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AN ATRIODIC SIMPLE-4-OD-LIKE CONTINUUM WHICH IS NOT SIMPLE-TRIOD-LIKE

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ABSTRACT. The paper contains an example of a continuum K such that K is the inverse limit of simple 4-ods, K cannot be represented as the inverse limit of simple triods and each proper subcontinuum of K is an arc.

1. INTRODUCTION

All topological spaces considered in this paper are metric. A *continuum* is a connected and compact space. A *simple n-od* is the union of n arcs meeting at a common endpoint and which are mutually disjoint otherwise. A simple 3-od is called a *simple* triod. If X and Y are continua, we say that Y is *X-like* provided that Y is the inverse limit of a sequence of copies of X. A continuum is *atriodic* if does not contain three subcontinua A, B and C such that none of them is contained in the union of the remaining two and $\varnothing \neq A \cap B \cap C = A \cap B = A \cap C = B \cap C$.

In 1972, W. T. Ingram gave his brilliant example of an atriodic continuum which is simple-triod-like and not arc-like [4]. Note that an arc-like continuum is simple-2-od-like. The Ingram continuum is not only atriodic, but each of its proper subcontinuums is an arc. S. Young asked whether there exists a simple-4-od-like continuum which is not simple-triod-like and whose every proper subcontinuum is an arc [7, Problem 115]. A similar question was asked by H. Cook, W. T. Ingram and A. Lelek. They asked whether there exists an atriodic simple-4-od-like continuum which is not simple-triod-like [1, Problem 5]. Of course, if every proper subcontinuum is an arc, then the continuum is atriodic [3], so a positive answer to Young's question implies a positive answer to the question by Cook, Ingram and Lelek. Even after a perfunctory glance at the problems, it becomes apparent that they should have a positive answer. It is very easy to get an example of a simple-4-od like continuum such that every proper subcontinuum is an arc. Most of such continua appear not to be simple-triod-like and it is very likely that they really are not. So the only difficulty is a proof. Ingram proved that his continuum [4] is not arc-like (chainable) by showing that it has a positive span, and it was proved earlier by Lelek that chainable continua have the span zero [6]. The same method was subsequently

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used by Ingram in [5] and by Davis and Ingram in [2]. A topological invariant different than the span is needed to distinguish between those continua which are simple-triod-like and those that are not. Another way of approaching the problem is to use a continuum with simplicial (piecewise linear) bonding maps and prove that they cannot be factored through a simple triod. In this paper we choose this way to prove that there is a simple-4-od-like but not simple-triod-like continuum K such that every proper subcontinuum of K is an arc. Recently, the author [8] introduced an operation d assigning to a simplicial map between graphs a simplicial map between another pair of graphs and using it characterized simplicial maps which can be factored through an arc. This characterization yielded an alternate proof [8, Examples 5.12 and 5.14] of non-chainability for the Ingram and Davis-Ingram continua. In this paper we adapt the same idea to show that some maps cannot be factored through a simple triod.

2. Simplicial maps

In this section we introduce the notion of simplicial maps and prove some auxiliary propositions.

By a *graph* we understand one dimensional, finite simplicial complex. If G is a graph then $\mathscr{V}(G)$ will denote the set of vertices and $\mathscr{E}(G)$ will denote the set of edges. By the order of a vertex v we understand the number of edges containing v. Two points belonging to an edge are called adjacent. A *simplicial* map of a graph G_1 into a graph G_0 is a function from $\mathscr{V}(G_1)$ into $\mathscr{V}(G_0)$ taking every two adjacent vertices either onto a pair of adjacent vertices or onto a single vertex. A simplicial map is *light* if the image of each edge is nondegenerate.

In this paper the same notation is kept for a graph and for its geometric realization. We will assume that every graph is a subset of three dimensional Euclidean space and every edge is a straight linear closed segment between its vertices. In this convention a simplicial map is understood as an actual continuous mapping (linearly extended to the edges). But it is important to note that a graph, either abstract or geometric, has a fixed collection of vertices and any change in this collection changes the graph.

A graph with a geometric realization homeomorphic to an arc is simply called an arc. Observe that two arcs are isomorphic if and only if they have the same number of vertices. A connected graph without a simple closed curve is called a tree. A tree T is a *triod* if it is the union of three arcs intersecting at a common endpoint. If u and v are two adjacent vertices of a graph, by $\langle u, v\rangle$ we will denote the edge between u and v. Additionally, if u and v are two vertices of a tree, by $\langle u, v\rangle$ we will denote the arc between u and v.

2.1 **Proposition.** *Let $\{G_j, \varphi_j^i\}$ be an inverse system of graphs with simplicial and surjective bonding maps $\varphi_j^i\colon G_i \to G_j$. Let K denote the inverse limit (in the topological sense) of $\{G_j, \varphi_j^i\}$. Suppose that K is simple-triod-like, i.e. K is the inverse limit of simple 3-ods with continuous and not necessarily simplicial bonding maps. Then for each positive integer j there is a positive integer i such that φ_j^i can be factored through a (simplicial) triod.*

Proof. Let p_n denote the projection of K onto G_n. For each $v \in \mathscr{V}(G_j)$, let $U(v)$ denote a small ball around v in G_j such that $U(v_1) \cap U(v_2) = \varnothing$ for each $v_1, v_2 \in \mathscr{V}(G_j)$ and $v_1 \neq v_2$. Let $\mathscr{U}$ be the open covering of G_j consisting of the sets $U(v)$ and all open edges of G_j. (By an open edge we understand an edge without its endpoints.) The collection $\mathscr{K} = \{p_j^{-1}(U) | U \in \mathscr{U}\}$ is an open covering of K. Since K is simple-triod-like, there is an open covering $\mathscr{T}$ of K such that $\mathscr{T}$ subdivides $\mathscr{K}$ and the nerve of $\mathscr{T}$ is a triod. By a chain of elements of $\mathscr{T}$ we understand a sequence of sets $t_1, t_2, \dots, t_k$ such that $t_n \cap t_m \neq \varnothing$ if and only if $|n-m| \leq 1$. Since the nerve of $\mathscr{T}$ is a triod, for any two elements a and b of $\mathscr{T}$ there is exactly one chain $\mathrm{ch}(a, b)$ with the first element a and the last b. ($\mathrm{ch}(a, a)$ denote the chain reduced to one element a.) If A is a subset of $\mathscr{T}$, then by $\mathrm{conv}(A)$ we will denote the union of all chains $\mathrm{ch}(a, b)$, where $a, b \in A$. Let t be the only element of $\mathscr{T}$ intersecting three other elements of $\mathscr{T}$, and let a_1, a_2 and a_3 be the three elements of $\mathscr{T}$ such that $\mathrm{ch}(t, a_1) \cup \mathrm{ch}(t, a_2) \cup \mathrm{ch}(t, a_3) = \mathscr{T}$.

For each $v \in \mathscr{V}(G_j)$, let $\mathscr{T}(v)$ be the set of the elements of $\mathscr{T}$ contained in $p_j^{-1}(U(v))$. Denote by $\widetilde{\mathscr{T}}$ the union of $\mathscr{T}(v)$, where $v \in \mathscr{V}(G_j)$. We will define an equivalence relation $\cong$ on $\widetilde{\mathscr{T}}$ in the following way: $t_1 \cong t_2$ if there is $v \in \mathscr{V}(G_j)$ such that $t_1, t_2 \in \mathscr{T}(v)$ and $\mathrm{ch}(t_1, t_2) \subset (\mathscr{T} \backslash \widetilde{\mathscr{T}}) \cup \mathscr{T}(v)$. Let Θ denote the set $\widetilde{\mathscr{T}} / \cong$. If $\tau \in \Theta$, then by $v(\tau)$ we will denote the vertex of $\mathscr{V}(G_j)$ such that $\tau \subset \mathscr{T}(v(\tau))$. Observe that if τ_1 and τ_2 are two distinct elements of Θ, then the sets $\mathrm{conv}(\tau_1)$ and $\mathrm{conv}(\tau_2)$ are disjoint. Note also that $\mathrm{conv}(\tau_1 \cup \tau_2) \subset (\mathscr{T} \backslash \widetilde{\mathscr{T}}) \cup \tau_1 \cup \tau_2$ if and only if there are elements $t_1 \in \tau_1$ and $t_2 \in \tau_2$ such that $\mathrm{ch}(t_1, t_2) \subset (\mathscr{T} \backslash \widetilde{\mathscr{T}}) \cup \tau_1 \cup \tau_2$.

