 4.3] v extends to a valuation on Σ, which we will also denote by v, such that $v(\alpha+\beta j) = \min\{v(\alpha), v(\beta)+\pi/2\}$ for $\alpha, \beta \in K$. Let B be the valuation ring of this valuation. Then $B = W \oplus Tj$ where $W = B \cap K$ and $T = \{\alpha \in K | v(\alpha) \geq -\pi/2\}$. If $B = B_f$ for some f then $B = W \oplus Wx_\sigma$ is a finitely generated V-module. Thus T is a finitely generated W-module, hence principal. But if $T = Wx$ then $v(x)$ is the least element of $v(T) = \{\gamma \in \Gamma | v(\gamma) \geq -\pi/2\}$. But $v(T)$ has no least element because $-\pi/2 \notin \Gamma$ and Γ is dense in $\mathbb{R}$. Hence B is not a crossed product order B_f for any f.

We end this section with a proposition that will be useful in the next section.

Proposition 2.19. *Suppose f is standard for W_n. Let w be the value function associated to f and assume that $B_w = \sum_{\sigma\in G} W_\sigma x_\sigma$ is in standard form (that is $B_w = B_f$). If H is any subgroup of G, then $\sum_{\sigma\in H} W_\sigma x_\sigma$ is a valuation ring in $\sum_{\sigma\in H} Kx_\sigma$.*

Proof. Let $E = \sum_{\sigma\in H} Kx_\sigma$ and let $w' = w|_E$. Then $C = B_w \cap E = \{z \in E | w'(z) \geq 0\}$. Let $J = \{z \in E | w'(z) > 0\}$. By [M, Theorem 2.4] we will be done if we can show w' is a value function and C/J is simple. Moreover to show w' is a value function, it suffices to show that if $y \in \operatorname{im}(w')$, then there exists $z \in E^\times$ such that $w'(z) = y$ and $w'(z^{-1}) = -w'(z)$. But the definition of w shows that $\operatorname{im}(w') = \{w(a) + w(x_\sigma) | a \in F,\ \sigma \in H\}$. Because $w(a) + w(x_\sigma) = w(ax_\sigma)$ and $w((ax_\sigma)^{-1}) = -w(ax_\sigma)$, we have shown w' is a value function.

We now proceed to show C/J is simple. Let $g = f_{|H\times H}$ and let $H_i' = H \cap H_i = \{\sigma \in H | f(\sigma, \sigma^{-1}) \in W_i^\times\}$ for $0 \leq i \leq n$. Then for $i < n$, $(H_i' \times H_i') \subseteq f(H_i \times H_i) \subseteq W_{i+1}$, so g is standard for W_n and $C = B_g$. By Proposition 2.6 C is primary and

$$J(C) = \sum_{\sigma\in H_n'} J(W_\sigma)x_\sigma + \sum_{\sigma\notin H_n'} W_\sigma x_\sigma.$$

But then $J = J(B) \cap C = J(C)$ so C/J is simple. □

3. Discrete rank N valuation rings

In this section we consider the case where V is unramified but not necessarily indecomposed in K. We restrict our attention to discrete rank n valuation rings V. Let $V = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_2 \subseteq V_1 \subseteq F$ be the overrings of V and let W_i be the integral closure of V_i in K.

Let $f \in Z^2(G, K^\times)$ be a normalized two-cocycle and let $(K/F, G, f) = \sum_{\sigma\in G} Kx_\sigma$. By Proposition 1.7 we may assume there is a valuation ring $B = \sum_{\sigma\in G} W_\sigma x_\sigma$ over V in $(K/F, G, f)$, where each W_σ is an overring of W_{id}.

In order to reduce the confusion caused by the too frequent use of the letter W in our notation, we will let $S = W_1$. That is S will denote the integral closure of V in K.

We want to determine the rings W_σ. Recall from §1 that each W_σ is a semilocal Prüfer ring and hence an intersection of valuation rings. More specifically $W_\sigma = \bigcap S_P$ where the intersection is over those prime ideals P of S such that $W_\sigma \subseteq S_P$. Also note that $V = B \cap F = W_{\mathrm{id}} \cap F$.

Definition. A prime Q of S is said to *belong* to B if $W_{\mathrm{id}} \subseteq S_Q$.

Observe that for every $i \le n$ there is a prime ideal of height i belonging to B: It suffices to show there is a maximal ideal M of S belonging to B because then any prime ideal contained in M also belongs to B. But if no maximal ideal belongs to B then $W_{\mathrm{id}} \supseteq W_2$ and so $W_{\mathrm{id}} \cap F = V_2$, a contradiction.

The following is the basic result of this section.

Theorem 3.1. *Let $B = \sum_{\sigma\in G} W_\sigma x_\sigma$ be a valuation ring over V. Then*

(a) *We have $J(B) = \sum_{\sigma\in G} J_\sigma x_\sigma$ where for each $\sigma \in G$, $J_\sigma = \{k \in W_\sigma | kf(\sigma, \sigma^{-1}) \in J(W_\sigma)\}$.*
(b) *If Q_1, Q_2 are prime ideals of S of the same height belonging to B, then there exists $\sigma \in G$ such that $Q_1^\sigma = Q_2$ and $f(\sigma^{-1}, \sigma) \notin Q_1 W_{\mathrm{id}}$.*

Proof. The proof is by induction on the rank. If the rank is one then the theorem is a consequence of [H, Proposition 3.1, Theorem 3.2]. Hence we may assume the rank n is greater than one. Because $B_{n-1} = BV_{n-1}$ is a valuation ring of rank $n-1$ the results may be assumed true for it. We will assume the theorem for valuation rings of rank less than n in the following lemmas.

If P is a prime ideal of S of height $i < n$ we will let S^P denote $\bigcap S_Q$ where the intersection is over those primes Q of S that contain P. Note that if U is any overring of S then $US^P = \bigcap S_Q$ where this intersection is over those primes Q such that $P \subseteq Q$ and $U \subseteq S_Q$.

Lemma 3.2. *Let P be a prime of height i belonging to B. Let $\sigma \in G$.*

(a) *If $f(\sigma, \sigma^{-1}) \notin PW_{\mathrm{id}}$ then $W_\sigma S^P = W_{\mathrm{id}} S^P$.*
(b) *If $Q \supseteq P$ is a prime of height $i+1$ belonging to B and $f(\sigma, \sigma^{-1}) \notin PW_{\mathrm{id}}$, then $W_{\mathrm{id}} \subseteq S_{Q^{\sigma^{-1}}}$ and $W^\sigma_{\sigma^{-1}} \subseteq S_Q$.*
(c) *If $f(\sigma, \sigma^{-1}) \in PW_{\mathrm{id}}$, then $W_\sigma S^P = W_\sigma S_P$.*

Proof. (a) Let Q be a height $i+1$ prime of S containing P. We need to show that $W_{\mathrm{id}} \subseteq S_Q$ if and only if $W_\sigma \subseteq S_Q$. We know $W_{\mathrm{id}} \subseteq W_\sigma$. Hence certainly $W_\sigma \subseteq S_Q$ implies $W_{\mathrm{id}} \subseteq S_Q$.

