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INDUCED CONNECTIONS ON S^1-BUNDLES OVER RIEMANNIAN MANIFOLDS

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ABSTRACT. Let (V, g) and (W, h) be Riemannian manifolds and consider two S^1-bundles $X \to V$ and $Y \to W$ with connections Γ on X and ∇ on Y respectively. We study maps $X \to Y$ which induce both connections and metrics. Our study relies on Nash's implicit function theorem for infinitesimally invertible differential operators. We show, for the case when $Y \to W = \mathbf{C}P^q$ is the Hopf bundle, that if $2q \geq n(n+1)/2 + 3n$ then there exists a nonempty open subset in the space of C^∞-pairs (g, Γ) on V which can be induced from (h, ∇) on $\mathbf{C}P^Q$.

0. INTRODUCTION

Let (V, g) and (W, h) be Riemannian C^∞-manifolds and consider two C^∞-smooth S^1-bundles $X \to V$ and $Y \to W$ with C^∞-connections Γ on X and ∇ on Y respectively. We look for a map $f\colon V \to W$ such that

(a) the induced metric $f^*(h)$ equals g;

(b) the induced bundle $f^*(Y)$ over V is isomorphic to X. Moreover, we want an isomorphism $f^*(Y) \to X$ which carries the connection in $f^*(Y)$ induced from ∇ to Γ.

Equivalently, this can be expressed by saying that we look for a map $\tilde{f}\colon X \to Y$ such that

$(\mathrm{a})_1$ the map $f\colon V \to W$ underlying $\tilde{f}$ is isometric, $f^*(h) = g$ as earlier.

$(\mathrm{b})_1$ $\tilde{f}$ is an S^1-bundle map. That is, each fiber of X goes to a fiber of Y and this map of the fibers, say
$$\tilde{f}_v\colon S^1_v \to S^1_w, \qquad v \in V,\ w \in W,$$
is S^1-equivariant.

$(\mathrm{b})_2$ $\tilde{f}$ is a connection preserving map, $\tilde{f}^*(\nabla) = \Gamma$.

It is clear that the map f underlying $\tilde{f}$ satisfies (a) and (b). On the other hand, if an f satisfies (a) and (b) then it can be lifted to an $\tilde{f}$ satisfying $(\mathrm{b})_{1-2}$. In fact, the isomorphism $X \to f^*(Y)$ implies by (b) composed with the tautological map $f^*(Y) \to Y$ is our $\tilde{f}$. Notice, that the lifted map $\tilde{f}$ is uniquely determined by f up to the S^1-action on X as the automorphism group of (X, Γ) over V is given by this action.

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0.1. **Yang-Mills motivation.** In this paper we study maps between arbitrary S^1-bundles which induce both connections and metrics. The problem naturally generalizes to higher dimensional bundles and in fact, our subject was motived by Yang-Mills equations (which involve both metric and connection) where one could hope to get some insight by inducing both from some universal object. For example, the $SU(2)$-instantons over S^4 can be induced by certain maps $f: S^4 \to \mathbf{H}P^q$ (see [AT]).

Unfortunately, our general method of inducing (g, Γ) does not apply in the Yang-Mills case. In fact, our method is based on Nash's *implicit function theorem* for *infinitesimally invertible* differential operators (see §2) which requires certain genericity assumptions on the partial differential equations expressing the inducing relations $f^*(h) = g$, $\tilde{f}^* f(\nabla) = \Gamma$. But in the Yang-Mills case these equations become very degenerate and the corresponding differential operator is *not infinitesimally invertible.* This degeneration phenomenon can be already seen for S^1-bundles and it is discussed further in §0.5.

0.2. **Connection and metric inducing maps** $V \to \mathbf{C}P^q$. An especially interesting case of our problem for S^1-bundles is where the manifold W is the complex projective space and $Y = S^{2q+1} \to \mathbf{C}P^q = W$ is the Hopf bundle. The connection ∇ here is defined by the (horizontal hyperplane) subbundle $\Sigma \subset T(S^{2q+1})$ consisting of the vectors normal to the Hopf circles. The relevant metric h on $\mathbf{C}P^q$ is the one which corresponds to twice the usual spherical metric $\tilde{h}$ on (the unit sphere) S^{2q+1} restricted to Σ. That is, the differential of the Hopf map is an isometry:

$$(\Sigma, 2\tilde{h}|\Sigma) \to (T(\mathbf{C}P^q), h).$$

Notice, that both h and ∇ are invariant under the obvious actions of the group $U(q+1)$ on $\mathbf{C}P^q$ and S^{2q+1}. In fact, the connection ∇ is uniquely characterized by this invariance and an invariant h is unique up to a scalar multiple. Denote by $\omega = \omega_\nabla$ the curvature form of ∇ and consider the $\mathbf{C}$-valued bilinear form $\mathbf{h} = h + \sqrt{-1}\omega_\nabla$ on $T(W = \mathbf{C}P^q)$. One can show that this form $\mathbf{h}$ is *hermitian* for the standard complex structure on $\mathbf{C}P^q$. In fact, one can easily identify $\mathbf{h}$ with the classical Fubini-Study (hermitian) metric on $\mathbf{C}P^q$. Now, for an arbitrary manifold V with a metric g and connection Γ on an S^1-bundle $X \to V$, one can take the curvature ω_Γ and then the $\mathbf{C}$-valued form $\mathbf{g} = g + \sqrt{-1}\omega_\Gamma$ on $T(V)$. As V has no distinguished complex structure one cannot say that $\mathbf{g}$ is hermitian. But, the form $\mathbf{g}$ uniquely extends to a hermitian form on the complexified bundle $\mathbf{C}T(V)$ (this follows by elementary linear algebra) and this extended form is still denoted by $\mathbf{g}$.

Now, if $\tilde{f}: X \to Y$ induces the connection Γ from ∇, then f induces the curvature form ω_Γ on V

$$\omega_\Gamma = f^*(\omega_\nabla) \tag{0.2.1}$$

and, if in addition f is (g, h)-isometric then f is also isometric for the corresponding hermitian forms.

In the case $W = \mathbf{C}P^q$ this isometry is seen in the *complexified differential* of f, denoted $d_{\mathbf{C}}f: \mathbf{C}T(V(\to T(W)$, as it is expressed by the relation

$$(d_{\mathbf{C}}f)^*(\mathbf{h}) = \mathbf{g}. \tag{0.2.2}$$

We see from this that our inducing problem (see $(\mathrm{a})_1$ and $(\mathrm{b})_2$) is closely related to the isometric immersion problem for hermitian metrics (compare with §0.6).

0.2.A. *Remarks.* (i) For general W one has a similar relation with $\mathbf{h}$ defined on $\mathbf{C}T(W)$ rather than on $T(W)$.

(ii) The interest in $W = \mathbf{C}P^q$ stems from the fact that the Hopf bundle $Y \to \mathbf{C}P^q$ is universal: every S^1-bundle over X with $\dim_{\mathbf{C}} X < q$ can be induced from Y. Moreover, one can choose the bundle inducing morphism $f: X \to Y$ such that it induces the connection as well provided $q \geq 2 \dim X$ (see 3.4.2 in [Gro] and also see [N-R]).

We shall show in this paper that one can sometimes induce the Riemannian metric as well and thus produce an isometric map for the hermitian forms $\mathbf{h}$ and $\mathbf{g}$. In fact, our Theorem 0.4.A says that for $2q \geq n(n+1)/2 + 3n$ there exists a *nonempty open* subset in the space of C^∞-pairs (g, Γ) on V which can be induced from (h, ∇) on $\mathbf{C}P^q$. For the purpose of the proof, it is convenient to reformulate the statement in the framework of the vector bundle terminology which we introduce in the next section.