Let T be the graph defined in the following way: $\mathscr{V}(T) = \Theta$ and two vertices τ_1 and τ_2 of T are adjacent if $\mathrm{conv}(\tau_1 \cup \tau_2) \subset (\mathscr{T} \backslash \widetilde{\mathscr{T}}) \cup \tau_1 \cup \tau_2$. Let $\beta \colon T \to G_j$ be defined by the formula $\beta(\tau) = v(\tau)$ for $\tau \in \mathscr{V}(T)$. Clearly, β is a simplicial map.

We will prove that T is a triod (possibly degenerate). Note that if $\widetilde{\mathscr{T}}$ is contained in the union of two of the chains $\mathrm{ch}(t, a_1)$, $\mathrm{ch}(t, a_2)$ and $\mathrm{ch}(t, a_3)$, then T is an arc (or a single vertex). So we can assume that for each $k = 1, 2, 3$, there is $b_k \in \Theta$ such that b_k intersects the chain $\mathrm{ch}(t, a_k)$ and no other element of Θ intersects $\mathrm{conv}(b_k \cup \{t\})$. Let V_k denote the set of elements of Θ contained in $\mathrm{conv}(b_k \cup \{a_k\})$, and let A_k be the subgraph of T spanned by V_k. Observe that A_k is an arc (possibly degenerate) and b_k is an end point of A_k. Suppose that T is not a triod. Then the vertices b_1, b_2 and b_3 are distinct and each of them is adjacent to the remaining two. Since for each two of the sets $\mathrm{conv}(b_1)$, $\mathrm{conv}(b_2)$ and $\mathrm{conv}(b_3)$ there is a chain in $\mathscr{T}$ between them which does not intersect the third, we may assume that $t \notin \widetilde{\mathscr{T}}$ and $b_k \subset \mathrm{ch}(t, a_k)$ for $k = 1, 2, 3$. Let s_k be the first element of $\mathrm{ch}(t, a_k)$ belonging to $\widetilde{\mathscr{T}}$. Clearly, $s_k \in b_k$. Since $t \notin \widetilde{\mathscr{T}}$, there is an open edge e of G_j such that $p_j(t) \subset e$. Let v' and v'' be the vertices of e. Observe that $p_j(z) \subset e$ for each $z \in \mathrm{ch}(t, s_k) \backslash \{s_k\}$. It follows that $v(s_k)$ is either v' or v'' and consequently two of b_1, b_2 and b_3 coincide. This contradiction proves that T is a triod.

Let ε be a Lebesgue number for the covering $\mathscr{T}$ (i.e. ε is a positive number such that for each subset Y of K, if the diameter of Y is less than ε, then Y is contained in some element of $\mathscr{T}$). There is a positive integer i such that the diameter $p_i^{-1}(z)$ is less than ε for each $z \in G_i$. Since the bonding maps of the inverse system defining K are surjective, we have that $p_i(p_i^{-1}(z)) = z$. For each $w \in \mathscr{V}(G_i)$, let $a(w)$ be an element of $\mathscr{T}$ containing $p_i^{-1}(w)$. Note that $a(w) \in \mathscr{T}(\varphi_j^i(w))$. Let $\alpha(w)$ be the vertex of T representing $a(w)$. Clearly, $\beta \circ \alpha = \varphi_j^i$. To complete the proof it is enough to show that α is a simplicial map.

Let w and w' be two adjacent vertices of G_i. We will prove that $\alpha(w)$ and $\alpha(w')$ are adjacent vertices of T. Let $I_1, I_2, \dots, I_n$ be a chain covering of $\langle w, w' \rangle$ such that $p_i^{-1}(I_1) \subset a(w)$, $p_i^{-1}(I_n) \subset a(w')$ and the diameter $p_i^{-1}(I_k)$ is less than ε for each $k = 1, \dots, n$. Let B_k be an element of $\mathscr{T}$ containing $p_i^{-1}(I_k)$ with $B_1 = a(w)$ and $B_n = a(w')$.

Consider two cases $\varphi_j^i(w) = \varphi_j^i(w')$ and $\varphi_j^i(w) \neq \varphi_j^i(w')$. If $\varphi_j^i(w) = \varphi_j^i(w')$, then $\varphi_j^i(I_k) = \varphi_j^i(w)$ and thus $B_k \in \mathscr{T}(\varphi_j^i(w))$ for each $k = 1, \dots, n$. It follows that in this case $\alpha(w) = \alpha(w')$. So we may assume that $\varphi_j^i(w) \neq \varphi_j^i(w')$. Let e be the edge between $\varphi_j^i(w)$ and $\varphi_j^i(w')$. Since $p_j(B_k) \cap e \neq \varnothing$, we have the result that $p_j(B_k) \subset e \cup U(\varphi_j^i(w)) \cup U(\varphi_j^i(w'))$ and thus $B_k \in (\mathscr{T} \setminus \widetilde{\mathscr{T}}) \cup \mathscr{T}(\varphi_j^i(w)) \cup \mathscr{T}(\varphi_j^i(w'))$. Let m be the greatest integer such that $B_m \in \mathscr{T}(\varphi_j^i(w))$. Since $\varphi_j^i(I_1 \cup \cdots \cup I_m) \subset U(\varphi_j^i(w))$ and $U(\varphi_j^i(w)) \cap U(\varphi_j^i(w')) = \varnothing$, we have the result that $B_k \in (\mathscr{T} \setminus \widetilde{\mathscr{T}}) \cup \mathscr{T}(\varphi_j^i(w))$ for each $k = 1, \dots, m$. It follows that $B_1 \cong B_m$. Let m' be the least integer such that $B_{m'} \in \mathscr{T}(\varphi_j^i(w'))$. Clearly, $m' > m$. By the same argument as the one above we infer that $B_{m'} \cong B_n$. Since the collection $B_m, B_{m+1}, \dots, B_{m'}$ contains the chain $\operatorname{ch}(B_m, B_{m'})$, we have the result that $\alpha(w)$ and $\alpha(w')$ are adjacent vertices of T. □

We need to recall the following definitions from [8, 5.1 and 5.3].

2.2 **Definition.** We will say that a graph G' *subdivides* a graph G if G' is a graph obtained from G by adding vertices on some of its edges. More precisely, G' is a graph such that $\mathscr{V}(G) \subset \mathscr{V}(G')$ and for every edge $e \in \mathscr{E}(G)$ there is an arc (e, G') contained in G' such that

(i) (e, G') has the same endpoints as e,
(ii) $(d, G') \cap (e, G') = d \cap e$ for $d, e \in \mathscr{E}(G)$ and $d \neq e$, and
(iii) every vertex from $\mathscr{V}(G')$ belongs to some (e, G') and every edge from $\mathscr{E}(G')$ is an edge of some (e, G').

If v is a vertex of G and e is an edge of G containing v, then by (v, e, G') we denote the edge of (e, G') containing v.

Let $\varphi\colon G_1 \to G_0$ be a simplicial map between graphs. Let G_0' be a graph subdividing G_0 and let φ' be a simplicial map of a graph G_1' subdividing G_1 onto G_0'. We will say that φ' is a *subdivision* of φ *matching* G_0' provided that $\varphi'(v) = \varphi(v)$ for each vertex $v \in \mathscr{V}(G_1)$, and for each edge $e \in \mathscr{E}(G_1)$ we have that

if $\varphi(e)$ is degenerate then $(e, G_1') = e$, and

if $\varphi(e)$ is an edge of G_0 then φ' is an isomorphism of (e, G_1') onto $(\varphi(e), G_0)$.

2.3 **Proposition.** *Suppose* $\psi\colon G_1 \to G_0$ *is a simplicial map between connected graphs. Let* G_0' *be a graph subdividing* G_0 *and let* $\psi'\colon G_1' \to G_0'$ *be a subdivision of* φ *matching* G_0'. *Then* ψ *can be factored through a triod if and only if* ψ' *can be factored through a triod.*

Proof. Observe that by [8, Proposition 5.4], if ψ can be factored through a triod, then φ' also can be factored through a triod. Suppose that there is a triod T' and there are simplicial maps $\alpha'\colon G_1' \to T'$ and $\beta'\colon T' \to G_0'$ such that $\beta' \circ \alpha' = \psi'$. In view of [8, Proposition 5.13] we can assume that α' is surjective. Let t denote the only order 3 vertex of T'. Let t_0, t_1 and t_2 be the endpoints of T'. Let $V = \{v \in \mathscr{V}(T') | \beta'(v) \in \mathscr{V}(G_0)\}$. Observe that $\alpha'(\mathscr{V}(G_1)) = V$ and $\alpha'(\mathscr{V}(G_1')\backslash\mathscr{V}(G_1)) = \mathscr{V}(T')\backslash V$. Let $V_i = V \cap \langle t, t_i\rangle$ and let w_i be the vertex of V_i which is the closest to t. Let A_i denote the graph with V_i as its set of vertices such that any two vertices of V_i are adjacent if there are no other points of V between them. Observe that A_i is an arc. We will prove the following claim.

