TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 339, Number 1, September 1993

THE CONSTRAINED LEAST GRADIENT PROBLEM IN R^n

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ABSTRACT. We consider the constrained least gradient problem

$$\inf\left\{\int_\Omega |\nabla u|\,dx : u \in C^{0,1}(\overline{\Omega}),\ |\nabla u| \le 1 \text{ a.e.}, u = g \text{ on } \partial\Omega\right\}$$

which arises as the relaxation of a nonconvex problem in optimal design. We establish the existence of a solution by an explicit construction in which each level set is required to solve an obstacle problem. We also establish the uniqueness of solutions and discuss their structure.

1. INTRODUCTION

Consider the following minimization problem

$$\inf_u \int_\Omega W(|\nabla u|)\,dx \tag{1.1}$$

where $\Omega \subset R^n$, $u\colon \Omega \to R^1$ and W is a nonnegative, nonconvex function. Models of this type arise, for example, in elasticity [BP] or in optimal design [KS1, KS2]. The nonconvexity of W is frequently a barrier to establishing the existence of a solution, necessitating the "relaxation" of the problem, a process which in this context, amounts to replacing W by its convexification W^{**}. In any region where the solution, u_0, to the relaxed problem

$$\inf_u \int_\Omega W^{**}(|\nabla u|)\,dx$$

satisfies $W^{**}(|\nabla u_0|) < W(|\nabla u_0|)$, W^{**} must be linear, so that in this region u_0 will satisfy a least-gradient problem; that is, $|\nabla u_0|$ will minimize $\int |\nabla u|$ in this region, subject to boundary conditions and additional constraints on $|\nabla u|_{L^\infty}$.

In [KS1] Kohn and Strang explore this phenomenon for a problem encountered in optimal design. The relaxed problem they obtain is the constrained least gradient problem

$$\inf\left\{\int_\Omega |\nabla u|\,dx : u \in C^{0,1}(\overline{\Omega}),\ |\nabla u| \le 1 \text{ a.e.}, u = g \text{ on } \partial\Omega\right\} \tag{1.2}$$

Received by the editors July 8, 1991.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 49F22; Secondary 49A29, 53A10.

The first author's research was supported in part by a grant from the National Science Foundation and an Indiana University Summer Faculty Fellowship.

The second author's research was conducted while visiting Indiana University.

The third author's research was supported in part by a National Science Foundation grant.

where g is a continuous function on $\partial\Omega$ satisfying the Lipschitz condition $|g(p) - g(q)| \leq d_\Omega(p, q)$ for all $p, q \in \partial\Omega$. Here the metric, d_Ω, on $\overline{\Omega}$ is defined by

$$d_\Omega(x, y) = \inf\{\text{length of } \gamma\}$$

where the infimum is taken over all rectifiable curves γ lying in $\overline{\Omega}$ joining x to y. One can easily establish existence of a solution to (1.2) using the direct method in the calculus of variations. From the standpoint of understanding (1.1), however, one is most interested in the more difficult question of characterizing the solution u_0 to (1.2). This is because, as is generally the case in a relaxation process, u_0 will be a weak limit of a minimizing sequence for the unrelaxed problem (1.1). In the absence of a minimizer to (1.1), then, knowledge of the structure of u_0 leads to knowledge of the structure of a minimizing sequence for (1.1) which in turn leads to "nearly optimal designs," [KS3].

In [KS1], the authors introduce a very interesting technique for actually constructing a solution to (1.2), by constructing each of its level sets. Their method is based on the observation that the level sets of the solution u to (1.2) without the constraint $|\nabla u| \leq 1$ are minimal surfaces. It is the co-area formula that provides the connection between functions of least gradient and minimal surfaces. The co-area formula (cf. [F1], [FR]) states that whenever $u\colon \Omega \to R^1$ is Lipschitz, then

$$\begin{aligned} \int_\Omega |\nabla u|\, dx &= \int_{-\infty}^{\infty} H^{n-1}(\Omega \cap u^{-1}(t))\, dt \\ &= \int_{-\infty}^{\infty} P(A_t, \Omega)\, dt \end{aligned} \tag{1.3}$$

where H^{n-1} denotes $(n-1)$-dimensional Hausdorff measure, $A_t = \Omega \cap \{x\colon u(x) \geq t\}$, and $P(A_t, \Omega)$ is the perimeter of A_t in Ω. In the work of [BDG] the area-minimizing property of level sets of functions of least gradient was exploited to further understand the structure of area-minimizing surfaces. In the present work and in our study of the unconstrained least gradient problem [SWZ1], the opposite point of view is adopted. That is, we use virtually the full strength of codimension-1 minimal surface theory to gain a better understanding of the structure of functions of least gradient.