Now suppose $W_{\mathrm{id}} \subseteq S_Q$, but W_σ is not contained in S_Q. Then $W_\sigma S_Q$ is a ring properly containing S_Q. Hence $W_\sigma S_Q$ is valuation ring. But P is the unique prime of height n contained in Q. It follows that $W_\sigma S_Q \supseteq S_P$. On the other hand we know $W_\sigma f(\sigma, \sigma^{-1}) \subseteq W_{\mathrm{id}}$, so we have $W_\sigma f(\sigma, \sigma^{-1}) S_Q \subseteq$

$W_{\mathrm{id}}S_Q = S_Q$. Thus

$$S_Q \supseteq W_\sigma f(\sigma, \sigma^{-1})S_Q = f(\sigma, \sigma^{-1})W_\sigma S_Q \supseteq f(\sigma, \sigma^{-1})S_P = S_P,$$

because $f(\sigma, \sigma^{-1}) \notin PW_{\mathrm{id}}$. Thus $S_P \subseteq S_Q$, a contradiction.

(b) By part (a) $W_\sigma S^P = W_{\mathrm{id}}S^P$. Hence $W_\sigma \subseteq S_Q$. But because the product $x_\sigma W_{\mathrm{id}}$ is in B we see that $W_{\mathrm{id}}^\sigma \subseteq W_\sigma$. Hence $W_{\mathrm{id}} \subseteq S_{Q^{\sigma^{-1}}}$. In particular $W_{\mathrm{id}} \subseteq S_{P^{\sigma^{-1}}}$, so $P^{\sigma^{-1}}$ belongs to B. Moreover $f(\sigma^{-1}, \sigma) = f^{\sigma^{-1}}(\sigma, \sigma^{-1}) \notin P^{\sigma^{-1}}W_{\mathrm{id}}$, so by part (a) $W_{\sigma^{-1}} \subseteq S_{Q^{\sigma^{-1}}}$. Hence $W_{\sigma^{-1}}^\sigma \subseteq S_Q$.

(c) Assume $f(\sigma, \sigma^{-1}) \in PW_{\mathrm{id}}$. Let Q be a prime of height $i+1$ that contains P. We *claim* W_σ is not contained in S_Q: There are two cases: If $PW_\sigma = W_\sigma$, then from $W_\sigma \subseteq S_Q$ we obtain $W_\sigma = PW_\sigma \subseteq PS_Q \subseteq QS_Q$, a contradiction. If $PW_\sigma \neq W_\sigma$, then $PW_\sigma V_i$ is a maximal ideal in $W_\sigma V_i$. Let $J = \{k \in W_\sigma V_i | kf(\sigma, \sigma^{-1}) \in J(W_\sigma V_i)\}$. We know by the induction hypothesis that $J \subseteq J(B_i)$. Because $B \subseteq B_i$ and both are valuation rings we infer that $J(B_i) \subseteq B$. Hence $J \subseteq W_\sigma$. Let $T = \prod N$ where the product is over those primes of S of height at most i not contained in P. Note that $T \subseteq J$ because $f(\sigma, \sigma^{-1}) \in PW_{\mathrm{id}}$. Hence $W_\sigma V_i T \subseteq W_\sigma V_i J \subseteq J \subseteq W_\sigma$.

Now assume $W_\sigma \subseteq S_Q$. Then $W_\sigma V_i T \subseteq S_Q$, so $W_\sigma V_i T S_Q \subseteq S_Q$. But $TS_Q = S_Q$: If not $TS_Q \subseteq QS_Q$, so $T \subseteq Q$. Hence there is a prime N of height at most i such that $N \subseteq Q$ but N is not contained in P. But $P \subseteq Q$, so this is not possible. Thus $TS_Q = S_Q$ and so if $W_\sigma V_i T S_Q \subseteq S_Q$, then $V_i \subseteq S_Q$, a contradiction. This proves the claim.

Because $W_\sigma S^P$ is a Prüfer ring it is the intersection of the valuation rings that contain it and each such valuation ring is a localization of $W_\sigma S^P$ at some prime ideal. We have just seen that any valuation ring that contains $W_\sigma S^P$ must have rank at most i. However every prime ideal of S^P of height at most i is contained in P and so every valuation ring of rank at most i that contains S^P must contain S_P. Hence $W_\sigma S^P \supseteq S_P$ and so $W_\sigma S^P = W_\sigma S_P$. □

Proposition 3.3. *Let P be a prime of height $i \leq n$ belonging to B. Let $P = P_i \supseteq P_{i-1} \supseteq \cdots \supseteq P_1 \supseteq P_0 = 0$ be the unique chain of prime ideals of S contained in P. If $\sigma \in G$ then $W_\sigma S_P = S_{P_j}$ where $j \leq n$ is the unique integer such that $f(\sigma, \sigma^{-1}) \in P_j W_{\mathrm{id}} - P_{j-1}W_{\mathrm{id}}$.*

Proof. This is an easy consequence of parts (a) and (c) of Lemma 3.2. □

Proposition 3.4. *We have $W_{\mathrm{id}} = \bigcap S_M$ where the intersection is over those maximal ideals M of S that belong to B.*

Proof. It suffices to show that every prime of height $i < n$ that belongs to B is contained in a height $i+1$ prime ideal belonging to B. By the remark immediately preceding Theorem 3.1, we know there is some height $i+1$ prime ideal Q of S belonging to B. Let P be the unique height i prime of S contained in Q. Then P belongs to B. By the induction hypothesis applied to part (b) of the theorem, if T is another height i prime of S belonging to B, then there is an element $\sigma \in G$ such that $f(\sigma^{-1}, \sigma) \notin P$ and $P^\sigma = T$. By part (b) of Lemma 3.2, $W_{\mathrm{id}} \subseteq S_{Q^\sigma}$. But Q^σ is a height $i+1$ prime ideal containing T, so we have proved the proposition. □

It should be observed that these propositions give, for each $\sigma \in G$, a prescription for finding W_σ in terms of W_{id}: If $M_1, M_2, \ldots, M_k$ are the maximal ideals that belong to B, that is for which $W_{\mathrm{id}} \subseteq S_{M_i}$, and if $M_i = P_{i,n} \supseteq P_{i,n-1} \supseteq \cdots \supseteq P_{i,1} \supseteq P_{i,0} = 0$ is the chain of prime ideals contained in M_i, then $W_\sigma = \bigcap_{i=1}^k S_{P_{i,j_i}}$ where for each i, $1 \le i \le k$, j_i is the unique integer such that $f(\sigma, \sigma^{-1}) \in P_{j_i} W_{\mathrm{id}} - P_{j_i - 1} W_{\mathrm{id}}$. The description of the rings W_σ is thus reduced to describing W_{id}, or in other words to finding the primes that belong to B.

