EXTENDING THE t -DESIGN CONCEPT

A. R. CALDERBANK AND P. DELSARTE

ABSTRACT. Let $\mathfrak{B}$ be a family of k -subsets of a v -set V , with $1 \leq k \leq v/2$. Given only the inner distribution of $\mathfrak{B}$, i.e., the number of pairs of blocks that meet in j points (with $j = 0, 1, \dots, k$), we are able to completely describe the regularity with which $\mathfrak{B}$ meets an arbitrary t -subset of V , for each order t (with $1 \leq t \leq v/2$). This description makes use of a linear transform based on a system of dual Hahn polynomials with parameters v, k, t . The main regularity parameter is the dimension of a well-defined subspace of $\mathbb{R}^{t+1}$, called the t -form space of $\mathfrak{B}$. (This subspace coincides with $\mathbb{R}^{t+1}$ if and only if $\mathfrak{B}$ is a t -design.) We show that the t -form space has the structure of an ideal, and we explain how to compute its canonical generator.

1. INTRODUCTION

Consider a set (a *design*) $\mathfrak{B}$ whose elements, called blocks, are k -subsets of a given v -set V of points, with $1 \leq k \leq v/2$. For a given integer t , in the range $0 \leq t \leq k$, we are interested in some regularity properties of $\mathfrak{B}$, of order t , which can be defined as follows. With any t -subset x^t of V and any integer $i \in [0, t]$ we associate the number $D_i(x^t)$ that counts the blocks in $\mathfrak{B}$ meeting x^t in $t - i$ points. Suppose that we have a linear relation

$$f_0 D_0(x^t) + f_1 D_1(x^t) + \cdots + f_t D_t(x^t) = c,$$

where $f_0, f_1, \dots, f_t$ and c are fixed real numbers. Then we say that the $(t+1)$ -tuple $(f_i)_{i=0}^t$ is a t -form for $\mathfrak{B}$. The set of t -forms clearly is a vector space, which will be called the t -form space of $\mathfrak{B}$. We propose to take the dimension of that space as a measure of the regularity of $\mathfrak{B}$ with respect to the t -subsets of V .

The idea originates mainly from a recent paper [2] in which linear relations as above, with the restrictive property $f_1 \neq 0, f_2 = \cdots = f_t = 0$, are deduced from a strengthening of the Assmus-Mattson theorem in coding theory. This investigation has been continued in a companion paper [1]. Recall that $\mathfrak{B}$ is said to be a t -design, of index c , when $D_0(x^t) = c$ for all t -subsets x^t of V . (For a thorough algebraic study, see especially [13].) This implies that every $(t+1)$ -tuple $(f_i)_{i=0}^t$ is a t -form for $\mathfrak{B}$. So, our definition of regularity can be viewed as an extension of the classical t -design concept.

Received by the editors November 3, 1989 and, in revised form, May 20, 1991.

1991 *Mathematics Subject Classification*. Primary 05B20, 05B30; Secondary 33A63, 20C30.

Key words and phrases. t -design, t -form space, t -distribution matrix, dual Hahn polynomials.

We shall see how the theory of t -form spaces provides us with a satisfactory “combinatorial interpretation” of the algebraic notion of a T -design [3, 4], with T a subset of $[0, t]$. One of the main results of this paper is an explicit description of the t -form space in terms of the *inner distribution* of $\mathfrak{B}$. In fact, this space is proved to be an ideal of a well-defined $(t + 1)$ -dimensional commutative algebra, and to admit a canonical generator which can be computed from the inner distribution. Furthermore, the theory gives us an efficient method to determine the t -distribution matrix of $\mathfrak{B}$, having the numbers $D_i(x^t)$ as its entries, from a minimum set of data. The methods of proof are based on a suitable matrix transform, involving a system of *dual Hahn polynomials* (with parameters v, k, t) [6, 7, 10]. Some combinatorial applications of the Hahn polynomials (quite different from those considered in what follows) can be found in [4, 12, and 13].

For several reasons, it is useful to replace the “combinatorial” notion of a design $\mathfrak{B}$ by the more general “algebraic” notion of a *design vector* β , which is defined simply as a real row vector, of size $\binom{v}{k}$, with components labelled by the k -subsets of V (see [4], for example). The concept of the t -form space can be extended naturally to this general definition of a design vector. (The situation described above corresponds to the special case where β is the characteristic vector of a combinatorial design $\mathfrak{B}$.) All our theorems are valid for general design vectors. Note that the theory would be essentially the same if we would consider only rational vectors β or, equivalently, integer vectors β , i.e., *signed designs* [8, 9, 11].

2. t -FORMS AND THE t -DISTRIBUTION MATRIX

Let V be a v -set (i.e., a set of finite cardinality v), and let $\mathcal{P}^i$ be the set of i -subsets of V , for $i = 0, 1, \dots, v$. Given an integer k , with $0 \leq k \leq \lfloor v/2 \rfloor$, we shall be interested in the real $\binom{v}{k}$ -dimensional vector space $\mathbb{R}\mathcal{P}^k$, with entries indexed by the elements x^k of $\mathcal{P}^k$. (In this paper, we always use **row** vectors.) Consider a vector β in $\mathbb{R}\mathcal{P}^k$. If $\beta(x^k) = 0$ or 1 for every x^k , then we can identify β with the set $\mathfrak{B} = \{x^k \in \mathcal{P}^k : \beta(x^k) = 1\}$ that has β as its characteristic vector. Such a subset $\mathfrak{B}$ of $\mathcal{P}^k$ is a “design” for the Johnson scheme $J(v, k)$; the elements of $\mathfrak{B}$ are often called blocks. By extension, we shall say that any vector β in $\mathbb{R}\mathcal{P}^k$ is a *design vector* for $J(v, k)$.

Let t be a fixed integer, with $t \in [0, k]$. We shall express the “regularity properties of order t ” of a given design vector β through the notions of the t -distribution matrix and the t -form space.