With the constraint $|\nabla u| \leq 1$ present, it is, of course, no longer true in general that the set $u^{-1}(t)$ is locally area-minimizing. Indeed, consider an arbitrary point $p \in \partial\Omega$, and a point $x \in \Omega$ such that $u(x) = t$. The condition on each competitor u in (1.2) that all difference quotients be bounded above by 1 leads to the requirement $|t - g(p)| \leq d_\Omega(x, p)$. This implies that x must avoid the union of all balls relative to the metric d_Ω, $\overline{\Omega} \cap \{x\colon d_\Omega(x, p) < |g(p) - t|,\ p \in \partial\Omega\}$ and led Kohn and Strang to conjecture that the solution u_0 could be constructed by requiring the set $\{u_0 \geq t\}$ to minimize perimeter among all competitors E for which

$$E \supset \left\{ \bigcup \{x\colon d_\Omega(x, p) \leq g(p) - t\},\ p \in \partial\Omega,\ g(p) \geq t \right\},$$

while E omits the interior of the set

$$\left\{ \bigcup \{x\colon d_\Omega(x, p) \leq t - g(p)\},\ p \in \partial\Omega,\ g(p) \leq t \right\}.$$

Thus, for each t, the set $\{u_0 \geq t\} \equiv E_t$ must solve the double obstacle problem. It was formally shown in [KS1] that this method succeeds in R^2 and the first objective of this paper is to provide a rigorous proof in R^n, $n \geq 2$.

In order to show that the function u constructed in this manner is a solution to (1.2), it is necessary to first show that u is Lipschitz with constant 1. One of the major results of this paper is the fact that the sets ∂E_s and ∂E_t are suitably separated. That is, the distance $d_\Omega(\partial E_s, \partial E_t)$ is no less than $|s-t|$ for all s, t. The obstacle construction would appear to only provide this separation at the boundary. Roughly speaking, however, the level sets will be semisolutions of the minimal surface equation, and interior separation can intuitively be seen as a consequence of a minimum principle for elliptic partial differential equations.

In §2, we introduce notation and basic information concerning BV functions and sets of finite perimeter. Section 3 introduces the obstacles that are employed in the construction of what will be the level sets of our solution. It also incorporates the results of Tamanini [T1] which allows us to establish preliminary $C^{1,1/2}$-regularity results for the level sets. Section 4 contains our proof that the construction yields a solution to our problem in R^n.

The proof of Theorem 4.5, which states that ∂E_s and ∂E_t are suitably separated, is rather long and involved. If one is willing to assume that Dirichlet data g satisfies the stronger condition $|g(p) - g(q)| \leq |p - q|$ where $|p - q|$ denotes the Euclidean distance between p and q, then the proof of this theorem simplifies considerably. We present the simpler proof under this stronger hypothesis in §5.

In §6 we construct a solution to (1.2). Then in §7, we establish the uniqueness of solutions to (1.2). Hence, the above mentioned construction yields the only solution to the problem. Finally, in §8, we present a reformulation of the least gradient problem in which the constraint $|\nabla u| \leq 1$ is replaced by the condition $f(x) \leq u(x) \leq F(x)$ for suitably defined Lipschitz functions f and F. This allows us to obtain detailed information on the structure of the solution u_0 at points x which avoid both obstacles.

The authors would like to thank Robert Kohn for suggesting this problem and providing useful background material. They are also indebted to Bruce Solomon who gave generous amounts of time during many interesting and helpful conversations related to this work.