3.1. **Proof of part (a) of the theorem.** Let $\sigma \in G$. We begin by giving an alternate description of J_σ. We have that $J(W_\sigma) = \bigcap QW_\sigma$ where the intersection is over those prime ideals Q of S such that QW_σ is a maximal ideal of W_σ. If $f(\sigma, \sigma^{-1}) \in QW_{\mathrm{id}}$ for all such Q, then $J_\sigma = W_\sigma$. Otherwise, that is if $f(\sigma, \sigma^{-1}) \notin J(W_\sigma)$, then $J_\sigma = \bigcap QW_\sigma$ where the intersection is over those prime ideals Q of S such that QW_σ is a maximal ideal of W_σ and $f(\sigma, \sigma^{-1}) \notin QW_{\mathrm{id}}$. We claim that it follows that any such Q must be a *maximal* ideal of S: Because $f(\sigma, \sigma^{-1}) \notin QW_{\mathrm{id}}$ we know from Lemma 3.2 that $W_\sigma S^Q = W_{\mathrm{id}} S^Q$. Applying Proposition 3.4 we see there is a maximal ideal M of S such that $M \supseteq Q$ and MW_σ is a maximal ideal of W_σ. But MW_σ contains QW_σ, so $MW_\sigma = QW_\sigma$ by the maximality of QW_σ. Hence $Q = M$ is maximal. We infer that either $J_\sigma = W_\sigma$ or $J_\sigma = \bigcap QW_\sigma$ where the intersection is over those maximal ideals Q of S such that QW_σ is a maximal ideal of W_σ and $f(\sigma, \sigma^{-1}) \notin QW_{\mathrm{id}}$.

Now let $I = \sum_{\sigma \in G} J_\sigma x_\sigma$. We begin by showing that I is an ideal of B. To see that it is a right ideal it suffices to show that for all $\sigma, \tau \in G$, $J_\sigma x_\sigma W_\tau x_\tau \subseteq J_{\sigma\tau} x_{\sigma\tau}$. This is equivalent to $J_\sigma W_\tau^\sigma f(\sigma, \tau) \subseteq J_{\sigma\tau}$. First observe that $J_\sigma W_\tau^\sigma f(\sigma, \tau) \subseteq W_{\sigma\tau}$. Hence if $J_{\sigma\tau} = W_{\sigma\tau}$ the inclusion is clear. We may thus assume $J_{\sigma\tau} = \bigcap QW_{\sigma\tau}$ where the intersection is over those maximal ideals Q of S such that $QW_{\sigma\tau}$ is a maximal ideal of $W_{\sigma\tau}$ and $f(\sigma\tau, \tau^{-1}\sigma^{-1}) \notin QW_{\mathrm{id}}$. Hence to show $J_\sigma W_\tau^\sigma f(\sigma, \tau) \subseteq J_{\sigma\tau}$ we need to show that if Q is a maximal ideal of S such that $QW_{\sigma\tau}$ is maximal in $W_{\sigma\tau}$ and $f(\sigma\tau, \tau^{-1}\sigma^{-1}) \notin QW_{\mathrm{id}}$, then $J_\sigma W_\tau^\sigma f(\sigma, \tau) \subseteq QW_{\sigma\tau}$. This last inclusion is equivalent to $J_\sigma W_\tau^\sigma f(\sigma, \tau) S_Q \subseteq QS_Q$.

So assume Q is chosen as above. Let P be the unique height $n-1$ prime of S such that $P \subseteq Q$. If $f(\sigma\tau, \tau^{-1}\sigma^{-1}) \in PW_{\mathrm{id}}$, then by Lemma 3.2, $W_{\sigma\tau} S^P = S_P$, contradicting the fact that $W_{\sigma\tau} \subseteq S_Q$. Hence we have $f(\sigma\tau, \tau^{-1}\sigma^{-1}) \notin PW_{\mathrm{id}}$.

Now if $f(\sigma, \sigma^{-1}) \notin QW_{\mathrm{id}}$, then in particular $f(\sigma, \sigma^{-1}) \notin PW_{\mathrm{id}}$ and so by Lemma 3.2 we have $W_\sigma S_Q = W_{\mathrm{id}} S_Q = S_Q$. Moreover $J_\sigma S_Q \subseteq QS_Q$. Hence $J_\sigma W_\tau^\sigma f(\sigma, \tau) S_Q \subseteq W_\tau^\sigma f(\sigma, \tau) QS_Q$. But $W_\tau^\sigma f(\sigma, \tau) \subseteq W_{\sigma\tau} \subseteq S_Q$, so we get the desired inclusion.

Now assume $f(\sigma, \sigma^{-1}) \notin QW_{\mathrm{id}}$. Let $P = P_{n-1} \supseteq P_{n-2} \supseteq \cdots \supseteq P_1 \supseteq P_0 = 0$ be the chain of prime ideals contained in P. Assume j is the unique integer such that $f(\sigma, \sigma^{-1}) \in P_j W_{\mathrm{id}} - P_{j-1} W_{\mathrm{id}}$. If $J_\sigma = W_\sigma$ when $J_\sigma S_Q = W_\sigma S_Q = S_{P_j}$ by Proposition 3.3. If $J_\sigma \ne W_\sigma$ then we know $J_\sigma = \bigcap NW_\sigma$ where the intersection is over those maximal ideals N of S such that NW_σ is a maximal ideal of W_σ and $f(\sigma, \sigma^{-1}) \notin NW_{\mathrm{id}}$. Because $f(\sigma, \sigma^{-1}) \in P_j W_{\mathrm{id}}$ we know that if N is such a maximal ideal then N does not contain P_j. In particular $J_\sigma \supseteq \prod N$ where the product is over those maximal ideals and so $J_\sigma S_Q = J_\sigma W_\sigma S_Q = J_\sigma S_{P_j} \supseteq (\prod N) S_{P_j} = S_{P_j}$ because otherwise some N would be contained in P_j. Hence in either case we have $J_\sigma S_Q = S_{P_j}$ and so

$$J_\sigma W_\tau^\sigma f(\sigma, \tau)S_Q = W_\tau^\sigma f(\sigma, \tau)S_{P_j} = W_\tau^\sigma f(\sigma, \tau)f(\sigma\tau, \tau^{-1}\sigma^{-1})S_{P_j},$$

because $f(\sigma\tau, \tau^{-1}\sigma^{-1}) \notin P_j W_{\mathrm{id}}$. Now we apply the following cocycle identity:

$$f^\sigma(\tau, \tau^{-1}\sigma^{-1})f(\sigma, \sigma^{-1}) = f(\sigma, \tau)f(\sigma\tau, \tau^{-1}\sigma^{-1}).$$

It follows that

$$W_\tau^\sigma f(\sigma, \tau)f(\sigma\tau, \tau^{-1}\sigma^{-1})S_{P_j} = W_\tau^\sigma f^\sigma(\tau, \tau^{-1}\sigma^{-1})f(\sigma, \sigma^{-1})S_{P_j}.$$

But $f(\sigma, \sigma^{-1}) \in P_j S_{P_j}$ and $W_\tau^\sigma f^\sigma(\tau, \tau^{-1}\sigma^{-1}) \subseteq W_{\sigma^{-1}}^\sigma$. Moreover $W_{\sigma^{-1}}^\sigma \subseteq S_{P_j}$ by part (b) of Lemma 3.2. Thus

$$W_\tau^\sigma f^\sigma(\tau, \tau^{-1}\sigma^{-1})f(\sigma, \sigma^{-1})S_{P_j} \subseteq P_j S_{P_j} \subseteq QS_Q$$

as desired.