Definition 2.1. For $i \in [0, t]$, the i th *adjacency matrix* of $\mathcal{P}^k$ with respect to $\mathcal{P}^t$ is the $\{0, 1\}$ matrix $C_i^{k,t}$, of size $\binom{v}{k} \times \binom{v}{t}$, with rows indexed by $x^k \in \mathcal{P}^k$ and columns indexed by $x^t \in \mathcal{P}^t$, defined by

$$C_i^{k,t}(x^k, x^t) = 1 \quad \text{if and only if} \quad |x^k \cap x^t| = t - i.$$

The t -distribution matrix of the design vector $\beta \in \mathbb{R}\mathcal{P}^k$ is the real matrix D^t , of size $(t + 1) \times \binom{v}{t}$, with i th row

$$D_i^t = \beta C_i^{k,t}, \quad \text{for } i \in [0, t].$$

The combinatorial meaning of this definition is the following. If β is the characteristic vector of a subset $\mathfrak{B}$ of $\mathcal{P}^k$, then the (i, x^t) entry $D_i^t(x^t)$ of the t -distribution matrix of β counts the blocks in $\mathfrak{B}$ that have $t-i$ points in common with x^t . (Note that the k -distribution matrix D^k coincides with the “outer distribution matrix” in [3].)

Definition 2.2. A vector $\mathbf{f}^t = (f_0^t, f_1^t, \dots, f_t^t)$ in $\mathbb{R}^{t+1}$ is a t -form for the design vector $\beta \in \mathbb{R}^{\mathcal{P}^k}$ if and only if it satisfies

$$\mathbf{f}^t D^t = c\mathbf{1}, \text{ with } c \in \mathbb{R},$$

where $\mathbf{1}$ denotes the all-one vector. The subspace of $\mathbb{R}^{t+1}$ formed by all the t -forms for β will be denoted by $\mathcal{F}^t$ and will be referred to as the t -form space of β . Finally, the t -degree of β , denoted by r^t (or simply by r), is defined as the codimension of the t -form space, that is,

$$r^t = r = t + 1 - \dim(\mathcal{F}^t).$$

By definition, $\mathbf{f}^t$ is a t -form iff the linear combination $\sum_{i=0}^t f_i^t D_i^t(x^t)$ is constant over the set $\mathcal{P}^t$. It seems therefore natural to consider that a design vector β is highly regular with respect to $\mathcal{P}^t$ when it admits a large number of linearly independent t -forms, i.e., when its t -degree r^t is small. One may say that r^t is a measure of the “ t -irregularity” of β . The most regular designs $\mathfrak{B}$ are the t -designs (in the usual sense), defined by the fact that $D_0^t(x^t)$ is a constant. This implies that $D_i^t(x^t)$ is a constant, for each $i \in [0, t]$. Therefore, a t -design $\mathfrak{B}$ is characterized by the fact that its t -form space coincides with $\mathbb{R}^{t+1}$, i.e., by the property $r^t = 0$.

Let us now emphasize the simple relationship between the t -degree r^t and the rank of D^t . (In what follows, we denote by $\langle \xi, \eta \rangle$ the inner product $\xi \eta^T$ of two vectors ξ and η in $\mathbb{R}^{\mathcal{P}^k}$.) We need a preliminary result.

Lemma 2.1. The constant c in the equation $\mathbf{f}^t D^t = c\mathbf{1}$ is determined from $\mathbf{f}^t$ and $\langle \beta, \mathbf{1} \rangle$ by

$$c \binom{v}{t} = \langle \beta, \mathbf{1} \rangle \sum_{i=0}^t f_i^t \binom{k}{t-i} \binom{v-k}{i}.$$

Proof. We have $D_i^t \mathbf{1}^T = \langle \beta, \mathbf{1} \rangle \binom{k}{t-i} \binom{v-k}{i}$, by simple counting of t -sets relative to a fixed k -set. Using this identity, we prove the lemma by computing the inner product of both sides of $\mathbf{f}^t D^t = c\mathbf{1}$ with $\mathbf{1}$. $\square$

Proposition 2.1. The t -form space $\mathcal{F}^t$ of any design vector β contains the all-one vector $\mathbf{1}$, and the t -distribution matrix D^t of β has rank r^t or $r^t + 1$ according as $\langle \beta, \mathbf{1} \rangle = 0$ or $\langle \beta, \mathbf{1} \rangle \neq 0$.

Proof. The first part is obvious, for $\mathbf{1} D^t = \langle \beta, \mathbf{1} \rangle \mathbf{1}$. The second part is proved as follows. If $\langle \beta, \mathbf{1} \rangle = 0$, then any t -form equation $\mathbf{f}^t D^t = c\mathbf{1}$ must satisfy $c = 0$, by Lemma 2.1, which means that $\mathcal{F}^t$ equals the orthogonal complement of the column space of D^t . On the other hand, if $\langle \beta, \mathbf{1} \rangle \neq 0$, then we see that the t -form $\mathbf{1}$ is not orthogonal to the column space of D^t . $\square$

Note that the distinction between the cases $\langle \beta, \mathbf{1} \rangle = 0$ and $\langle \beta, \mathbf{1} \rangle \neq 0$ is inconsequential, since replacing β by $\beta + a\mathbf{1}$, with any $a \in \mathbb{R}$, does not affect the t -form space.

3. HARMONIC ANALYSIS AND DUAL HAHN POLYNOMIALS

To determine the t -form space of a design vector we shall use a transform method involving dual Hahn polynomials. We need some preliminaries about the harmonic analysis of the symmetric group S_v (see [5, 6, 7]). Under the natural action of S_v , the space $\mathbb{R}\mathcal{P}^k$ decomposes as the orthogonal sum of $k+1$ irreducible S_v -invariant subspaces, called the harmonic subspaces of $\mathbb{R}\mathcal{P}^k$. A “constructive definition” is given below.