2. Notation and preliminaries

The Lebesgue measure of a set $E \subset R^n$ will be denoted by $|E|$ and $H^\alpha(E)$, $\alpha > 0$, will denote α-dimensional Hausdorff measure of E. Throughout, we almost exclusively employ H^{n-1}. The Euclidean distance between points $x, y \in R^n$ will be denoted by $|x - y|$. If $\Omega \subset R^n$ is an open set, the class of functions $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is denoted by $BV(\Omega)$ and is called the space of *functions of bounded variation in* Ω. The space $BV(\Omega)$ is endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{1;\Omega} + \|\nabla u\|(\Omega) \tag{2.1}$$

where $\|\nabla u\|$ is the total variation of the vector-valued measure ∇u defined for each nonnegative, continuous function f on Ω with compact support by

$$\|\nabla u\|(f) = \sup\left\{ \int_\Omega u \operatorname{div} v \, dx \colon \ v = (v_1, \dots, v_n) \in C_0^\infty(\Omega; R^n), \right. \tag{2.2}$$
$$\left. |v(x)| \le f(x) \text{ for } x \in \Omega \right\}.$$

The following compactness result for $BV(\Omega)$ will be needed later, cf. [GI] or [Z].

Theorem 2.1. *If $\Omega \subset R^n$ is a bounded Lipschitz domain, then*

$$BV(\Omega) \cap \{u \colon \|u\|_{BV(\Omega)} \le 1\}$$

is compact in $L^1(\Omega)$. Moreover, if $u_i \to u$ in $L^1(\Omega)$, and $U \subset \Omega$ is open, then

$$\liminf_{i\to\infty} \|\nabla u_i\|(U) \ge \|\nabla u\|(U).$$

A Borel set $E \subset R^n$ is said to have *finite perimeter in* Ω provided the characteristic function of E, χ_E, is a function of bounded variation in Ω. Thus, the partial derivatives of χ_E are Radon measures on Ω and the perimeter of E in Ω is defined as

$$P(E, \Omega) = \|\nabla \chi_E\|(\Omega). \tag{2.3}$$

A set E is said to be of *locally finite perimeter* if $P(E, \Omega) < \infty$ for every bounded open set $\Omega \subset R^n$.

One of the fundamental results of the theory of sets of finite perimeter is that they possess a measure-theoretic exterior normal which is suitably general to ensure the validity of the Gauss-Green theorem. A unit vector ν is defined as the exterior normal to E at x provided

$$\lim_{r\to 0} r^{-n} |B(x, r) \cap \{y \colon (y - x)\cdot \nu < 0, \ y \notin E\}| = 0$$

and

$$\lim_{r\to 0} r^{-n} |B(x, r) \cap \{y \colon (y - x)\cdot \nu > 0, \ y \in E\}| = 0, \tag{2.4}$$

where $B(x, r)$ denotes the open ball of radius r centered at x. The measure-theoretic normal of E at x will be denoted by $\nu(x, E)$ and we define

$$\partial^* E = \{x \colon \nu(x, E) \text{ exists}\}. \tag{2.5}$$

Clearly, $\partial^* E \subset \partial E$, where ∂E denotes the topological boundary of E. Also, the *topological interior of* E is denoted by $E^i = (R^n - \partial E) \cap E$ and the *topological exterior* by $E^e = (R^n - \partial E) \cap (R^n - E)$.

If $E \subset R^n$ is a Borel set, we define the *measure-theoretic boundary of* E as

$$\partial_M E = \left\{x \colon 0 < \limsup_{r\to 0} \frac{|E \cap B(x, r)|}{|B(x, r)|}\right\} \cap \left\{x \colon \liminf_{r\to 0} \frac{|E \cap B(x, r)|}{|B(x, r)|} < 1\right\}. \tag{2.6}$$

In other words, the measure-theoretic boundary of E is all points at which the metric density of E is neither 1 nor 0. Clearly, $\partial^* E \subset \partial_M E \subset \partial E$. Moreover, it is well known that