The computations to show I is a left ideal are similar and will be omitted.

Having shown I is an ideal we proceed to show $I = J(B)$. Because B is a valuation ring we know $J(B)$ is the unique maximal ideal of B, so $I \subseteq J(B)$. Moreover $J(B)$ is an S-S bimodule, so we can write $J(B) = \sum_{\sigma\in G} T_\sigma x_\sigma$ for some ideals T_σ in W_σ. We have $T_\sigma \supseteq J_\sigma$ for all $\sigma \in G$, and we want to show equality.

First observe that

$$J_{\mathrm{id}} = \{k \in W_{\mathrm{id}} | kf(\mathrm{id},\ \mathrm{id}) \in J(W_{\mathrm{id}})\} = J(W_{\mathrm{id}}).$$

Because T_{id} consists of quasiregular elements in W_{id} it follows that $T_{\mathrm{id}} \subseteq J(W_{\mathrm{id}})$. Hence $T_{\mathrm{id}} = J_{\mathrm{id}}$.

Now let $\sigma \in G$. If $J_\sigma = W_\sigma$ then certainly $T_\sigma = J_\sigma$. Hence we may assume $J_\sigma \neq W_\sigma$ and so $J_\sigma = \bigcap QW_\sigma$ where the intersection is over those maximal ideals Q of S such that QW_σ is maximal in W_σ and $f(\sigma, \sigma^{-1}) \notin QW_{\mathrm{id}}$. Let Q be such a maximal ideal. Now because $J(B)$ is an ideal we have $T_\sigma x_\sigma x_{\sigma^{-1}} \subseteq T_{\mathrm{id}} \subseteq J(W_{\mathrm{id}})$ and so $T_\sigma f(\sigma, \sigma^{-1}) \subseteq J(W_{\mathrm{id}})$. Moreover if Q is a maximal ideal in S such that QW_σ is maximal in W_σ, then QW_{id} is a maximal ideal of W_{id}. Hence $J(W_{\mathrm{id}}) \subseteq J_\sigma$ and so $T_\sigma f(\sigma, \sigma^{-1})^2 \subseteq J(W_\sigma) \subseteq QW_\sigma$. But $f(\sigma, \sigma^{-1})^2 \notin QW_\sigma$ so $T_\sigma \subseteq QW_\sigma$. Because Q was arbitrary, we see that $T_\sigma \subseteq J_\sigma$, as desired. □

3.2. **Proof of part (b) of the theorem.** If the height of Q_1 is less than n then the result follows by induction. Hence we may assume that Q_1 and Q_2 are maximal ideals of S. Let $Q_1, Q_2, \ldots, Q_r, Q_{r+1}, \ldots, Q_m$ be all the maximal ideals of S and assume $Q_1, Q_2, \ldots, Q_r$ are the ones that belong to B. By Proposition 3.4 we know $W_{\mathrm{id}} = \bigcap_{i=1}^r S_{Q_i}$. We also know from part (a) of the theorem that

$$J(B)_{\mathrm{id}} = J(W_{\mathrm{id}}) = \bigcap_{i=1}^r Q_i W_{\mathrm{id}}.$$

Let $T = \prod_{i=2}^m Q_i$. Then $TW_{\mathrm{id}} = \prod_{i=2}^r Q_i W_{\mathrm{id}}$ because if $i > r$ then $Q_i W_{\mathrm{id}} = W_{\mathrm{id}}$. It follows that T is not contained in $J(W_{\mathrm{id}})$. Because B is primary we infer that $BTB = B$. Computing the identity component of BTB we see that we must have $\sum_{\sigma\in G} W_\sigma x_\sigma T W_{\sigma^{-1}} x_{\sigma^{-1}} = W_{\mathrm{id}}$.

For each i, $i = 1, 2, \ldots, m$, there is an element $h_i \in G$ such that $Q_1^{h_i} = Q_i$. Let $D = D(Q_1)$. The elements $h_1, h_2, \ldots, h_m$ form a set of left coset representatives of D in G. Hence we can write

$$W_{\mathrm{id}} = \sum_{\sigma \in G} W_\sigma x_\sigma T W_{\sigma^{-1}} x_{\sigma^{-1}} = \sum_{\sigma \in G} W_\sigma T^\sigma W_{\sigma^{-1}}^\sigma f(\sigma, \sigma^{-1})$$
$$= \sum_{i=1}^{m} \sum_{d \in D} W_{h_i d} T^{h_i d} (W_{(h_i d)^{-1}}^{h_i d} f(h_i d, (h_i d)^{-1})).$$

Note that for all $d \in D$, $T^{h_i d} = T^{h_i}$ because $T^d = T$. Also observe that if $i \neq 2$ then $T^{h_i} \subseteq Q_2$ and so

$$\sum_{d \in D} W_{h_i d} T^{h_i d} (W_{(h_i d)^{-1}}^{h_i d} f(h_i d, (h_i d)^{-1})) \subseteq Q_2 W_{\mathrm{id}}$$

because for every $\sigma \in G$, $W_\sigma W_{\sigma^{-1}}^\sigma f(\sigma, \sigma^{-1}) \subseteq W_{\mathrm{id}}$.

Now assume that for all $d \in D$, $f(h_2 d, (h_2 d)^{-1}) \in Q_1^{h_2}$ $(= Q_2)$. We *claim* that it follows that

$$\sum_{d \in D} W_{h_2 d} T^{h_2 d} (W_{(h_2 d)^{-1}}^{h_2 d} f(h_2 d, (h_2 d)^{-1})) \subseteq Q_2 W_{\mathrm{id}}.$$

If so then $BTB \subseteq Q_2 W_{\mathrm{id}}$, a contradiction. It will then follow that for some $d \in D$, $f(h_2 d, (h_2 d)^{-1}) \notin Q_1^{h_2} = Q_1^{h_2 d}$ and so that

$$f((h_2 d)^{-1}, h_2 d) = f^{(h_2 d)^{-1}}(h_2 d, (h_2 d)^{-1}) \notin Q_1.$$

Because $Q_1^{h_2 d} = Q_2$ that will finish the proof.

