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LOOP SPACE HOMOLOGY OF SPACES OF SMALL CATEGORY

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ABSTRACT. Only little is known concerning $H_*(\Omega X; \mathbf{k})$, the loop space homology of a finite CW complex X with coefficients in a field $\mathbf{k}$. A space X is called an r-cone if there exists a filtration $* = X_0 \subset X_1 \subset \cdots \subset X_r = X$, such that X_i has the homotopy type of the cofibre of a map from a wedge of sphere into X_{i-1}. Denote by A_X the sub-Hopf algebra image of $H_*(\Omega X_1)$. We prove then that for a graded r-cone, $r \leq 3$, there exists an isomorphism $A_X \otimes T(U) \xrightarrow{\cong} H_*(\Omega X)$.

INTRODUCTION

The structure of $H_*(\Omega X; \mathbf{k})$, the loop space homology of a finite CW complex X with coefficients in a field $\mathbf{k}$, is an exciting and interesting subject. In particular it has been conjectured that either the growth of $H_*(\Omega X; \mathbf{k})$ is polynomial or else that $H_*(\Omega X; \mathbf{k})$ contains a free Lie algebra on two generators. On the other hand some important results concerning the depth of $H_*(\Omega X; \mathbf{k})$ have been recently obtained by Halperin, Lemaire, and the authors [8].

In this note we precise the structure of the Pontrjagin algebra, $H_*(\Omega X; \mathbf{k})$, when X is an r-cone with r small.

Recall that an r-cone, $r \geq 0$ (resp. a *finite r-cone*) X is a sequence

$$* = X_0 \subset X_1 \subset \cdots \subset X_r = X,$$

where X_i, $i \geq 1$, is the cofiber of a map g_i, $W_i \xrightarrow{g_i} X_{i-1} \xrightarrow{f_i} X_i$, with W_i a wedge of spheres (resp. a wedge of finitely many spheres) and $(f_i)_*\colon \widetilde{H}_*(X_{i-1}; \mathbf{k}) \to \widetilde{H}_*(X_i; \mathbf{k})$ is trivial.

The image A_X of $H_*(\Omega X_1; \mathbf{k})$ into $H_*(\Omega X; \mathbf{k})$ is a graded sub-Hopf algebra of $H_*(\Omega X; \mathbf{k})$, isomorphic to the quotient Hopf algebra $H_*(\Omega X_1; R)/\mathrm{Ker}(\Omega f_2)_*$ when X is a 2-cone.

An r-cone is a space of Lusternik-Schnirelmann category (LS-category) less than or equal to r.

A space X of LS-category one is a co-H-space. The Bott-Samelson theorem shows that $H_*(\Omega X; \mathbf{k})$ is then a tensor algebra, $T(V)$, with

$$V = \bigoplus_{i \geq 0} V_i \quad \text{and} \quad V_i = \widetilde{H}_{i+1}(X; \mathbf{k}).$$

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In the study of the loop space homology of spaces, the first nontrivial case is given by the 1-connected CW-complex of finite type of category 2. In this case:

(a) the numbers $\sum_{i=0}^{n} \dim H_i(\Omega X ; \mathbf{k})$ grow either polynomially or else the algebra $H_*(\Omega X ; \mathbf{k})$ contains a tensor algebra on two generators [9].

(b) When $\mathbf{k} = \mathbb{Q}$, there exists a short exact sequence of graded Lie algebras,

$$0 \to \mathbb{L}(U) \to \pi_*(\Omega X) \otimes \mathbb{Q} \to L_X \to 0,$$

with $A_X = \mathbf{U}L_X$ [11].

We will precise (a) (Theorem 1), and generalize (b) for some spaces of LS-category less or equal to 4 (Theorems 2 and 3).

In order to make our results more precise, we have to introduce what we shall call algebraic r-cones.

Recall that a *free model* for an augmented chain R-algebra A is a chain algebra map $\varphi\colon (T(V), d) \to A$ such that

(i) $V = \bigoplus_{i \geq 0} V_i$

(ii) φ induces an isomorphism $\varphi_*\colon H(T(V), d) \to H(A)$.

When X is a 1-connected CW-complex, Adams and Hilton [1] have already established the existence of a model of the chain algebra, $A = C_*(\Omega X ; \mathbf{k})$, consisting of the singular cubical chains modulo degeneracies. As shown in [12, Appendix], one can choose V such that

$$V_i \cong \widetilde{H}_{i-1}(X ; \mathbf{k}), \qquad i \geq 2.$$

In this case the chain algebra $(T(V), d)$ is called *minimal*.