(2.7) $\quad E$ is of finite perimeter if and only if $H^{n-1}(\partial_M E) < \infty$

and that

$$P(E, \Omega) = H^{n-1}(\Omega \cap \partial_M E) = H^{n-1}(\Omega \cap \partial^* E), \tag{2.8}$$

cf. [F2, §4.5]. From this it easily follows that

$$P(E \cup F, \Omega) + P(E \cap F, \Omega) \leq P(E, \Omega) + P(F, \Omega), \tag{2.9}$$

thus implying that sets of finite perimeter are closed under finite unions and intersections. The Gauss-Green theorem in this context states that if E is a set of locally finite perimeter and $V\colon R^n \to R^n$ is a Lipschitz vector field, then

$$\int_E \operatorname{div} V(x)\, dx = \int_{\partial^* E} V(x) \cdot \nu(x, E)\, dH^{n-1}(x), \tag{2.10}$$

cf. [F2, §4.5.6]. This result allows us to identify sets of finite perimeter as n-dimensional *integral currents.* We shall use only a few basic facts concerning currents and refer the reader to [S1] or [F2] for further details.

By definition, sets of finite perimeter are determined only up to sets of measure zero. In other words, each set determines an equivalence class of sets of finite perimeter. In order to avoid this ambiguity, whenever a set of finite perimeter, E, is considered, we shall employ always the measure-theoretic closure as the set to represent E. Thus, with this convention, we have

$$x \in E \text{ if and only if } \limsup_{r \to 0} \frac{|E \cap B(x, r)|}{|B(x, r)|} > 0. \tag{2.11}$$

With this convention in force, it can be shown that

$$\overline{\partial^* E} = \partial E, \tag{2.12}$$

where $\overline{A}$ denotes the closure of a set A, cf. [GI, Theorem 4.4].

Of particular importance to us are sets of finite perimeter whose boundaries are area-minimizing. If E is a set of locally finite perimeter and U a bounded, open set, let

$$\psi(E, U) = \|\nabla \chi_E\|(U) - \inf\{\|\nabla \chi_F\|(U) \colon E \Delta F \Subset U\} \tag{2.13}$$

where $E \Delta F$ denotes the symmetric difference of E and F. ∂E is said to be *area-minimizing in* U if $\psi(E, U) = 0$ and *locally area-minimizing* if $\psi(E, U) = 0$ whenever U is bounded.

The regularity of ∂E will play a crucial role in our development. In particular, we will employ the notion of tangent cone. Suppose ∂E is area-minimizing in U and for convenience of notation, suppose $0 \in U \cap \partial E$. For each $r > 0$, let $E_r = R^n \cap \{x \colon rx \in E\}$. It is known [S1, §35, MM, §2.6] that for each sequence $\{r_i\} \to 0$ there exists a subsequence (denoted by the full sequence) such that $\chi_{E_{r_i}}$ converges in $L^1_{\mathrm{loc}}(R^n)$ to χ_C, where C is a set of locally finite perimeter. In fact, ∂C is area-minimizing and is called the tangent cone to E at 0. Although it is not immediate, C is a cone and therefore a union of half-lines issuing from 0. It follows from [S1, §37.6] that if $\overline{C}$ is contained in $\overline{H}$ where H is any half-space in R^n with $0 \in \partial H$, then ∂E is regular at 0. That is, there exists $r > 0$ such that

$$B(0, r) \cap \partial E \text{ is a real analytic hypersurface.} \tag{2.14}$$

Furthermore, if we denote by $\operatorname{reg}\partial E$ the set of points of ∂E where ∂E is regular, then

$$H^{\alpha}((\partial E - \operatorname{reg}\partial E) \cap U) = 0, \quad \text{for all } \alpha > n = 8. \tag{2.15}$$

We will need the following result which is a direct consequence of a maximum principle for area-minimizing hypersurfaces recently established in [S2].

Theorem 2.2. *Let $E \subset F$ and suppose ∂F and ∂E are area-minimizing in an open set $U \subset R^n$. Further, suppose $x \in (\partial E) \cap (\partial F) \cap U$. Then the components of $U \cap \partial F$ and $U \cap \partial E$ that contain x are equal.*

Proof. For $B(x, r) \subset U$, consider the set of all components S of $B(x, r) \cap \operatorname{reg}\partial E$ and recall from the proof of [S2, Corollary 1] that only a finite number of such S can intersect $B(x, r/2)$. Moreover, since $\operatorname{reg}\partial E$ is dense in ∂E [S1, §36.2], it follows that there exists a component S_1 whose closure contains x. With a similar description pertaining to the components T of $B(x, r) \cap \operatorname{reg}\partial F$, let T_1 correspondingly denote that component whose closure contains x. Then, it follows from [S2, Corollary 1] that $\overline{S_1} = \overline{T_1}$. The result now readily follows. □

We will need a further result related to the regularity of ∂E.