0.3. **Some definitions and notations.** Let $S^2(V)$ denote the symmetric square of the cotangent bundle of V. Thus the Riemannian metrics g on V are C^∞-smooth sections $g: V \to S^2(V)$. Connections on X are viewed as C^∞-sections of the fibration $E \to V$ whose fiber $E_v \subset E$ for $v \in V$ can be described as follows:

Denote by X_v^1 the space of 1-jets (or differentials) of (germs of) sections $V \to X$ at $x \in V$. Namely, X_v^1 consists of linear maps $T_v(V) \to T(X)$ which project to the identity $\mathrm{Id}: T_v(V) \hookleftarrow$ by the differential (of the projection map) of the fibration $X \to V$. The group S^1 naturally acts on this X_v^1 and the fiber E_v, $v \in V$, equals X_v^1/S^1.

Our basic object over V is the pair $G = (g, \Gamma)$ which is also denoted by $G = g \oplus \Gamma$. In fact, one should think of G as the section of the Whitney sum $S^2(V) \oplus E$, which is the fibration over V with the fibers $(S^2(V) \oplus E)_v = S_v^2(V) \oplus E_v$, $v \in V$. Also, we denote by $H = (h, \nabla)$ (or $H = h \oplus \nabla$) the pair (metric, connection) defined on the manifold W. Throughout the sequel, we shall express the inducing relations $(\mathrm{a})_1$ and $(\mathrm{b})_2$ by

$$\tilde{f}^*(H) + G \tag{0.3.1}$$

and sometimes by

$$f^*(H) = G. \tag{0.3.2}$$

Equation (0.3.1) (or (0.3.2)) is, by definition, the system of equations:

$$f^*(h) = g, \qquad \tilde{f}^*(\nabla) = \Gamma. \tag{0.3.3}$$

The "operation" $H \to G = \tilde{f}^*(H)$ can be thought of as a map (or operator) relating to $\tilde{f}$ the induced structure. We denote this by

$$\tilde{f} \mapsto G = \mathscr{D}_H(\tilde{f}) \overset{\text{def}}{=} \tilde{f}^*(H)$$

and interpret it as an operator from $\{\tilde{f}\}$ to $\{G\}$ where $\{\tilde{f}\}$ is the space of C^∞-bundle morphisms $\tilde{f}: X \to Y$ and $\{G\}$ is the space of C^∞ pairs (g, Γ) (see the discussion in §1.1).

0.4. **The statement of the main result.** In the theorem below we refer to the respective *fine* (also called Whitney) C^∞-topologies in the function spaces $\{\tilde{f}\}$

and $\{G\}$. (We recall that if the manifold V is compact the fine C^∞-topology coincides with the ordinary C^∞-topology.) Now assume that $W = \mathbf{C}P^q$. Our main result can be stated as

0.4.A. **Theorem.** *If $2q \geq n(n+1)/2 + 3n$, $n = \dim V$ then there exists a C^∞-vector bundle map $\tilde{f}_0 \colon X \to Y$ and a C^∞-neighborhood $\tilde{U}$ of the induced structure $\tilde{f}_0^*(H) \in \{G\}$ such that every $G \in \tilde{U}$ is induced by some C^∞-map $\tilde{f} \colon X \to Y$.*

0.4.B. *Remarks.* (i) An alternative formulation of Theorem 0.4.A is as follows: if $2q \geq n(n+1)/2 + 3n$, *then there exists a nonempty open set $\tilde{U} \subset \{G\}$ such that $\tilde{U} \subset \mathscr{D}_H\{\tilde{f}\}$*. This says essentially that there exists on V a "substantial amount" of structures $G \subset \{G\}$ which are induced from $H = (h, \nabla)$ on $W = \mathbf{C}P^q$ by some $\tilde{f} \colon X \to Y$. Equivalently, the differential system (0.3.1) has a solution for "many" right-hand sides. (Here the words "substantial" and "many" refer to the fact that a nonempty open subset in a Hausdorff space contains quite a few points.) Notice in this regard that if $2q < n(n+1)/2+n-1$ (where $n = \dim V$, $2q = \dim W$) namely, if the number of equations exceeds the number of unknown functions, then the image $\mathscr{D}_H\{\tilde{f}\} \subset \{G\}$ contains *no* nonempty open subset.

(ii) For the intermediate range of $\dim W$, i.e., for $n(n+1)/2 + 3n > 2q \geq n(n+1)/2+n-1$ the situation is more complicated and not completely clear. By taking into account Gromov's results on the infinitesimal invertibility of *generic underdetermined* nonlinear differential operators (see 2.3.8 in [Gro]) it seems reasonable to conjecture that Theorem 0.4.A is true for $2q \geq n(n+1)/2 + n$.

(iii) Our statement 0.4.A can be strengthened by replacing the C^∞-topology with the C^3-topology in the space $\{G\}$ of C^∞-pairs (g, Γ) on V.

0.5. **Noninducible structures.** The following question naturally arises: Is it possible to prove that the system (0.3.1) is solvable for all $G \subset \{G\}$? Indeed, the answer is yes if we replace (0.3.1) by one of the equations in (0.3.3) because these are separately solvable for q sufficiently large (see [Na, Gro, D'A]). However, the system (0.3.1) is not always solvable even for large q. One can see this by looking at standard examples where not all pairs (metric, connection) are inducible unless certain restrictions on (g, Γ) are added. To give an illustration, we may take $W = \mathbf{C}P^q$ with the standard metric h and connection ∇ already described in §0.2. As before, we denote by $\omega = \omega_\nabla$ the curvature of ∇. Then, by Wirtinger's inequality this 2-form satisfies

$$\|\omega\|_h \leq 1 \tag{0.5.1}$$

where $\|\omega\|_h$ denotes the following:

$$\|\omega(\xi, \eta)\|_h \overset{\text{def}}{=} \sup_{\xi, \eta} \|\omega(\xi, \eta)\|_h$$

where ξ, η are orthogonal unit vectors in $T_w(W)$, $w \in W$.

Notice, that the inequality (0.5.1) is equivalent to the positive semidefiniteness of the hermitian form $\mathbf{h} = h + \sqrt{-1}\omega_\nabla$ (see §0.2). This can be easily checked by using elementary linear algebra.

Now, let g and Γ be the metric and the connection on the manifold V and let ω_Γ be the curvature of Γ. Clearly, if $G = (g, \Gamma)$ is induced from

$H = (h, \nabla)$ then (0.5.1) implies that

$$\|\omega_\Gamma\|_g \le 1 \tag{0.5.1$'$}$$

also holds true. It follows that if the condition (0.5.1$'$) is violated, then G cannot be induced from H.

To see an example, take $\mathbf{C}P^m$ for V with the connection $\Gamma = \nabla$ and $g = \lambda h$ for the standard h on $\mathbf{C}P^m$ corresponding to the Fubini-Study metric and for some $0 < \lambda < 1$. Here, if we take the vectors $\xi, \eta \in T_v(V = \mathbf{C}P^m)$, such that $\eta = \sqrt{-1}\xi$, $\|\xi\|_h = 1$, then $\|\eta\|_h$ also equals 1 and η is h-orthogonal to ξ. On the other hand, clearly,

$$\omega_\Gamma(\xi, \eta) = 1.$$

Thus the pair $\lambda^{-1/2}\xi$, $\lambda^{-1/2}\eta$ is g-orthonormal; yet

$$\omega(\lambda^{-1/2}\xi, \lambda^{1/2}\eta) = \lambda^{-1} > 1.$$

0.5.A. A particular situation of interest is when the inequality (0.5.1) is not strict (i.e., when we have $\|\omega\|_h = 1$). In this case, the pair (g, Γ) is of the same nature of (h, ∇). Assume, for example, that the manifolds V and W are both hermitian (e.g., Kähler) for the metrics g and h. Then the solutions $f: V \to W$ of (0.3.2) are isometries for these metrics. Moreover, it is not difficult to show that these isometric maps $V \to W$ are *holomorphic* (see Lemma 3.1).

