TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 338, Number 2, August 1993

AN INVERSE BOUNDARY VALUE PROBLEM FOR SCHRÖDINGER OPERATORS WITH VECTOR POTENTIALS

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ABSTRACT. We consider the Schrödinger operator for a magnetic potential $\vec{A}$ and an electric potential q, which are supported in a bounded domain in $\mathbb{R}^n$ with $n \geq 3$. We prove that knowledge of the Dirichlet to Neumann map associated to the Schrödinger operator determines the magnetic field $\operatorname{rot}(\vec{A})$ and the electric potential q simultaneously, provided $\operatorname{rot}(\vec{A})$ is small in the L^∞ topology.

1. INTRODUCTION

In this paper we consider the Schrödinger operator

$$H_{\vec{A},q} = \sum_{j=1}^{n}\left(-i\frac{\partial}{\partial x_j} + A_j(x)\right)^2 + q(x), \tag{1.1}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$, $i = \sqrt{-1}$. The vector function $\vec{A} = (A_1, A_2, \dots, A_n)$ is the magnetic potential and the scalar function q is the electric potential. We assume that $A_j \in W^{1,\infty}(\mathbb{R}^n)$, $1 \leq j \leq n$, $q \in L^\infty(\mathbb{R}^n)$, and that they are real-valued.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. If zero is not a Dirichlet eigenvalue of (1.1) on Ω, then for any boundary value $f \in H^{1/2}(\partial\Omega)$ there exists a unique solution $u \in H^1(\Omega)$ which solves

$$H_{\vec{A},q}u = 0 \quad \text{in } \Omega \qquad \text{and} \qquad u|_{\partial\Omega} = f. \tag{1.2}$$

Variational principles show that the solution u in (1.2) can be obtained by minimizing the functional

$$\mathbf{I}_{\vec{A},q}(w) = \int_\Omega (\nabla w \nabla \overline{w} + (\vec{A}^2 + q) w\overline{w} + i\vec{A}\cdot(w\nabla\overline{w} - \overline{w}\nabla w))\,dx \tag{1.3}$$

over functions w with $w|_{\partial\Omega} = f$ in $H^1(\Omega)$. More precisely,

$$\mathbf{I}_{\vec{A},q}(u) = \inf_{\substack{w\in H^1(\Omega)\\ w|_{\partial\Omega}=f}} \mathbf{I}_{\vec{A},q}(w). \tag{1.4}$$

Received by the editors May 29, 1991.
1991 *Mathematics Subject Classification*. Primary 35R30; Secondary 35Q40.
Partly supported by NSF DMS 9015621.

In terms of boundary value f, the functional $\mathbf{I}_{\vec{A},q}$ can be expressed also as

$$\mathbf{I}_{\vec{A},q}(u) = \int_{\partial\Omega} \overline{f} \Lambda_{\vec{A},q}(f)\, ds,$$

where the operator $\Lambda_{\vec{A},q}$, mapping $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$, is defined as

$$\Lambda_{\vec{A},q} \colon f \to \left.\frac{\partial u}{\partial N}\right|_{\partial\Omega} + i(\vec{A}\cdot N)f, \qquad f \in H^{1/2}(\partial\Omega), \tag{1.5}$$

with u solution of (1.2) and N the outer normal on $\partial\Omega$.

The operator $\Lambda_{\vec{A},q}$, which is the main subject of this paper, is called the Dirichlet to Neumann map of $H_{\vec{A},q}$ on $\partial\Omega$. In this paper we assume that $\operatorname{supp}\vec{A}$, $\operatorname{supp} q \subset \overline{\Omega}$. Thus in this case

$$\Lambda_{\vec{A},q}(f) = \left.\frac{\partial u}{\partial N}\right|_{\partial\Omega}. \tag{1.6}$$

When Ω is given, the Dirichlet to Neumann map $\Lambda_{\vec{A},q}$ is uniquely determined by $H_{\vec{A},q}$, i.e., by potentials $\vec{A}$ and q. The problem under discussion in this paper is whether the converse is true. More specifically, we ask whether the potentials $\vec{A}$ and q are uniquely determined by $\Lambda_{\vec{A},q}$. A resolution to this problem would have important applications to the problem of the inverse scattering at fixed energy. On the other hand, inverse boundary value problems for general elliptic operators are of independent interest. Our study is partly devoted to understand exactly what the Dirichlet to Neumann map does determine if an elliptic operator involves a first-order term. One shall note that a selfadjoint elliptic operator of second order with Δ as its principal symbol can always be written as a Schrödinger operator with vector potentials.

In recent years significant progress has been made on this problem in the case of $\vec{A} = 0$. It has been shown that in dimension $n \geq 3$, an L^∞ potential q is uniquely determined by the Dirichlet to Neumann map Λ_q [NSU]. The L^∞ hypothesis on q can even be relaxed to L^s with $s > n/2$ [Ch] and to $L^{n/2}$ [LN], and the smoothness assumption on $\partial\Omega$ can be relaxed to $C^{1,1}$ [N]. It has also been shown that in dimension $n = 2$, a $W^{1,\infty}$ potential q is uniquely determined by Λ_q provided q is close to zero [SU-II] and close to "most potentials" [SuU-I]. More recently, it has been shown that singularities of an arbitrary two-dimensional potential q are uniquely determined by Λ_q [SuU-II]. We refer readers to [C, KV-I, KV-II, SU-I, A, I], and [Su-I] for results on the inverse isotropic conductivity problem which is closely related to the problem discussed here and to [NH, N, R], and [W] for applications to inverse scattering.

In the case of $\vec{A} \neq 0$, however, there is an obstruction to uniqueness. In fact a change of the magnetic potential $\vec{A}$ to its gauge equivalence $\vec{A'} = \vec{A} + \nabla g$ for some $g \in W^{1,\infty}$ with $g = \partial g/\partial N = 0$ on $\partial\Omega$ would not change the Dirichlet to Neumann map $\Lambda_{\vec{A},q}$. Indeed, it is a straightforward computation to show that replacing $\vec{A}$ by $\vec{A'}$ in (1.1) is equivalent to replacing the solution u in (1.2) by $u' = ue^{-ig}$. Since u' carries the same boundary value and the normal derivative as u, it follows that $\Lambda_{\vec{A'},q} = \Lambda_{\vec{A},q}$.

It is easy to see that the above gauge transformation $\vec{A} \to \vec{A}'$ preserves the rotation $\operatorname{rot}(\vec{A}) = \operatorname{rot}(\vec{A}')$, where

$$\operatorname{rot}(\vec{A}) = \sum_{j,l=1}^{n} \left(\frac{\partial A_l}{\partial x_j} - \frac{\partial A_j}{\partial x_l} \right) dx_j \wedge dx_l.$$

Physically, $\operatorname{rot}(\vec{A})$ is called the magnetic field induced by $\vec{A}$. On the other hand, one shows easily that if $\operatorname{rot}(\vec{A}) = \operatorname{rot}(\vec{A}')$ holds for two magnetic potentials $\vec{A}$ and $\vec{A}'$ satisfying our basic hypotheses and if Ω is simply connected, then $\vec{A}$ and $\vec{A}'$ are gauge equivalent and therefore $\Lambda_{\vec{A}',q} = \Lambda_{\vec{A},q}$.