Definition 3.1. For $s \in [0, k]$, the *homogeneous subspace* of degree s of $\mathbb{R}\mathcal{P}^k$, denoted by Hom_s^k , is defined as the row space of $(C_0^{k,s})^T$ (which is the characteristic matrix of the inclusion relation between s -subsets and k -subsets of V). The *harmonic subspace* of degree s of $\mathbb{R}\mathcal{P}^k$ is denoted by Harm_s^k and is defined as the orthogonal complement of Hom_{s-1}^k in Hom_s^k .

The latter definition makes sense since Hom_{s-1}^k is a subspace of Hom_s^k , as a consequence of $C_0^{k,s}C_0^{s,s-1} = (k-s+1)C_0^{k,s-1}$. Note the properties $\text{Hom}_k^k = \mathbb{R}\mathcal{P}^k$ and $\text{Hom}_0^0 = \text{Harm}_0^0 = \mathbb{R}\mathbf{1}$. As the columns of $C_0^{k,s}$ are linearly independent, Hom_s^k has dimension $\binom{v}{s}$ and Harm_s^k has dimension $\binom{v}{s} - \binom{v}{s-1}$.

Definition 3.2. Let $\mathcal{A}^{k,t}$ be the real $(t+1)$ -dimensional vector space generated by the adjacency matrices $C_i^{k,t}$ with $i \in [0, t]$. In particular, $\mathcal{A}^{k,k}$ is the *Bose-Mesner algebra* of the Johnson scheme $J(v, k)$. For $s \in [0, k]$, let E_s^k be the orthogonal projection matrix from $\mathbb{R}\mathcal{P}^k$ onto Harm_s^k . The matrices E_s^k are the *irreducible idempotents* of $\mathcal{A}^{k,k}$.

Let us now give two useful results concerning, first, the relation between the spaces $\mathcal{A}^{k,t}$ and the algebra $\mathcal{A}^{k,k}$ and, next, the relation between the eigenspaces of both algebras $\mathcal{A}^{k,k}$ and $\mathcal{A}^{t,t}$.

Lemma 3.1. The space $\mathcal{A}^{k,t}$ is generated by the $t+1$ matrices $E_s^k C_0^{k,t}$ with $s \in [0, t]$.

Proof. A straightforward argument shows that if X belongs to $\mathcal{A}^{k,k}$, then $XC_0^{k,t}$ belongs to $\mathcal{A}^{k,t}$. It remains to show that the matrices $E_s^k C_0^{k,t}$ are linearly independent. This follows from the well-known fact (easily deduced from the information above) that $C_0^{k,t}(C_0^{k,t})^T$ is a linear combination of $E_0^k, E_1^k, \dots, E_t^k$ with positive coefficients. More precisely, as shown in [3], the coefficient of E_i^k is $\binom{k-i}{k-t} \binom{v-t-i}{k-t}$ for $i = 0, 1, \dots, t$. $\square$

Lemma 3.2. For $s \in [0, t]$, the harmonic subspaces of degree s of $\mathbb{R}\mathcal{P}^k$ and $\mathbb{R}\mathcal{P}^t$ are related by

$$\text{Harm}_s^k = \text{Harm}_s^t (C_0^{k,t})^T, \quad \text{Harm}_s^t = \text{Harm}_s^k C_0^{k,t}.$$

Hence, the irreducible idempotents of the Bose-Mesner algebras $\mathcal{A}^{k,k}$ and $\mathcal{A}^{t,t}$ are related by $C_0^{k,t} E_s^t = E_s^k C_0^{k,t}$.

Proof. We start from $C_0^{k,t} C_0^{t,s} = \binom{k-s}{t-s} C_0^{k,s}$. This shows that the first statement of the lemma is satisfied when Harm is replaced by Hom (in view of the fact that $C_0^{k,t} (C_0^{k,t})^T$ preserves Hom_s^k). The statement about harmonic spaces can then be deduced (by use of Definition 3.1) from the fact that Harm_s^k is an eigenspace of $C_0^{k,t} (C_0^{k,t})^T$, with a positive eigenvalue. The second statement readily follows from the first one. Further details are omitted. $\square$

Next, we introduce a three-parameter family of dual Hahn polynomials [10] (with a normalization adapted to our subject), and recall some of their well-known properties.

Definition 3.3. For $i \in [0, t]$, the dual Hahn polynomial $Q_i^{k,t}(z)$, with $0 \leq t \leq k \leq \lfloor v/2 \rfloor$, is defined by

$$\binom{k}{t} Q_i^{k,t}(z) = \binom{k}{t-i} \binom{v-k}{i} \sum_{l=0}^i \frac{(-i)_l (-z)_l (z-v-1)_l}{(1)_l (-t)_l (k-v)_l},$$

where $(a)_l = a(a+1) \cdots (a+l-1)$. Thus, $Q_i^{k,t}(z)$ has degree $2i$ in z and degree i in the variable $\lambda^t(z) = Q_1^{t,t}(z) = t(v-t) - z(v+1-z)$.

Theorem 3.1. The dual Hahn polynomials satisfy the following orthogonality relation and three-term recurrence relation:

$$\begin{aligned} \sum_{s=0}^t \frac{v-2s+1}{v-s+1} \binom{v}{s} \binom{k-s}{k-t} \binom{v-t-s}{k-t} Q_i^{k,t}(s) Q_j^{k,t}(s) \\ = \binom{v}{k} \binom{k}{t-i} \binom{v-k}{i} \delta_{i,j}, \end{aligned}$$

$$\lambda^t(z) Q_i^{k,t}(z) = \sum_{j=i-1}^{i+1} w_{i,j}^{k,t} Q_j^{k,t}(z),$$

with $w_{i,i-1}^{k,t} = (t+1-i)(v+1-k-i)$, $w_{i,i}^{k,t} = (t-i)(k-t+i) + i(v-k-i)$, $w_{i,i+1}^{k,t} = (i+1)(k+1-t+i)$, and $\lambda^t(z) = t(v-t) - z(v+1-z)$.