The same inductive construction as in [1], following the cone filtration instead of the skeleton filtration gives a filtered free model for a 1-connected r-cone X. In this case, V is a bigraded vector space satisfying

(a) $V = \bigoplus_{0 \leq i \leq r-1,\, 0 \leq j} V_{i,j}$, $d(V_{i,j}) \subset \bigoplus_{s<i,\, s+t=i+j-1} (T(V))_{s,t}$,

(b) $H_*(\Omega X_i ; \mathbf{k}) = H(T(V_{\leq i}), d)$ for $i \geq 0$,

(c) $A_X = T(V)/(d(V) + V_{+,*})$ is a sub-Hopf algebra of the graded connected cocommutative Hopf algebra $H(T(V), d)$.

By definition, a space X such that $C_*(\Omega X ; \mathbf{k})$ admits a free model, $(T(V), d)$, satisfying the conditions (a) and (c) is called an *algebraic r-cone over* $\mathbf{k}$ *with associated model* $(T(V), d)$.

If, moreover each V_i is a graded vector space of finite type (resp. if $\dim V_i < \infty$ for any i) we shall say that X is an *algebraic r-cone of finite type* (*resp. a finite algebraic r-cone*).

Clearly, any r-cone is an algebraic r-cone over any field $\mathbf{k}$. The converse is false since, for instance for $r = 1$, there exists co-H-spaces which are not wedges of spheres.

Among the class of algebraic r-cones we distinguish those satisfying the strengthened condition:

(a′) $d(V_{i,j}) \subset (T(V))_{i-1,j}$.

These algebraic r-cones will be called *graded r-cones*. Clearly an algebraic 2-cone is always a graded 2-cone.

For a graded r-cone, X, over $\mathbf{k}$, the homology of ΩX is a bigraded vector space:

$$H_*(\Omega X ; \mathbf{k}) = \bigoplus H_{i,j}(\Omega X ; \mathbf{k}).$$

The integer i is called the filtered degree and $i+j$ the total degree. Later on, we shall simply denote $H_{i,*}(\Omega X;\mathbf{k})$ by H_i. In particular $H_0 = A_X$. We can now state

Theorem 1. *If X is a 1-connected algebraic 2-cone of finite type over a field $\mathbf{k}$, then there exists a graded subspace U of H_1 such that*

(a) *$T(U)$ is a subalgebra of $H_*(\Omega X;\mathbf{k})$,*
(b) *The multiplication law induces two linear isomorphisms*

$$T(U)\otimes A_X \xrightarrow{\cong} H_*(\Omega X;\mathbf{k}) \xleftarrow{\cong} A_X\otimes T(U),$$

(c) *If $\dim H_*(X;\mathbf{k})\geq 3$ then either $U=0$ or else $\dim U\geq 2$.*

Notice that $U=0$ if and only if there is an isomorphism of Hopf algebras

$$H_*(\Omega X;\mathbf{k}) \xrightarrow{\cong} H_*(\Omega X_0;\mathbf{k})/\mathrm{Ker}(\Omega f)_* = A_X.$$

In this case the attaching map g_2 is called $\mathbf{k}$-*inert* [12] or *strongly free* [3].

Example 1. Let X be the 6-skeleton of the product $S^3_a\times S^3_b\times S^3_c$, i.e., the *fat wedge* of the three spheres S^3_a, S^3_b, S^3_c. Then $H_*(\Omega X;\mathbf{k})\cong A_X\otimes T(U)$; $A_X=\mathbf{k}[a,b,c]$, with $\deg(a)=\deg(b)=\deg(c)=2$ and $U=\alpha\cdot\mathbf{k}\otimes A_X$, with $\deg(\alpha)=7$.

Example 2. The restriction of the suspension homomorphism

$$\sigma_*\colon H_*(\Omega X;\mathbf{k})\to H_{*-1}(X;\mathbf{k})$$

to V_0 is an injective map. Denote by K_σ the kernel of the linear homomorphism induced by σ_* on the vector space of indecomposable elements of $H_*(\Omega X)$:

$$0\to K_\sigma\to Q(H_*(\Omega X))\xrightarrow{\sigma} H_{*-1}(X).$$

This yields the following corollary.

Corollary. *The subalgebra of $H_*(\Omega X;\mathbf{k})$ generated by K_σ is free: $T(K_\sigma)$ injects into $H_*(\Omega X)$.*

Theorem 2. *If X is a 1-connected finite graded 3-cone over a field $\mathbf{k}$ with associated free model $(T(V),d)$, then there exist U_1 and U_2 respectively subspaces of H_1 and H_2 such that $T(U_1\oplus U_2)$ is a free subalgebra of $H_*(\Omega X;R)$ and such that the multiplication law induces two linear isomorphisms*

$$A_X\otimes T(U_1\oplus U_2)\xrightarrow{\cong} H_*(\Omega X;\mathbf{k})\xleftarrow{\cong} T(U_1\oplus U_2)\otimes A_X.$$