The above analysis shows that in general the best one can expect in the case of $\vec{A} \neq 0$ is that $\Lambda_{\vec{A},q}$ determines $\operatorname{rot}(\vec{A})$ and q uniquely. The goal of this paper is to show that in dimension $n \geq 3$, and under the a priori hypothesis that $\operatorname{rot}(\vec{A})$ is small in the L^∞ topology, $\Lambda_{\vec{A},q}$ determines $\operatorname{rot}(\vec{A})$ and q uniquely. In what follows we use $W_\Omega^{m,\infty}$ to denote the space of functions f in $W^{m,\infty}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \overline{\Omega}$.

Theorem. *Let* $\vec{A}_j \in W_\Omega^{2,\infty}$, $q_j \in L^\infty(\Omega)$, $j = 1, 2$. *Assume that zero is not a Dirichlet eigenvalue of* $H_{\vec{A}_j,q_j}$, $j = 1, 2$. *Then there exists a positive constant* $\varepsilon = \varepsilon(\Omega)$ *such that if*

$$\| \operatorname{rot}(\vec{A}_j) \|_{L^\infty(\Omega)} < \varepsilon, \qquad j = 1, 2,$$

and

$$\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_2,q_2},$$

then

$$\operatorname{rot}(\vec{A}_1) = \operatorname{rot}(\vec{A}_2) \quad \textit{and} \quad q_1 = q_2.$$

In §2 we shall construct a special class of expotentially growing solutions in the null space of (1.1), which are analogous to the special solutions constructed by Sylvester and Uhlmann in [SU-I]. These solutions shall serve as a basic tool in this paper. The presence of the magnetic potential $\vec{A}$ in (1.1) makes the construction much more difficult especially when $\operatorname{rot}(\vec{A})$ is not small. This is the only reason which leads to the smallness assumption on $\operatorname{rot}(\vec{A})$ in the theorem. If one converts the differential equation $H_{\vec{A},q}u = 0$ into an integral equation using Faddeev's Green's function, one sees that the set of exceptional points for that integral equation may not be bounded in $\mathbb{C}^n$ when $\operatorname{rot}(\vec{A})$ is not small [NH]. Therefore it remains as an open question whether such solutions can still be constructed when $\operatorname{rot}(\vec{A})$ is large.

Section 3 is devoted to establish an orthogonality identity which relates $\vec{A}$ and q with $\Lambda_{\vec{A},q}$.

Section 4 is the heart of the proof. The main difficulty, which one did not encounter in the case of $\vec{A} = 0$ treated in [NSU], is that one has to determine $\operatorname{rot}(\vec{A})$ from a nonlinear integral functional rather than the Fourier transform of $\operatorname{rot}(\vec{A})$. Once $\operatorname{rot}(\vec{A})$ has been recovered from $\Lambda_{\vec{A},q}$, we can go further to recover q using gauge invariant property of $\Lambda_{\vec{A},q}$ and the method in [NSU].

We remark that this method does not apply in dimension 2. We shall study the two-dimensional case in a forthcoming paper [Su-II].

2. Construction of solutions

Following the idea of the geometric optics construction of solutions to hyperbolic equations, we look for solutions of the form given below in the null space of $H_{\vec{A},q}$.

$$u(x, \xi) = e^{\xi x + \phi(x, \xi)}(1 + \omega(x, \xi)), \tag{2.1}$$

where $\xi \in \mathbb{C}^n$ is a complex vector satisfying $\xi \cdot \xi = 0$ and the function $\omega(x, \xi)$ behaves like $|\xi|^{-1}$ as $|\xi|$ tends to ∞ in an appropriate function space. We shall show that it is always possible to construct such solutions provided $\|\operatorname{rot}(\vec{A})\|_{L^\infty(\Omega)}$ is sufficiently small and $|\xi|$ is sufficiently large.

Substituting (2.1) into the equation $H_{\vec{A},q}u = 0$ and equating coefficients of powers of $|\xi|$ to zero, we get two equations

$$\xi \cdot \nabla\phi = -i\xi \cdot \vec{A}, \tag{2.2}$$

$$\Delta\omega + 2(\xi + \nabla\phi + i\vec{A}) \cdot \nabla\omega - G\omega = G, \tag{2.3}$$

where

$$G = \vec{A}^2 - i\nabla \cdot \vec{A} + q - 2i\vec{A} \cdot \nabla\phi - \nabla\phi \cdot \nabla\phi - \Delta\phi. \tag{2.4}$$

We divide the rest of the section into two parts, where ϕ and ω will be constructed separately. In what follows we assume

$$|\xi| \geq 1, \quad \vec{A} \in W_\Omega^{2,\infty}, \quad q \in L^\infty(\mathbb{R}^n), \quad \operatorname{supp} q \subset \overline{\Omega}. \tag{2.5}$$

2.1. Construction of ϕ.

Fourier transforming (2.2) gives

$$-i\xi \cdot \eta\hat{\phi}(\eta, \xi/|\xi|) = -i\xi \cdot \vec{A}^\wedge(\eta),$$

where $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ is the dual coordinates and $\wedge$ denotes the Fourier transform with respect to x. We denote by $\vee$ its inverse. We construct

$$\phi\left(x, \frac{\xi}{|\xi|}\right) = \left(\frac{\xi \cdot \vec{A}^\wedge(\eta)}{\xi \cdot \eta}\right)^\vee = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-ix\eta}\left(\frac{\xi \cdot \vec{A}^\wedge(\eta)}{\xi \cdot \eta}\right) d\eta. \tag{2.6}$$

We shall show that the solution $\phi(x, \xi)$ has the following three properties:

$$\|\phi(\cdot, \xi/|\xi|)\|_{W^{2,\infty}(\Omega)} \leq C\|\vec{A}\|_{W_\Omega^{2,\infty}}, \tag{2.7}$$

$$\|\nabla\phi + i\vec{A}\|_{L^\infty(\Omega)} \leq C\|\operatorname{rot}(\vec{A})\|_{L^\infty(\Omega)}, \tag{2.8}$$

(2.9) If $\xi(s)\colon (a, b) \to \mathbb{C}^n$ is a differentiable map with $\xi(s)\cdot\xi(s) = 0$ and $|\xi(s)| \geq 1$ for all s, then $s \to \phi(\cdot, \xi(s)/|\xi(s)|)$ is differentiable as a map from (a, b) to $L^\infty(\Omega)$.

The constant C involved in (2.7)–(2.9) depends only on Ω.

Lemma 2.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$, $n \geq 2$. *Then*

$$L(f) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix\eta} \left(\frac{\hat{f}(\eta)}{\eta_1 + i\eta_2} \right) d\eta \tag{2.10}$$

defines a bounded map from $W_\Omega^{m,\infty}$ *to* $W^{m,\infty}(\Omega)$ *for any nonnegative integer* m.