The fundamental reason why the dual Hahn polynomials occur in our investigation can be explained in a group theoretic setting by the fact that these polynomials give the spherical and intertwining functions involved in the harmonic analysis relative to the symmetric group [6, 7]. Not surprisingly, the spherical case, $t = k$, plays a special role in our theory (see [5]). The following theorem shows, in explicit terms, how the dual Hahn polynomials $Q_i^{k,t}(z)$ appear on the scene.

Theorem 3.2. The adjacency matrices $C_i^{k,t}$ are related to the irreducible idempotents E_s^k of the Bose-Mesner algebra of $J(v, k)$ by

$$C_i^{k,t} = \left(\sum_{s=0}^t Q_i^{k,t}(s) E_s^k \right) C_0^{k,t}, \quad \text{for } i \in [0, t].$$

Proof. In view of Lemma 3.1, we may write the relation above if we replace $Q_i^{k,t}(s)$ by an unknown real number $X_i^{k,t}(s)$. To prove the desired coincidence, we start from the matrix identity

$$C_i^{k,t} C_1^{t,t} = \sum_{j=0}^t w_{i,j}^{k,t} C_j^{k,t}.$$

Here, $w_{i,j}^{k,t}$ counts the elements y^t of $\mathcal{P}^t$ that satisfy $|x^k \cap y^t| = t - i$ and $|y^t \cap x^t| = t - 1$, for a given pair $(x^k, x^t) \in \mathcal{P}^k \times \mathcal{P}^t$ with $|x^k \cap x^t| = t - j$. In particular, $w_{i,j}^{k,t} = 0$ if $|i - j| \geq 2$. A counting argument shows that the intersection numbers $w_{i,j}^{k,t}$ thus defined (for $|i - j| \leq 1$) coincide with the recurrence parameters for the dual Hahn polynomials (see Theorem 3.1). Post-multiplying the identity above by E_s^t (with $s \in [0, t]$) and using the last statement in Lemma 3.2, together with the well-known formula $C_1^{t,t} E_s^t = \lambda^t(s) E_s^t$, we readily obtain the three-term relation

$$\lambda^t(s) X_i^{k,t}(s) = \sum_{j=i-1}^{i+1} w_{i,j}^{k,t} X_j^{k,t}(s).$$

This uniquely determines the unknowns $X_i^{k,t}(s)$, from the initial values $X_0^{k,t}(s) = 1$ and $X_{-1}^{k,t}(s) = 0$. A comparison with the second part of Theorem 3.1 yields $X_i^{k,t}(s) = Q_i^{k,t}(s)$ as was to be proved. $\square$

Definition 3.4. Let $Q^{k,t}$ be the rational square matrix of order $t + 1$ with (i, s) entry $Q_i^{k,t}(s)$ for $i, s \in [0, t]$. Given any vector $\mathbf{f} = (f_0^t, f_1^t, \dots, f_t^t)$ in $\mathbb{R}^{t+1}$, the *dual Hahn transform* of $\mathbf{f}$, of type (k, t) , is defined as the vector $\hat{\mathbf{f}} = \mathbf{f} Q^{k,t}$. In other terms, $\hat{\mathbf{f}} = (\hat{f}^t(0), \hat{f}^t(1), \dots, \hat{f}^t(t))$ where $\hat{f}^t(s)$ is the value at point s of the *dual Hahn transform polynomial*

$$\hat{f}^t(z) = \sum_{i=0}^t f_i^t Q_i^{k,t}(z).$$

4. β -REGULARITY AND T -DESIGNS

This section contains some further results about T -designs, introduced in [3]. In particular, we shall discover a close connection between the T -design concept and the t -form concept.

Definition 4.1. Given a design vector β in $\mathbb{R}^{\mathcal{P}^k}$, a subspace Φ of $\mathbb{R}^{\mathcal{P}^k}$ is β -regular if and only if

$$\langle \beta, \phi \rangle = \frac{\langle \beta, \mathbf{1} \rangle}{\langle \mathbf{1}, \mathbf{1} \rangle} \langle \mathbf{1}, \phi \rangle \quad \text{for all } \phi \in \Phi.$$

Let T be a subset of the integer interval $[0, k]$. A vector $\beta \in \mathbb{R}^{\mathcal{P}^k}$ is a T -design vector for the Johnson scheme $J(v, k)$ if and only if

$$\langle \beta, \phi \rangle = 0 \quad \text{for all } \phi \in \text{Harm}_s^k \text{ with } 0 \neq s \in T.$$

Let Φ be an S_v -invariant subspace of $\mathbb{R}^{\mathcal{P}^k}$. Then Φ decomposes as an orthogonal sum $\Phi = \bigoplus_{s \in T} \text{Harm}_s^k$ for a well-determined subset T of $[0, k]$. This set T will be referred to as the *harmonic support* of Φ .

The β -regularity property means that β behaves like the all-one vector $\mathbf{1}$ with respect to the considered “test space” Φ . When β is the characteristic vector of a subset $\mathfrak{B}$ of $\mathbb{R}^{\mathcal{P}^k}$, our β -regularity concept reduces to the $\mathfrak{B}$ -regularity introduced in [2]. Note that $\mathfrak{B}$ is a t -design if and only if the space Hom_t^k is $\mathfrak{B}$ -regular.

Let us now introduce the important notion of the inner distribution [3, 4], with a suitable normalization.

Definition 4.2. The *inner distribution* of a design vector $\beta \in \mathbb{R}^{\mathcal{P}^k}$ is the $(k+1)$ -vector $\mathbf{b}^k = (b_0^k, b_1^k, \dots, b_k^k)$ with

$$\binom{k}{i} \binom{v-k}{i} b_i^k = \langle \beta, C_i^{k,k} \beta \rangle.$$

(If β is the characteristic vector of a set $\mathfrak{B}$, then the right-hand side counts the pairs of blocks in $\mathfrak{B}$ that have $k-i$ points in common.)