Over $\mathbb{Q}$, we have

$$H_*(T(V),d)\cong H_*(\Omega X;\mathbb{Q})\cong \mathrm{U}(\pi_*(\Omega X\otimes\mathbb{Q}))\quad\text{and}\quad A_X=\mathrm{U}L_X.$$

In particular, the rational homotopy Lie algebra of a graded 3-cone appears as an extension of L_X by a free graded Lie algebra,

$$0\to\mathbb{L}(U_1\oplus U_2)\to\pi_*(\Omega X)\otimes\mathbb{Q}\to L_X\to 0.$$

Theorem 3. *If X is a 1-connected finite graded 4-cone over $\mathbb{Q}$, then the structure of $\pi_*(\Omega X)\otimes\mathbb{Q}$ is given by the following two short exact sequences:*

$$0\to P\to\pi_*(\Omega X)\otimes\mathbb{Q}\to L_X\to 0,$$
$$0\to\mathbb{L}(U)\to P\to N\to 0,$$

where N is the quotient of a free Lie algebra by an ideal generated by quadratic elements. Moreover, if X is a 4*-cone the generators of N are in the image of the canonical map $H_*(\Omega X_2; \mathbb{Q}) \to H_*(\Omega X_4; \mathbb{Q}) = H_*(\Omega X; \mathbb{Q})$.*

To prove these theorems, we use the I-adic spectral sequence [5]. For the convenience of the reader some of its properties are recalled in §2. In the first section we prove that algebraic r-cones X satisfy $\text{M-cat}(X) \leq r$ (see below for the definition). The other sections are devoted to the proofs of Theorems 1 to 3.

I. M-CATEGORY OF AN r-CONE

It is well known that the LS-category of an r-cone is less or equal to r. In this chapter we establish an analogous result for algebraic r-cones.

In [12], Halperin and Lemaire have introduced a new and very powerful approximation of the category. They called it M-cat. We first recall the definition of M-cat X for a simply connected space X.

Consider a free model $(T(W), d)$ of $C^*(X; \mathbf{k})$ in the category of cochain algebras. Any cochain algebra map $(T(W), d) \to (A, d_A)$ factors as $(T(W), d) \to (T(W \oplus U), d) \xrightarrow{\cong} (A, d_A)$. In particular, for any m we can factor the quotient map

$$\rho_m : (T(W), d) \to (T(W)/T^{>m}(W), \bar{d})$$

as

$$(T(W), d) \xrightarrow{j} (T(W \oplus U_m), d) \xrightarrow{\cong} (T(W)/T^{>m}(W), \bar{d}).$$

Definition. $\text{M-cat}(X) = \text{M-cat}((T(V), d))$ is the least integer m such that there exists a morphism

$$r: (T(W \oplus U_m), d) \to (T(W), d)$$

of left $(T(V), d)$ differential modules such that $rj = \mathrm{Id}_{T(V)}$.

The M-category is an approximation of the category in the sense that [12] $\text{M-cat}(X) \leq \text{cat}(X)$. We now consider a graded algebra,

$$R = \mathbf{k} \oplus \bigoplus_{k \geq 1} R^k.$$

The *global dimension of* R (possibly ∞) is the largest integer n such that $\text{Ext}_R^n(\mathbf{k}, \mathbf{k}) \neq 0$. The *depth of* R (possibly ∞) is the least integer n such that $\text{Ext}_R^n(\mathbf{k}, R) \neq 0$.

Now, by [8, Theorem A′] we have

Proposition 1.1. *Let X be a simply connected space, $\mathbf{k}$ a field, and assume that each $H_i(X; \mathbf{k})$ is finite dimensional. Then*

(1) $\operatorname{depth} H_*(\Omega X; \mathbf{k}) \leq \text{M-cat}\, X \leq \operatorname{gl\,dim} H_*(\Omega X; \mathbf{k})$.
(2) *If* $\operatorname{depth} H_*(\Omega X; \mathbf{k}) = \text{M-cat}\, X$ *then*

$$\operatorname{depth} H_*(\Omega X; \mathbf{k}) = \operatorname{gl\,dim} H_*(\Omega X; \mathbf{k}).$$

Here we prove

Proposition 1.2. *Let X be a simply connected space, $\mathbf{k}$ a field, and assume that each $H_i(X;\mathbf{k})$ is finite dimensional. If X is an algebraic n-cone then* $\text{M-cat}(X) \leq n$.