Proof. Rewriting (2.10) in terms of convolution respect to variable (x_1, x_2), we get

$$L(f) = \int_{\mathbb{R}^2} \frac{f(x_1 - z_1, x_2 - z_2, x_3, \dots, x_n)}{z_1 + iz_2} \, dz_1 \, dz_2 \tag{2.11}$$

and thus

$$\|L(f)\|_{L^\infty(\Omega \cap T)} \leq C\|f\|_{L^\infty(\Omega \cap T)} \tag{2.12}$$

for any two-dimensional plane T that is parallel to (x_1, x_2)-plane. Therefore

$$\|L(f)\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}. \tag{2.13}$$

This proves Lemma 2.1 in the case of $m = 0$. Differentiating both sides of (2.11) and repeating using (2.13) yield desired results. □

Proof of (2.7). Without loss of generality, we assume that $\xi = \gamma_1 + i\gamma_2$, where $\gamma_1, \gamma_2 \in \mathbb{R}^n$, $|\gamma_1| = |\gamma_2| = 1$, and $\gamma_1 \cdot \gamma_2 = 0$. Making a rotation of coordinates if necessary one can rewrite (2.6) in terms of convolution as follows:

$$\phi\left(x, \frac{\xi}{|\xi|}\right) = \int_{\mathbb{R}^2} \frac{\xi \cdot \vec{A}(x - (z_1\gamma_1 + z_2\gamma_2))}{z_1 + iz_2} \, dz_1 \, dz_2. \tag{2.14}$$

Clearly, (2.7) follows from Lemma 2.1. If $\xi(s) = \gamma_1(s) + i\gamma_2(s)$ is differentiable in s and $|\gamma_1(s)| = |\gamma_2(s)| = 1$ for all $s \in (a, b)$, then for a fixed $x \in \Omega$,

$$\begin{aligned} \dot{\phi}\left(x, \frac{\xi}{|\xi|}\right) &= \int_{\mathbb{R}^2} \frac{\dot{\xi} \cdot \vec{A}(x - (z_1\gamma_1 + z_2\gamma_2))}{z_1 + iz_2} \, dz_1 \, dz_2 \\ &\quad + \int_{\mathbb{R}^2} \frac{(z_1\dot{\gamma}_1 + z_2\dot{\gamma}_2) \cdot \nabla(\xi \cdot \vec{A})(x - (z_1\gamma_1 + z_2\gamma_2))}{z_1 + iz_2} \, dz_1 \, dz_2, \end{aligned} \tag{2.15}$$

where the dot means d/ds. Using Lemma 2.1 again and noting that $\operatorname{supp} \vec{A} \subset \overline{\Omega}$, one sees that the right-hand side of (2.15) is a function in $L^\infty(\Omega)$. Hence (2.8) follows.

By a change of coordinates we need only to prove (2.9) in the case that $\gamma_1 = (1, 0, \dots, 0)$ and $\gamma_2 = (0, 1, \dots, 0)$. In this case

$$\nabla\phi + i\vec{A} = \left(-i\eta \frac{\widehat{A}_1 + i\widehat{A}_2}{\eta_1 + i\eta_2} + i\vec{A}^\wedge \right)^\vee .$$

We now compute explicitly the components of $\nabla\phi + i\vec{A}$. We have

$$\begin{aligned} \left(\frac{-\eta_1(\widehat{A}_1 + i\widehat{A}_2)}{\eta_1 + i\eta_2} + i\widehat{A}_1 \right)^\vee &= \left(\frac{-i\eta_1(\widehat{A}_1 + i\widehat{A}_2) + i\widehat{A}_1(\eta_1 + i\eta_2)}{\eta_1 + i\eta_2} \right)^\vee \\ &= iL\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right), \end{aligned} \tag{2.16}$$

$$\left(\frac{-i\eta_2(\widehat{A}_1+i\widehat{A}_2)}{\eta_1+i\eta_2}+i\widehat{A}_2\right)^{\vee}=\left(\frac{-i\eta_2(\widehat{A}_1+i\widehat{A}_2)+i\widehat{A}_2(\eta_1+i\eta_2)}{\eta_1+i\eta_2}\right)^{\vee}=L\left(\frac{\partial A_1}{\partial x_2}-\frac{\partial A_2}{\partial x_1}\right), \tag{2.17}$$

and for $3\leq j\leq n$,

$$\left(\frac{-i\eta_j(\widehat{A}_1+i\widehat{A}_2)}{\eta_1+i\eta_2}+i\widehat{A}_j\right)^{\vee}=\left(\frac{-i\eta_j(\widehat{A}_1+i\widehat{A}_2)+i\widehat{A}_j(\eta_1+i\eta_2)}{\eta_1+i\eta_2}\right)^{\vee}=L\left(\frac{\partial A_1}{\partial x_j}-\frac{\partial A_j}{\partial x_1}\right)-iL\left(\frac{\partial A_2}{\partial x_j}-\frac{\partial A_j}{\partial x_2}\right). \tag{2.18}$$

Applying Lemma 2.1 to (2.16)–(2.18) we get (2.8).

2.2. **Construction of** ω.

Proposition 2.2. *Let $\vec{A}$ and q be potential functions satisfying (2.5). Then there exist positive constants $\delta=\delta(\Omega)$ and K such that if $\|\operatorname{rot}(\vec{A_j})\|_{L^\infty(\Omega)}<\varepsilon$ and $|\xi|>K$, then equation (2.3) has a solution $\omega\in H^1(\Omega)$. Moreover,*

$$\|\omega\|_{L^2(\Omega)}\leq C|\xi|^{-1} \tag{2.19}$$

and

$$\|\nabla\omega\|_{L^2(\Omega)}\leq C, \tag{2.20}$$

where K and C depend only on Ω, $\|\vec{A}\|_{W^{2,\infty}(\Omega)}$, and $\|q\|_{L^\infty(\Omega)}$.

The proof of this proposition is based on the following two lemmas.

Lemma 2.3. *Let $L_\xi=\Delta+2\xi\cdot\nabla$. Then the operator L_ξ admits a bounded inverse $L_\xi^{-1}\colon L^2(\Omega)\to H^1(\Omega)$. If $f\in L^\infty(\Omega)$ and $v=L_\xi^{-1}(f)\in H^1(\Omega)$, then*

$$\|v\|_{L^2(\Omega)}\leq C|\xi|^{-1}\|f\|_{L^2(\Omega)}, \tag{2.21}$$

$$\|\nabla v\|_{L^2(\Omega)}\leq C\|f\|_{L^2(\Omega)}, \tag{2.22}$$

where C depends only on Ω.

Proof. The existence of L_ξ^{-1} as well as the estimate (2.21) follows from a fundamental result obtained by Sylvester and Uhlmann [SU-I]. (Also see [I] for a more direct proof.) The estimate (2.22) was proven in the two dimensional case [SU-II]. We now give a proof of (2.22) in the case of $n\geq 3$.

Extending f to be zero outside Ω and letting Ω' be a bounded domain that contains Ω we can construct a solution v satisfying (2.21). We shall show that the following estimate must hold:

$$\|\nabla v\|^2_{L^2(\Omega)}\leq C(|\xi|^2\|v\|^2_{L^2(\Omega')}+\|f\|^2_{L^2(\Omega)}), \tag{2.23}$$

where $C=(\Omega,\Omega')$. Clearly, the restriction of v to Ω is a solution satisfying (2.21) and (2.22).