Consider the dual Hahn transform $\hat{\mathbf{b}}^k = \mathbf{b}^k Q^{k,k}$, of spherical type (k, k) , of the inner distribution $\mathbf{b}^k$ (see Definition 3.4). By specializing Theorem 3.2 to the spherical case ($C_0^{k,k} = I$), and by using the orthogonality relation in Theorem 3.1, we obtain the formula

$$\frac{v-2s+1}{v-s+1} \binom{v}{s} \hat{b}^k(s) = \binom{v}{k} \langle \beta, E_s^k \beta \rangle.$$

We now are in a position to give two characterizations of the T -design property, in terms of the inner distribution [3] and in terms of the β -regularity concept [2], respectively.

Theorem 4.1. (i) *The vector β is a T -design vector if and only if the dual Hahn transform of its inner distribution satisfies $\hat{b}^k(s) = 0$ for all $s \in T \setminus \{0\}$.* (ii) *The vector β is a T -design vector if and only if the S_v -invariant subspace of $\mathbb{R}^{\mathcal{P}^k}$ with harmonic support T is β -regular.*

Proof. Note first that $\langle \beta, E_s^k \beta \rangle = \|E_s^k \beta\|^2$, with $E_s^k \beta$ the orthogonal projection of β in Harm_s^k . Hence the formula above yields $\hat{b}^k(s) \geq 0$, for all $s \in [0, k]$, and $\hat{b}^k(s) = 0$ if and only if β is orthogonal to Harm_s^k . This proves the first statement. The second part is immediate (from Definition 4.1), since $\langle \mathbf{1}, \phi \rangle = 0$ for every $\phi \in \text{Harm}_s^k$ with $s \geq 1$. $\square$

We now point out an interesting application of Lemma 3.2 concerning the relationship between T -design vectors for two different Johnson schemes (with the same point set).

Proposition 4.1. *If β is a T -design vector for $J(v, k)$, then $\beta C_0^{k,t}$ is a $(T \cap [0, t])$ -design vector for $J(v, t)$. Conversely, if γ is a T -design vector for $J(v, t)$, then $\gamma(C_0^{k,t})^T$ is a $(T \cup [t+1, k])$ -design vector for $J(v, k)$.*

We now come back to the t -form concept (Definition 2.2). Let $\mathbf{f}^t \in \mathbb{R}^{t+1}$. With each t -subset x^t of V we associate the vector ξ^t (in $\mathbb{R}^{\mathcal{P}^k}$) defined by

$$\binom{v-t}{k-t} \xi^t = \sum_{i=0}^t f_i^t (C_i^{k,t}(\cdot, x^t))^T,$$

where $C_i^{k,t}(\cdot, x^t)$ denotes the x^t -column of the adjacency matrix $C_i^{k,t}$. It follows from Lemma 3.1 that ξ^t belongs to the homogeneous space Hom_t^k of degree t . Let $\Phi(\mathbf{f}^t)$ be the subspace of Hom_t^k generated by the vectors ξ^t with $x^t \in \mathcal{P}^t$. It is obvious that $\Phi(\mathbf{f}^t)$ is S_v -invariant.

By definition, $\mathbf{f}^t$ is a t -form for β if and only if the inner product $\langle \beta, \xi^t \rangle$ is independent of the choice of x^t . This means that $\Phi(\mathbf{f}^t)$ is a β -regular space (see Definition 4.1). It follows from Theorem 4.1 that $\mathbf{f}^t$ is a t -form for β if and only if β is an $S(\mathbf{f}^t)$ -design vector, where $S(\mathbf{f}^t)$ denotes the harmonic support of $\Phi(\mathbf{f}^t)$.

To identify this set $S(\mathbf{f}^t)$ we shall obtain another “ T -design characterization” of the t -form property. We argue as follows. From a given β we construct the vectors $\gamma_s^t = \beta E_s^k C_0^{k,t}$ in $\mathbb{R}^{\mathcal{P}^t}$, for $s \in [0, t]$. We have $\beta E_s^k \in \text{Harm}_s^k$, whence $\gamma_s^t \in \text{Harm}_s^t$ by Lemma 3.2. Since $D_i^t = \sum_{s=0}^t Q_i^{k,t}(s) \gamma_s^t$, by Theorem 3.2, we observe that $\mathbf{f}^t$ is a t -form for β if and only if its dual Hahn transform $\hat{\mathbf{f}}^t = \mathbf{f}^t Q^{k,t}$ satisfies the equation $\sum_{s=0}^t \hat{f}^t(s) \gamma_s^t = c \mathbf{1}$, with $c \in \mathbb{R}$. This amounts to $\hat{f}^t(s) \gamma_s^t = \mathbf{0}$ for each $s \in [1, t]$. Since $\gamma_s^t = \mathbf{0}$ if and only if $\beta E_s^k = \mathbf{0}$, this means that β is orthogonal to Harm_s^k for every s in $[1, t]$ that satisfies $\hat{f}^t(s) \neq 0$. As a conclusion we obtain the following result.