Proof of Proposition 1.2. Let $(T(V), d) \to C_*(\Omega X;\mathbf{k})$ be a free model associated to the structure of an algebraic n-cone. For every graded vector space E, we will denote by $E^\vee$ the graded dual of E. The letter $\underline{B}$ will denote the (reduced) bar construction. We then have the following quasi-isomorphisms of differential graded coalgebras:

$$\underline{B}(T(V), d) \to \underline{B}(C_*(\Omega X;\mathbf{k})) \leftarrow C_*(X;\mathbf{k}).$$

Write $(T(W), d^\vee) = (\underline{B}(T(V), d))^\vee$ the dual cochain algebra. By definition of $\underline{B}$, $W^{i,j} \cong (T^+(V))_{i-1,j}$ and $d^\vee$ is a differential of bidegree $(1, 0)$.

Let $M = \mathbf{k} \oplus T^{n+1}(W)$, then $M^{i,*} = 0$ for $0 < i \leq n$. On the other hand, by [8, proof of Theorem A′], we have a differential D and a quasi-isomorphism of $(T(W), d^\vee)$-differential modules:

$$(T(W) \otimes M, D) \xrightarrow{\varphi} (T(W)/T^{>n}(W), \overline{d^\vee}).$$

$\text{M-cat}(X) \leq n$ if and only if there exists a morphism

$$r\colon (T(W) \otimes M, D) \to (T(W), d^\vee)$$

of differential $(T(W), d^\vee)$-modules such that the composite

$$(T(W), d^\vee) \rightarrowtail (T(W) \otimes M, D) \xrightarrow{r} (T(W), d^\vee)$$

equals $\mathrm{Id}_{T(V)}$.

By [10] we know that the composite q:

$$\begin{aligned}\underline{B}(T(V), d)^\vee &= (T(W), d^\vee) \to (T(W)/T^{>2}(W), d^\vee)\\ &= (\mathbf{k} \oplus W, d') \to (\mathbf{k} \oplus sV, sd_V)^\vee\end{aligned}$$

is a quasi-isomorphism of bigraded differential vector spaces. Thus

$$H^{i,*}(T(W), d^\vee) = 0$$

for $i > n$ and any cocycle in $T^{n+1}(W)$ is a coboundary. This makes possible the construction of a retraction r. □

II. The I-adic spectral sequence

The I-adic spectral sequence is defined in [5]. For the convenience of the reader, we recall here its construction and main properties.

Let $(T(V), d)$ be a chain algebra over a field $\mathbf{k}$ such that $V = \bigoplus_{i\geq 0} V_i$ is a graded vector space satisfying $d(V_i) \subset T(V)_{i-1}$. We denote by I the kernel of the composite $T(V) \to T(V_0) \to H_0$. This gives a short exact sequence of graded algebras: $I \to T(V) \to H_0$. It appears that I is a free left (and right) $T(V)$-module: $I = T(V) \otimes M$ for some subspace M of I.

The powers of the ideal I define an increasing filtration

$$F_0 = T(V), \qquad F_{-p} = I^p, \quad p > 0,$$

which generates a second quadrant spectral sequence of H_0-algebras satisfying:

(1) $E^0_{-p,q} = (I^p/I^{p+1})_{-p+q} \cong (H_0 \otimes (\bigotimes^p M))_{-p+q}$.

(2) The multiplication law induces an isomorphism of H_0-modules $\bigotimes^p_{H_0}(I/I^2) \to (I^p/I^{p+1}) = E^0_{p,*}$ and $(I/I^2)_n \cong H_0 \otimes V_n \otimes H_0$ for any $n > 0$.
(3) $E^1_{-p,q} = 0$ if $q < 2p$, therefore the I-adic spectral sequence converges.
(4) If $V = V_{\leq n}$, then $E^0_{-p,q} = 0$ for $q > p \cdot (n+1)$.
(5) If $H_r(T(V), d) = 0$ for $0 < r < m$, then $E^1_{-p,q} = 0$ for $q < r \cdot (m+1)$.

Remark that property (3) gives for $n \leq 2p$ the short exact sequences

$$E^0_{-p,n} \cap \ker d_0 \rightarrowtail E^0_{-p,n} \xrightarrow{d_0} E^0_{-p,n-1} \cap \ker d_0.$$

As I^n/I^{n+1} is a free H_0-module, $E^0_{-n,0} = E^0_{-n,0} \cap \ker d_0$ is a projective H_0-module. An induction on n shows then that, for each $n \leq 2p$, $E^0_{-p,n} \cap \ker d_0$ is a projective H_0-module.

Lemma 2.1. *Let $(T(V), d)$ be a chain algebra over a field $\mathbf{k}$ such that $V = V_0 \oplus V_1$ is a vector space, $d(V_i) \subset T(V)_{i-1}$. Then there exists a natural isomorphism of algebras $T_{H_0}(H_1(T(V))) \to E^1_{*,*}$.*

Proof of Lemma 2.1. By properties (3) and (4) $E^1_{p,q} = 0$ if $q \neq 2p$. Therefore the I-adic spectral sequence collapses at the E^1 level and

$$H_p(T(V), d) = E^1_{-p,2p}.$$

Now, $E^1_{-1,2} = E^0_{-1,2} \cap \ker d_0$ is a projective H_0-module.