To prove the claim it suffices to show that for each $d \in D$

$$W_{h_2 d} T^{h_2 d} (W_{(h_2 d)^{-1}}^{h_2 d} f(h_2 d, (h_2 d)^{-1})) \subseteq Q_2 W_{\mathrm{id}},$$

or equivalently that

$$W_{h_2 d} T^{h_2 d} (W_{(h_2 d)^{-1}}^{h_2 d} f(h_2 d, (h_2 d)^{-1})) S_{Q_2} \subseteq Q_2 S_{Q_2}.$$

To simplify the notation let $Q = Q_2$. Let $Q = P_n \supseteq P_{n-1} \supseteq \cdots \supseteq P_1 \supseteq P_0 = 0$ be the chain of prime ideals contained in Q. Let $d \in D$ and let j be the unique integer such that $f(h_2 d, (h_2 d)^{-1}) \in P_j W_{\mathrm{id}} - P_{j-1} W_{\mathrm{id}}$. Let $\tau = h_2 d$. We need to show $W_\tau T^\tau W_{\tau^{-1}}^\tau f(\tau, \tau^{-1}) S_Q \subseteq Q S_Q$. We have $f(\tau, \tau^{-1}) \subseteq P_j W_{\mathrm{id}}$ and by Proposition 3.3 $W_\tau S_Q = S_{P_j}$. Moreover $W_{\tau^{-1}}^\tau \subseteq S_{P_j}$ by Lemma 3.2. Hence $W_\tau T^\tau W_{\tau^{-1}}^\tau f(\tau, \tau^{-1}) S_Q \subseteq P_j S_{P_j}$. But $P_j S_{P_j} \subseteq Q S_Q$, so we are done. □

Corollary 3.5. *Let P be a prime of height $i < n$ belonging to B and let $T = \{Q | Q$ is a prime of height $i+1$ belonging to B and containing $P\}$.*

(a) *The group $H(P)$ acts transitively on T. In particular the order of T is the index $(H(P) \colon D(Q) \cap H(P))$, where Q is any element of T.*

(b) *If P' is another height i prime belonging to B and $T' = \{Q | Q$ is a prime of height $i+1$ belonging to B and containing $P'\}$ then there is an element $\sigma \in G$ such that $f(\sigma^{-1}, \sigma) \notin P W_{\mathrm{id}}$ and $T' = T^\sigma$.*

Proof. (a) If $Q \in T$ and $\sigma \in H(P)$ then $f(\sigma, \sigma^{-1}) \notin P W_{\mathrm{id}}$ so by Lemma 3.2 we know $W_{\mathrm{id}} \subseteq S_{Q^{\sigma^{-1}}}$. Hence the group $H(P)$ acts on T. Moreover if

Q_1 and Q_2 are in T then by the theorem there is an element σ in G such that $f(\sigma^{-1}, \sigma) \notin Q_1 W_{\mathrm{id}}$ and $Q_2 = Q_1^\sigma$. Because P is the unique prime of height i contained in Q_1 and the unique prime of height i contained in Q_2 it follows that $P^\sigma = P$, that is $\sigma \in D(P)$. But $f(\sigma, \sigma^{-1}) = f^\sigma(\sigma^{-1}, \sigma) \notin Q_1^\sigma W_{\mathrm{id}} = Q_2 W_{\mathrm{id}}$, so in particular $f(\sigma, \sigma^{-1}) \notin P W_{\mathrm{id}}$. Hence $\sigma \in H(P)$. The last statement is immediate.

(b) By Theorem 3.1 there is an element $\sigma \in G$ such that $f(\sigma^{-1}, \sigma) \notin P W_{\mathrm{id}}$ and $P' = P^\sigma$. By Lemma 3.2 it follows immediately that every element of T^σ belongs to B, so $T^\sigma \subseteq T'$. The opposite inclusion follows by considering $f(\sigma, \sigma^{-1}) = f^\sigma(\sigma^{-1}, \sigma)$. □

Corollary 3.6. (a) *If P is a prime of height $i < n$ that belongs to B, and Q is a prime of height $i+1$ that belongs to B and contains P, then there is a set of right coset representatives of $D(Q) \cap H(P)$ in $H(P)$ such that for each representative g, $f(g, g^{-1}) \notin Q$.*

(b) *Let P be a prime ideal of height $i < n$ that belongs to B. Let $H = H(P)$. The ring $W_{\mathrm{id}} S^P \cap K^H$ is a valuation ring of rank i. Moreover if ρ is the prime ideal in S^H of height i such that $W_{\mathrm{id}} S^P \cap K^H = (S^H)_\rho$, then the prime ideals in S of height $i+1$ that belong to B and contain P are precisely those that lie over ρ.*

Proof. (a) This is an easy consequence of part (a) of Corollary 3.5.

(b) This is also a consequence of part (a) of Corollary 3.5: In the notation of that corollary, because H acts transitively on T, it follows that all the prime ideals in T lie over the same prime ideal in S^H. If we let ρ denote that prime ideal in S^H, then T consists of precisely the primes of S lying over ρ. But $W_\sigma S^P = \bigcap_{Q \in T} S_Q$ and so $W_\sigma S^P \cap K^H = (S^H)_\rho$. □

The result of Corollary 3.6 and [H, Corollary 3.11] lead one to suspect that there should be a Dubrovin valuation ring "involved" with B whose center is $W_{\mathrm{id}} S^P \cap K^H$. We are now headed for such a result. Let P be a prime of height $i < n$ belonging to B. Let $D = D(P)$ and let $H = H(P)$. Let $B' = \sum_{\sigma \in D} (W_\sigma S^P) x_\sigma$ and let $B'' = \sum_{\sigma \in H} (W_\sigma S^P) x_\sigma$. Similarly if M is a maximal ideal of S belonging to B and $E = D(M)$, let $C = \sum_{\sigma \in E} (W_\sigma S_M) x_\sigma$.

Proposition 3.7. *The rings B', B'', and C are Dubrovin.*

Proof. We proceed by induction on n, the rank of B. If the rank of B is one, then the rank of P must be zero, so $B' = B'' = B$. The fact that C is Dubrovin is the content of [H, Corollary 3.11]. Hence we may assume the rank of B is greater than one.

We begin with B'. If P is a prime of rank $i < n-1$ then for all $\sigma \in G$ $W_\sigma S^P = W_\sigma V_{n-1} S^P$ and so the result follows by induction applied to B_{n-1}. Hence we may assume P is a prime of rank $n-1$. It follows that

$$C' = B' V_{n-1} = \sum_{\sigma \in D} (W_\sigma V_{n-1} S^P) x_\sigma = \sum_{\sigma \in D} (W_\sigma S_P) x_\sigma$$

is Dubrovin by induction. Moreover we *claim* $J(C') \subseteq B'$: We know by Theorem 3.1 that

$$J(C') = \sum_{\sigma \in H} (P S_P) x_\sigma + \sum_{\substack{\sigma \in D \\ \sigma \notin H}} (W_\sigma S_P) x_\sigma.$$

But if $\sigma \notin H$ then $W_\sigma S^P = W_\sigma S_P$ by Lemma 3.2 and if $\sigma \in H$ then $PS_P \subseteq MS_M$ for every maximal ideal of S that contains P. Hence if $\sigma \in H$, then $PS_P \subseteq S^P \subseteq W_\sigma S^P$. This proves the claim.