Let $\chi\in C_0^\infty(\Omega')$ so that $\chi(x)=1$ for $x\in\Omega$ and $0\leq\chi(x)\leq 1$ for $x\in\Omega'\backslash\Omega$. Let $u=\chi v$. It follows from an elliptic regularity theorem that u is a $H^2(\Omega)$ solution of the equation

$$\Delta u+2\xi\cdot\nabla u=f\chi+v\Delta\chi+2\nabla v\nabla\chi+2v\xi\cdot\nabla\chi. \tag{2.24}$$

Multiplying $\overline{u}$ to both sides of (2.24) and integrating by parts we get

$$\begin{aligned}\int_{\Omega'} |\nabla u|^2\,dx &\le 2\int_{\Omega'} (|\overline{u}\xi\cdot\nabla u| + |f\overline{u}\chi + \overline{u}\chi + \overline{u}v\Delta\chi + 2\overline{u}v\xi\cdot\nabla\chi|)\,dx \\ &\quad + 2\left|\int_{\Omega'} \overline{u}\nabla v\nabla\chi\,dx\right| \\ &= I_1 + I_2.\end{aligned} \tag{2.25}$$

Denoting $M = \|\chi\|_{C^2(\Omega')}$ and using Schwartz's inequality we get

$$\begin{aligned} I_1 &\le (4|\xi|^2 + M(1+|\xi|^2))\int_{\Omega'} |u|^2\,dx \\ &\quad + \frac{1}{4}\int_{\Omega'} |\nabla u|^2\,dx + M\left(2\int_{\Omega'} |v|^2\,dx + \int_{\Omega'} |f|^2\,dx\right).\end{aligned} \tag{2.26}$$

Using integration by parts one has

$$\int_{\Omega'} \overline{u}\nabla v\nabla u\,dx = -\int_{\Omega'} v(\overline{u}\Delta\chi + \nabla\chi\nabla u)\,dx.$$

Hence

$$\begin{aligned} I_2 &\le 2M\int_{\Omega'} (|v\overline{u}| + |v\nabla\overline{u}|)\,dx \\ &\le (M+4M^2)\int_{\Omega'} |v|^2\,dx + M\int_{\Omega'} |u|^2\,dx + \frac{1}{4}\int_{\Omega'} |\nabla u|^2\,dx.\end{aligned} \tag{2.27}$$

Combining (2.25) with (2.26) and (2.27) yields

$$\begin{aligned} \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega')} &\le (|\xi|^2(M+4) + 2M)\int_{\Omega'} |u|^2\,dx \\ &\quad + (3M+4M^2)\int_{\Omega'} |v|^2\,dx + M\int_{\Omega'} |f|^2\,dx.\end{aligned} \tag{2.28}$$

Since $u = v$ in Ω and $|u| \le |v|$ in Ω', it follows from (2.28) that

$$\|\nabla v\|^2_{L^2(\Omega)} \le 2(|\xi|(M+4) + 5M + 4M^2)\|v\|^2_{L^2(\Omega')} + M\|f\|^2_{L^2(\Omega')}.$$

This leads to (2.23) immediately. $\square$

We set $\widetilde{G} = G_{\chi_\Omega}$ and $\tilde{\phi} = \phi\chi_\Omega$, with G and ϕ as in (2.2) and (2.4), where χ_Ω is the indicator function of Ω. To solve (2.3) it suffices to solve

$$(L_{\xi,\vec{A}} - \widetilde{G})\omega = \widetilde{G}, \tag{2.29}$$

where

$$L_{\xi,\vec{A}} = L_\xi + (\nabla\tilde{\phi} + i\vec{A})\cdot\nabla = \Delta + 2(\xi + \nabla\tilde{\phi} + i\vec{A})\cdot\nabla.$$

Lemma 2.4. *If $\|\operatorname{rot}\vec{A}\|_{L^\infty(\Omega)}$ is sufficiently small, then $L_{\xi,\vec{A}}$ has a bounded inverse $L^{-1}_{\xi,\vec{A}}\colon L^2(\Omega)\to H^1(\Omega)$. Moreover, if $f\in L^2(\Omega)$ and $v = L^{-1}_{\xi,\vec{A}}(f)$, then*

$$\|v\|_{L^2(\Omega)} \le C|\xi|^{-1}\|f\|_{L^2(\Omega)}, \tag{2.30}$$

$$\|\nabla v\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}, \tag{2.31}$$

where C depends only on Ω and $\|\operatorname{rot}\vec{A}\|_{L^\infty(\Omega)}$.

Proof. Applying L_ξ^{-1} to both sides of $L_{\xi,\vec{A}}v = f$ yields an integral equation

$$(I + 2F_1)v = L_\xi^{-1} f, \tag{2.32}$$

where

$$F_1 = L_\xi^{-1} \circ (\nabla\tilde{\phi} + i\vec{A}) \circ \nabla,$$

where I denotes the identity operator and $(\ )$ denotes the multiplication operator.

From Lemma 2.3 one sees clearly that the right-hand side of (2.32) is in $H^1(\Omega)$. Moreover,

$$\|L_\xi^{-1} f\|_{L^2(\Omega)} \le C|\xi|^{-1} \|f\|_{L^2(\Omega)}, \tag{2.33}$$

$$\|L_\xi^{-1} f\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{2.34}$$

From (2.8), (2.21), and (2.22) we have that F_1 maps $H^1(\Omega)$ into $H^1(\Omega)$ and moreover,

$$\|F_1\|_{H^1, H^1} \le C\|\operatorname{rot}\vec{A}\|_{L^\infty(\Omega)} \tag{2.35}$$

where C depends only on Ω. Thus, if we view (2.32) as an integral equation in $H^1(\Omega)$ and use (2.34) and (2.35), we conclude that (2.32) has a unique solution $v \in H^1(\Omega)$ provided $\|\operatorname{rot}\vec{A}\|_{L^\infty(\Omega)}$ is sufficiently small and thus (2.31) follows. Clearly,

$$v = L_\xi^{-1}(f + (\nabla\tilde{\phi} + i\vec{A}) \circ \nabla v).$$

Hence by (2.21),

$$\|v\|_{L^2(\Omega)} \le C|\xi|^{-1}(\|f\|_{L^2(\Omega)} + \|\operatorname{rot}\vec{A}\|_{L^\infty(\Omega)}\|\nabla v\|_{L^2(\Omega)}).$$

This estimate together with (2.31) leads to (2.30). □

Proof of Proposition 2.2. Applying $L_{\xi,\vec{A}}^{-1}$ to both sides of (2.29) yields an integral equation for ω:

$$(I + F_2)\omega = L_{\xi,\vec{A}}^{-1}(\widetilde{G}), \tag{2.36}$$

where

$$F_2 = L_{\xi,\vec{A}}^{-1} \circ (\widetilde{G}).$$

By Lemma 2.4 one sees clearly that the right-hand side of (2.36) is in $H^1(\Omega)$. Moreover,

$$\|L_{\xi,\vec{A}}^{-1}\widetilde{G}\|_{L^2(\Omega)} \le C|\xi|^{-1}\|\widetilde{G}\|_{L^2(\Omega)}, \tag{2.37}$$

$$\|L_{\xi,\vec{A}}^{-1}\widetilde{G}\|_{H^1(\Omega)} \le C\|\widetilde{G}\|_{L^2(\Omega)}. \tag{2.38}$$

We view (2.36) as an integral equation in $L^2(\Omega)$. From (2.30) and (2.31) we see clearly that F_2 maps $L^2(\Omega)$ into $L^2(\Omega)$ and moreover,

$$\|F_2\|_{L^2, L^2} \le C\|\widetilde{G}\|_{L^\infty(\Omega)}|\xi|^{-1}, \tag{2.39}$$

where C depends only on Ω and $\|\operatorname{rot}\vec{A}\|_{L^\infty(\Omega)}$. Therefore (2.36) has a unique L^2 solution ω provided $|\xi|$ is large enough. Using (2.37) one sees that ω satisfies (2.19). On the other hand (2.36) can be rewritten as

$$\omega = L^{-1}_{\xi,\vec{A}}(\widetilde{G}(1+\omega)).$$

Now it follows from (2.31) and (2.19) that the L^2 solution ω is also a H^1 solution of (2.36). Moreover,

$$\|\omega\|_{H^1(\Omega)} \le C\|G\|_{L^\infty(\Omega)}(1+\|\omega\|_{L^2(\Omega)}),$$

from which (2.20) follows. □

3. An identity

The main purpose of this section is to present an orthogonality identity which relates potential functions $\vec{A}$ and q to the Dirichlet to Neumann map $\Lambda_{\vec{A},q}$.