Now, the isometric holomorphic embeddings of Kähler manifolds are "rare" maps: namely, by the standard jet counting argument (see §3.2) *a generic Kähler metric on a complex manifold* V, $\dim_{\mathbf{C}} V \ge 1$, *cannot be induced by a holomorphic map into a fixed Kähler manifold* W *of any dimension.* In fact, thanks to a paper by Calabi (see [Ca]) one has a precise description of the Kähler metrics on V inducible from $W = \mathbf{C}P^q$.

0.6. *Remark.* We have indicated in §0.2 a close relation between the two structure inducing problems: one for the pairs (Riemannian metric, S^1-connection) and the second for hermitian metrics. Namely, whenever (g, Γ) is induced from (h, ∇) the hermitian form $\mathbf{g} = g + \sqrt{-1}\omega_\Gamma$ on $\mathbf{C}T(V)$ is induced from $\mathbf{h} = h + \sqrt{-1}\omega_\nabla$. Conversely, if a map $V \to W$ induces $\mathbf{g}$ from $\mathbf{h}$ then it is, of course, (g, h)-isometric and moreover the induced connection on $f^*(Y)$ is isomorphic to that of X. In fact, if two S^1-bundles over V with some connections say, (X, Γ) and (Y^*, ∇^*) have equal curvature forms $\omega_\Gamma = \omega_{\nabla^*}$ then there exists an isomorphism $X \to Y^*$ over V sending Γ to ∇^*. Yet the two inducing problems are not quite equivalent, because not every closed 2-form ω comes as the curvature of some connection over V. In fact, ω appears as the curvature ω_Γ of a connection Γ on some vector bundle X over V, if and only if the cohomology class $[\omega/2\pi] \in H^2(V; \mathbf{R})$ is *integral*. This means that the integral of ω over every closed oriented surface in V is an integer multiple of 2π. (The nontrivial part of the above claim, "if", is due to Kostant: see [Ko].)

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1. A CRITERION FOR THE INFINITESIMAL INVERTIBILITY OF $\mathscr{D}_H$

1.1. Let us now explain why $\mathscr{D}_H$ is an actual differential operator between spaces of sections of certain fibrations over V.

We already mentioned (see §0.3) that $G = (g, \Gamma)$ is a section of some bundle, namely it is a section of $S^2(V) \oplus E \to V$.

We also view the bundle morphisms $X \to Y$ as sections of a bundle over V. This bundle, called $\widetilde{\mathscr{F}} \to V$ is the bundle associated with the principle S^1-bundle $X \to V$ with the fiber Y. This makes sense as Y (being a principal circle bundle itself) comes with an S^1-action.

To see the picture, observe that the fiber $\widetilde{\mathscr{F}_v}$ over $v \in V$ consists of all maps of the circle S_v^1 (that is the fiber $X_v \subset X$) into Y such that S_v^1 goes onto some circle $S_w^1 = Y_w$ by an S^1-equivariant map. Thus $\widetilde{\mathscr{F}_v}$ can be identified (noncanonically) with Y, since every map $S_v^1 \to Y$ is determined just by where a given point $s \in S_v^1$ goes (noncanonicity is due to the freedom in the choice of s). Notice, that our $\widetilde{\mathscr{F}}$ naturally fibers over the product $V \times W$ with the fiber $X_v \times Y_w/S^1$ canonically isomorphic to the space of S^1-equivariant maps $X_v \to Y_w$. Also, every $\tilde{f}: X \to Y$ by definition is given by $f: V \to W$ and a family of S^1-equivariant maps $\tilde{f}_v: X_v \to Y_{f(v)}$ which are C^∞-smooth in $v \in V$. Thus $\tilde{f}$ becomes a section $V \to \widetilde{\mathscr{F}}$ covering the graph of f.

Now, for the above bundles $\widetilde{\mathscr{F}}$ with $S^2(V) \oplus E$ we denote by $\{\tilde{f}\}$ and $\{G\}$ respectively the sets of C^∞-sections with the fine C^∞-topology and view our $\mathscr{D}_H$ as a map (operator) between these spaces of sections, $\mathscr{D}_H: \{\tilde{f}\} \to \{G\}$. In fact, this $\mathscr{D}_H$ is a first order nonlinear differential operator which can be described as follows.

First, since $S^2(V) \oplus E$ is the sum of two fibrations $\mathscr{D}_H$ naturally splits into the sum of two operators

$$\mathscr{D}_H = \mathscr{D}_h \oplus \mathscr{D}_\nabla$$

where

$$\mathscr{D}_h(\tilde{f}) = \mathscr{D}_h(f) = f^*(h) \tag{1.1.1}$$

and

$$\mathscr{D}_\nabla(\tilde{f}) = \tilde{f}^*(\nabla). \tag{1.1.1'}$$

From this we see that in order to understand the differential nature of our operator $\mathscr{D}_H$ we need to analyze $\mathscr{D}_h$ and $\mathscr{D}_\nabla$ separately.

1.1.A. *Remark.* The study of $\mathscr{D}_H$ (in particular, the solvability problem for the equation $D_H(\tilde{f}) = G$) *cannot* be reduced to separate problems for $\mathscr{D}_h$ and $\mathscr{D}_\nabla$ as they depend on the same argument $\tilde{f}$.

1.2. **The operators $\mathscr{D}_h$ and $\mathscr{D}_\nabla$.** Now, we fix local coordinates $u_1, \dots, u_n$, $n = \dim V$, around $v \in V$ and observe that every section $\tilde{f}: V \to \widetilde{\mathscr{F}}$ is locally given by maps $f: V \to W$ and $\varphi: V \to S^1$.

Our first operator $\mathscr{D}_h$ only depends on f and in a neighborhood of $v \in V$ it can be expressed by

$$\mathscr{D}_h(f(u_1, \dots, u_n)) = \{g_{ij} = \langle \partial_i f, \partial_j f \rangle\}, \qquad i, j = 1, \dots, n, \tag{1.2.1}$$

where $\partial_i f = Df(\partial/\partial u_i)$, $i = 1, \dots, n$, denote the images of the vector fields $\partial/\partial u_i$ on V under the differential of f, $\langle\ ,\ \rangle$ denotes the scalar product with respect to the metric h on W and where g_{ij} are the components of a quadratic differential form on V $G = \sum_{i,j=1}^n g_{ij} du_i du_j$ in our local coordinates.

To describe our second operator $\mathscr{D}_\nabla$, we also act locally and fix some sections α and β of the fibrations $X \to V$ and $Y \to W$. Then connections on X and Y become (ordinary) 1-forms on V and W respectively.

Furthermore, those sections allow our interpretation of $\tilde{f}$ as pairs (f, φ) where $\varphi: V \to S^1$ is thought of as the "rotation" of $X \to V$ which moves the given section α of X to the $\tilde{f}$ pull-back of the section $\beta: W \to Y$.