Hence by [D_2, §1, Proposition 2] it suffices to show $\widetilde{B}' = B'/J(C')$ is a valuation ring. To do this we first consider the valuation rings $\widetilde{B} = B/J(B_{n-1}) \subseteq \overline{B}_{n-1} = B_{n-1}/J(B_{n-1})$. Note that by Theorem 3.1, $J(B_{n-1})_{\mathrm{id}} = J(W_{\mathrm{id}}V_{n-1})$. Hence

$$\begin{aligned}\overline{B}_{n-1} = \overline{W_{\mathrm{id}}V_{n-1}} &= W_{\mathrm{id}}V_{n-1}/J(W_{\mathrm{id}}V_{n-1}) \\ &= \bigoplus \sum_{i=1}^{k} (W_{\mathrm{id}}V_{n-1})/(P_i W_{\mathrm{id}}V_{n-1}),\end{aligned}$$

where $P = P_1, P_2, \dots, P_k$ are the primes of height $n-1$ that belong to B. Let e be the minimal idempotent in $\overline{W_{\mathrm{id}}V_{n-1}}$ corresponding to

$$(W_{\mathrm{id}}V_{n-1})/(PW_{\mathrm{id}}V_{n-1}).$$

Next observe that

$$J(W_{\mathrm{id}}V_{n-1}) = \bigcap_{i=1}^{k} P_i W_{\mathrm{id}}V_{n-1} \subseteq \bigcap_{i=1}^{k} P_i S_{P_i} \subseteq \bigcap MS_M$$

where this last intersection is over all the maximal ideals that belong to B. Hence $J(W_{\mathrm{id}}V_{n-1}) \subseteq J(W_{\mathrm{id}})$. Moreover

$$W_{\mathrm{id}}/J(W_{\mathrm{id}}V_{n-1}) = \bigoplus \sum_{i=1}^{k} W_{\mathrm{id}}/P_i W_{\mathrm{id}}$$

and for each i, $W_{\mathrm{id}}/P_i W_{\mathrm{id}}$ is a Dedekind domain with field of fractions

$$(W_{\mathrm{id}}V_{n-1})/(P_i W_{\mathrm{id}}V_{n-1}).$$

In particular note that $e \in W_{\mathrm{id}}/P_i W_{\mathrm{id}} \subseteq \widetilde{B}$. Because $\widetilde{B}$ is a valuation ring in the simple algebra $\overline{B}_{n-1}$, it follows from [D_1, §1, Theorem 7] that $e\widetilde{B}e$ is a valuation ring in the simple algebra $e\overline{B}_{n-1}e$.

We want to compute $e\widetilde{B}e$ and $e\overline{B}_{n-1}e$. To simplify notation let $T_\sigma = W_\sigma V_{n-1}$ and let $I_\sigma = J(B_{n-1})_\sigma$, the σ-component of the radical of B_{n-1}. We have

$$e\overline{B}_{n-1}e = \sum_{\sigma \in G} e(T_\sigma/I_\sigma)\sigma(e)x_\sigma.$$

If $\sigma \notin D$ then $\sigma(e) \neq e$ and so $\sigma(e)e = 0$. Hence

$$e\overline{B}_{n-1}e = \sum_{\sigma \in D} e(T_\sigma/I_\sigma)\sigma(e)x_\sigma.$$

Moreover we *claim* if $\sigma \notin H$ then $e(T_\sigma/I_\sigma) = 0$: If $I_\sigma = T_\sigma$ then the claim is certainly true. If $I_\sigma \neq T_\sigma$ then by the description of I_σ given in the proof of part (a) of Theorem 3.1 we know that $I_\sigma = \bigcap QT_\sigma$ where the intersection is over those prime ideals Q of height $n-1$ that belong to B and for which $f(\sigma, \sigma^{-1}) \notin QT_\sigma$. But $f(\sigma, \sigma^{-1}) \in PT_\sigma$ and so P is not among those primes. The claim follows. Hence $e\overline{B}_{n-1}e = \sum_{\sigma \in H} e(T_\sigma/I_\sigma)x_\sigma$. It follows that $e\widetilde{B}e = \sum_{\sigma \in H} e(W_\sigma/I_\sigma)x_\sigma$.

We want to show that $\widetilde{B}' \cong e\widetilde{B}e$. If so then $\widetilde{B}'$ is a valuation ring, as desired. We have $\widetilde{B}' = \sum_{\sigma\in H}(W_\sigma S^P/PS_P)x_\sigma$. If $\sigma \in H$ then we know by Proposition 3.3 that $W_\sigma \subseteq S_P$ and so PW_σ is a proper ideal of W_σ. Moreover $I_\sigma \subsetneq PW_\sigma$. Hence there is a canonical homomorphism ϕ from W_σ/I_σ to S_P/PS_P. It is easy to see that $e(W_\sigma/I_\sigma)$ is precisely the image of ϕ. On the other hand, $W_\sigma \subseteq W_\sigma S^P \subseteq S_P$ and $PS_P \subseteq W_\sigma S^P$. We have then the following commutative diagram of ring homomorphisms, where ρ is the canonical ring homomorphism from $W_\sigma S^P/PS_P$ to S_P/PS_P.

$$\begin{tikzcd}
W_\sigma/I_\sigma \arrow[r] \arrow[dr, "\phi"'] & W_\sigma S^P/PS_P \arrow[d, "\rho"] \\
& S_P/PS_P
\end{tikzcd}$$

To show that $\widetilde{B}'$ is isomorphic to $e\widetilde{B}e$, it suffices to show that in this diagram ϕ and ρ have the same image. Hence it is enough to show that the map from W_σ/I_σ to $W_\sigma S^P/PS_P$ is surjective. Thus we are reduced to showing that $W_\sigma + PS_P = W_\sigma S^P$. We have $W_\sigma \subseteq W_\sigma + PS_P \subseteq W_\sigma S^P$. Because $W_\sigma \subseteq S_P$ we see that PS_P is a W_σ-submodule of S_P and so $W_\sigma + PS_P$ is a ring. By the properties of Prüfer rings, we know that $W_\sigma + PS_P = \bigcap S_Q$ where the intersection is over those prime ideals Q of S such that $W_\sigma + PS_P \subseteq S_Q$. Moreover $W_\sigma S^P = \bigcap S_M$ where the intersection is over those maximal ideals M of S such that $M \supseteq P$ *and* $W_\sigma \subseteq S_M$. Hence it suffices to show that if Q is a prime ideal of S such that $PS_P \subseteq S_Q$, then $Q \supseteq P$. To see this statement we may assume $Q \neq P$. If $PS_P \subseteq S_Q$, then $QPS_P \subseteq QS_Q$. But $QS_P = S_P$ because Q is not contained in P. Hence $PS_P \subseteq QS_Q$, so $P = PS_P \cap S \subseteq QS_Q \cap S = Q$. Hence we have shown that $\widetilde{B}'$ is isomorphic to $e\widetilde{B}e$ and so $\widetilde{B}'$ is a valuation ring. As we have seen it follows that B' is a valuation ring.