On the other hand, the natural inclusion $E^1_{-1,2} \to E^0_{-1,2}$ extends to a homomorphism of differential R-algebras

$$(T_{H_0}(E^1_{-1,2}), 0) \to (T_{H_0}(E^0_{-1,2}), d_0) \cong (E^0_{*,*}, d_0).$$

Thus, an induction on the tensor powers and the Künneth spectral sequence show that this homomorphism induces an isomorphism $T_{H_0}(E^1_{-1,2}) \xrightarrow{\cong} E^1_{*,*}$ and thus an isomorphism of algebras

$$T_{H_0}(H_1(T(V))) \xrightarrow{\cong} E^1_{*,*}. \quad \square$$

III. Proof of Theorem 1

Consider a minimal free model associated to an algebraic 2-cone X. We thus have

(i) $V = V_0 \oplus V_1$, $d(V_0) = 0$,
(ii) $V_0 = \bigoplus_{1 \leq i \leq g} V_{0,i}$, $V_1 = \bigoplus_{1 \leq j \leq r} V_{1,j-1}$,
(iii) $d(V_0) = 0$, $d(V_{1,j}) \subset (T(V))_{0,j}$,
(iv) $H_0(T(V), d) \cong A_X$.

The first two parts of Theorem 2 are now easy consequences of Lemma 2.1.

Indeed,

$$H_*(\Omega X; \mathbf{k}) \cong T_{H_0}(H_1(T(V))),$$

and since $H_1(T(V))$ is a free left and right H_0-module [14, Lemma 5], we put $H_1(T(V)) = U \otimes A_X$ or $A_X \otimes U$.

The last part of Theorem 1 requires some preliminaries:

Lemma 3.1. *If A_X is an algebra of global dimension* 2 *then* $A_X = H(T(V), d)$.

Proof of Lemma 3.1. Any graded connected chain algebra with trivial differential, $(B, 0)$ admits a free bigraded model [12], $f: (T(Z), d) \xrightarrow{\simeq} (B, 0)$ which is a bigraded differential chain algebra satisfying:

(1) $Z = \bigoplus_{p\geq 0, q\geq 0} Z_{p,q}$.

(2) $d(Z_{p,q}) \subset T^{\geq 2}(Z)_{p-1,q}$.

(3) $H_{0,*}(T(Z), d) \cong B$, $H_{i,*}(T(Z), d) = 0$, $i > 0$.

(4) $(T(Z), d)$ is unique up to an isomorphism of bigraded differential algebras.

Following Adams and Hilton [1], we can associate to $(T(Z), d)$ an acyclic $(T(Z), d)$-module $C = T(Z) \otimes (\mathbf{k} \oplus sZ)$ with a differential D extending d and defined by

(1) $D(sz) = z - s(dz)$, $z \in Z$,

(2) $s(sz) = 0$, $z \in Z$, $s(xy) = (-1)^{\deg(x)} x s(y)$, $x, y \in T(Z)$.

The chain complex

$$B \otimes (\mathbf{k} \oplus sZ) = B \otimes_{T(Z)} C$$

is then a resolution of the field $\mathbf{k}$ by free B-modules:

$$\cdots B \otimes s(Z_2) \xrightarrow{d_2} B \otimes s(Z_1) \xrightarrow{d_1} B \otimes s(Z_0) \xrightarrow{d_0} B \to \mathbf{k}.$$

Denote by z an element of Z_p. We have

$$D(sz) = z - s(dz).$$

Write $d(z) = \sum_i \nu_i \beta_i + \sum_j \gamma_j \cdot z_j$, with β_i a basis of Z_0, and z_j a basis of $\bigoplus_{i\geq 1} Z_i$. This implies

$$d_p s(z) = \sum_j f(\gamma_j) \cdot s(z_j), \qquad f(\gamma_i) \in B_+.$$

Thus, $\mathrm{Tor}_p^B(\mathbf{k}, \mathbf{k}) = sZ_{p-1}$.

Since the global dimension of B is less or equal to 2 and since $d(Z) \subset T^{\geq 2}(Z)$, this last relation implies that $Z_p = 0$ for $p \geq 2$.

Now, for $B = A_X$, the two minimal models $(T(Z_{\leq 1}), d)$ and $(T(V), d)$ are isomorphic [12, Appendix], and therefore $H_p = 0$ for any $p > 0$. □

Lemma 3.2. *Let G be a graded connected cocommutative Hopf algebra of finite type over a field* $\mathbf{k}$. *Then the depth of G is zero if and only if* $\dim G < \infty$.