Claim. We can assume that $\beta'(w_0) = \beta'(w_2)$.

If $t \in V$, then $t = w_0 = w_1 = w_2$ and the claim is true. So we can assume that $t \notin V$. Since α' is surjective and G_1 is connected there are two pairs a, a' and b, b' of vertices of G_1 such that a and a' are adjacent in G_1, b and b' are adjacent in G_1, and the set $\{\alpha'(a), \alpha'(a'), \alpha'(b), \alpha'(b')\}$ consists of all three vertices w_0', w_1 and w_2.

Without loss of generality we can assume that $\alpha'(a) = w_0$, $\alpha'(a') = w_1 = \alpha'(b)$ and $\alpha'(b') = w_2$. Let e_0 be the edge of G_0 joining $\beta'(w_0) = \beta'(\alpha'(a)) = \psi(a)$ and $\beta'(w_1) = \beta'(\alpha'(a')) = \psi(a')$. Let e_1 be the edge of G_0 joining $\beta'(w_1) = \beta'(\alpha')b)) = \psi(b)$ and $\beta'(w_2) = \beta'(\alpha'(b')) = \psi(b')$. Since (e_0, G_0') and (e_1, G_0') have two common vertices $\beta'(w_1)$ and $\beta'(t)$, e_0 and e_1 coincide. Thus $\beta'(w_0) = \beta'(w_2)$.

We will define T considering two cases $\beta'(w_0) = \beta'(w_2) = \beta'(w_1)$ and $\beta'(w_0) = \beta'(w_2) \neq \beta'(w_1)$. In the first case T is the union of A_0, A_1 and A_2 with w_0, w_1 and w_2 identified to one vertex. If $\beta'(w_0) = \beta'(w_2) \neq \beta'(w_1)$, T is the union of A_0, A_1 and A_2 with w_0 and w_2 identified to one vertex, and with an edge between the result of the identification and w_1.

Let $\beta\colon T \to G_0$ be such that $\beta(v) = \beta'(v)$ for each $v \in V$. Note that β is a simplicial map. Let $\alpha\colon G_1 \to T$ be such that $\alpha(v) = \alpha'(v)$ for each $v \in \mathscr{V}(G_1)$. One can verify that α is a simplicial map and $\beta \circ \alpha = \psi$. □

3. Construction of the example

In this section we will construct a simplicial map φ between two subdivisions of a simple 4-od. Then we consider the inverse system with subdivisions of φ as bonding maps. We define K to be the inverse limit of this system. We show in this section that each proper subcontinuum of K is an arc. We will show later than K is not simple-triod-like.

Let X be a tree with its vertices named as in Figure 1.

Let X' be a subdivision of X with twelve new vertices $u_0, u_1, \dots, u_{11}$ added as shown in Figure 2. Let $\varphi\colon X' \to X$ be the simplicial map defined by Table 1 and the following equality $\varphi(u_0) = \varphi(u_2) = \varphi(u_4) = \varphi(u_5) = \varphi(u_7) = \varphi(u_9) = \varphi(u_{10}) = v_0$.

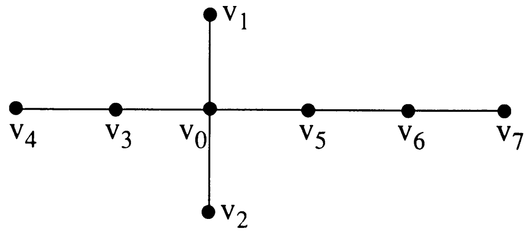

FIGURE 1

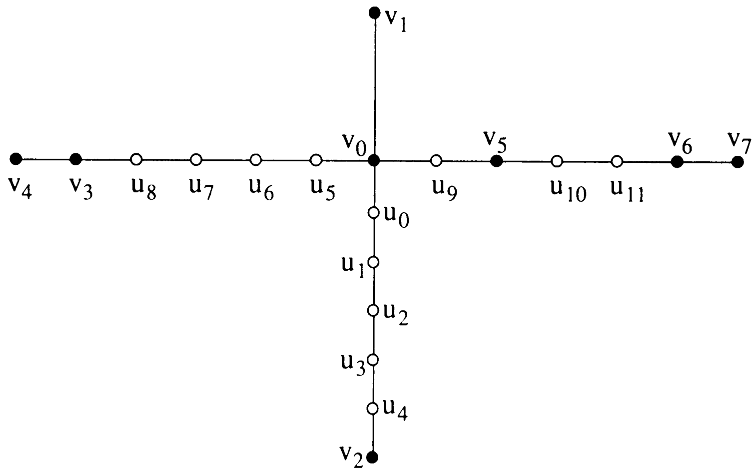

FIGURE 2

TABLE 1

x:	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	u_1	u_3	u_6	u_8	u_{11}
$\varphi(x)$:	v_3	v_4	v_1	v_6	v_7	v_1	v_6	v_7	v_5	v_2	v_2	v_5	v_5

The map φ is indicated in Figure 3. The dotted line graph represents the domain, X', while the solid black represents the range, X, and each vertex of the domain is mapped onto the nearest vertex of the range. Only vertices $v_0, v_1, \ldots, v_7$ of X' are labeled. Not labeled are the remaining vertices of X' and the vertices of X, which are named exactly the same as shown in Figure 1.

3.1 **Proposition.** *$\varphi^4(\langle u_{11}, v_6\rangle) = X$ and for each vertex x of X', $\varphi^4(x)$ is either v_6 or v_7. (φ^n is the nth iteration of φ.)* □

We need to recall another definition from [8].

3.2 **Definition.** Let n be a positive integer and let N denote either the set $\{0, 1, \ldots, n\}$ or the set of all nonnegative integers. Denote by N_1 the set $N\backslash\{0\}$. Let $G_0, G_1, G_2, \ldots$ be a sequence of graphs with N as the set of indices. Let Σ be a sequence of simplicial maps $\psi_1, \psi_2, \ldots$ such that for each $j \in N_1$, ψ_j maps a graph G'_j subdividing G_j into G_{j-1}. Using inductively Proposition 5.4 from [8], we can define a sequence of simplicial maps $\sigma_1, \sigma_2, \ldots$ such that $\sigma_1 = \psi_1$ and for each $j \in N_1\backslash\{1\}$, σ_j subdivides ψ_j

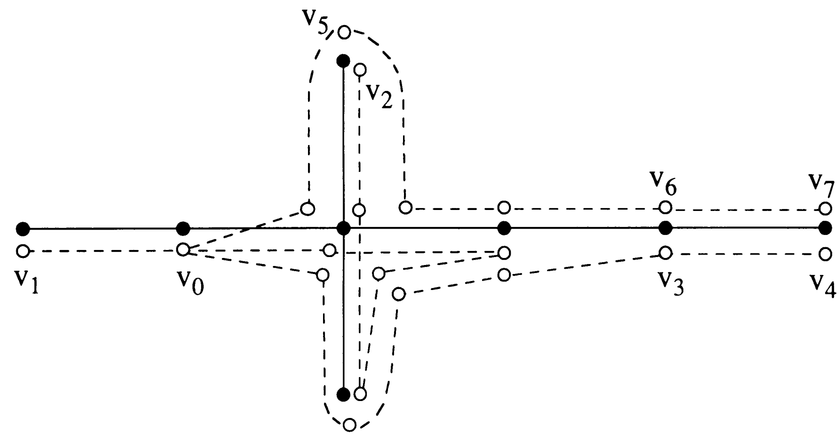

FIGURE 3

matching the domain of σ_{j-1}. For each $j \in N_1$, denote by Σ_j the domain of ψ_j. Set $\Sigma_0 = G_0$. For every two integers i and j from N such that $i > j$, let Σ_j^i denote the composition $\sigma_{j+1} \circ \cdots \circ \sigma_i$ mapping Σ_i into Σ_j. By Σ_i^i we will understand the identity on Σ_i. We will say that the inverse system $\{\Sigma_j, \Sigma_j^i\}$ is generated by the sequence Σ.

Let Φ be an infinite sequence of simplicial maps $\varphi_1, \varphi_2, \ldots$ each of which is φ. By $\{\Phi_j, \Phi_j^i\}$ we denote the inverse system generated by Φ. Let K be the inverse limit of $\{\Phi_j, \Phi_j^i\}$. Denote by p_j the projection of K onto Φ_j.