Lemma 2.3. *Suppose ∂E is area-minimizing in an open set U. For $x_0 \in U - \overline{E}$, let $y_0 \in \overline{E}$ have the property that*

$$|x_0 - y_0| = \inf\{|x_0 - y| : y \in \overline{E}\}. \tag{2.16}$$

Then $y_0 \in \operatorname{reg}\partial E$.

Proof. Let H be the half-space defined by

$$H = \{x : (x_0 - y_0) \cdot (x - y_0) < 0\}.$$

If $y_0 \notin \operatorname{reg}\partial E$, then from the fact preceding (2.14), we would conclude that the closure of the tangent cone to E at y_0 is not contained in $\overline{H}$. For convenience of notation, let $y_0 = 0$. This would imply the existence of $x \notin \overline{H}$ and a sequence $\{x_i\} \to x$, $x_i \notin \overline{H}$, where x_i is of the form $x_i = y_{r_i}/r_i$ with $r_i \to 0^+$ and $y_{r_i} \in \overline{E}$. Because of (2.16) and since $y_{r_i} \in \overline{E}$, it follows that

$$\limsup_{i\to\infty} x_0 \cdot \frac{y_{r_i}}{|y_{r_i}|} \leq 0. \tag{2.17}$$

However, $x \cdot x_0/|x_0| > 0$ and therefore $\lim_{i\to\infty} x_0/|x_0| \cdot x_i/|x_i| > 0$ which implies that

$$\liminf_{i\to\infty} x_0 \cdot \frac{y_{r_i}}{|y_{r_i}|} > 0,$$

a contradiction to (2.17). □

3. Obstacles and area-minimizing hypersurfaces

We now begin the construction of the solution to the problem stated in (1.2). Throughout the remainder of this paper, Ω will be taken as a bounded, Lipschitz domain. Also, we may take Ω to be connected, for otherwise we can consider a distinct least gradient problem on each component. Recall the definition of the metric, d_Ω, given by

$$d_\Omega(x, y) = \inf\{\text{length of } \gamma\}$$

where the infimum is taken over all rectifiable curves γ lying in $\overline{\Omega}$ joining x to y. Since any such curve has a Lipschitz parameterization, an application of the Arzela-Ascoli Compactness Theorem yields the existence of a curve (geodesic) that attains the infimum above. Observe also that if $x \in \Omega$ and $\{x_i\}$ is a sequence in Ω converging to x, then

$$\lim_{i\to\infty} \frac{d_\Omega(x_i, x)}{|x_i - x|} = 1. \tag{3.1}$$

We assume that the Dirichlet data $g\colon \partial\Omega \to R^1$ is a Lipschitz function with constant 1 relative to the metric d_Ω. That is,

$$|g(p) - g(q)| \le d_\Omega(p, q) \quad \text{for } p, q \in \partial\Omega. \tag{3.2}$$

We will let

$$[a, b] = \left\{\bigcap I\colon I \text{ an interval containing } g(\partial\Omega)\right\}. \tag{3.3}$$

For each real number $t \in [a, b]$ we define obstacles L_t and M_t as follows:

$$L_t = \left\{\bigcup\{x\colon d_\Omega(x, p) \le g(p) - t\},\ p \in \partial\Omega,\ g(p) \ge t\right\}, \tag{3.4}$$

$$M_t = \left\{\bigcup\{x\colon d_\Omega(x, p) \le t - g(p)\},\ p \in \partial\Omega,\ g(p) \le t\right\}. \tag{3.5}$$

Clearly, L_t and M_t are closed sets for suppose $x_i \to x$ where $d_\Omega(x_i, p_i) \le g(p_i) - t$. Then, after passing to a suitable subsequence, $p_i \to p$ for some $p \in \partial\Omega$ and therefore, $d_\Omega(x, p) \le g(p) - t$ since g is continuous. Note that

$$L_t \cap (M_t)^i = \varnothing, \tag{3.6}$$

for if $x \in L_t \cap (M_t)^i$, there would exist $p, q \in \partial\Omega$ such that $d_\Omega(x, p) \le g(p) - t$ and $d_\Omega(x, q) \le t - g(q)$. Hence,

$$d_\Omega(p, q) \le d_\Omega(x, p) + d_\Omega(x, q) \le g(p) - t + t - g(q) \le d_\Omega(p, q).$$