Proof of Lemma 3.2. As

$$\mathrm{Ext}_G^0(\mathbf{k}; \mathbf{k}) = \mathrm{Hom}_G(\mathbf{k}, G),$$

one see that $\mathrm{depth}(G) = 0$ if and only if the annihilator of G_+, $\mathrm{Ann}\, G$, is nonzero.

$$\mathrm{Ann}\, G = \{x \in G | gx = 0 \text{ for all } g \in G_+\}.$$

Clearly, if $\dim G < \infty$ then $\mathrm{depth}(G) = 0$.

We now prove the converse, and we suppose $\mathrm{depth}(G) = 0$.

(a) If $\mathbf{k} = \mathbb{Q}$. The Milnor-Moore theorem implies that $G = \mathbf{U}L$ for some Lie algebra $L = \bigoplus_{i\geq 0} L_i$. In this case $\mathrm{depth}(G) = \dim_{\mathbf{k}} L_{\mathrm{even}}$ [7]. Thus, L is concentrated in odd degrees and the Poincaré-Birkoff-Witt theorem [14] implies that $\dim_{\mathbf{k}} UL < \infty$.

(b) If $\mathbf{k} = \mathbb{Z}_p$ for a fixed prime p. Let $\overline{\Delta}\colon G_+ \to G_+ \otimes G_+$ be the reduced diagonal. Define $\overline{\Delta}_n\colon G_+ \to G_+ \otimes G_+ \otimes \cdots \otimes G_+$ ($n+1$ times) by

$$\overline{\Delta}_0 = \overline{\Delta}, \qquad \overline{\Delta}_{n+1} = (\overline{\Delta} \otimes 1 \cdots 1 \otimes 1) \circ \overline{\Delta}_n$$

and

$$F^0 G = R, \qquad F^{n+1} G = \operatorname{Ker} \overline{\Delta}_n, \quad n \geq 0.$$

$(F^p G)$ is an increasing filtration of G: $F^p G \subset F^{p+1} G$, $\bigcup_{p \geq 0} F^p G = G$.

Set $A^p = F^p G / F^{p-1} G$, $A = \bigoplus_p A^p$. Then by [6, 1.3] the Hopf algebra A is both commutative and cocommutative and $\xi(A_+) = 0$ where $\xi(x) = x^p$.

The Borel theorem implies that the algebra A is isomorphic to $\bigotimes A(i)$ where the x_i are generators of A and $A(i) = \mathbb{Z}_p[x_i]$ or $\mathbb{Z}_p[x_i]/x_i^p$ if x_i has even degree and $\bigwedge x_i$ if x_i has odd degree or $p = 2$.

Now, by our assumption, the annihilator of A is nonzero which implies that $\dim_{\mathbf{k}} A = \dim_{\mathbf{k}} G$ is finite. $\square$

End of the proof of Theorem 1. We denote by $(T(V), d)$ a free model that gives to X its algebraic 2-cone structure. We put $A = H_*(\Omega X; \mathbf{k}) = H(T(V), d)$ and recall that A_X is a sub-Hopf algebra of A which is not necessarily a normal subalgebra.

By Propositions 1.1 and 1.2, we reduce the proof to three cases.

First case. The global dimension of A is 2 and $\dim A_X = \infty$.

The global dimension of A_X is then 2 and Lemma 3.1 implies that $H_1 = H_1(T(V), d) = 0$ and thus part (b) of Theorem 1 implies $U = 0$. $\square$

Second case. Depth of $A = 1$ and $\dim A_X = \infty$.

We suppose that U is generated by only one element u and we will construct a contradiction. As $T(u)$ is a normal subalgebra, we have isomorphisms

$$A_X \otimes T(u) \xrightarrow{\cong} A \xleftarrow{\cong} T(u) \otimes A_X.$$

This defines a linear automorphism of A_X, $b \to b^u$ satisfying $u \cdot b - b^u \cdot u \in (u^2)$ where (u^2) denotes the two-sided ideal of A generated by u^2.

On the other hand,

$$\operatorname{Ext}^1_A(\mathbf{k}, A) = \operatorname{Hom}_A(A_+, A)/\operatorname{Hom}_A(A, A) \neq 0.$$

We denote by $\beta\colon A_+ \to A$ a morphism of left A-modules representing a nontrivial element of $\operatorname{Ext}^1_A(\mathbf{k}, A)$, i.e., β does not extend into an A-linear endomorphism of A.