3.3 **Proposition.** *Every proper nondegenerate subcontinuum of K is an arc.*

Proof. Let P be a proper subcontinuum of K. Suppose that P is not an arc. Since φ is simplicial on X', there is a vertex x of X' such that $x \in p_j(P)$ for infinitely many j. In view of Proposition 3.1, we have that either $v_6 \in p_j(P)$ for all j or $v_7 \in p_j(P)$ for all j. Since φ restricted to $\langle u_{11}, v_7\rangle$ is an embedding, $u_{11} \in p_j(P)$ for infinitely many j. Since $\langle u_{11}, v_6\rangle \subset \langle u_{11}, v_7\rangle$ and $\varphi^4(\langle u_{11}, v_6\rangle) = X$, $p_j(P) = X$ for each j. Thus P is not proper. □

3.4 **Proposition.** *Let n be a positive integer. Let e be an edge of X. Let $\tilde{e}$ denote the interior of e. Suppose $C \subset \Phi_n$ is a component of $(\Phi_1^n)^{-1}(\tilde{e})$. Then the closure of C is mapped by Φ_1^n isomorphically onto e.*

Proof. Since Φ_0^{n-1} is a simplicial map onto X, any component of $(\Phi_0^{n-1})^{-1}(\tilde{e})$ is the interior of an edge of Φ_{n-1}. Since Φ_1^n is a simplicial subdivision of Φ_0^{n-1} matching X', C is a subdivision of the interior of an edge of Φ_{n-1}. Since Φ_0^{n-1} maps isomorphically edges of Φ_{n-1} onto edges of X, C is mapped by Φ_1^n isomorphically onto e. □

3.5 **Proposition.** *Let n be a positive integer. Suppose that a is a vertex of Φ_n such that $\Phi_0^n(a) = v_0$. Then a is a vertex of order 2 in Φ_n. Moreover, if b and c are the two vertices of Φ_n adjacent to a, then Φ_0^n is an embedding of $\langle b, c\rangle$ into X.*

Proof. Since $\Phi_1^n(a)$ is a vertex of X' but not of X, the proposition follows from Proposition 3.4.

3.6 **Proposition.** *Suppose that Y is a triod which is the union of three arcs A_1, A_2 and A_3 meeting at a common endpoint y. Let $y, y_1^i, y_2^i, \dots, y_{k(i)}^i$ denote the sequence of consecutive vertices of A_i. Let $\beta: Y \to X$ be a simplicial map such that $\beta(y) = v_0$ and the points $\beta(y_1^1)$, $\beta(y_1^2)$ and $\beta(y_1^3)$ are three different vertices from the set $\{v_1, v_2, v_3, v_5\}$. Suppose there is a positive integer n and there is a simplicial map $\alpha: \Phi_n \to Y$ such that $\beta \circ \alpha = \Phi_0^n$. Then there is a triod Y' with its vertex of order three denoted by y' and there are simplicial maps $\beta': Y' \to X$ and $\alpha': \Phi_n \to Y'$ such that $\beta' \circ \alpha' = \Phi_0^n$ and $\beta'(y') \neq v_0$.*

Proof. Let $W_{1,2}$, $W_{1,3}$ and $W_{2,3}$ denote the sets $\alpha^{-1}(A_1) \cap \alpha^{-1}(A_2)$, $\alpha^{-1}(A_1) \cap \alpha^{-1}(A_3)$ and $\alpha^{-1}(A_2) \cap \alpha^{-1}(A_3)$, respectively. By Proposition 3.5, the sets $W_{1,2}$, $W_{1,3}$ and $W_{2,3}$ are mutually exclusive and $\alpha^{-1}(y)$ is their union.

3.6.1 *Claim.* If one of the sets $W_{1,2}$, $W_{1,3}$ and $W_{2,3}$ is empty then the proposition is true.

Since A_1, A_2 and A_3 play the same role in the statement of the proposition, we may assume that $W_{1,2} = \varnothing$. Define $\mathscr{V}(Y')$ to be $\mathscr{V}(Y)$ with y replaced by two points w_1 and w_2. Let $\mathscr{E}(Y')$ consists of $\langle y_1^3, w_1\rangle$, $\langle y_1^3, w_2\rangle$, $\langle w_1, y_1^1\rangle$, $\langle w_2, y_1^2\rangle$ and all edges of Y not containing y. Define $\beta'(w_1) = \beta'(w_2) = v_0$ and $\beta'(w) = \beta(w) = \alpha(v)$ for $w \in \mathscr{V}(Y') \cap \mathscr{V}(Y)$. Define $\alpha'(v)$ for $v \in \mathscr{V}(\Phi_n) \backslash \alpha^{-1}(y)$, $\alpha'(v) = w_1$ for $v \in W_{1,3}$ and $\alpha'(v) = w_2$ for $v \in W_{2,3}$. One can verify that the hypothesis of the proposition is satisfied with $y' = y_1^3$.

3.6.2 *Claim.* Suppose that $\beta(y_1^1) = v_1$ and $\beta(y_1^2) = v_2$. Suppose also that either $\beta(y_1^3) = v_3$ or $\beta(y_1^3) = v_5$ and $\beta(y_2^3) = v_6$. If $W_{1,2} \neq \varnothing$, then $\alpha(\Phi_n) \subset A_1 \cup A_2$.

By Proposition 3.5, there are three vertices w_1, w and w_2 of Φ_n such that w_1 is adjacent to w, w is adjacent to w_2, $\alpha(w_1) = y_1^1$, $\alpha(w) = y$ and $\alpha(w_2) = y_1^2$. Since $v_2 - u_4 - u_3$ is the only pair of intersecting edges of X' mapped by φ onto $v_1 - v_0 - v_2$, we have the result that $\Phi_1^n(w_1) = v_2$, $\Phi_1^n(w) = u_4$ and $\Phi_1^n(w_2) = u_3$. Let x be an arbitrary vertex of Φ_n. Let $s_0 = w$, $s_1, \dots, s_k = x$ be vertices of $\langle w, x\rangle$ such that for $i = 0, 1, \dots, k-1$, $\alpha(s_i) = y$ and $\alpha(a) \neq y$ for any vertex a from the interior of $\langle s_i, s_{i+1}\rangle$.

In order to prove the claim, it is enough to show that if $\Phi_1^n(s_i) = u_4$, then $\alpha(\langle s_i, s_{i+1}\rangle) \subset A_1 \cup A_2$ and if $i + 1 < k$, then also $\Phi_1^n(s_{i+1}) = u_4$. Observe that $\alpha(\langle s_i, s_{i+1}\rangle)$ is contained in one of the arcs A_1, A_2 and A_3. Let $a_0 = s_i$, $a_1, a_2, \dots, a_m = s_{i+1}$ be vertices of $\langle s_i, s_{i+1}\rangle$ listed in the natural order. Clearly, either $\Phi_1^n(a_1) = u_3$ or $\Phi_1^n(a_1) = v_2$. We will consider each of these cases separately.

Suppose $\Phi_1^n(a_1) = u_3$. Since $\beta \circ \alpha = \Phi_0^n = \varphi \circ \Phi_1^n$, $\alpha(a_1) = y_1^2$ and consequently $\alpha(\langle s_i, s_{i+1}\rangle) \subset A_2$. By Proposition 3.4, there is a vertex b of Φ_n such that Φ_0^n maps consecutive vertices of $\langle a_1, b\rangle$ onto the sequence v_2, v_0, v_5, v_0, v_3. To match this pattern, α must map $\langle a_1, b\rangle$ into A_2 and we must have that $\beta(y_2^2) = v_0$, $\beta(y_3^2) = v_5$, $\beta(y_4^2) = v_0$ and $\beta(y_5^2) = v_3$. Now, assume that $\alpha(s_{i+1}) = y$. Clearly, $m > 5$ and $\alpha(a_2) = y_2^2$, $\alpha(a_3) = y_3^2$, $\alpha(a_4) = y_4^2$ and $\alpha(a_5) = y_5^2$. Let $j < m$ be the greatest integer such that $\alpha(a_j) = y_4^2$. Note that $m \geq j + 4$. Since $\beta \circ \alpha = \Phi_0^n$, it follows from Proposition 3.5 that $\alpha(a_{j-1}) = y_5^2$ and $\alpha(a_{j+1}) = y_3^2$. Observe that $\Phi_0^n(a_{j-1}) = v_3$, $\Phi_0^n(a_j) = v_0$ and $\Phi_0^n(a_{j+1}) = v_5$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_5$, $\Phi_1^n(a_{j+1}) = u_1$. By Proposition 3.4, there is a vertex c of Φ_n such that $a_j \in \langle a_{j-1}, c\rangle$ and $\langle a_{j-1}, c\rangle$ is mapped by Φ_1^n isomorphically onto $\langle v_0, v_2\rangle$. It follows from Proposition 3.4 that the vertices a_{j+1}, a_{j+2}, a_{j+3} and a_{j+4} belong to $\langle a_{j-1}, c\rangle$. Since $\langle a_{j-1}, c\rangle$ has seven vertices, a_{j+4} is adjacent to c. Since $\alpha(a_{j-1}) = y_5^2$, $\alpha(a_j) = y_4^2$, $\alpha(a_{j+1}) = y_3^2$ and Φ_0^n maps consecutive vertices of $\langle a_{j-1}, c\rangle$ onto the sequence $v_3, v_0, v_5, v_0, v_2, v_0, v_1$, we have the result that $\alpha(c) = y_1^1$. Thus $\alpha(a_{j+4}) = y$, $s_{i+1} = a_{j+4}$ and consequently $\Phi_1^n(s_{i+1}) = u_4$. This completes the proof of the claim in the case where $\Phi_1^n(a_1) = u_3$.