Proposition 3.1. *Let $\vec{A}_j$ and q_j, $j = 1, 2$, be potential functions satisfying* (2.5). *Then*

$$i\int_\Omega (\vec{A}_1 - \vec{A}_2)\cdot(u_1\nabla\overline{u}_2 - \overline{u}_2\nabla u_1)\,dx + \int_\Omega (\vec{A}_1^2 - \vec{A}_2^2 + q_1 - q_2)u_1\overline{u}_2\,dx$$
$$= -\int_{\partial\Omega}\overline{u}_2(\Lambda_{\vec{A}_1,q_1} - \Lambda_{\vec{A}_2,q_2})u_1\,ds,$$

holds for arbitrary u_j solution of $H_{\vec{A}_j,q_j}u_j = 0$, $j = 1, 2$.

Proof. We have

$$\Delta u_1 + 2i\vec{A}_1\cdot\nabla u_1 - Q_1u_1 = 0 \quad \text{in } \Omega, \tag{3.1}$$

$$\Delta\overline{u}_2 - 2i\vec{A}_2\cdot\nabla\overline{u}_2 - \overline{Q}_2\overline{u}_2 = 0 \quad \text{in } \Omega, \tag{3.2}$$

where

$$Q_j = \vec{A}_j^2 - i\nabla\cdot\vec{A}_j + q_j, \qquad j = 1, 2. \tag{3.3}$$

Multiplying (3.1) by $\overline{u}_2$ and (3.2) by u_1 and then adding them, we get

$$\int_\Omega (\overline{u}_2\Delta u_1 + u_1\Delta\overline{u}_2 + 2i\overline{u}_2\vec{A}_1\cdot\nabla u_1 - 2iu_1\vec{A}_2\cdot\nabla\overline{u}_2 - Q_1\overline{u}_2u_1 - \overline{Q}_2u_1\overline{u}_2)\,dx = 0.$$

Integrating by parts yields

(3.4)
$$-2i\int_\Omega u_1(\vec{A}_1 - \vec{A}_2)\cdot\nabla\overline{u}_2\,dx - \int_\Omega u_1(Q_1 - \overline{Q}_2)\overline{u}_2\,dx - 2i\int_\Omega \nabla\cdot\vec{A}_1u_1\overline{u}_2\,dx$$
$$+\int_{\partial\Omega}\left(\frac{\partial u_1}{\partial N}\overline{u}_2 - \frac{\partial\overline{u}_2}{\partial N}u_1 + 2i(\vec{A}_1\cdot N)u_1\overline{u}_2\right)ds = 0.$$

Exchanging positions of u_j and $\vec{A}_j$, $j = 1, 2$, in (3.4) and then taking the complex conjugate, we get

(3.5)
$$-2i\int_\Omega \overline{u}_2(\vec{A}_1 - \vec{A}_2)\cdot\nabla u_1\,dx + \int_\Omega u_1(Q_1 - \overline{Q}_2)\overline{u}_2\,dx + 2i\int_\Omega \nabla\cdot\vec{A}_2u_1\overline{u}_2\,dx$$
$$+\int_{\partial\Omega}\left(\frac{\partial u_1}{\partial N}\overline{u}_2 + \frac{\partial\overline{u}_2}{\partial N}u_1 + 2i(\vec{A}_2\cdot N)u_1\overline{u}_2\right)ds = 0.$$

Substracting (3.5) from (3.4) and using (3.3) we get

$$\begin{aligned}(3.6)\qquad & -i\int_\Omega(\vec{A}_1-\vec{A}_2)\cdot(u_1\nabla\overline{u}_2-\overline{u}_2\nabla u_1)\,dx-\int_\Omega(\vec{A}_1^2-\vec{A}_2^2+q_1-q_2)u_1\overline{u}_2\,dx\\ &+\int_{\partial\Omega}\left(\frac{\partial u_1}{\partial N}\overline{u}_2-\frac{\partial\overline{u}_2}{\partial N}u_1+2i((\vec{A}_1+\vec{A}_2)\cdot N)u_1\overline{u}_2\right)ds=0.\end{aligned}$$

Since (3.6) holds for any $\vec{A}_j$ and q_j, $j=1,2$, it holds in particular for $\vec{A}_1=\vec{A}_2$ and $q_1=q_2$. In this case (3.6) becomes

$$(3.7)\qquad \int_{\partial\Omega}\left(\frac{\partial u}{\partial N}\overline{v}-\frac{\partial\overline{v}}{\partial N}u+4i(\vec{A}_2\cdot N)u\overline{v}\right)ds=0,$$

where u and v satisfy

$$H_{\vec{A}_2,q_2}u=H_{\vec{A}_2,q_2}v=0\quad\text{in }\Omega,$$
$$v|_{\partial\Omega}=u_2|_{\partial\Omega}\quad\text{and}\quad u|_{\partial\Omega}=u_1|_{\partial\Omega}.$$

Hence

$$(3.8)\qquad \int_{\partial\Omega}\frac{\partial\overline{u}_2}{\partial N}u_1dx=\int_{\partial\Omega}\left(\frac{\partial u}{\partial N}\overline{u}_2+4i(\vec{A}_2\cdot N)u_1\overline{u}_2\right)ds.$$

Using (3.8) and recalling (1.5) we find that the third integral in (3.6) is equal to

$$\begin{aligned}&\int_{\partial\Omega}\left(\overline{u}_2\left(\frac{\partial u_1}{\partial N}-\frac{\partial u}{\partial N}\right)+2i((\vec{A}_1-\vec{A}_2)\cdot N)u_1\overline{u}_2\right)ds\\ &=\int_{\partial\Omega}\overline{u}_2(\Lambda_{\vec{A}_1,q_1}-\Lambda_{\vec{A}_2,q_2})u_1\,ds,\end{aligned}$$

from which Proposition 3.1 follows. □

Corollary 3.2. *Let $\vec{A}_j$ and q_j, $j=1,2$, be potential functions satisfying (2.5). Assume that $\Lambda_{\vec{A}_1,q_1}=\Lambda_{\vec{A}_2,q_2}$. Then*

$$(3.9)\quad i\int_\Omega(\vec{A}_1-\vec{A}_2)\cdot(u_1\nabla\overline{u}_2-\overline{u}_2\nabla u_1)\,dx+\int_\Omega(\vec{A}_1^2-\vec{A}_2^2+q_1-q_2)u_1\overline{u}_2\,dx=0$$

holds for arbitrary u_j solution of $H_{\vec{A}_j,q_j}u_j=0$, $j=1,2$.

In the rest of this section we shall replace u_j in (3.9) with the exponentially growing solutions constructed in the previous section.