Now, once the section β is given, the connection ∇ on Y can be represented by a 1-form on Y, say ∇'_β, for $\nabla'_\beta = \nabla - \nabla_\beta$. That is, ∇'_β is the difference of two connections where ∇_β denotes the trivial connection for which the section β is parallel. It follows (see below) that the inducing connection relation may be written as

$$\Gamma = \mathscr{D}_\nabla(\tilde{f}) \overset{\text{def}}{=} \tilde{f}^*(\nabla) = f^*(\nabla'_\beta) + d\varphi + \nabla_\alpha \tag{1.2.2}$$

where $f^*(\nabla'_\beta)$ is the induced 1-form on V, $d\varphi = \varphi^*(d\theta)$ for the cyclic parameter θ on S^1 and where ∇_α is the trivial connection on X associated to the section $\alpha: V \to X$. To prove (1.2.2) we first assume that $\tilde{f}$ sends α to β. Then $\varphi = \text{id}$ and $d\varphi = 0$. In this case, we have

$$\tilde{f}^*(\nabla_\beta) = \nabla_\alpha \quad (\text{as } \alpha \text{ goes to } \beta)$$

and

$$f^*(\nabla'_\beta) = \tilde{f}^*(\nabla) - \tilde{f}^*(\nabla_\beta) \quad \text{since } \nabla'_\beta = \nabla - \nabla_\beta .$$

Hence,

$$\tilde{f}^*(\nabla) = f^*(\nabla'_\beta) + \nabla_\alpha$$

which is exactly (1.2.2) for $\varphi = \text{id}$.

Now, a general $\tilde{f}$ is obtained from the special one (where $\varphi = \text{id}$) by composing it with φ thought of as a rotation (or better as a gauge transformation) of X. The effect of this on connections Γ on X is $\Gamma \mapsto \Gamma + d\varphi$, and thus the general (1.2.2) follows from the special one.

From formula (1.2.2) we see that the connection inducing operator $\mathscr{D}_\nabla(\tilde{f}) = \tilde{f}^*(\nabla)$ amounts to inducing 1-forms

$$f \mapsto f^*(\nabla'_\beta) \tag{1.2.3}$$

and taking differentials of maps $\varphi: V \to S^1$. The former is a differential operator of the same nature as $\mathscr{D}_h$ (since $\mathscr{D}_h$ induces symmetric 2-forms) and $d\varphi$ is (obviously) a first order differential operator as well. In a fixed system of local coordinates $u_1, \dots, u_n$ at $v \in V$, (1.2.2) is expressed by

$$\mathscr{D}_h((f, \varphi)(u_1, \dots, u_n)) = \{\Gamma_i = \nabla'_\beta(\partial_i f) + \partial\varphi/\partial u_i\}_{i=1,\dots,n}$$

where Γ_i denote the components of the 1-form on V corresponding (via α) to the connection Γ.

1.3. Linearization of the operators $\mathscr{D}_h$ and $\mathscr{D}_\nabla$. Our next objective is the construction of an infinitesimal inverse for both the operators $\mathscr{D}_h$ and $\mathscr{D}_\nabla$.

To do this, we have first need to define their *linearization.* We shall follow here the same approach and shall use the same terminology as in [Gro, §2.3.1], to where the reader is referred for the pertinent definitions and for a general discussion on infinitesimally invertible differential operators and their basic properties. The reader may also consult [Na, Gre, G-J, Ja, Ha] where similar techniques to those presented in [Gro] have been used for the isometric immersion problem.

1.3.A. The operator $\mathscr{D}_h$ sends the space of maps $V \to W$ to the space of quadratic differential forms on V and for a fixed $f: V \to W$ its linearization say, $L_h(\partial)$, is a linear operator assigning to each tangent vector field ∂ on $f(V) \subset W$ tangent to W a quadratic form g on V. Recall, that a vector field on $f(V)$ tangent to W is, by definition, a section of the induced vector bundle $f^*(T(W)) \to V$. Now, we take a smooth 1-parametric family of maps $f_t: V \to W$, $t \in [0, 1]$, such that $f_0 = f$ and $df_t/dt|_{t=0} = \partial$ for a given $\partial: V \to f^*(T(W))$. Then, by definition, the linearization of $\mathscr{D}_h$ acts on ∂ by

$$L_h(\partial) = \frac{d}{dt}(\mathscr{D}_h(f_t))|_{t=0}$$

or, for brevity,

$$L_h(\partial) = \frac{d}{dt} g_t$$

for the family g_t of the induced metrics. We introduce local coordinates $u_1, \dots, u_n, t$ on $V \times [0, 1]$ and denote by $\partial_i f_t$, $\partial_t f_t$ the derivatives of f_t with respect to the fields $\partial_i = \partial/\partial u_i$ and $\partial_t = \partial/\partial t$.

In other words, $\partial_i f_t = Df_t(\partial_i)$ and $\partial_t f_t = Df_t(\partial_t)$ for the differential D of the map $V \times [0, 1] \to W$ defined by $(v, t) \mapsto f_t(v)$. We think of these derivatives as vector fields in W along the mapped manifold $V \times [0, 1] \to W$.

Next, we abbreviate the previous notation by setting $\overline{\partial}_i = Df_t(\partial_i)$, $\overline{\partial}_t = Df_t(\partial/\partial t)$ and as before we denote by $\langle\ ,\ \rangle$ the h-scalar product in $T_w(W)$, $w = f(v)$. Then we have

$$\frac{d}{dt}(g_{ij}) = \langle \nabla^h_{\overline{\partial}_t} \overline{\partial}_i, \overline{\partial}_j \rangle + \langle \overline{\partial}_i, \nabla^h_{\overline{\partial}_t} \overline{\partial}_j \rangle = \langle \nabla^h_{\overline{\partial}_i} \overline{\partial}_t, \overline{\partial}_j \rangle + \langle \overline{\partial}_i, \nabla^h_{\overline{\partial}_j} \overline{\partial}_t \rangle$$

as the fields ∂_i commute with ∂_t. (Here ∇^h is the Levi-Civita connection in (W, h).) To simplify our notation, we denote by ∇^h_i the covariant derivatives in the induced bundle $f^*(T(W)) \to V$ with respect to the ∂_i and rewrite the above equality as

$$\frac{d}{dt}(g_{ij}) = \langle \nabla^h_i \overline{\partial}_t, \overline{\partial}_j \rangle + \langle \overline{\partial}_i, \nabla^h_j \overline{\partial}_t \rangle. \tag{1.3.1}$$

Finally, we restrict (1.3.1) to $V = V \times 0$ and obtain with our old $\partial = \overline{\partial}|_{t=0}$ the following expression for $L_h(\partial)$ in the local coordinates $u_1, \dots, u_n$ on V:

$$\partial \mapsto \langle \nabla^h_i \partial, \overline{\partial}_j \rangle + \langle \overline{\partial}_i, \nabla^h_j \partial \rangle. \tag{1.3.1$'$}$$

1.3.B. Now we linearize the operator $\mathscr{D}_\nabla$ at some morphism $\tilde{f}: X \to Y$. This linearization, denoted L_∇, acts on fields $\tilde{\partial}$ which are sections of the induced tangent bundle $\tilde{f}^*(T(Y)) \to X$. These are rather special fields as they represent tangent vectors to the space of our *morphisms* $X \to Y$ rather than of all maps $X \to Y$. These vectors are characterized by the S^1-invariance property for the action of S^1 on $\tilde{f}^*(T(Y)) \to X$ induced by the action of S^1 on X and (the differential of) the action of S^1 on Y. This is better seen with our description of $\mathscr{D}_\nabla$ by

$$\mathscr{D}_\nabla(\tilde{f}) = f^*(\nabla'_\beta) + d\varphi + \nabla_\alpha$$

for given frames α in X and β in Y (see (1.2.2)). Now, the relevant fields $\tilde{\partial}$ are given by pairs $\tilde{\partial} = (\partial, \varphi')$ where $\partial\colon V \to f^*(T(W))$ is the field along V underlying $\tilde{\partial}$ and φ' is the function on V which corresponds to the vertical S^1-invariant component of $\tilde{\partial}$.