By linearity, $u \cdot \beta(b) - b^u \cdot \beta(u)$ is an element of the right A_X-submodule $(u^2) + T^+(u) \otimes A_X$ for any $b \in (A_X)_+$. We decompose the element $\beta(u)$ as

$$\beta(u) = b_0 + ua, \qquad b_0 \in A_X, \ a \in A.$$

Necessarily,

$$b^u \cdot b_0 = 0 \quad \text{for any } b \in (A_X)_+.$$

As $\dim A_X = \infty$, Lemma 3.2 implies $b_0 = 0$. Now the relation $ub = b'u$ associates to any $b \in A_X$ some element $b' \in A$. This leads to the relations

$$u\beta(b) = b'\beta(u) = b'ua = uba.$$

As A is a free left $T(u)$-module, this gives $\beta(b) = ba$ for any $b \in A_X$. This yields a contradiction with the assumption of nontriviality of the cocycle β.

Thus, $b_0 \neq 0$. By the above relation, the element b_0 belongs to the annihilator of $(A_X)_+$ and by Lemma 2, $\dim A_X < \infty$. $\square$

Third case. $\operatorname{Dim} A_X < \infty$.

The Poincaré series of a graded vector space $M = \bigoplus_i M_i$ will be denoted by $P(M) = \sum_i \dim M_i t^i$. In the same way, the Koszul-Poincaré series of a bigraded vector space N will be denoted by

$$KP(N) = \sum_{i,j} (-1)^i \dim N_{i,j} t^{j-i}.$$

From the obvious relation, $KP(T(V)) = KP(H(T(V), d))$, we deduce

$$\sum_i (-t)^{-i} P(T(V)_i) = \sum_i (-t)^{-i} P(H_i).$$

On the other hand, as $V = V_0 \oplus V_1$, with $V_0 = \bigoplus_{1 \le i \le g} v_i \mathbf{k}$, degree of $v_i = m_i$ and $V_1 = \bigoplus_{1 \le j \le r} v'_j \mathbf{k}$, degree of $v'_j = n_j + 1$, we have

$$\sum_r (-t)^r P(T(V)_r) = (Q(t))^{-1}, \quad \text{with } Q(t) = 1 - \sum_{i \in I} t^{m_i} + \sum_{j \in J} t^{n_j}.$$

We have also (since $u \in H_{1,s}$ and A_X is finite dimensional):

$$\sum_i (-t)^i P(H_i) = P(A_X)/(1 + t^s),$$

where $P(A_X)$ is a polynomial. Thus,

$$P(A_X) Q(t) = (1 + t^s).$$

So, as $P(A_X)$ has nonnegative coefficients, the relation

$$P(A_X)(1) Q(1) = 2, \qquad Q(1) = 1 - g + r,$$

implies

$$\dim A_X = P(A_X)(1) = 1 \text{ or } 2.$$

If $A_X = \mathbf{k}$ then X is an algebraic 1-cone thus the only case to consider is $A_X = \bigwedge x$. In this case $g = 1$, $r = 1$, and A is the homology of the chain algebra $(T(x, y), dy = x^2)$, i.e., $\dim H_+(X, \mathbf{k}) = 2$. $\square$

IV. Proof of Theorem 2

Let $(T(V_0 \oplus V_1 \oplus V_2), d) \to C_*(\Omega X; \mathbf{k})$ be a free model which defines the structure of graded 3-cone on X. The associated I-adic spectral sequence satisfies then

$$E^1_{-p,q} = 0 \quad \text{if } 2p > q \text{ or } q > 3p.$$

This implies

(i) $H_1 = E^1_{-1,2}$, $H_2 = E^1_{-1,3} \oplus E^2_{-2,4}$,

(ii) $0 = d_2 = d_3 = \cdots$.

On the other hand, H_0 is a sub-Hopf algebra of $H(T(V), d) = \bigoplus_{i \ge 0} H_i$. By [5, Lemma 5], the H_0-modules $E^1_{-1,2}$ and $E^1_{-1,3}$ are free.

The differential module $(I/I^2, d_0) = (E^0_{-1,2} \oplus E^0_{-1,3}, d_0)$ is quasi-isomorphic to the free module $(E^1, 0) = (E^1_{-1,2} \oplus E^1_{-1,3}, 0)$ and then, using the Kunneth spectral sequence and the isomorphisms of H_0-modules $\bigotimes_H^n I/I^2 \xrightarrow{\cong} I^n/I^{n+1}$, we obtain an isomorphism of H_0-modules: $E^1 \xrightarrow{\cong} T_{H_0}(E^1_{-1,*})$. From (ii) and (i), we obtain an isomorphism of H_0-module,

$$H_*(\Omega X;\mathbf{k}) \to T_{H_0}(H_1 \oplus M)$$

where M is a direct summand of the H_0-module H_2.