Let k, γ_1, and γ_2 be three mutually orthogonal vectors in $\mathbb{R}^n$ with $|\gamma_1|=|\gamma_2|=1$. Let $\zeta,\xi\in\mathbb{C}^n$ be given by

$$(3.10)\qquad \zeta=\gamma_1+i\gamma_2,\qquad \xi=s\zeta+g(s,k)\gamma_1,$$

where s is a positive real parameter and

$$(3.11)\qquad g(s,k)=2^{-1}|k|^2((|k|^2+4s^2)^{1/2}+4s)^{-1}.$$

Let $\xi_1,\xi_2\in\mathbb{C}^n$ be given by

$$(3.12)\qquad \xi_1=ik/2+\xi,\qquad \overline{\xi}_2=ik/2-\xi.$$

One checks directly that

$$(3.13)\qquad \xi_1\cdot\xi_1=\xi_2\cdot\xi_2=0,\quad \xi_1+\overline{\xi}_2=ik,\quad \xi_1-\overline{\xi}_2=2\xi,$$

$$(3.14)\qquad \xi_1/s\to\zeta,\quad \overline{\xi}_2/s\to-\zeta,\quad \xi/s\to\zeta,\qquad \text{as } s\to\infty.$$

Following the construction in §2 we can construct

$$u_j(x, \xi_j) = e^{\xi_j \cdot x + \phi_j(x, \xi_j/|\xi_j|)}(1 + \omega_j(x, \xi_j)) \tag{3.15}$$

solution of $H_{\vec{A}_j, q_j} u_j = 0$, $j = 1, 2$, where ϕ_j solves

$$\xi_j \cdot \nabla \phi_j = -i\xi_j \cdot \vec{A}_j, \qquad j = 1, 2, \tag{3.16}$$

and ω_j, $j = 1, 2$, satisfies

$$\|\omega_j\|_{L^2(\Omega)} \le C|\xi_j|^{-1} \tag{3.17}$$

and

$$\|\nabla \omega_j\|_{L^2(\Omega)} \le C, \tag{3.18}$$

where C depends only on Ω, $\|\vec{A}_j\|_{W^{2,\infty}(\Omega)}$, and $\|q_j\|_{L^\infty(\Omega)}$, $j = 1, 2$.

Substituting (3.15) into (3.9) yields

$$F_1 + F_2 = 0, \tag{3.19}$$

where F_1 and F_2 are functions of s, k, γ_1, and γ_2 and they are given by the following formulas:

$$F_1 = -2i \int_\Omega e^{ikx + \phi_1 + \overline{\phi}_2} \xi \cdot (\vec{A}_1 - \vec{A}_2)\, dx, \tag{3.20}$$

$$\begin{aligned} F_2 = i &\int_\Omega e^{ikx + \phi_1 + \overline{\phi}_2}((\vec{A}_1 - \vec{A}_2) \cdot \nabla(\overline{\phi}_2 - \phi_1) - i(\vec{A}_1^2 - \vec{A}_2^2 + q_1 - q_2))\, dx \\ &- 2i \int_\Omega e^{ikx + \phi_1 + \overline{\phi}_2} \xi \cdot (\vec{A}_1 - \vec{A}_2)(\omega_1 + \overline{\omega}_2 + \omega_1 \overline{\omega}_2)\, dx \\ &+ i \int_\Omega e^{ikx + \phi_1 + \overline{\phi}_2}(\vec{A}_1 - \vec{A}_2) \cdot (\nabla \overline{\omega}_2 - \nabla \omega_1 + \omega_1 \nabla \overline{\omega}_2 - \overline{\omega}_2 \nabla \omega_1)\, dx \\ &+ i \int_\Omega e^{ikx + \phi_1 + \overline{\phi}_2}(\vec{A}_1 - \vec{A}_2) \cdot (\nabla \overline{\phi}_2 - \nabla \phi_1)(\omega_1 + \overline{\omega}_2 + \omega_1 \overline{\omega}_2)\, dx \\ &+ \int_\Omega e^{ikx + \phi_1 + \overline{\phi}_2}(\vec{A}_1^2 - \vec{A}_2^2 + q_1 - q_2)(\omega_1 + \overline{\omega}_2 + \omega_1 \overline{\omega}_2)\, dx. \end{aligned} \tag{3.21}$$

If we fix k, γ_1, and γ_2, view F_1 and F_2 as functions of s and apply (2.19) and (2.20) to (3.20) and (3.21), then it is clear that

$$F_1 = O(s), \quad F_2 = O(1), \qquad \text{as } s \to \infty. \tag{3.22}$$

Thus

$$\lim_{s \to \infty} s^{-1} F_1 = -2i \lim_{s \to \infty} s^{-1} \int_\Omega e^{ikx + \phi_1 + \overline{\phi}_2} \xi \cdot (\vec{A}_1 - \vec{A}_2)\, dx = 0. \tag{3.23}$$

Since ϕ_j is continuous in $\xi_j/|\xi_j|$ (see (2.9)) and $\xi_j/|\xi_j|$ is continuous in s, it follows from (3.10), (3.11), (3.14), and (3.23) that

$$\int_\Omega e^{ikx + \phi_1^* + \overline{\phi_2^*}} \zeta \cdot (\vec{A}_1 - \vec{A}_2)\, dx = 0, \tag{3.24}$$

where $\phi_j^* = \phi_j^*(x, \zeta)$ solves

$$\zeta \cdot \nabla \phi_1^* = -i\zeta \cdot \vec{A}_1, \qquad \zeta \cdot \nabla \overline{\phi_2^*} = i\zeta \cdot \vec{A}_2, \tag{3.25}$$

and by (2.14)

$$\phi_1^*(x,\zeta)=\int_{\mathbb{R}^2}\frac{\zeta\cdot\vec{A}_1(x-(z_1\gamma_1+z_2\gamma_2))}{z_1+iz_2}\,dz_1\,dz_2, \tag{3.26}$$

$$\overline{\phi}_2^*(x,\zeta)=-\int_{\mathbb{R}^2}\frac{\zeta\cdot\vec{A}_2(x-(z_1\gamma_1+z_2\gamma_2))}{z_1+iz_2}\,dz_1\,dz_2. \tag{3.27}$$

We summarize the result of this section in the following proposition.

Proposition 3.3. *Let $\vec{A}_j$ and q_j, $j=1,2$, be potential functions satisfying* (2.5). *Assume that $\Lambda_{\vec{A}_1,q_1}=\Lambda_{\vec{A}_2,q_2}$. Then* (3.24) *holds with ϕ_j^*, $j=1,2$, as in* (3.26) *and* (3.27).

4. Proof of theorem

We divide this section into two parts. In the first part we shall prove $\mathrm{rot}(\vec{A}_1)=\mathrm{rot}(\vec{A}_2)$ and in the second part we shall show $q_1=q_2$.

4.1. **Proof of** $\mathrm{rot}(\vec{A}_1)=\mathrm{rot}(\vec{A}_2)$. We shall assume throughout this subsection that Ω is a ball, i.e.,

$$\Omega=\{x\in\mathbb{R}^n,\ |x|<R\} \tag{4.1}$$

for some $R>0$. This additional assumption would have no influence to our result. Suppose that Ω is not a ball, then we can choose a ball Ω' so that $\Omega\subset\Omega'$ and extend $\vec{A}_j$ and q_j to be zero in $\Omega'\backslash\Omega$. Clearly, $\vec{A}_j$ and q_j still satisfy (2.5) with $\Omega=\Omega'$ after the extension. Standard arguments show that $\Lambda_{\vec{A}_1,q_1}=\Lambda_{\vec{A}_2,q_2}$ on $\partial\Omega$ implies $\Lambda_{\vec{A}_1,q_1}=\Lambda_{\vec{A}_2,q_2}$ on $\partial\Omega'$. Therefore it suffices to prove the theorem with assumption (4.1). See [SU-I] for relevant arguments.