It is now easy to see that B'' is also a valuation ring: We have

$$B'' \subseteq C'' = \sum_{\sigma\in H}(W_\sigma S^P V_{n-1})x_\sigma = \sum_{\sigma\in H}(W_\sigma S_P)x_\sigma = \sum_{\sigma\in H} S_P x_\sigma.$$

The algebra C'' is Azumaya, hence a valuation ring. Moreover $J(C'') = \sum_{\sigma\in H} PS_P x_\sigma$. To show B'' is a valuation ring it therefore suffices to show that the quotient ring $B''/J(C'')$ is a valuation ring. But clearly $B''/J(C'') = \widetilde{B}'$, which we just proved is a valuation ring.

Finally we need to show $C = \sum_{\sigma\in E}(W_\sigma S_M)x_\sigma$ is a valuation ring. Let P be the unique prime of height $n-1$ contained in M. Again we consider

$$C' = CV_{n-1} = \sum_{\sigma\in E}(W_\sigma S_M V_{n-1})x_\sigma = \sum_{\sigma\in E}(W_\sigma S_P)x_\sigma \subseteq \sum_{\sigma\in D}(W_\sigma S_P)x_\sigma.$$

By the induction hypothesis, $\sum_{\sigma\in D}(W_\sigma S_P)x_\sigma$ is a valuation ring and so by Proposition 2.19 the ring $C' = \sum_{\sigma\in E} W_\sigma S_P x_\sigma$ is a valuation ring. Hence we know

$$J(C') = \sum_{\substack{\sigma\in E\\ \sigma\in H}} PS_P x_\sigma + \sum_{\substack{\sigma\in E\\ \sigma\notin H}} W_\sigma S_P x_\sigma.$$

The quotient ring $C/J(C')$ is

$$\sum_{\substack{\sigma\in E\\ \sigma\in H}} (W_\sigma S_M/PS_P)x_\sigma = \sum_{\substack{\sigma\in E\\ \sigma\in H}} (S_M/PS_P)x_\sigma.$$

It suffices to show $C/J(C')$ is a valuation ring.

Recall that

$$\widetilde{B}' = \sum_{\sigma\in H}(W_\sigma S^P/PS_P)x_\sigma = \sum_{\sigma\in H}(W_{\mathrm{id}}S^P/PS_P)x_\sigma,$$

by Lemma 3.2. Let $M = M_1, M_2, \dots, M_m$ be the maximal ideals of S that contain P and belong to B. Then $W_{\mathrm{id}}S^P = \bigcap_i S_{M_i}$, and so $W_{\mathrm{id}}S^P/PS_P = \bigoplus\sum_i S_{M_i}/PS_{M_i}$. If we let f be the minimal idempotent in $W_{\mathrm{id}}S^P/PS_P$ corresponding to S_M/PS_M, then an argument similar to that given above shows that $f\widetilde{B}'f$ is isomorphic to $C/J(C')$. It follows that $C/J(C')$ and hence C are valuation rings. □

Corollary 3.8. *For every $i \le n$ if P is a prime of height i that belongs to B, then $H(P)$ is a normal subgroup of $D(P)\cap H(Q)$ where Q is the unique prime of height $i-1$ contained in P. Moreover, the quotient group $D(P)\cap H(Q)/H(P)$ is cyclic and there is an element $\sigma \in D(P)\cap H(Q)$ such that the following conditions hold:*

(i) *The coset $\sigma H(P)$ generates $D(P)\cap H(Q)/H(P)$ and $f(\sigma, \sigma^{-1}) \in PW_{\mathrm{id}} - (PW_{\mathrm{id}})^2$.*

(ii) *The partial ordering on $D(P)\cap H(Q)/H(P)$ is the chain $H(P) \le \sigma H(P) \le \sigma^2 H(P) \le \cdots \le \sigma^{m-1}H(P)$, where $m = |D(P)\cap H(Q)/H(P)|$.*

Proof. Let $D = D(P)$ and $H = H(P)$. The ideal $PW_{\mathrm{id}}V_i$ is a maximal ideal of $W_{\mathrm{id}}V_i$. By Proposition 3.7, $C = \sum_{\sigma\in D} S_P x_\sigma$ is a valuation ring. The result now follows from Theorem 2.14. □

We end this section with an example designed to display some of the various phenomena we have discussed. Let k be a field of characteristic not two and let s, t, x, y be indeterminates over k. Set $F = k(s,t)(x,y)$ and $K = F(\sqrt{1+x}, \sqrt{1+y}, \sqrt{1+2x})$. Let Σ_f be the F-algebra given by

$$\Sigma_f = \left(\frac{1+x, s}{F}\right)\otimes_F\left(\frac{1+y, t}{F}\right)\otimes_F\left(\frac{1+2x, y}{F}\right),$$

the tensor product of three quaternion algebras. Then $\Sigma_f = (K/F, G, f)$ where $G = \langle\sigma, \tau, \rho\rangle$ with

$$\begin{array}{lll}
\sigma(\sqrt{1+x}) = -\sqrt{1+x}, & \sigma(\sqrt{1+y}) = \sqrt{1+y}, & \sigma(\sqrt{1+2x}) = \sqrt{1+2x},\\
\tau(\sqrt{1+x}) = \sqrt{1+x}, & \tau(\sqrt{1+y}) = -\sqrt{1+y}, & \tau(\sqrt{1+2x}) = \sqrt{1+2x},\\
\rho(\sqrt{1+x}) = \sqrt{1+x}, & \rho(\sqrt{1+y}) = \sqrt{1}, & \rho(\sqrt{1+2x}) = -\sqrt{1+2x}+y
\end{array}$$

and the cocycle is given by Table 1.