Suppose $\Phi_1^n(a_1) = v_2$. The proof in this case is essentially the same as in the previous case. Since $\beta \circ \alpha = \Phi_0^n = \varphi \circ \Phi_1^n$, $\alpha(a_1) = y_1^1$ and consequently $\alpha(\langle s_i, s_{i+1}\rangle) \subset A_1$. Now, suppose that $\alpha(s_{i+1}) = y$. Clearly, $m \geq 2$ and $\Phi_1^n(a_2) = u_4$. Observe that either $\alpha(a_2) = y$ or $\alpha(a_2) = y_2^1$. In the first case the claim is satisfied, so we can assume that $\alpha(a_2) = y_2^1$. By Proposition 3.4, there is a vertex b of Φ_n such that Φ_0^n maps consecutive vertices of $\langle a_1, b\rangle$ onto the sequence $v_1, v_0, v_2, v_0, v_5, v_0, v_3$. To match this pattern, α must map $\langle a_1, b\rangle$ into A_1 and must have that $\beta(y_2^1) = v_0$, $\beta(y_3^1) = v_2$, $\beta(y_4^1) = v_0$, $\beta(y_5^1) = v_5$, $\beta(y_6^1) = v_0$ and $\beta(y_7^1) = v_3$. Clearly, $m > 7$ and $\alpha(a_3) = y_3^1$, $\alpha(a_4) = y_4^1$, $\alpha(a_5) = y_5^1$, $\alpha(a_6) = y_6^1$ and $\alpha(a_7) = y_7^1$. Let $j < m$ be the greatest integer such that $\alpha(a_j) = y_6^1$. Note that $m \geq j+6$. Since $\beta\circ\alpha = \Phi_0^n$, it follows from Proposition 3.5 that $\alpha(a_{j-1}) = y_7^1$ and $\alpha(a_{j+1}) = y_5^1$. Observe that $\Phi_0^n(a_{j-1}) = v_3$, $\Phi_0^n(a_j) = v_0$ and $\Phi_0^n(a_{j+1}) = v_5$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_5$, $\Phi_1^n(a_{j+1}) = u_1$. By Proposition 3.4, $\langle a_{j-1}, a_{j+5}\rangle$ is mapped by Φ_1^n isomorphically onto $\langle v_0, v_2\rangle$. In particular $\Phi_1^n(a_{j+5}) = v_2$ and consequently $\Phi_1^n(a_{j+6}) = u_4$. Since $\alpha(a_{j-1}) = y_7^1$, $\alpha(a_j) = y_6^1$, $\alpha(a_{j+1}) = y_5^1$ and Φ_0^n maps consecutive vertices of $\langle a_{j-1}, a_{j+5}\rangle$ onto the sequence $v_3, v_0, v_5, v_0, v_2, v_0, v_1$, we have the result that $\alpha(a_{j+5}) = y_1^1$. If $\alpha(a_{j+6}) = y$, then $s_{i+1} = a_{j+6}$ and the claim is true. Suppose that $\alpha(a_{j+6}) \neq y$ and thus $\alpha(a_{j+6}) = y_2^2$. By Proposition 3.4, there is a vertex c of Φ_n such that $a_{j+6} \in \langle a_{j+5}, c\rangle$ and $\langle a_{j+5}, c\rangle$ is mapped by Φ_1^n isomorphically onto $\langle v_0, v_2\rangle$. Since $\alpha(a_{j+5}) = y_1^2$, $\alpha(a_{j+6}) = y_2^1$ and Φ_0^n maps consecutive vertices of $\langle a_{j+5}, c\rangle$ onto the sequence $v_1, v_0, v_2, v_0, v_5, v_0, v_3$, we have the result that $\alpha(\langle a_{j+5}, c\rangle) = \langle y_1^1, y_7^1\rangle$. It follows that $m > j+11$ and $\alpha(a_{j+10}) = y_6^1$. This contradicts the choice of j. So the claim is true.

We will consider the following four cases: Case (i). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_2$ and $\beta(y_1^3) = v_3$, Case (ii). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_2$ and $\beta(y_1^3) = v_5$, Case (iii). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$, and Case (iv). $\beta(y_1^1) = v_2$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$.

Case (i). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_2$ and $\beta(y_1^3) = v_3$.

By 3.6.2, if $W_{1,2} \neq \varnothing$, then $\alpha(\Phi_n)$ is an arc and the proposition is trivially satisfied. So, we can assume that $W_{1,2} = \varnothing$ and infer the proposition from 3.6.1.

Case (ii). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_2$ and $\beta(y_1^3) = v_5$.

If $W_{1,3} = \varnothing$, then the proposition is true by 3.5.1. So we can assume that $W_{1,3} \neq \varnothing$. By Proposition 3.5, there are three vertices w_1, w and w_3 of Φ_n such that w_1 is adjacent to w, w is adjacent to w_3, $\alpha(w_1) = y_1^1$, $\alpha(w) = y$ and $\alpha(w_3) = y_1^3$. Since $v_5 - u_{10} - u_{11}$ is the only pair of intersecting edges of X' mapped by φ onto $v_1 - v_0 - v_5$, we have the result that $\Phi_1^n(w_1) = v_5$, $\Phi_1^n(w) = u_{10}$ and $\Phi_1^n(w_3) = u_{11}$. By Proposition 3.4, there is a vertex c of Φ_n

such that c is adjacent to w_3 and $\Phi_1^n(c) = v_6$. Since $\Phi_0^n(c) = v_6$, $\alpha(c) = y_2^3$ and $\beta(y_2^3) = v_6$. Now, we can use 3.6.2. If $W_{1,2} \neq \varnothing$, then $\alpha(\Phi_n)$ is an arc and the proposition is trivially satisfied. So, we can assume that $W_{1,2} = \varnothing$ and infer the proposition from 3.6.1.

Case (iii). $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$.

If $W_{1,3} = \varnothing$, then the proposition is true by 3.6.1. So we can assume that $W_{1,3} \neq \varnothing$. By Proposition 3.5, there are three vertices w_1, w and w_3 of Φ_n such that w_1 is adjacent to w, w is adjacent to w_3, $\alpha(w_1) = y_1^1$, $\alpha(w) = y$ and $\alpha(w_3) = y_1^3$. Since $v_5 - u_{10} - u_{11}$ is the only pair of intersecting edges of X' mapped by φ onto $v_1 - v_0 - v_5$, we have the result that $\Phi_1^n(w_1) = v_5$, $\Phi_1^n(w) = u_{10}$ and $\Phi_1^n(w_3) = u_{11}$. By Proposition 3.4, there is a vertex c of Φ_n such that c is adjacent to w_3 and $\Phi_1^n(c) = v_6$. Since $\Phi_0^n(c) = v_6$, $\alpha(c) = y_2^3$ and $\beta(y_2^3) = v_6$.

If $W_{2,3} = \varnothing$, then the proposition is true by 3.6.1. So we can assume that $W_{2,3} \neq \varnothing$. By Proposition 3.5, there are three vertices u_2, u and u_3 of Φ_n such that u_2 is adjacent to u, u is adjacent to u_3, $\alpha(u_2) = y_1^2$, $\alpha(u) = y$ and $\alpha(u_3) = y_1^3$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_5$, we have the result that $\Phi_1^n(u_2) = v_0$, $\Phi_1^n(u) = u_0$ and $\Phi_1^n(u_3) = u_1$. By Proposition 3.4, there are vertices a and b of Φ_n such that a is adjacent to u_3, b is adjacent to a, $\Phi_1^n(a) = u_2$ and $\Phi_1^n(b) = u_3$. Since $\Phi_0^n(a) = v_0$ and $\beta(y_2^3) = v_6$, we have the result that $\alpha(a) = y$. Thus $\alpha(b)$ must be one of the points y_1^1, y_1^2 and y_1^3. This is a contradiction, because $\beta(\alpha(b)) = \Phi_0^n(b) = v_2$, $\beta(y_1^1) = v_1$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$.