Theorem 4.2. *For any vector $\mathbf{f}^t \in \mathbb{R}^{t+1}$, let $T(\mathbf{f}^t)$ be the set containing the indices of the nonzero components of its dual Hahn transform $\hat{\mathbf{f}}^t = \mathbf{f}^t Q^{k,t}$, that is,*

$$T(\mathbf{f}^t) := \{s \in [0, t] \mid \hat{f}^t(s) \neq 0\}.$$

(i) *The set $T(\mathbf{f}^t)$ equals the harmonic support $S(\mathbf{f}^t)$ of the S_v -invariant subspace $\Phi(\mathbf{f}^t)$ of Hom_t^k generated by the vectors ξ^t with $x^t \in \mathcal{P}^t$. (ii) *The vector $\mathbf{f}^t$ is a t -form for a given β in $\mathbb{R}^{\mathcal{P}^k}$ if and only if β is a $T(\mathbf{f}^t)$ -design vector or, equivalently, if and only if the space $\Phi(\mathbf{f}^t)$ is β -regular.**

Proof. The essential part of the argument was given above. It remains only to show that we have $0 \in S(\mathbf{f}^t)$ if and only if $0 \in T(\mathbf{f}^t)$. This follows from the identity $\langle \mathbf{1}, \xi^t \rangle = \hat{f}^t(0)$, which is easily proved from the definition of ξ^t . $\square$

5. STRUCTURE OF THE t -FORM SPACE

In this section, it is shown how the t -form space of a design vector β can be characterized in terms of the inner distribution $\mathbf{b}^k$ of β (Theorems 5.1 and 5.3), and how the t -distribution matrix of β can be determined from its first r rows, with r denoting the t -degree of β (Theorem 5.2).

Definition 5.1. The t -annihilator set A^t for β is defined from the dual Hahn transform $\hat{\mathbf{b}}^k = \mathbf{b}^k Q^{k,k}$ of $\mathbf{b}^k$ by

$$A^t := \{s \in [1, t] \mid \hat{b}^k(s) \neq 0\}.$$

The t -annihilator polynomial $\hat{g}^t(z)$ for β , of degree $|A^t|$ in the variable

$$\lambda^t(z) = t(v - t) - z(v + 1 - z),$$

is defined by

$$\hat{g}^t(z) = \prod_{s \in A^t} (\lambda^t(z) - \lambda^t(s)).$$

Theorem 5.1. The t -degree r of a design vector β is equal to the cardinality of the annihilator set A^t for β . The t -form space $\mathcal{F}^t$ of β consists of the vectors $\mathbf{f}^t$ in $\mathbb{R}^{t+1}$ that satisfy $(\mathbf{f}^t Q^{k,t})(s) = 0$ for all $s \in A^t$. Equivalently, $\mathbf{f}^t$ belongs to $\mathcal{F}^t$ if and only if its dual Hahn transform polynomial $\hat{f}^t(z) = \sum_{i=0}^t f_i^t Q_i^{k,t}(z)$ is divisible by the t -annihilator polynomial $\hat{g}^t(z)$ for β .

Proof. From Theorems 4.1 and 4.2 it follows that $\mathbf{f}^t$ is a t -form for β if and only if $\hat{f}^t(s)\hat{b}^k(s) = 0$ for all $s \in [1, t]$, i.e., if and only if $\hat{f}^t(s) = 0$ for all $s \in A^t$. This means that $\hat{f}^t(z)$ is divisible by $\hat{g}^t(z)$. $\square$

Theorem 5.1 tells us that the dual Hahn transform image of the space $\mathcal{F}^t$ is an ideal of the algebra of real polynomials in the variable $\lambda^t(z)$, reduced modulo $m^t(z) = \prod_{s=0}^t (\lambda^t(z) - \lambda^t(s))$. This ideal contains a unique monic polynomial of minimum degree, namely the t -annihilator $\hat{g}^t(z)$. Note that $\hat{g}^t(z)$ divides the modulus polynomial $m^t(z)$. (In that respect, the only restriction is $\hat{g}^t(0) \neq 0$.) Conversely, any ideal of this type is the dual Hahn transform image of every design vector β that satisfies the condition $\hat{b}^k(s) = 0$ if and only if $\hat{g}^t(s) \neq 0$ for $s \in [1, t]$.

Definition 5.2. Let $\hat{g}^t(z)$ be the t -annihilator polynomial for a given design vector β . Consider the dual Hahn expansion $\hat{g}^t(z) = \sum_{i=0}^t g_i^t Q_i^{k,t}(z)$, with $g_r^t \neq 0$ and $g_i^t = 0$ for $i \in [r+1, t]$. The real vector $\mathbf{g}^t = (g_i^t)_{i=0}^t$ thus defined will be called the generator t -form for β (in agreement with Theorem 5.1).

The following theorem is an extension of a known result concerning the outer distribution matrix [3] to the more general concept of the t -distribution matrix D^t (see Definition 2.1).

Theorem 5.2. Let r be the t -degree of a given design vector β . For each $i \in [r, t]$, the row D_i^t of the t -distribution matrix of β is a rational linear combination of the $r+1$ vectors $\langle \beta, \mathbf{1} \rangle \mathbf{1}$, $D_0^t, \dots, D_{r-1}^t$, the coefficients of which can be calculated from the annihilator set for β .

Proof. Consider the generator t -form $\mathbf{g}^t$. In view of Definition 2.2 and Lemma 2.1, the row D_r^t of D^t can be expressed as follows:

$$D_r^t = (g_r^t)^{-1} \left[\binom{v}{t}^{-1} \binom{k}{t} \hat{g}^t(0) \langle \beta, \mathbf{1} \rangle \mathbf{1} - \sum_{i=0}^{r-1} g_i^t D_i^t \right].$$

We can obtain a similar result for the remaining rows of D^t by considering “shifted versions” $(\lambda^t(z))^j \hat{g}^t(z)$ of the t -annihilator polynomial, with $j \in [1, t - r]$. Details are omitted. $\square$

Remark 5.1. We briefly mention a combinatorial application. Let $\mathfrak{B}$ be a non-empty subset of $\mathcal{P}^k$. Define the t -covering radius of $\mathfrak{B}$ as the maximum for $x^t \in \mathcal{P}^t$ of the minimum for $x^k \in \mathfrak{B}$ of the integer $t - |x^k \cap x^t|$. It follows from Theorem 5.2 that the t -covering radius of $\mathfrak{B}$ is bounded from above by the t -degree of (the characteristic vector of) $\mathfrak{B}$. The reason is that the assumption $D_0^t(x^t) = \cdots = D_r^t(x^t) = 0$, for a given x^t , leads to a contradiction (when $\langle \beta, \mathbf{1} \rangle \neq 0$).