Adding two equations in (3.25) together gives

$$\zeta\cdot(\vec{A}_1-\vec{A}_2)=i\zeta\cdot\nabla(\phi_1^*+\overline{\phi}_2^*). \tag{4.2}$$

Substituting (4.2) into (3.24) and noticing $k\perp\zeta$ we have

$$\begin{aligned}\int_\Omega e^{ikx+\phi_1^*+\overline{\phi}_2^*}\zeta\cdot(\vec{A}_1-\vec{A}_2)\,dx&=i\int_\Omega e^{ikx}\zeta\cdot\nabla(e^{\phi_1^*+\overline{\phi}_2^*})\,dx\\&=i\int_\Omega\zeta\cdot\nabla(e^{ikx+\phi_1^*+\overline{\phi}_2^*})\,dx.\end{aligned}$$

Then integrating by parts gives

$$\int_{\partial\Omega}e^{ikx}(\zeta\cdot N)e^{\Psi(x,\zeta)}\,ds=0, \tag{4.3}$$

where

$$\Psi(x,\zeta)=\phi_1^*(x,\zeta)+\overline{\phi}_2^*(x,\zeta). \tag{4.4}$$

We shall use (4.3) to prove our result. We divide the remaining proof into three steps.

Step 1. For any $k\in\mathbb{R}^n$, $\zeta=\gamma_1+i\gamma_2\in\mathbb{C}^n$ with γ_1 and γ_2 as in (3.10) and integer $m\geq 0$,

$$\int_{\partial\Omega}e^{ikx}(\zeta\cdot N)(\zeta\cdot x)^m e^{\Psi(x,\zeta)}\,ds=0. \tag{4.5}$$

Proof. Fix $\zeta = \gamma_1 + i\gamma_2$ in (4.3). Since (4.3) holds for any $k \perp \gamma_1, \gamma_2$, it follows by using the inverse Fourier transform that

$$\int_{\partial\Omega\cap T} (\zeta \cdot N) e^{\Psi(x,\zeta)}\, ds_T = 0 \tag{4.6}$$

for any two-dimensional plane T that is parallel to γ_1 and γ_2, where ds_T is the usual surface measure on $\partial\Omega \cap T$.

Given k, γ_1, and γ_2, three mutually orthgonal vectors in $\mathbb{R}^n$ with $k \neq 0$ and $|\gamma_1| = |\gamma_2| = 1$. We view $\{\gamma_1, \gamma_2, k\}$ as a right-handed three-dimensional frame in $\mathbb{R}^n$. Then

$$\{\gamma_1, \gamma_2, k\} \to \int_{\partial\Omega} e^{ikx} (\zeta \cdot N) e^{\Psi(x,\zeta)}\, ds$$

defines a map from the collection of all such frames to $\mathbb{C}$. We shall prove (4.5) by differentiating this map in certain directions.

We construct two families of such frames, $\{\gamma_1^{(1)}(\theta), \gamma_2^{(1)}(\theta), k^{(1)}(\theta)\}$ and $\{\gamma_1^{(2)}(\theta), \gamma_2^{(2)}(\theta), k^{(2)}(\theta)\}$, depending smoothly on $\theta \in [0, \pi/4]$, as follows. Define $\gamma_2^{(1)}(\theta) = \gamma_2$ for all θ and define $\gamma_1^{(1)}(\theta)$ and $k^{(1)}(\theta)$ to be the resulting position vectors after we rotate the (right-handed) two-dimensional frame $\{\gamma_1, k\}$ clockwise with an angle θ in the plane spanned by γ_1 and k. Similarly, define $\gamma_1^{(2)}(\theta) = \gamma_1$ for all θ and define $\gamma_2^{(2)}(\theta)$ and $k^{(2)}(\theta)$ to be the resulting position vectors after we rotate the (right-handed) two-dimensional frame $\{\gamma_2, k\}$ clockwise with an angle θ in the plane spanned by γ_2 and k. It is easy to show that

$$\left.\frac{dk^{(1)}}{d\theta}\right|_{\theta=0} = |k|\gamma_1, \qquad \left.\frac{dk^{(2)}}{d\theta}\right|_{\theta=0} = |k|\gamma_2, \tag{4.7}$$

$$\left.\frac{d\gamma_1^{(1)}}{d\theta}\right|_{\theta=0} = \left.\frac{d\gamma_2^{(2)}}{d\theta}\right|_{\theta=0} = -|k|^{-1}k, \qquad \left.\frac{d\gamma_2^{(1)}}{d\theta}\right|_{\theta=0} = \left.\frac{d\gamma_1^{(2)}}{d\theta}\right|_{\theta=0} = 0. \tag{4.8}$$

If we define

$$\zeta^{(j)}(\theta) = \gamma_1^{(j)}(\theta) + i\gamma_2^{(j)}(\theta), \qquad j = 1, 2, \tag{4.9}$$

then (4.8) implies

$$\left.\frac{d\zeta^{(1)}}{d\theta}\right|_{\theta=0} + i\left.\frac{d\zeta^{(2)}}{d\theta}\right|_{\theta=0} = 0. \tag{4.10}$$

Next, we compute $(d/d\theta)(\Psi(x, \zeta^{(1)}(\theta)) + i\Psi(x, \zeta^{(2)}(\theta)))|_{\theta=0}$. Using formula (3.26) we have

(4.11)

$$\Psi(x, \zeta^{(j)}(\theta)) = \int_{\mathbb{R}^2} \frac{\zeta^{(j)}(\theta) \cdot (\vec{A}_1 - \vec{A}_2)(x - (z_1\gamma_1^{(j)}(\theta) + z_2\gamma_2^{(j)}(\theta)))}{z_1 + iz_2}\, dz_1\, dz_2$$

for $j = 1, 2$. Using (4.8), (4.10), and (4.11) we obtain

$$\begin{aligned} &\frac{d}{d\theta}(\Psi(x, \zeta^{(1)}(\theta)) + i\Psi(x, \zeta^{(2)}(\theta)))|_{\theta=0} \\ &= |k|^{-1} \int_{\mathbb{R}^2} k \cdot \nabla(\zeta \cdot (\vec{A}_1 - \vec{A}_2))(x - (z_1\gamma_1 + z_2\gamma_2))\, dz_1\, dz_2 \\ &= |k|^{-1} k \cdot \nabla \int_{\mathbb{R}^2} (\zeta \cdot (\vec{A}_1 - \vec{A}_2))(x - (z_1\gamma_1 + z_2\gamma_2))\, dz_1\, dz_2. \end{aligned} \tag{4.12}$$

It is easy to see that the last integral in (4.12) is a function of x which is equal to a constant on any two-dimensional plane that is parallel to γ_1 and γ_2.

Finally, letting $k = k^{(j)}(\theta)$ and $\zeta = \zeta^{(j)}(\theta)$, $j = 1, 2$, in (4.3), differentiating with respect to θ and using (4.7) and (4.10), we get

(4.13)
$$\begin{aligned} 0 &= \frac{d}{d\theta}\left(\int_{\partial\Omega} e^{ik^{(1)}(\theta)x}(\zeta^{(1)}(\theta)\cdot N)e^{\Psi(x,\zeta^{(1)}(\theta))}\,ds\right)\Bigg|_{\theta=0} \\ &\quad + i\frac{d}{d\theta}\left(\int_{\partial\Omega} e^{ik^{(2)}(\theta)x}(\zeta^{(2)}(\theta)\cdot N)e^{\Psi(x,\zeta^{(2)}(\theta))}\,ds\right)\Bigg|_{\theta=0} \\ &= |k|i\int_{\partial\Omega} e^{ikx}(\zeta\cdot N)(\zeta\cdot x)e^{\Psi(x,\zeta)}\,ds \\ &\quad + \int_{\partial\Omega} e^{ikx}(\zeta\cdot N)e^{\Psi(x,\zeta)}\,\frac{d}{d\theta}(\Psi(x,\zeta^{(1)}(\theta)) + i\Psi(x,\zeta^{(2)}(\theta)))\Big|_{\theta=0}\,ds. \end{aligned}$$