Case (iv). $\beta(y_1^1) = v_2$, $\beta(y_1^2) = v_3$ and $\beta(y_1^3) = v_5$.

If $W_{2,3} = \varnothing$, then the proposition is true by 3.6.1. So we can assume that $W_{2,3} \neq \varnothing$. By Proposition 3.5, there are three vertices w_2, w and w_3 of Φ_n such that w_2 is adjacent to w, w is adjacent to w_3, $\alpha(w_2) = y_1^2$, $\alpha(w) = y$ and $\alpha(w_3) = y_1^3$. Since $v_0 - u_0 - u_1$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_5$, we have the result that $\Phi_1^n(w_2) = v_0$, $\Phi_1^n(w) = u_0$ and $\Phi_1^n(w_3) = u_1$. By Proposition 3.4, there is a vertex b of Φ_n such that $w_3 \in \langle w_2, b\rangle$ and Φ_1^n maps $\langle w_2, b\rangle$ isomorphically onto $\langle v_0, v_2\rangle$. Note that Φ_0^n maps the consecutive vertices of $\langle w_3, b\rangle$ onto the sequence v_5, v_0, v_2, v_0, v_1. To match this pattern we must have that

$$\text{(3.6.3)} \qquad \text{either } \beta(y_2^3) = v_0, \text{ or } \beta(y_2^1) = v_0 \text{ and } \beta(y_3^1) = v_1.$$

If $W_{1,2} = \varnothing$, then the proposition is true by 3.6.1. So we can assume that $W_{1,2} \neq \varnothing$. By Proposition 3.5, there are three vertices z_1, z and z_2 of Φ_n such that z_1 is adjacent to z, z is adjacent to z_2, $\alpha(z_1) = y_1^1$, $\alpha(z) = y$ and $\alpha(z_2) = y_1^2$. Since $v_0 - u_5 - u_6$ is the only pair of intersecting edges of X' mapped by φ onto $v_3 - v_0 - v_2$, we have the result that $\Phi_1^n(z_2) = v_0$, $\Phi_1^n(z) = u_5$ and $\Phi_1^n(z_1) = u_6$. By Proposition 3.4, there is a vertex c of Φ_n such that $z_1 \in \langle z_2, c\rangle$ and Φ_1^n maps $\langle z_2 c\rangle$ isomorphically onto $\langle v_0, v_3\rangle$. Note that Φ_0^n maps the consecutive vertices of $\langle z_1, c\rangle$ onto the sequence v_2, v_0, v_5, v_6. To match this pattern we must have that

$$\text{(3.6.4)} \qquad \text{either } \beta(y_2^1) = v_0 \text{ and } \beta(y_3^1) = v_5, \text{ or } \beta(y_2^3) = v_6.$$

We will prove that

$$\text{(3.6.5)} \qquad \text{either } \beta(y_2^1) = v_0 \text{ and } \beta(y_3^1) = v_1, \text{ or } \beta(y_2^3) = v_6.$$

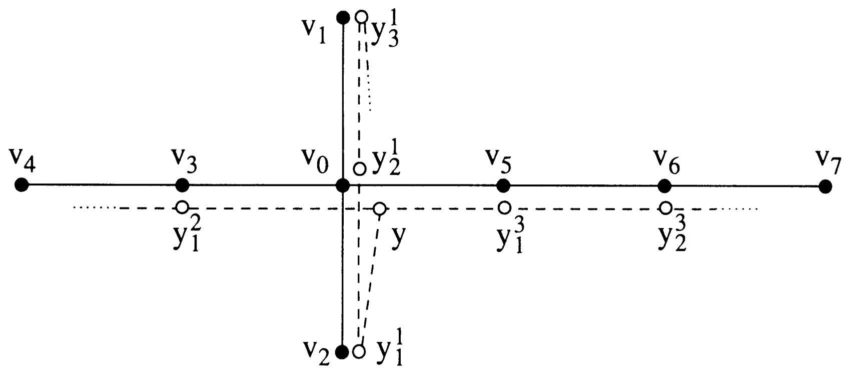

FIGURE 4

Again, by 3.6.1, we can assume that $W_{1,3} \neq \varnothing$. By Proposition 3.5, there are three vertices x_1, x and x_3 of Φ_n such that x_1 is adjacent to x, x is adjacent to x_3, $\alpha(x_1) = y_1^1$, $\alpha(x) = y$ and $\alpha(x_3) = y_1^3$. Since $u_6 - u_7 - u_8$ and $u_3 - u_2 - u_1$ are the only two pairs of intersecting edges of X' mapped by φ onto $v_2 - v_0 - v_5$, we have the result that either $\Phi_1^n(x_1) = u_6$, $\Phi_1^n(x) = u_7$ and $\Phi_1^n(x_3) = u_8$, or $\Phi_1^n(x_1) = u_3$, $\Phi_1^n(x) = u_2$ and $\Phi_1^n(x_3) = u_1$. Suppose that $\Phi_1^n(x_3) = u_8$. Then by Proposition 3.4, there is a vertex s of Φ_n such that s is adjacent to x_3 and $\Phi_1^n(s) = v_3$. This forces $\beta(y_2^3) = v_6$ and (3.6.5) is true. So, we can assume that $\Phi_1^n(x_1) = u_3$, $\Phi_1^n(x) = u_2$ and $\Phi_1^n(x_3) = u_1$. By Proposition 3.4, there is a vertex q of Φ_n such that $x_1 \in \langle x_3, q\rangle$ and Φ_1^n maps $\langle x_1, q\rangle$ isomorphically onto $\langle u_3, v_2\rangle$. Note that Φ_0^n maps the consecutive vertices of $\langle x_1, q\rangle$ onto the sequence v_2, v_0, v_1. To match this pattern we must have that $\beta(y_2^1) = v_0$ and $\beta(y_3^1) = v_1$. So (3.6.5) is true.

Combining (3.6.3), (3.6.4) and (3.6.5) we get the result that

$$(3.6.6) \qquad \beta(y_2^1) = v_0, \ \beta(y_3^1) = v_1 \text{ and } \beta(y_2^3) = v_6.$$

Figure 4 shows the map β on the "central" part of Y. As before, the dotted line graph represents the domain, Y, while the solid black represents the range, X, and each vertex of the domain is mapped onto the nearest vertex of the range.

Define $\mathscr{V}(Y')$ to be $\mathscr{V}(Y)$ with y replaced by four points g_0, g_2, g_5 and g_7, with y_1^1 replaced by two points g_3 and g_6, and with y_1^3 replaced by two points g_1 and g_8. Let $\mathscr{E}(Y')$ consists of $\langle y_2^1, g_3\rangle$, $\langle y_2^3, g_8\rangle$, $\langle y_1^2, g_0\rangle$, $\langle y_1^2, g_5\rangle$, $\langle g_0, g_1\rangle$, $\langle g_1, g_2\rangle$, $\langle g_2, g_3\rangle$, $\langle g_5, g_6\rangle$, $\langle g_6, g_7\rangle$, $\langle g_7, g_8\rangle$ and all edges of Y not containing any of the vertices y, y_2^1 and y_1^3. Observe that Y' is a triod and y_1^2 is its vertex of order 3. (See Figure 5.) Define $\beta'(g_0) = \beta'(g_2) = \beta'(g_5) = \beta'(g_7) = v_0$, $\beta'(g_3) = \beta'(g_6) = v_2$, $\beta'(g_1) = \beta'(g_8) = v_5$ and $\beta'(w) = \beta(w)$ for $w \in \mathscr{V}(Y')\cap\mathscr{V}(Y)$. Figure 5 shows the map β' on the "central" part of Y'. As usual, the dotted line graph represents the domain, Y', while the solid black represents the range, X, and each vertex of the domain is mapped onto the nearest vertex of the range. Compare this figure with Figures 3 and 4.