Recall, that the range of L_∇ consists of the space of 1-forms on V as the difference between two S^1-connections is such a form. Now, by applying (1.2.2) to a family of maps $\tilde{f}_t\colon X \to Y$, $t \in [0, 1]$, and differentiating at $t = 0$ (or specializing the linearization formula (2) on page 71 in [D'A]) we have

$$L_\nabla(\tilde{\partial} = (\partial, \varphi'))(\tau) = \omega_\nabla(\partial, \tau f) + d\varphi'(\tau) \tag{1.3.2}$$

where the following notations are used:

τ is the tangent vector on V where we evaluate the 1-form $L_\nabla(\tilde{\partial})$;

$\tau f \overset{\text{def}}{=} Df(\tau)$;

∂ is the vector field in W along $f(V)$ given by $\partial = \overline{\partial}_{t=0}$. (We recall from §1.3.A that $\overline{\partial}_t = Df_t(\partial/\partial t)$).)

The formula for the linearization L_∇ is expressed in local coordinates $u_1, \dots, u_n$ on V by substituting the n fields $\partial_i = \partial/\partial u_i$ in place of τ in (1.2.3). Thus we get

$$\tilde{\partial} \mapsto \{\omega_\nabla(\partial, \overline{\partial}_i) + \partial_i\varphi'\}_{i=1,\dots,n} \tag{1.3.2$'$}$$

where $\overline{\partial}_i = \partial_i f$ and $\partial_i\varphi' = d\varphi'(\partial_i)$.

1.4. **Inversion of the operator** $L_H = (L_h, L_\nabla)$. Now we want to infinitesimally invert our "mixed" operator $\mathscr{D}_H$, i.e. we want to invert the linear operator $L_H = (L_h, L_\nabla)$. This means that we want to solve the equation

$$L_H(\tilde{\partial}) = H' \tag{1.4.1}$$

where the right-hand side $H' = (g', \Gamma')$ is arbitrary, where g' is a quadratic form on V and Γ' is a 1-form. In local coordinates u_i on V, (1.4.1) becomes the following system of P.D.E. in the unknowns ∂ and φ':

$$\begin{gathered} \langle \nabla_i^h \partial, \overline{\partial}_j \rangle + \langle \overline{\partial}_i, \nabla_j^h \partial \rangle = g'_{ij}, \\ \omega_\nabla(\partial, \overline{\partial}_i) + \partial\varphi'/\partial u_i = \Gamma'_i. \end{gathered} \tag{1.4.2}$$

The number of equations in the system (1.4.2) is $n(n+1)/2 + n$. To solve (1.4.2) we follow Nash [Na] and add the auxiliary equations

$$\langle \partial, \overline{\partial}_i \rangle = 0. \tag{1.4.3}$$

As in [D'A], we also let $\varphi' = 0$ in (1.3.2$'$) (notice that this corresponds to seek a solution of (1.3.2$'$) among horizontal fields $\tilde{\partial}$). By differentiating (covariantly) (1.4.3) we get

$$\langle \nabla_i^h \partial, \overline{\partial}_i \rangle + \langle \partial, \nabla_j^h \overline{\partial}_i \rangle = 0.$$

Next, we alternate i and j and then (1.3.1′) and (1.4.3) become equivalent to the system

$$(1.4.4) \qquad \langle \nabla_j^h \overline{\partial}_i, \partial \rangle = -\tfrac{1}{2} g'_{ij}, \qquad \langle \partial, \overline{\partial}_i \rangle = 0.$$

In particular, every solution of the system of linear algebraic equations (1.4.4) also solves the linearized system (1.3.1′). Thus the construction of an infinitesimal inverse for the operator $\mathscr{D}_H$ is reduced to the solution for ∂, φ' of the system

$$(1.4.5) \qquad \langle \overline{\partial}_i, \partial \rangle = 0, \quad \langle \nabla_j^h \overline{\partial}_i, \partial \rangle = g'_{ij}, \quad \varphi' = 0, \quad \omega_\nabla(\overline{\partial}_i, \partial) = \Gamma'_i,$$

where g'_{ij} and Γ'_i, $i, j = 1, \dots, n$, are arbitrarily given right-hand sides which are functions on V representing in the local coordinates u_i the components of a metric tensor and of a connection form respectively. The system (1.4.5) is *algebraic* in the unknown field ∂. On the other hand, every solution of (1.4.5) also gives a solution of the original linearized system (1.4.2) (with the extra equations $\langle \overline{\partial}_i, \partial \rangle = 0$, $\varphi' = 0$). This discussion prepares the reader for the proof of the following

1.4.A. **Proposition.** *If the covectors*

$$(1.4.6) \qquad \langle \overline{\partial}_i, \cdot \rangle, \langle \nabla_j^h, \cdot \rangle, \omega_\nabla(\overline{\partial}_i, \cdot), \qquad i, j = 1, \dots, n,$$

are linearly independent at all points of V then the linear operator L_H is invertible over all of V by some differential operator $M = M_{\tilde{f}}$, i.e., $L_H \circ M = \mathrm{id}$.

Proof. First, we observe that the independence of the covectors (1.4.6) is independent of the coordinate system we use. Secondly, we note that in the independent case the solution ∂ of the corresponding system (1.4.5) form an affine bundle over V of rank $2q - n(n+1)/2 - 2n$. Now, every affine bundle admits a section over V. To choose it in a canonical way one may use any fixed auxiliary Riemannian metric on W (e.g., we can use h) and then take as canonical solution say, ∂_{can}, the solution ∂ of (1.4.5) which has the minimal length (norm) with respect to this metric at every point $w = f(v) \in W$ (see, e.g., [Na, G-R, Gro]). Finally, we define the infinitesimal inversion $M = M_{\tilde{f}}$ of $\mathscr{D}_H$ by

$$M_{\tilde{f}}(g', \Gamma') = (\partial_{\mathrm{can}}, 0)$$

where 0 corresponds to the choice $\varphi' = 0$.

Now, using the terminology of §2.3.1 in [Gro] we say that *the operator $\mathscr{D}_H$ is infinitesimally invertible* at those $\tilde{f}$ where the independency condition required by Proposition 1.4.A is satisfied. This allows us to apply Nash's implicit function theorem to our $\mathscr{D}_H$ so that we arrive at the following

1.4.B. **Corollary.** *If the morphism $\tilde{f}\colon X \to Y$ satisfies the conditions of Proposition* 1.4.A *then the operator $\mathscr{D}_H$ is an open operator from $\{\tilde{f}\}$ to $\{G\}$ at $\tilde{f}$ and therefore all the structures G in a small neighborhood U of the induced structure $\mathscr{D}_H(\tilde{f}) = \tilde{f}^*(H)$ are inducible from H. (Recall that our function spaces $\{G\}$ and $\{\tilde{f}\}$ are endowed with the fine C^∞-topology.)*

In view of Corollary 1.4.B, to prove our Theorem 0.4.A we now have to show the existence of some morphism $\tilde{f}\colon X \to Y$ satisfying the independency condition required by Proposition 1.4.A. This is done in §2 on the basis of an analysis of this independence from a purely algebraic point of view.

2. (h, ω)-REGULAR MAPS AND THE PROOF OF THEOREM 0.4.A

2.1. Assume that we are given a quadratic form h and an exterior 2-form ω on a linear space S and let T_1, T_2 be two linear subspaces $T_1 \subset T_2 \subset S$ of dimension $n' = \dim T_1$ and $n' + s' = \dim T_2$.

2.1.A. **Definition.** The pair (T_1, T_2) is called *(h, ω)-regular* if one of the following equivalent conditions (i)–(ii) is satisfied:

(i) for some (and hence for every) basis $\tau_1, \dots, \tau_{n'}, \tau_{n'+1}, \dots, \tau_{n'+s'}$ in T_2 such that $\tau_1, \dots, \tau_{n'}$ form a basis in T_1, the equations

$$h(\tau_i, \partial) = a_i, \qquad i = 1, \dots, n' + s',$$
$$\omega(\tau_i, \partial) = b_i, \qquad i = 1, \dots, n',$$

are solvable in $\partial \in \mathscr{S}$ for arbitrarily given a_i, b_i.