It is interesting to see how the structure of the t -form space can be derived in a purely combinatorial framework, without explicit reference to the dual Hahn polynomials. Consider the $(t+1) \times (t+1)$ tridiagonal matrix $W^{k,t} = [w_{i,j}^{k,t}]_{i,j=0}^t$ constructed from the intersection numbers $w_{i,j}^{k,t}$ met in the proof of Theorem 3.2. By use of Definition 2.1 we obtain $W^{k,t} D^t = D^t C_1^{t,t}$. If $\mathbf{f}^t$ is a t -form for β , i.e., if $\mathbf{f}^t D^t = c \mathbf{1}$, then this identity shows that $\mathbf{f}^t W^{k,t}$ also is a t -form for β . Therefore, $W^{k,t}$ is an endomorphism of the t -form space of any design vector β . In fact, we can state the following result, which is equivalent to Theorem 5.1. (The proof is omitted; see the comments below.)

Theorem 5.3. Let $W^{k,t} = [w_{i,j}^{k,t}]_{i,j=0}^t$. The t -form space $\mathcal{F}^t$ is $W^{k,t}$ -invariant. More precisely, if $\mathbf{g}^t$ is the generator t -form for β , then the $t+1-r$ vectors $\mathbf{g}^t (W^{k,t})^j$ with $j \in [0, t-r]$ form a basis of $\mathcal{F}^t$.

The connection between the settings of Theorems 5.1 and 5.3 can be described as follows. The recurrence relation for dual Hahn polynomials (Theorem 3.1) amounts to the matrix identity $W^{k,t} Q^{k,t} = Q^{k,t} \Lambda^t$, with $\Lambda^t = \text{diag}(\lambda^t(0), \dots, \lambda^t(t))$. Consider the $W^{k,t}$ -image $\mathbf{f}^t W^{k,t}$ of a given vector $\mathbf{f}^t$ in $\mathbb{R}^{t+1}$. In view of the identity above, the dual Hahn transform of this image vector equals $\hat{\mathbf{f}}^t \Lambda^t$, and the associated polynomial is the product $\hat{f}^t(z) \lambda^t(z)$ reduced modulo $m^t(z)$.

Remark 5.2. It is possible to work out the theory of t -forms for any value of t with $0 \leq t \leq \lfloor v/2 \rfloor$. Let us briefly examine the case $t \geq k$ (instead of $t \leq k$ as above). In this situation, we have to replace t by k in the definition of the adjacency matrices, roughly speaking. Thus, the t -form space $\mathcal{F}^t$ of a design vector β in $\mathbb{R}^{\mathcal{P}^k}$ now is a subspace of $\mathbb{R}^{k+1}$. It can be verified that the t -degree of β is independent of t , and counts the integers $s \in [1, k]$ that satisfy $\hat{b}^k(s) \neq 0$. The dual Hahn transform image of the t -form space of β is the same for all t (in the range $k \leq t \leq \lfloor v/2 \rfloor$). The appropriate transform is given by the polynomials $Q_i^{t,k}(z)$ with $i \in [0, k]$. This can be proved from the preceding results by use of the fact that the t -form space of β coincides with the k -form space of $\beta (C_0^{t,k})^T$, a vector in $\mathbb{R}^{\mathcal{P}^t}$.

Example 5.1. Let β be the design vector for the Steiner system $S(5, 8, 24)$, that is, the unique 5-design $\mathfrak{B}$ of index 1 with $v = 24$, $k = 8$, and $|\mathfrak{B}| = 759$. Its inner distribution vector (Definition 3.4) is well known and easily calculated as $\mathbf{b}^8 = 759(1, 0, 0, 0, 280, 0, 448, 0, 30)$. From Definition 3.4,

the components of the dual Hahn transform $\hat{\mathbf{b}}^8 = \mathbf{b}^8 Q^{8,8}$ of $\mathbf{b}^8$ are $\hat{\mathbf{b}}^8(0) = (3.11.23)^2$, $\hat{\mathbf{b}}^8(6) = (8.9.17.23)/13$, $\hat{\mathbf{b}}^8(8) = 5.8.23$, and $\hat{\mathbf{b}}^8(i) = 0$ for $i = 1, 2, 3, 4, 5, 7$. So from Theorem 4.1, $\boldsymbol{\beta}$ is a T -design vector for any $T \subseteq \{0, 1, 2, 3, 4, 5, 7\}$, and in particular, is a 5-design. From Definition 5.1, $A^8 = \{6, 8\}$, $A^7 = A^6 = \{6\}$, and $A^t = \emptyset$ for $0 \leq t \leq 5$. Hence from Definition 2.2 the respective t -degrees are $r^8 = 2$, $r^7 = r^6 = 1$, and $r^t = 0$ for $0 \leq t \leq 5$. For $t = 6, 7, 8$ the t -annihilator polynomials of type $(8, t)$ are, from Definitions 5.1, 5.2, and 3.3,

$$\hat{g}^6(z) = (6 - z)(19 - z) = 18Q_0^{8,6}(z) + 3Q_1^{8,6}(z) ,$$

$$\hat{g}^7(z) = (6 - z)(19 - z) = 2Q_0^{8,7}(z) + 2Q_1^{8,7}(z) ,$$

$$\hat{g}^8(z) = (6 - z)(19 - z)(8 - z)(17 - z) = 16Q_0^{8,8}(z) + 16Q_1^{8,8}(z) + 4Q_2^{8,8}(z) .$$