Let G denote the set $\{v \in \mathscr{V}(\Phi_n) | \alpha(v)$ is either y, or y_1^1, or $y_1^3\}$. Let

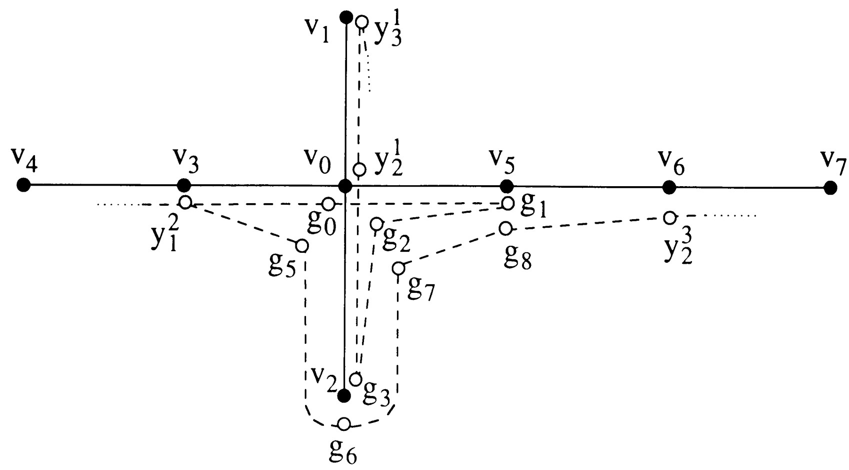

FIGURE 5

G_i denote the set $\{v \in G | \Phi_1^n(v) = u_i\}$ where $i = 0, 1, 2, 3, 5, 6, 7$ or 8. Using 3.5.6 and Proposition 3.4 one can prove that $G = G_0 \cup G_1 \cup G_2 \cup G_3 \cup G_5 \cup G_6 \cup G_7 \cup G_8$. Define $\alpha'(v) = \alpha(v)$ for $v \in \mathscr{V}(\Phi_n)\backslash G$ and $\alpha'(v) = g_i$ for $v \in G_i$, $i = 0, 1, 2, 3, 5, 6, 7, 8$. Clearly, α' is a simplicial map and $\beta' \circ \alpha' = \Phi_0^n$. $\square$

4. THE OPERATION d

In this section we will recall combinatorial methods introduced in [8] and apply them to the map φ.

4.1 **Definition.** For a graph G_0, Let $D(G_0)$ denote the graph such that

(i) the set of vertices of $D(G_0)$ consists of edges of G_0 and

(ii) two vertices of $D(G_0)$ are adjacent if and only if they intersect (as edges of G_0).

Let $\psi: G_1 \to G_0$ be a simplicial map between graphs. For every (closed) edge $e \in \mathscr{E}(G_0)$, let $\mathscr{K}(e)$ denote the set of components of $\psi^{-1}(e)$ which are mapped by ψ onto e. Denote by $\mathscr{K}(\psi)$ the union of all $\mathscr{K}(e)$. Let $D(\psi, G_1)$ be the graph such that

(i) the vertices of $D(\psi, G_1)$ are elements of $\mathscr{K}(\psi)$, and

(ii) two vertices of $D(\psi, G_1)$ are adjacent if and only if they intersect (as subgraphs of G_1).

Let $d[\psi]: D(\psi, G_1) \to D(G_0)$ be the map defined by the formula $d[\psi](v) = \psi(v)$ for every vertex v of $D(\psi, G_1)$.

Every vertex $v \in \mathscr{V}(D(\psi, G_1))$ is also a subgraph of G_1. To avoid confusion we will denote this subgraph by v^*.

Let σ be simplicial maps of a graph G_2 into G_1. Let $d[\psi, \sigma]: D(\psi \circ \sigma, G_2) \to D(\psi, G_1)$ be the map such that for every vertex v of $D(\psi \circ \sigma, G_2)$, $d[\psi, \sigma](v)$ is the vertex of $D(\psi, G_1)$ containing $\sigma(v^*)$.

4.2 **Proposition.** *Suppose that Y is a triod which is the union of three arcs A_1, A_2 and A_3 meeting at a common endpoint y. Let $y, y_1^i, y_2^i, \dots, y_{k(i)}^i$ denote the sequence of consecutive vertices of A_i. Suppose ψ is a simplicial map of Y into a graph G. Let p be the least integer such that $\psi(y_p^1) \neq \psi(y)$, and let q*

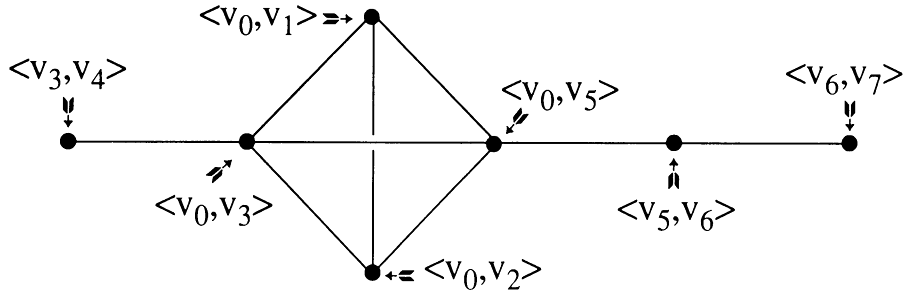

FIGURE 6

be the least integer such that $\psi(y_q^2) \neq \psi(y)$. *If* $\psi(y_p^1) = \psi(y_q^2)$, *then* $D(\psi, Y)$ *is a triod* (*possibly degenerate*).

Proof. Clearly, $\psi(\langle y, y_p^1\rangle) = \psi(\langle y, y_q^2\rangle)$ is an edge of G. Denote this edge by e. Let t be the vertex of $D(\psi, Y)$ representing the component of $\psi^{-1}(e)$ containing $\langle y, y_p^1\rangle \cup \langle y, y_q^2\rangle$. Observe that for any vertex $z \neq t$ of $D(\psi, Y)$, z^* is contained in one of the arcs A_1, A_2 and A_3. Let $\mathcal{Z}_1 = \{z \in D(\psi, Y) | z \neq t$ and $z^* \subset A_i\}$. Let z be an arbitrary point of $\mathcal{Z}_i$ and let j be an index such that $\langle y_j^i, y_{j+1}^i\rangle \subset z^*$ and $\psi(\langle y_j^i, y_{j+1}^i\rangle)$ is a nondegenerate edge of G. Observe that, if s is an element of $D(\psi, Y)$ different than z, then either $s^* \subset \langle y_{j+1}^i, y_{k(i)}^i\rangle$ or $s^* \subset Y \backslash \langle y_{j+1}^i, y_{k(i)}^i\rangle$. It follows that $\mathcal{Z}_i$ can be arranged into a sequence $z_1^i, z_2^i, \dots, z_{m(i)}^i$ such that $(z_1^i)^* \cap t^* \neq \varnothing$ and $(z_j^i)^* \cap (z_n^i) \neq \varnothing$ if and only if $|j-n| \leq 1$. Observe that $\langle t, z_{m(1)}^1\rangle$, $\langle t, z_{m(2)}^2\rangle$ and $\langle t, z_{m(3)}^3\rangle$ are three arcs intersecting at t. Clearly, $D(\psi, Y)$ is the union of these arcs. □

The following proposition follows immediately from 4.2.

4.3 **Proposition.** *Suppose that* Y *is a triod with its point of order* 3 *denoted by* y. *Suppose* β *is a simplicial map of* Y *into* X. *If* $\beta(y) \neq v_0$, *then* $D(\beta, Y)$ *is a triod* (*possibly degenerate*). □

Figure 6 shows $D(X)$ with its vertices labeled by the corresponding to them edges of X. Figure 7 indicates $d[\varphi]\colon D(\varphi, X') \to D(X)$. The dotted line graph represents the domain, $D(\varphi, X')$, while the solid black is the range, $D(X)$, and each vertex of the domain is mapped onto the nearest vertex of the range. The vertices of $D(\varphi, X')$ are labeled $t_0, t_1, \dots, t_{12}$ as shown in Figure 7. Table 2 shows the subgraphs of X' corresponding to the vertices of $D(\varphi, X')$.

Let S be a function assigning to every vertex of X a set of edges of X defined in the following way: $S(v_0) = \{\langle v_0, v_2\rangle, \langle v_0, v_3\rangle, \langle v_0, v_5\rangle\}$, $S(v_1) = \{\langle v_0, v_1\rangle\}$, $S(v_2) = \{\langle v_0, v_2\rangle\}$, $S(v_3) = \{\langle v_0, v_3\rangle\}$, $S(v_4) = \{\langle v_3, v_4\rangle\}$, $S(v_5) = \{\langle v_0, v_5\rangle\}$, $S(v_6) = \{\langle v_5, v_6\rangle\}$ and $S(v_7) = \{\langle v_6, v_7\rangle\}$. Note that v_i belongs to each edge from $S(v_i)$ for $i = 0, \dots, 6$. So, S is an edge selection on X according to [8, Definition 5.5]. Observe that

(i) $\varphi((v_i e, X')) \in S(\varphi(v_i))$ for each $v_i \in \mathscr{V}(X)$ and each $e \in S(v_i)$, where (v_i, e, X') denote the edge of (e, X') containing v_i.