Then we obtain

$$H_1 = H_0 \otimes U_1 = U_1 \otimes H_0 \quad \text{and} \quad E^1_{-1,3} = H_0 \otimes U_2 = U_2 \otimes H_0$$

and thus the inclusion $U_1 \oplus U_2 \to H_*(\Omega X;\mathbf{k})$ uniquely extends to an algebra homomorphism $T(U_1 \oplus U_2) \to H_*(\Omega X;\mathbf{k})$, and to two linear isomorphisms

$$B \otimes T(U_1 \oplus U_2) \xrightarrow{\cong} H_*(\Omega X;\mathbf{k}) \xleftarrow{\cong} T(U_1 \oplus U_2) \otimes B. \quad \square$$

V. Proof of Theorem 3

Let

$$(T(V_0 \oplus V_1 \oplus V_2 \oplus V_4), d) \to C_*(\Omega X;\mathbb{Q})$$

be a free model which defines a structure of graded 4-cone on X. The associated I-adic spectral sequence satisfies then

$$E^1_{-p,q} = 0 \quad \text{if } 2p > q \text{ or } q > 4p.$$

Over the rationals, the I-adic spectral sequence is a spectral sequence of Hopf algebras, thus each term $E^1_{r,s}$ is a free H_0-module. This gives an isomorphism of H_0-modules: $E^1 \cong T_{H_0}(E^1_{-1,*})$. Now, the differential d_1 is completely defined by its restriction on $E^1_{-1,4}$:

$$d_1\colon E^1_{-1,4} \to E^2_{-2,4} \cong E^1_{-1,2} \otimes_{H_0} E^1_{-1,2}.$$

By [13, Appendix A], $(T(V), d)$ is the enveloping algebra of a differential free bigraded Lie algebra. Hence $E^1 = H_0 \otimes \mathbb{UL}(W_{*,*})$, with

$$W = W_{-1,2} \oplus W_{-1,3} \oplus W_{-1,4},$$

$$H_0 = \mathbf{U}L_X, \qquad E^1_{1,2} = H_0 \otimes W_{-1,2},$$

$$E^1_{-1,3} \cong H_0 \otimes W_{-1,3}, \qquad E^1_{-1,4} \cong H_0 \otimes W_{-1,4}.$$

As, $d_1\colon W_{-1,*} \to \mathbb{L}_{-2,*}(W)$. It is clear that

$$E^2_{*,*} \cong H_0 \otimes H_*(T(W_{-1,*}), d_1).$$

To compute $H_*(T(W_{-1,*}), d_1)$, we decompose $W_{-1,*}$ in the form

$$W_{-1,*} = \overline{W}_0 \oplus \overline{W}_1, \quad \overline{W}_0 = W_{-1,2}, \quad \overline{W}_1 = W_{-1,3} \oplus W_{-1,4}.$$

The first part of Theorem 1 gives an isomorphism

$$H_*(T(W_{-1,*}), d_1) \cong [T(W_{-1,2})/d_1(W_{-1,4})] \otimes T(U).$$

The preceding discussion yields thus the two following exact sequences of Hopf algebras [14]:

$$\mathbb{Q} \to H_*(T(W_{-1,*}), d_1) \to E^2 \to H_0 \to \mathbb{Q},$$

$$\mathbb{Q} \to T(U) \to H_*(T(W_{-1,*}), d_1) \to T(L_{-1,2})/d_1(L_{-1,4}) \to \mathbb{Q}.$$

Moreover, as U is concentrated in bidegrees $(-n-1, 2n+4)$, the differentials $d_2, d_3, \ldots$ are all trivial on U by property (4) in §II. The spectral sequence collapses thus at the E^2-term.

Denote by P the Lie algebra of primitive elements of $H_*(T(W_{-1,*}), d_1)$ one get a short exact sequences of Lie algebras:

$$0 \to P \to \pi_*(\Omega X) \otimes \mathbb{Q} \to L_X \to 0,$$

$$0 \to \mathbb{L}(U) \to P \to \mathbb{L}(W_{-1,*})/d_1(\mathbb{L}(W_{-1,4})) \to 0,$$

Thus, $N = \mathbb{L}(W_{-1,*})/d_1(\mathbb{L}(W_{-1,4}))$ is the quotient of a free Lie algebra by an ideal generated by quadratic elements.

Moreover, if X is a 4-cone, it is clear that the generators of N are in the image of the canonical map $H_*(\Omega X_2; \mathbb{Q}) \to H_*(\Omega X_4; \mathbb{Q}) = H_*(\Omega X; \mathbb{Q})$.

Remark finally that

$$H_*(T(W_{-1,*}), d_1) \cong T(W_{-1,3}) * (H_*(T(W_{-1,2} \oplus W_{-1,4}), d_1)),$$

and applying again Theorem 1 shows that

$$H_*(T(W_{-1,*}), d_1) \cong T(W_{-1,3}) * (T(W_{-1,2})/d(W_{-1,4}) \otimes T(U')),$$

for some graded $\mathbb{Q}$-vector spaces U'. □

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