(ii) The homogeneous system

$$h(\tau_i, \partial) = 0, \qquad i = 1, \dots, n' + s',$$
$$\omega(\tau_i, \partial) = 0, \qquad i = 1, \dots, n',$$

is nonsingular. Namely, the dimension of the space of solutions equals $\dim S - (2n' + s') \geq 0$.

2.1.B. *Remarks.* (i) We shall use Definition 2.1.A in the case when $S = T_{w_0}(W)$ and when $T_1 \subset T_2 \subset T_{w_0}(W)$ are the first and the second osculating space respectively of a map $f\colon V \to W$ at a given point $v_0 \in V$. That is, we take $T_1 = T_f^1(v_0) = Df(T_{v_0}(V))$ and $T_2 = T_f^2(v_0) \subset T_{w_0}(W)$, $w_0 = f(v_0)$ where $T_f^2(v_0)$ denotes the subspace spanned by T_1 and by the second covariant derivatives $\nabla_i^h \overline{\partial}_j$, $1 \leq i, j \leq n$, at $w_0 = f(v_0)$ with respect to some (fixed) local coordinates $u_1, \dots, u_n$ at $v_0 \in V$. (We remind to the reader that we use as before the notation $\overline{\partial}_i = \partial_i f$ and $\nabla_i^h = \nabla_{\overline{\partial}_i}^h$.)

(ii) Note that the subspaces T_f^1, T_f^2, are independent of the choice of coordinates. Also, notice that the dimension of T_f^2 can vary between zero and $\min(\dim W, n+s)$, for $s = \frac{1}{2}n(n+1)$, which is the dimension of the symmetric square $S^2(T_v(V))$ of $T_v(V)$, $v \in V$.

2.1.C. **Definition.** A map $f\colon V \to W$ is called *(h, ω)-regular* if $\dim T_f^1 = n$, $\dim T_f^2 = \frac{1}{2}n(n+3) = n+s$, and if the pair $T_f^1, T_f^2 \subset T_w(W)$ is (h, ω)-regular at all points $w = f(v) \in W$.

Accordingly in our case we call the map $f\colon V \to W$ (h, ω)-regular at a point $v \in V$ if the full differential $D \oplus D^2$ mapping $T(V) \oplus S^2(T(V))$ into $T(W)$ is (h, ω)-regular at v, where $S^2(T(V))$ is the symmetric square of $T(V)$ and where D^2 acts by the second covariant derivatives $D^2(\tau_i \otimes \tau_j) \overset{\text{def}}{=} \nabla_i^h \overline{\tau}_j$, where $\tau_i, \tau_j \in T_v(V)$, $\overline{\tau}_i = Df(\tau_i)$.

2.1.D. *Remark.* The condition $\dim T_2 = \frac{1}{2}n(n+3)$ implies that $\dim T_1 = n$, and the equality $\dim T_2 = \frac{1}{2}n(n+3)$ is called *freedom* of f at $v_0 \in V$. This is equivalent to the linear independence of the vectors $\overline{\partial}_j$ and $\nabla_i^h \overline{\partial}_j$, $1 \leq i$, $j \leq n$, at $f(v_0)$.

With the above terminology, the following lemma is immediate.

2.1.E. **Lemma.** *If the map* $f\colon V \to W$ *is* (h, ω)*-regular at all points* $v \in V$ *then the corresponding linearized system* (1.4.5) *satisfies the assumptions of Proposition* 1.4.A *and hence the operator* $\mathscr{D}_H$ *is infinitesimally invertible at* $\tilde{f}$.

2.1.F. **Proposition.** *Let* (h, ω) *be a pair of forms on a complex manifold* W *such that* $\mathbf{h} = h + \sqrt{-1}\omega$ *is a hermitian form on* W. *Then for* $\dim_{\mathbf{R}} W \geq n(n+1)/2 + 3n = s + 3n$, *generic maps* $V \to W$ *are* (h, ω)*-regular.*

Proof. The basic (and standard) idea in the proof of this proposition is to interpret non-(n, ω)-regularity as a singularity in the space $J^2(V, W)$ of 2-jets of our maps $V \to W$, so that one can use *Thom's transversality theorem.* Recall that $J^2 = J^2(V, W)$ forms a bundle over $V \times W$ whose fibers are denoted by $J^2_{v,w}$. If we fix local coordinates $u_1, \dots, u_n$ around $v \in V$, then $J^2_{v,w}$ can be identified with the vector space of linear maps $T_v(V) \oplus S^2(T_v(V)) \to T_w(W)$ such that the jet of a given map $f\colon V \to W$ is given by the first and second covariant derivatives

$$J_v^2 f(v) = (\overline{\partial}_i, \nabla_i^h \overline{\partial}_j), \qquad 1 \leq i \leq j \leq n.$$

Here $S^2(T_v(V))$ denotes the symmetric square of $T_v(V)$ and one should notice that the identification $J^2_{v,w} = \mathrm{Hom}(T_v(V)) \oplus S^2(T_v(V)) \to T_w(W)$ depends on the local coordinates where the "second differential" $\{\nabla_i^h \overline{\partial}_j\}$ is not invariantly defined. Also notice that (h, ω)*-regularity* at $v \in V$ only depends on $J_v^2 f$ as it is expressed (see Definition 2.1.C) in terms of $\overline{\partial}_i$ and $\nabla_i^h \overline{\partial}_j$. Thus we can define the subspace $\Sigma_{v,w} \subset J^2_{v,w}$ consisting of the jets of non-(h, ω)-regular maps. Now, we need the following

2.1.G. **Algebraic Lemma.** *The set* $\Sigma_{v,w} \subset J^2_{v,w}$ *is a stratified subset of codimension* $c = \dim W - s - 2n + 1$.

We shall prove this lemma later (see Proof 2.1.H). Now, we note that $\Sigma = \bigcup_{v,w \in V \times W} \Sigma_{v,w} \subset J^2(V, W)$ fibers over $V \times W$ and so, by the Algebraic Lemma 2.1.G, Σ also is a stratified subset in $J^2(V, W)$ of codimension $c = \dim W - s - 2n + 1$. Finally, we observe that a map $f\colon V \to W$ is non-(h, ω)-singular if and only if $J_f^2(V) \subset J^2(V, W)$ does not meet Σ. (This follows by the very definition of Σ.) Hence, by (the special case of) Thom's transversality theorem (see, e.g., [Gro, Corollary (D′), p. 33]) generic maps $f\colon V \to W$ do have the property $J_f^2(V) \cap \Sigma = \varnothing$ as our $c = \dim W - s - 2n + 1$ and $\dim V = n$. This concludes the proof of Proposition 2.1.F.

2.1.H. *Proof of the Algebraic Lemma* 2.1.G. We first identify the space $J^2_{v,w}$ with $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^{n+s} \to \mathbf{C}^q)$, where we assume $\dim_{\mathbf{C}} W = q$. Next, we denote by $\tau_1, \dots, \tau_n, \tau_{n+1}, \dots, \tau_{n+s} \in \mathbf{C}^q$ the images of the standard basis of $\mathbf{R}^{n+s}$ in $\mathbf{C}^q$. Now, the form $\mathbf{h} = h + \sqrt{-1}\omega$ (see the statement of Proposition 2.1.F) is the standard hermitian form $\sum_{i=1}^q z_i \overline{z}_i$ on $\mathbf{C}^q$ where $h = \sum_{i=1}^q x_i^2 + y_i^2$ and $\omega = \sum_{i=1}^q x_i \wedge y_i$.