From Definition 2.2, the generator t -form identities directly deduced from these are:

$$6D_0^6(x^6) + D_1^6(x^6) = 6 ,$$

$$D_0^7(x^7) + D_1^7(x^7) = 1 ,$$

$$4D_0^8(x^8) + 4D_1^8(x^8) + D_2^8(x^8) = 4 ,$$

for every $x^t \in \mathcal{P}^t$. Note that the three equations above have 2, 2, and 3 nonnegative integer solutions respectively, namely $(1, 0)$ or $(0, 6)$, $(1, 0)$ or $(0, 1)$, and $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 4)$. Theorem 5.2 shows how to calculate the whole t -distribution matrix D^t . Note that D^t is a $(t+1) \times \binom{24}{t}$ matrix, but by the calculations above, has only $r^t + 1$ distinct columns! (It is interesting to note here that any of the three equations above would imply that $\mathfrak{B}$ is a Steiner system.) As in Remark 5.2, it is also possible to calculate the t -form space for $9 \leq t \leq 12$, with t -degree $r^t = 2$. As an illustration,

$$20D_0^{12}(x^{12}) + 6D_1^{12}(x^{12}) + D_2^{12}(x^{12}) = 132 .$$

(Remember that $D_i^{12}(x^{12})$ counts the blocks of $\mathfrak{B}$ that have $8 - i$ points in common with a given 12-subset x^{12} of the 24-set V .)

Example 5.2. Let us briefly examine the simple situation characterized by $r^t = 1$, i.e., by $A^t = \{s_0\}$ for some s_0 with $1 \leq s_0 \leq t \leq k$. According to Theorem 5.1, this corresponds to T -design vectors $\boldsymbol{\beta}$ with $T = [1, t] \setminus \{s_0\}$. The t -annihilator polynomial is $\hat{g}^t(z) = (s_0 - z)(v + 1 - s_0 - z)$. From its dual Hahn expansion we obtain the following generator t -form identity:

$$\begin{aligned} [s_0(v + 1 - s_0) - t(v - k)]D_0^t(x^t) + (k + 1 - t)D_1^t(x^t) \\ = \binom{v}{t}^{-1} \binom{k}{t} s_0(v + 1 - s_0) \langle \boldsymbol{\beta} , \mathbf{1} \rangle . \end{aligned}$$

The first two cases of Example 5.1 above illustrate this result, with $(t, s_0) = (6, 6)$ and $(7, 6)$, respectively.

Example 5.3. We now give a “combinatorial construction” in the special case $s_0 = 1$. Suppose that $\mathfrak{B}_0$ is a t -design for $J(v - 1, k)$ and $\mathfrak{B}_1$ is a t -design for $J(v - 1, k - 1)$, with the same point set V_0 (of cardinality $v - 1$). Consider

the combinatorial structure in $J(v, k)$ with the point set $V = V_0 \cup \{\infty\}$ and with the block set $\mathfrak{B} = \mathfrak{B}_0 \cup \mathfrak{B}'_1$, where $\mathfrak{B}'_1$ consists of the k -sets $x \cup \{\infty\}$ with $x \in \mathfrak{B}_1$. We suppose that $\mathfrak{B}$ is not a 1-design. Then, the pair $(D'_0(x'), D'_1(x'))$ for $\mathfrak{B}$ assumes exactly two distinct values, one value in the case $\infty \in x'$ and one (different) value in the other case. Hence there exists a nontrivial identity $g'_0 D'_0(x') + g'_1 D'_1(x') = c$ (for all $x' \in \mathcal{P}'$). The pair (g'_0, g'_1) is uniquely determined, within normalization, and g'_1 is not zero. Elementary "design computation" yields $g'_0 = v - t(v - k)$ and $g'_1 = k + 1 - t$. Therefore, our construction exemplifies the situation examined above, with $s_0 = 1$. As a conclusion, $\mathfrak{B}$ is a $[2, t]$ -design for $J(v, k)$, although it is not a t -design.

ACKNOWLEDGMENT

The authors express their gratitude to the anonymous referee whose constructive remarks led to a more compact and systematic presentation of their results.

REFERENCES

1. A. R. Calderbank and P. Delsarte, *On error-correcting codes and invariant linear forms*, SIAM J. Discrete Math. **6** (1993), 1–23.
2. A. R. Calderbank, P. Delsarte, and N. J. A. Sloane, *A strengthening of the Assmus-Mattson theorem*, IEEE Trans. Information Theory **IT-37** (1991), 1261–1268.
3. P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. **10** (1973).
4. ———, *Pairs of vectors in the space of an association scheme*, Philips Res. Rep. **32** (1977), 373–411.
5. ———, *Hahn polynomials, discrete harmonics, and t -designs*, SIAM J. Appl. Math. **34** (1978), 157–166.
6. C. F. Dunkl, *Spherical functions on compact groups and applications to special functions*, Sympos. Math. **22** (1977), 145–161.
7. ———, *An addition theorem for Hahn polynomials: the spherical functions*, SIAM J. Math. Anal. **9** (1978), 627–637.
8. R. L. Graham, S.-Y. R. Li, and W.-C. W. Li, *On the structure of t -designs*, SIAM J. Algebraic Discrete Methods **1** (1980), 8–14.
9. J. E. Graver and W. B. Jurkat, *The module structure of integral designs*, J. Combin. Theory Ser. A **15** (1973), 75–90.
10. S. Karlin and J. L. McGregor, *The Hahn polynomials, formulas and an application*, Scripta Math. **26** (1961), 33–46.
11. D. K. Ray-Chaudhuri and N. M. Singhi, *On existence of t -designs with large v and λ* , SIAM J. Discrete Math. **1** (1988), 98–104.
12. R. M. Wilson, *Inequalities for t -designs*, J. Combin. Theory Ser. A **34** (1983), 313–324.
13. ———, *On the theory of t -designs*, Enumeration and Designs (D. M. Jackson and S. A. Vanstone, eds.), Academic Press, New York, 1984, pp. 19–49.

MATHEMATICAL SCIENCES RESEARCH CENTER, AT&T BELL LABORATORIES, MURRAY HILL, NEW JERSEY 07974

E-mail address: rc@research.att.com

DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF LOUVAIN, PLACE SAINTE BARBE 2, B1348 LOUVAIN LA NEUVE, BELGIUM

E-mail address: delarte@info.ucl.ac.be