Thus, equality holds and therefore x lies on a geodesic joining p and q. Points on this geodesic closer to p than x cannot be in M_t for otherwise the argument above could be repeated with x replaced by this closer point to yield a strict inequality, an impossibility. Thus, $x \notin (M_t)^i$. A similar argument yields

$$M_t \cap (L_t)^i = \varnothing. \tag{3.7}$$

We also note that

$$|\Omega \cap \partial L_t| = |\Omega \cap \partial M_t| = 0. \tag{3.8}$$

Letting $\overline{B}(p, r)$ denote the closed ball of radius $r \ge 0$, we employ the Vitali Covering Theorem to find a sequence of closed balls $\{\overline{B}_i\}$ such that each $\overline{B}_i$ is contained within some $\overline{B}(p, g(p) - t)$ and that $|L_t - \bigcup_{i=1}^{\infty}(\overline{B}_i \cap \overline{\Omega})| = 0$. Therefore, almost all of $|\Omega \cap \partial L_t|$ is contained within $\bigcup_{i=1}^{\infty} \partial\overline{B}_i$. Finally, note that if $x \in \partial L_t$, so that $d_\Omega(x, p) = g(p) - t$, then it is possible to find a point $p' \in \partial\Omega$ such that

$$|x - p'| = g(p') - t \tag{3.9}$$

for some $p' \in \partial\Omega$. To see this, note that if a geodesic joining x to p is not a line segment, then there is a point $p' \in \partial\Omega$ on this geodesic such that

$d_\Omega(x, p') = |x - p'|$. The conclusion follows from the triangle inequality. Similarly, if $x \in \partial M_t$, then there exists a point $p' \in \partial\Omega$ such that

$$|x - p'| = t - g(p') \tag{3.10}$$

for some $p' \in \partial\Omega$.

The analysis of the following minimization problem is one of the major concerns of this paper. For each t employed in (3.4) and (3.5), consider

$$\lambda_t = \inf\{P(E, \Omega) : \overline{\Omega} \supset E \supset L_t,\ \overline{E} \cap (M_t)^i = \varnothing\} \tag{3.11}$$

and the related problem

$$\mu_t = \sup\{|E| : E \text{ a solution of } (3.11)\}. \tag{3.12}$$

We begin the analysis by observing that there is a set E that attains the infimum in (3.11), for if $\{E_i\}$ is a minimizing sequence, then Theorem 2.1 provides a subsequence such that $\chi_{E_i} \to \chi_E$ a.e. with $\liminf_{i\to\infty} \|\nabla\chi_{E_i}\|(\Omega) \geq \|\nabla\chi_E\|(\Omega)$. If we define $E_0 = E^* \cup (L_t - E)$ (see (2.11)) then clearly $E_0 \supset L_t$ and $P(E, \Omega) = P(E_0, \Omega)$ since $|L_t - E| = 0$. Reference to (3.6) shows that $\overline{E_0} \cap (M_t)^i = \varnothing$ and therefore E_0 is a minimizer for (3.11).

We now proceed to investigate the regularity of such minimizers. For this, we begin with the following.

Lemma 3.1. *If E is a minimizer of* (3.11) *and F a competing set in* (3.11), *then $P(E, U) \leq P(F, U)$ whenever U is an open subset of Ω and $E\Delta F \Subset U$.*

Proof. We may as well assume that $P(F, U) < \infty$, in which case $P(F, \Omega) < \infty$ by (2.7). The hypotheses imply that

$$\|\nabla\chi_E\|(\Omega) \leq \|\nabla\chi_F\|(\Omega) \tag{3.13}$$

with

$$\|\nabla\chi_E\|(\Omega) = \|\nabla\chi_E\|(U) + \|\nabla\chi_E\|(\Omega - U) \tag{3.14}$$

and

$$\|\nabla\chi_F\|(\Omega) = \|\nabla\chi_F\|(U) + \|\nabla\chi_F\|(\Omega - U). \tag{3.15}$$