We observe that the (h, ω)-regularity for this $\mathbf{h}$ is equivalent to the regularity of the following system of linear equations

$$\begin{aligned} \mathbf{h}(\tau_i, \partial) &= (\cdot)_i, & i &= 1, \dots, n, \\ h(\tau_j, \partial) &= (\cdot)_j, & j &= n+1, \dots, n+s, \end{aligned}$$

as the first equations are equivalent to

$$\left.\begin{aligned} h(\tau_i, \partial) &= (\cdot)'_i, \\ \omega(\tau_i, \partial) &= (\cdot)''_i, \end{aligned}\right. \quad i = 1, \dots, n.$$

Hence, it is easy to see (the verification only requires elementary linear algebra) that the (h, ω)-regularity of maps $V \to W$ is characterized by the following equivalent properties of the vectors $\tau_1, \dots, \tau_n, \tau_{n+1}, \dots, \tau_{n+s}$.

(1) The dimension of $\mathrm{Span}_{\mathbf{R}}(\tau_1, \dots, \tau_n, \dots, \tau_{n+s}, \sqrt{-1}\tau_1, \dots, \sqrt{-1}\tau_n)$ is maximal, i.e., equals $2n+s$.

(2) The vectors $\tau_1, \dots, \tau_n$ are linearly independent over $\mathbf{C}$, while all together the vectors $\tau_1, \dots, \tau_n, \dots, \tau_{n+s}, \sqrt{-1}\tau_1 \dots, \sqrt{-1}\tau_n$ are linearly independent over $\mathbf{R}$, which means that $\tau_{n+1}, \dots, \tau_{n+s}$ are $\mathbf{R}$-independent modulo $\mathrm{Span}_{\mathbf{C}}(\tau_1 \dots, \tau_n)$.

The next step will be to prove the following statement which measures what happens when the above independence fails to be true.

2.1.G′. **Sublemma.** *Let $\Sigma_0 \subset \mathbf{C}^{qn} \oplus \mathbf{C}^{qs}$ denote the subset of those $(n+s)$-tuples $\tau_1, \dots, \tau_n, \dots, \tau_{n+s} \in \mathbf{C}^q$ such that either $\tau_1, \dots, \tau_n$ are $\mathbf{C}$-dependent or $\tau_{n+1}, \dots, \tau_{n+s}$ are $\mathbf{R}$-dependent modulo* $\mathrm{Span}_{\mathbf{C}}(\tau_1, \dots, \tau_n)$. *Then* $\operatorname{codim}\Sigma_0 = 2q - 2n - s + 1$.

Proof. Denote by $\Sigma' \subset \mathbf{C}^{qn}$ the set of $\mathbf{C}$-linearly dependent n-tuples $\tau_1, \dots, \tau_n$ and let p be the natural projection $\mathbf{C}^{qn} \oplus \mathbf{C}^{qs} \to \mathbf{C}^{qn}$. Then $\Sigma_0 \subset p^{-1}(\Sigma') \cup (p^{-1}(\Omega') \cap \Sigma_0)$ for $\Omega' = \mathbf{C}^{qn} - \Sigma'$. The singularity subset $\Sigma' \subset \mathbf{C}^{qn}$ has $\operatorname{codim}_{\mathbf{R}} = 2\operatorname{codim}_{\mathbf{C}} = 2(q-n)+2$ (see, e.g., [A-V-Z]) and hence $p^{-1}(\Sigma')$ also has codimension $2(q-n)+2$. Now, if we write $\Sigma_0 = (\Sigma_0 \cap p^{-1}(\Sigma')) \cup (p^{-1}(\Omega') \cap \Sigma_0)$ and take into account that $\operatorname{codim}(A \cup B) = \min(\operatorname{codim} A, \operatorname{codim} B)$, then it remains to show that $\Sigma'' = p^{-1}(\Omega') \cap \Sigma_0$ has codimension $2q - 2n - s + 1$. To see this, we consider the natural map $\pi \colon \Sigma'' \to \mathrm{Gr}_n \mathbf{C}^q$ (where $\mathrm{Gr}_n \mathbf{C}^q$ is the set of all n-dimensional complex subspaces in $\mathbf{C}^q$) which sends $\tau_1, \dots, \tau_n, \dots, \tau_{n+s}$ into $\mathrm{Span}_{\mathbf{C}}(\tau_1, \dots, \tau_n)$. This map is obviously a smooth fibration. Next, we enlarge Σ'' to the set Ω'' consisting of those $\tau_1, \dots, \tau_n, \dots, \tau_{n+s}$ where $\tau_1, \dots, \tau_n$ are $\mathbf{C}$-independent (and we do not make any restriction on the τ_i's). Then Ω'' (which is a smooth manifold, in fact an open subset in $\mathbf{C}^{qn} \oplus \mathbf{C}^{qs}$) smoothly fibers over $\mathrm{Gr}_n \mathbf{C}^q$. Let us now evaluate the codimension of the fibers of $\Sigma'' \to \mathrm{Gr}_n \mathbf{C}^q$ in the fibers of the fibration $\Omega'' \to \mathrm{Gr}_n \mathbf{C}^q$. Take a fiber of Ω'', say Ω''_a, which corresponds to a fixed n-dimensional subspace $a \in \mathbf{C}^q$ and observe that $\Sigma''_a = \Sigma'' \cap \Omega''_a$ consists of those $\tau_1, \dots, \tau_{n+s} \in \mathbf{C}^q$ whose projections to the factor space $\mathbf{C}^q/a$ are $\mathbf{R}$-linearly dependent. Now, $\mathbf{C}^q/a = \mathbf{C}^{q-n}$ and the codimension of the (singularity) subset $\Sigma^* \subset \mathbf{C}^{s(q-n)}$ of those s-tuples of vectors in $\mathbf{C}^{q-n} = \mathbf{R}^{2(q-n)}$ which are $\mathbf{R}$-dependent equals $2(q-n) - s + 1$ (see [A-V-Z]) and hence $\Sigma'' \cap \Omega_a$ has codimension $2(q-n) - s + 1$ in Ω''_a. Clearly, this applies to all Ω''_a, $a \in \mathbf{C}^q$ and thus we see that

$$\operatorname{codim}(\Sigma'' \subset \mathbf{C}^{qn} \oplus \mathbf{C}^{qs}) = \operatorname{codim}(\Sigma'' \subset \Omega'') = 2(q-n) - s + 1. \quad \square$$

2.1.I. *Remark.* Our conventions concerning the dimension and the codimension of stratified sets are those usually accepted (see, e.g., 1.3.2 in [Gro]).

2.2. **Conclusion of the proof of Theorem 0.4.A.** We return to the structure inducing problem by morphisms $\tilde{f}: X \to Y$ for the Hopf bundle $Y \to W = \mathbf{C}P^q$, where $2q \geq n(n+1)/2 + 3n$ and observe that the corresponding form $\mathbf{h} = h + \sqrt{-1}\omega$ is hermitian in this case. (Compare with Proposition 2.1.F.) We are going to show the existence of a nonempty open subset $\widetilde{U}$ of $\{G\}$ which is contained in the image $\mathscr{D}_H\{\tilde{f}\} \subset \{G\}$. To do this, we only have to prove the existence of a single morphism $\tilde{f}: X \to Y$. Indeed, if the (underlying) maps $f: V \to W$ satisfying the (h, ω)-regularity condition form an open *dense* set in the space of all f's, the same is true for the $\tilde{f}$'s, as the natural map between the function spaces $\{\tilde{f}\} \to \{f\}$ (where $\{\tilde{f}\}$ is the space of C^∞-bundle morphisms $X \to Y$ and $\{f\}$ is the space of C^∞-maps $V \to W$) is a Serre fibration. (In fact, if some map $f: V \to W$ is "covered" by $\tilde{f}: X \to Y$, then this is true for all the maps which are close (and hence homotopic) to f.)