Let V_1 be the y-adic valuation ring of F and $V_2 \subseteq V_1$ the (x,y)-adic valuation ring of F. Let W_i be the integral closure of V_i in K, for $i = 1, 2$. An easy calculation shows $W_i = V_i[\sqrt{1+x}, \sqrt{1+y}, \sqrt{1+2x}]$. Write

TABLE 1

f	1	σ	τ	ρ	$\sigma\tau$	$\sigma\rho$	$\tau\rho$	$\sigma\tau\rho$
1	1	1	1	1	1	1	1	1
σ	1	s	1	1	s	s	1	s
τ	1	1	t	1	t	1	t	t
ρ	1	1	1	y	1	y	y	y
$\sigma\tau$	1	s	t	1	st	s	t	st
$\sigma\rho$	1	s	1	y	s	sy	y	sy
$\tau\rho$	1	1	t	y	t	y	ty	ty
$\sigma\tau\rho$	1	s	t	y	st	sy	ty	sty

$\Sigma_f = \sum_{\gamma\in G} Kx_\gamma$ be the crossed product algebra determined by f. We want to construct for $i = 1, 2$ valuation rings B_i in Σ_f with $B_i \cap F = V_i$. Let

$$\Sigma_1 = \left(\frac{1+x\,,\,s}{F}\right); \quad \Sigma_2 = \left(\frac{1+y\,,\,t}{F}\right); \quad \Sigma_3 = \left(\frac{1+2x\,,\,y}{F}\right).$$

For each j let $K_j = \Sigma_j \cap K$ and let $S = W_2$.

Because $1+x$, s, $1+y$, $t \in V_1^\times$,

$$\left(\frac{1+x\,,\,s}{V_1}\right) \otimes_{V_1} \left(\frac{1+y\,,\,t}{V_1}\right) \subseteq \Sigma_1 \otimes_F \Sigma_2$$

is Azumaya over V_1. Now $\overline{V}_1 = k(s\,,\,t)(x)$. Because $1+2x \in \overline{V}_1 - (\overline{V}_1)^2$ and $v(y) \notin 2\Gamma_{v_1}$, where v is the y-adic valuation of F, it follows by [JW, Example 4.3] that there is an invariant valuation ring A in Σ_3 extending V_1. It can be seen that $A = (W_1 \cap K_3) + (W_1 \cap K_3)x_\rho$. Therefore by [W, Proposition 3.3] we may take B_1 to be

$$B_1 = \left(\frac{1+x\,,\,s}{V_1}\right) \otimes_{V_1} \left(\frac{1+y\,,\,t}{V_1}\right) \otimes_{V_1} A.$$

It follows that $B_1 = \sum_{\gamma\in G} W_1 x_\gamma$. Also by [W, Proposition 3.3]

$$J(B_1) = \left(\frac{1+x\,,\,s}{V_1}\right) \otimes_{V_1} \left(\frac{1+y\,,\,t}{V_1}\right) \otimes_{V_1} J(A),$$

where

$$J(A) = (J(W_1) \cap K_3) + (W_1 \cap K_3)x_\rho.$$

Hence

$$J(B_1) = \sum_{\gamma\in\langle\sigma\,,\,\tau\rangle} J(W_1)x_\gamma + \sum_{\gamma\notin\langle\sigma\,,\,\tau\rangle} W_1 x_\gamma.$$

For B_2 again we see $((1+x\,,\,s)/V_2) \otimes_{V_2} ((1+y\,,\,t)/V_2)$ is Azumaya. For Σ_3 we see that because $A/J(A) = \overline{V}_1(\sqrt{1+2x})$ and $V_2/J(V_1)$ extends in two ways to $A/J(A)$, it follows that if C is the preimage in A of one of these two extensions, then C is a (noninvariant) total valuation ring in Σ_3. Hence by [W, Proposition 3.3]

$$B_2 = \left(\frac{1+x\,,\,s}{V_2}\right) \otimes_{V_2} \left(\frac{1+y\,,\,t}{V_2}\right) \otimes_{V_2} C$$

is a valuation ring of Σ_f with $B_2 \cap F = V_2$ and $B_2 \subseteq B_1$. One can check that $C = (S_M \cap K_3) + (S_P \cap K_3)x_\rho$ for some maximal ideal $M \supseteq P$ of S.

Because V_1 extends uniquely to $F(\sqrt{1+x}\,,\,\sqrt{1+2x})$ but not to $F(\sqrt{1+y})$ we have $D(P) = \langle \sigma\,,\,\rho \rangle$ and $H(P) = \langle \sigma \rangle$. Because V_2 is completely split in K, $D(M) = \langle 1 \rangle$ and so $H(M) = \langle 1 \rangle$. From the description of B_1 we see that

$$B_2 = \sum_{\gamma \in \langle \sigma\,,\,\tau \rangle} W_{\mathrm{id}} x_\gamma + \sum_{\gamma \notin \langle \sigma\,,\,\tau \rangle} W_\rho x_\gamma .$$

To determine W_{id} we see that because $D(M) = \langle 1 \rangle$ and $G(K/K_3) = \langle \sigma\,,\,\tau \rangle$, $S_M \cap K_3 \subseteq S_M \cap S_{M^\sigma} \cap S_{M^\tau} \cap S_{M^{\sigma\tau}}$ and so $W_{\mathrm{id}} = S_M \cap S_{M^\sigma} \cap S_{M^\tau} \cap S_{M^{\sigma\tau}}$. Moreover because $J(B_1) \subseteq B_2$, the description of $J(B_1)$ forces $W_1 \subseteq W_\rho$ and so $W_\rho = W_1$. Thus

$$B_2 = \sum_{\gamma \in \langle \sigma\,,\,\tau \rangle} (S_M \cap S_{M^\sigma} \cap S_{M^\tau} \cap S_{M^{\sigma\tau}}) x_\gamma + \sum_{\gamma \notin \langle \sigma\,,\,\tau \rangle} W_1 x_\gamma .$$

If $J(B_2) = \sum_{\gamma \in G} J_\gamma x_\gamma$, we see that $J_\gamma = W_1$ for $\gamma \notin \langle \sigma\,,\,\tau \rangle$ because $J(B_1) \subseteq J(B_2)$. If $\gamma \in \langle \sigma\,,\,\tau \rangle$ then because $J_\gamma = \{a \in K \mid af(\gamma\,,\,\gamma^{-1}) \in J(W_{\mathrm{id}})\}$ and $f(\gamma\,,\,\gamma^{-1}) \in V_2^\times$, we have

$$J_\gamma = J(W_{\mathrm{id}}) = MS_M \cap M^\sigma S_{M^\sigma} \cap M^\tau S_{M^\tau} \cap M^{\sigma\tau} S_{M^{\sigma\tau}}$$

and so

$$J(B_2) = \sum_{\gamma \in \langle \sigma\,,\,\tau \rangle} J(W_{\mathrm{id}}) x_\gamma + \sum_{\gamma \notin \langle \sigma\,,\,\tau \rangle} W_1 x_\gamma .$$

If $B_1'' = \sum_{\gamma \in H(P)} W_1 x_\gamma$ and $B_2'' = \sum_{\gamma \in H(P)} S^P x_\gamma$, where as usual $S^P = \bigcap_{M \supseteq P} S_M$, then B_i'' is a valuation ring over V_i and $\overline{B_1}$ is Brauer equivalent to $\overline{B_1''} = \overline{S_P^{H(P)}} = \overline{V_1}(\sqrt{1+2x})$ while $\overline{B_2}$ is Brauer equivalent to $\overline{B_2''} = \overline{S_M^{H(M)}} = \overline{V_2}$.

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