By (4.6) and (4.12)

$$\begin{aligned} &\int_{\partial\Omega\cap T}(\zeta\cdot N)e^{\Psi(x,\zeta)}\,\frac{d}{d\theta}(\Psi(x,\zeta^{(1)}(\theta)) + i\Psi(x,\zeta^{(2)}(\theta)))\Big|_{\theta=0}\,ds_T \\ &\quad = \frac{d}{d\theta}(\Psi(x,\zeta^{(1)}(\theta)) + i\Psi(x,\zeta^{(2)}(\theta)))\Big|_{\theta=0}\int_{\partial\Omega\cap T}(\zeta\cdot N)e^{\Psi(x,\zeta)}\,ds_T = 0 \end{aligned}$$

for any two-dimensional plane T that is parallel to γ_1 and γ_2. Therefore the last integral in (4.13) must be equal to zero. This proves (4.5) with $m = 1$. Repeating the above procedure gives (4.5) with arbitrary positive integer m. □

Step 2. Let T be any two-dimensional plane that is parallel to γ_1 and γ_2. Then

$$\int_{\partial\Omega\cap T}(\zeta\cdot N_T)^m\Psi(x,\zeta)\,ds_T = 0 \tag{4.14}$$

for any integer $m \geq 1$, where N_T is the outer normal of $\partial\Omega\cap T$ in T.

Proof. Using the same argument as the one which gave (4.6) we obtain from (4.5) that

$$\int_{\partial\Omega\cap T}(\zeta\cdot N)(\zeta\cdot x)^m e^{\Psi(x,\zeta)}\,ds_T = 0 \tag{4.15}$$

for any integer $m \geq 0$.

Recall that $\Omega = \{x \in \mathbb{R}^n,\ |x| < R\}$ is a ball and thus $\partial\Omega\cap T$ is a circle with origin as its center in the plane T. Therefore

$$\zeta\cdot N = \zeta\cdot N_T, \qquad \zeta\cdot x = R(\zeta\cdot N_T) \tag{4.16}$$

for $x \in \partial\Omega\cap T$. Combining (4.15) with (4.16) yields

$$\int_{\partial\Omega\cap T}(\zeta\cdot N_T)^m e^{\Psi(x,\zeta)}\,ds_T = 0 \tag{4.17}$$

for any integer $m \geq 1$.

If we denote by θ, $0 \leq \theta < 2\pi$, the angle between γ_1 and N_T, then $\zeta\cdot N_T = e^{i\theta}$. Hence (4.17) can be rewritten as

$$\int_0^{2\pi} e^{im\theta}e^{f(\theta)}\,d\theta = 0, \quad \forall \text{ integer } m \geq 1, \tag{4.18}$$

where $f = \Psi|_{\partial\Omega\cap T}$. Equation (4.18) implies that there exists a holomorphic function u defined on $D = \{x \in \mathbb{R}^2;\, |x| < 1\}$ such that $u|_{\partial D} = e^f$. By the mapping property of the exponential function one sees that $\log u$ is well defined on D. Since $\log u$ is also holomorphic in D and $\log u = f$ in ∂D, we must have

$$\int_0^{2\pi} e^{im\theta} f(\theta)\, d\theta = 0, \quad \forall \text{ integer } m \geq 1, \tag{4.19}$$

which leads to (4.17). □

Step 3. $\operatorname{rot}(\vec{A}_1) = \operatorname{rot}(\vec{A}_2)$.

Proof. It suffices to show that

$$\gamma_1 \cdot \int_\Omega e^{ikx}(\vec{A}_1 - \vec{A}_2)\, dx = 0 \tag{4.20}$$

for k and γ_1 as in (3.10). From (4.2) and (4.4) we have

$$\gamma_1 \cdot (\vec{A}_1 - \vec{A}_2) = -\operatorname{Im}(\zeta \cdot \nabla\Psi). \tag{4.21}$$

Multiplying both sides of (4.21) by e^{ikx} and integrating by parts we find that

$$\gamma_1 \cdot \int_\Omega e^{ikx}(\vec{A}_1 - \vec{A}_2)\, dx = -\int_{\partial\Omega} e^{ikx} \operatorname{Im}((\zeta \cdot N)\Psi(x, \zeta))\, dx. \tag{4.22}$$

We now show that the right-hand side of (4.22) must be equal to zero. It suffices to show that

$$\int_{\partial\Omega\cap T} \operatorname{Im}((\zeta \cdot N)\Psi(x, \zeta))\, ds_T = 0 \tag{4.23}$$

for any two-dimensional plane T that is parallel to γ_1 and γ_2. Since this is just a consequence of (4.14) (with $m = 1$), the proof is complete. □

Corollary 4.1. *There exists* $p \in W_\Omega^{1,\infty}$ *so that* $\vec{A}_1 - \vec{A}_2 = \nabla p$ *in* Ω.

Proof. $\operatorname{rot}(\vec{A}_1) = \operatorname{rot}(\vec{A}_2)$ implies that there exist $p \in W^{1,\infty}(\Omega)$ so that $\vec{A}_1 - \vec{A}_2 = \nabla p$ in Ω. The fact of $\operatorname{supp} \vec{A}_j \subset \overline{\Omega}$, $j = 1, 2$, implies that $p =$ constant in $\partial\Omega$. Hence, by substracting a constant we can adjust the function p so that $p \in W_\Omega^{1,\infty}$. □

4.2. **Proof of** $q_1 = q_2$. From Corollary 4.1 and the fact that $\Lambda_{\vec{A},q}$ is invariant under gauge transformations

$$\vec{A} \to \vec{A} + \nabla p, \qquad p \in W_\Omega^{1,\infty},$$

we deduce that

$$\Lambda_{\vec{A}_1, q_2} = \Lambda_{\vec{A}_2, q_2}.$$

Then by the hypothesis we must have

$$\Lambda_{\vec{A}_1, q_1} = \Lambda_{\vec{A}_1, q_2}.$$

Thus, we may assume without loss of generality that $\vec{A}_1 = \vec{A}_2 = \vec{A} \in W_\Omega^{2,\infty}$ in the rest of this section. Under this assumption (3.9) reads

$$\int_\Omega (q_1 - q_2) u_1 \overline{u}_2\, dx = 0. \tag{4.24}$$

Substituting the solution (3.15) into (4.24) gives

$$(4.25)\quad \int_\Omega e^{ikx+\phi_1+\overline{\phi}_2}(q_1-q_2)\,dx = \int_\Omega e^{ikx+\phi_1+\overline{\phi}_2}(q_1-q_2)(\omega_1+\overline{\omega}_2+\omega_1\overline{\omega}_2)\,dx.$$

From (4.2) and the statement following (3.23) we see that

$$(4.26)\qquad \phi_1+\overline{\phi}_2 \to \phi_1^*+\overline{\phi}_2^* = 0, \quad \text{in } L^\infty(\Omega) \qquad \text{as } s\to\infty.$$

Using this fact as well as (2.19) we deduce that the left-hand side of (2.25) tends to $(q_1-q_2)^\wedge(k)$ while the right-hand side tends to zero as s goes to ∞. Therefore $q_1=q_2$ in Ω. □

Acknowledgment

It is a pleasure for the author to thank Victor Isakov for his interests in this work and for valuable suggestions. The author would also like to thank Adrian Nachman, John Sylvester, and Gunther Uhlmann for stimulating discussions.

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