Now, the existence of a single morphism $\tilde{f}: X \to Y$ is a purely topological question which has a positive solution if Y is an n-universal S^1-bundle. In particular, the standard S^1-bundle over $\mathbf{C}P^q$ is q-universal (see [St]) and then Theorem 0.4.A is completely proved.

2.2.A. *Final Remark.* Theorem 0.4.A remains true for a general (inducing) S^1-bundle $Y \to W$ $(W \neq \mathbf{C}P^q)$ under the following conditions:

(i) (h, ω) constitute a hermitian form,
(ii) $\dim_{\mathbf{R}} W \geq n(n+1)/2 + 3n$,
(iii) The S^1-principal fibration $Y \to W$ is n-universal. (Of course, one only needs a single morphism $X \to Y$.)

3. Appendix: Isometric immersions between Kähler manifolds

We begin this Appendix with the following

3.1. **Lemma.** *Let V and W be two hermitian (e.g., Kähler) manifolds. If the map $f: V \to W$ is an isometry, then f is necessarily holomorphic.*

Proof. The proof consists in showing that every $\mathbf{R}$-linear map $f: \mathbf{C}^n \to \mathbf{C}^q$ which is isometric is $\mathbf{C}$-linear. Then the lemma will follow by applying this to the differential Df. Let us show that our f satisfies $f(\sqrt{-1}z) = \sqrt{-1}f(z)$ for all $z \in \mathbf{C}^n$.

$$\begin{aligned}
&\langle f(\sqrt{-1}z) - \sqrt{-1}f(z), f(\sqrt{-1}z) - \sqrt{-1}f(z)\rangle \\
&\quad = \langle f(\sqrt{-1}z), f(\sqrt{-1}z)\rangle - \langle f(\sqrt{-1}z), \sqrt{-1}f(z)\rangle \\
&\qquad - \langle \sqrt{-1}f(z), f(\sqrt{-1}z)\rangle + \langle \sqrt{-1}f(z), \sqrt{-1}f(z)\rangle \\
&\quad = \langle \sqrt{-1}z, \sqrt{-1}z\rangle + \sqrt{-1}\langle \sqrt{-1}z, z\rangle - \sqrt{-1}\langle z, \sqrt{-1}z\rangle + \langle z, z\rangle = 0
\end{aligned}$$

where $\langle\ ,\ \rangle$ is the hermitian scalar product in $\mathbf{C}^q$.

3.2. **Jet counting.** Let us compare the dimension of the jets space of holomorphic maps $f: \mathbf{C} \to \mathbf{C}^q$ with that for Kähler metrics $\tilde{h}$ on $\mathbf{C}$. The first of our spaces, i.e., the space of jets of order r of holomorphic maps $\mathbf{C} \to \mathbf{C}^q$ at the origin, call it $\mathscr{H}_r$, is given by $q(r+1)$ complex numbers which are the (complex) derivatives $d^i f(0)/dz^i \in \mathbf{C}^q$, $i = 0, 1, \ldots, r$. (There is no other derivatives as f is holomorphic.) Next, there is a 1-1 correspondence between

Kähler metrics $\tilde{h}$ on $\mathbf{C}$ and exact positive 2-forms ω given by

$$\tilde{h}(a, b) = \omega(\sqrt{-1}a, b) + \sqrt{-1}\omega(a, b)$$

where the $(1, 1)$ condition on the 2-form ω is automatic as $\dim_{\mathbf{C}} \mathbf{C} = 1$. (Recall that every 2-form ω on $\mathbf{C}$ equals $\psi\, dx \wedge dy$ where ψ is some function and ω is positive if and only if ψ is positive.)

Now, we have the exterior differential acting from 1-forms to exact 2-forms on $\mathbf{C}$. On the level of r-jets we have $d_r^1 \colon \Omega_r^1 \to E_{r-1}^2$ where Ω_r^1 denotes the space of r-jets at the origin of real 1-forms on $\mathbf{C}$ and E_{r-1}^2 denotes the space of $(r-1)$-jets of exact 2-forms.

The kernel of d_r consists of the jets of exact 1-forms which are differentials of functions:

$$\operatorname{Ker} d_r = \operatorname{Im} d_r^0 \quad \text{for } d_r^0 \colon \Omega_{r+1}^0 \to \Omega_r^1 .$$

Thus $\dim E_{r-1}^2 \geq \dim \Omega_r^1 - \dim \Omega_{r+1}^1$. Now, each 1-form is given by two functions having together $2((r+1)(r+2)/2)$ partial derivatives of order $\leq r$. Similarly, 0-forms (functions) have $(r+2)(r+3)/2$ derivatives of order $\leq r+1$.

Thus $\dim E_{r-1}^2 \geq (r+1)(r+2) - (r+2)(r+3)/2$ which, for large r, is clearly greater than the dimension of $\mathscr{H}_r$ which is equal to $2q(r+1)$.

3.3. **Noninducing conclusion.** Now we can see that a generic exact form ω on $\mathbf{C}$ cannot be induced by a holomorphic map $f \colon \mathbf{C} \to \mathbf{C}^q$. In fact, the inducing operator gives rise to a smooth map between the jet spaces $\mathscr{D}_r \colon \mathscr{H}_r \to E_{r-1}^2$ which has a nowhere dense image for large r by the above dimension inequality. This easily implies (see, e.g., [G-R, Sp]) that $\mathscr{D}$ itself has nowhere dense image. Then this conclusion extends to the general manifolds V and W in place of $\mathbf{C}$ and $\mathbf{C}^q$ since our considerations are purely local and because (germs at the origin $0 \in \mathbf{C}$ of) perturbations of Kähler metrics extend from complex curves ($= \mathbf{C}$) in V (which is locally $\mathbf{C}^n$) to all of V.

References

[A-V-Z] V. Arnold, A. Varchenko, and S. Goussein Zadé, *Singularités des applications différentiables* I, "Mir", Moscow, 1986.

[At] M. F. Atiyah, *Geometry of Yang-Mills fields*, Lezioni Fermiane, Scuola Normale Superiore, Pisa, 1979.

[Ca] E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. (2) **58** (1953), 1–23.

[D'A] G. D'Ambra, *Construction of connection inducing maps between principal bundles*, J. Differential Geom. **26** (1987), 67–79.

[Gre] R. Greene, *Isometric embeddings of Riemannian and pseudoriemannian manifolds*, Mem. Amer. Math. Soc. **97** (1970).

[G-J] R. Greene and H. Jacobowitz, *Analytic isometric embeddings*, Ann. of Math. (2) **93** (1971), 189–203.

[Gro] M. Gromov, *Partial differential relations*, Springer-Verlag, Berlin and New York, 1986.

[G-R] M. Gromov and V. Rokhlin, *Embeddings and immersions in Riemannian geometry*, Uspekhi Mat. Nauk. **25** (1970), no. 5, 3–62.

[Ha] R. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7** (1982), 65–222.

[Ja] H. Jacobowitz, *Implicit function theorems and isometric embeddings*, Ann. of Math. (2) **95** (1972), 191–225.

[Ko] B. Kostant, *Quantization and unitary representations*, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin and New York, 1970, pp. 87–207.

[N-R] M. Narasimhan and S. Ramanan, *Existence of universal connections*, Amer. J. Math. **83** (1961), 563–572.

[Na] J. Nash, *The embedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–63.

[Sp] M. Spivak, *A comprehensive introduction to differential geometry*, vol. 5, Publish or Perish, Berkeley, Calif., 1979.

[St] N. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, N.J., 1970.

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