Observe also that

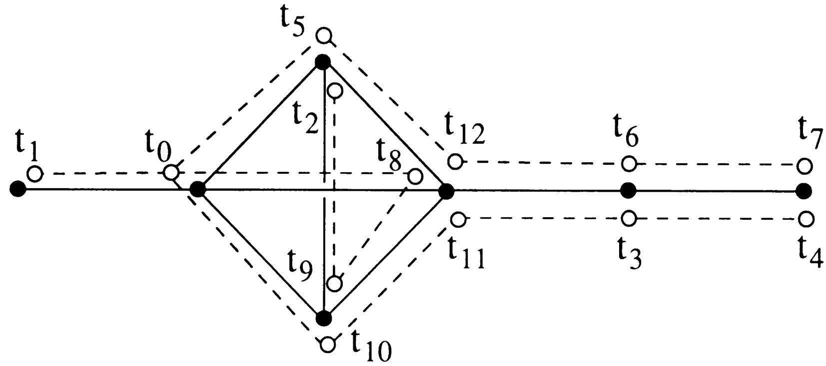

FIGURE 7

(ii) if e and e' are two different edges of X' intersecting at a common vertex q then at least one of the edges $\varphi(e)$ and $\varphi(e)$ belongs to $S(\varphi(q))$.

The above two conditions mean exactly that

4.4 **Proposition.** *φ preserves (S, S) (in the sense of* [8, Definition 5.7]). □

Observe that

(iii) $(v_i, e, X') \subset t_i^*$ for each $i = 0, 1, \dots, 7$ and each $e \in S(v_i)$ and

(iv) $t_8^* = \langle u_0, u_1\rangle \cup \langle u_1, u_2\rangle \subset \langle v_0, v_2\rangle$, $t_9^* = \langle u_2, u_3\rangle \cup \langle u_3, u_4\rangle \subset \langle v_0, v_2\rangle$, $t_{10}^* = \langle u_5, u_6\rangle \cup \langle u_6, u_7\rangle \subset \langle v_0, v_3\rangle$, $t_{11}^* = \langle u_7, u_8\rangle \subset \langle v_0, v_3\rangle$ and $t_{12}^* = \langle u_{10}, u_{11}\rangle \subset \langle v_5, v_6\rangle$.

Let X'' be a subdivision of X with five new vertices: v_8 added between v_0 and v_2, v_9 added between v_8 and v_2, v_{10} added between v_0 and v_3, and v_{12} added between v_{10} and v_3, and v_{12} added between v_5 and v_6. Let $\lambda : X'' \to D(\varphi, X')$ be defined by the formula $\lambda(v_i) = t_i$ for $i = 0, 1, \dots, 12$. Observe that λ is an isomorphism. Conditions (iii) and (iv) mean exactly that

TABLE 2

t_i	$d[\varphi](t_i)$	t_i^*
t_0	$\langle v_0, v_3\rangle$	$\langle v_0, u_0\rangle \cup \langle v_0, u_5\rangle \cup \langle v_0, u_9\rangle$
t_1	$\langle v_3, v_4\rangle$	$\langle v_0, v_1\rangle$
t_2	$\langle v_0, v_1\rangle$	$\langle u_4, v_2\rangle$
t_3	$\langle v_5, v_6\rangle$	$\langle u_8, v_3\rangle$
t_4	$\langle v_6, v_7\rangle$	$\langle v_3, v_4\rangle$
t_5	$\langle v_0, v_1\rangle$	$\langle u_9, v_5\rangle \cup \langle v_5, u_{10}\rangle$
t_6	$\langle v_5, v_6\rangle$	$\langle u_{11}, v_6\rangle$
t_7	$\langle v_6, v_7\rangle$	$\langle v_6, v_7\rangle$
t_8	$\langle v_0, v_5\rangle$	$\langle u_0, u_1\rangle \cup \langle u_1, u_2\rangle$
t_9	$\langle v_0, v_2\rangle$	$\langle u_2, u_3\rangle \cup \langle u_3, u_4\rangle$
t_{10}	$\langle v_0, v_2\rangle$	$\langle u_5, u_6\rangle \cup \langle u_6, u_7\rangle$
t_{11}	$\langle v_0, v_5\rangle$	$\langle u_7, u_8\rangle$
t_{12}	$\langle v_0, v_5\rangle$	$\langle u_{10}, u_{11}\rangle$

4.5 **Proposition.** *φ is consistent on S and λ is a consistency isomorphism (see [8, Definition 5.7]).* □

The following proposition can be readily verified by Figure 7.

4.6 **Proposition.** *$D[\varphi]$ is light and for each $e \in \mathscr{E}(D(X))$, each component of $(D[\varphi])^{-1}(e)$ is either a vertex or an edge of $D(\varphi, X')$.* □

5. K IS NOT SIMPLE-4-OD-LIKE

5.1 **Proposition.** *Suppose Φ_0^n can be factored through a triod. Then the map $d[\varphi, \Phi_1^n] : D(\Phi_0^n, \Phi_n) \to D(\varphi, X')$ can also be factored through a triod.*

Proof. Let Y be a triod with its point of order 3 denoted by y. Let $\alpha\colon \Phi_n \to Y$ and $\beta\colon Y \to X$ be simplicial maps such that $\alpha \circ \beta = \Phi_0^n$. By Proposition 3.5, we can assume that $\beta(y) \neq v_0$. It follows from Proposition 4.3, $D(\beta, Y)$ is a triod.

Observe that $d[\beta, \alpha] : D(\Phi_0^n, \Phi_n) \to D(\beta, Y)$, $D[\beta] : D(\beta, Y) \to D(X)$, $d[\varphi, \Phi_1^n] : D(\Phi_0^n, \Phi_n) \to D(\varphi, X')$ and $d[\varphi] : D(\varphi, X') \to D(X)$ are light simplicial maps. By Proposition 4.6, it follows from [8, Theorem 4.3] that there is $\beta'\colon D(\beta, Y) \to D(\varphi, X')$ such that $\beta' \circ d[\beta, \alpha] = d[\varphi, \Phi_1^n]$. □

5.2 **Proposition.** *The map Φ_0^n cannot be factored through a triod.*

Proof. Clearly, the proposition is true if $n = 0$. Now, suppose that the proposition is true for $n-1$. Let Γ denote the sequence $D[\varphi_1]\circ\lambda$, $\varphi_2, \varphi_3, \dots, \varphi_n$, where $\varphi_i = \varphi$, and let $\{\Gamma_j, \Gamma_j^i\}_{j=0}^n$ we denote the system generated by Γ. It follows from Propositions 4.4, 4.5 and [8, Theorem 5.11] that the system $\{D(\Phi_0^j, \Phi_j), d[\Phi_0^j, \Phi_j^i]\}_{j=0}^n$ is isomorphic to $\{\Gamma_j, \Gamma_j^i\}_{j=0}^n$.

Suppose Φ_0^n can be factored through a triod. Then, by Proposition 5.1, $d[\Phi_0^1, \Phi_1^n]$ and consequently Γ_1^n can be factored through a triod. Since the system $\{\Gamma_j, \Gamma_j^i\}_{j=1}^n$ is generated by subdivisions of $\varphi_2, \dots, \varphi_n$, according to the inductive assumption and Proposition 2.3, Γ_1^n cannot be factored through a triod. This contradiction proves the proposition. □

5.3 **Theorem.** *K is a simple-4-od-like but not simple-triod-like continuum and each proper nondegenerate subcontinuum of K is an arc.*

Proof. K is simple-4-od-like as the inverse limit of the system $\{\Phi_j, \Phi_j^i\}$ of subdivisions of X. By Proposition 3.3, each proper nondegenerate subcontinuum of K is an arc. Suppose that K is triod-like. Then by Proposition 2.1, Φ_0^n can be factored through a triod for some n, contrary to Proposition 5.2. □

REFERENCES

1. H. Cook. W. T. Ingram and A. Lelek, *Eleven annotated problems about continua*, Open Problems in Topology, North-Holland, Amsterdam, 1990, pp. 295–302.
2. J. F. Davies and W. T. Ingram, *An atriodic tree-like continuum with positive span which admits a monotone mapping to a chainable continuum*, Fund. Math. **131** (1988), 13–24.
3. W. T. Ingram, *Decomposable circle-like continua*, Fund. Math. **58** (1968), 193–198.
4. ——, *An atriodic tree-like continuum with positive span*, Fund. Math. **77** (1972), 99–107.
5. ——, *An uncountable collection of mutually exclusive planar atriodic tree-like continuua with positive span*, Fund. Math. **85** (1974), 73–78.

6. A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. **55** (1964), 199–214.
7. W. Lewis, *Continuum theory problems*, Topology Proc. **8** (1983), 361–394.
8. P. Minc, *On simplicial maps and chainable continua* (preprint).

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