Now $E = F$ in $\Omega - E\Delta F$ and therefore, by (2.2)

$$\|\nabla\chi_E\|(V) = \|\nabla\chi_F\|(V)$$

for every open set $V \subset \Omega - (E\Delta F)$. Hence, by the outer regularity of the measures $\|\nabla\chi_E\|$ and $\|\nabla\chi_F\|$, we have

$$\|\nabla\chi_E\|(\Omega - U) = \|\nabla\chi_F\|(\Omega - U).$$

Reference to (3.13), (3.14), and (3.15) yields

$$\|\nabla\chi_E\|(U) \leq \|\nabla\chi_F\|(U). \quad \square$$

Corollary 3.2. *If E_t is a minimizer of* (3.11) *and $U_t \subset \Omega - (L_t \cap M_t)$ an open set, then $\psi(E, U_t) = 0$. That is, ∂E_t is area-minimizing in U_t.*

Proof. Let F be a set with $E\Delta F \Subset U_t$. It is an easy matter to verify that F is admissible in (3.11) and thus, the result follows from the previous lemma. $\square$

This result implies that ∂E_t is real analytic everywhere in $\Omega - (L_t \cup M_t)$ except for a small singular set, see (2.14) and (2.15). In order to obtain some regularity of ∂E_t near the obstacle L_t, we will invoke the following work of Tamanini [T1]. For each set E of finite perimeter and each open set U we recall (2.13) and define

$$\begin{aligned}\psi(E, U) &= P(E, U) - \inf\{P(F, U) \colon F \Delta E \Subset E\},\\ \psi_0(E, U) &= P(E, U) - \inf\{P(F, U) \colon F \Delta E \Subset U, \ F \subset E\},\\ \psi_1(E, U) &= P(E, U) - \inf\{P(F, U) \colon F \Delta E \Subset U, \ E \subset F\}.\end{aligned}$$

A major result proved in [T1] is the following.

Theorem 3.3. *E is locally of class $C^{1,\alpha}$ if and only if it is of class C^1 and*

$$\psi(E, B(x, r)) \leq C r^{n-1+2\alpha}$$

for every $x \in \partial E$ and every small $r > 0$ where C is some local constant independent of r.

An obstacle L is said to satisfy an *interior ball condition of radius R* if for each $x \in L$, there is a ball $B \subset L$ of radius $r \geq R$ such that $x \in B$. We will prove that the obstacles defined in (3.4) and (3.5) satisfy an interior ball condition of radius $\varepsilon > 0$ at all points in Ω that are at least a distance of 3ε from $\partial\Omega$.

Lemma 3.4. *For each $x \in L_t \cap \{x \colon \operatorname{dist}(x, \partial\Omega) \geq 3\varepsilon\}$ there exists a ball $B(\varepsilon)$ such that $x \in \overline{B}(\varepsilon) \subset L_t$. The obstacle M_t satisfies a similar condition.*

Proof. Let $x \in L_t \cap \{x \colon \operatorname{dist}(x, \partial\Omega) \geq 3\varepsilon\}$. Then there exists $p \in \partial\Omega$ such that $d_\Omega(x, p) \leq g(p) - t$. Let γ be a geodesic in $\overline{\Omega}$ joining x and p, and let $x' \in \gamma$ satisfy

$$d_\Omega(x, x') = |x - x'| = \varepsilon.$$

Note that $\overline{B}(x', \varepsilon) \subset \Omega$ since $\operatorname{dist}(x, \partial\Omega) \geq 3\varepsilon$. Now let $y \in \overline{B}(x', \varepsilon)$. Then, $|x' - y| = d_\Omega(x', y) \leq \varepsilon$ and

$$d_\Omega(p, x') = d_\Omega(p, x) - |x - x'| \leq g(p) - t - |x - x'| = g(p) - t - \varepsilon.$$

Hence,

$$d_\Omega(p, y) \leq d_\Omega(p, x') + |x' - y| \leq g(p) - t - \varepsilon + \varepsilon = g(p) - t.$$

Thus, $y \in L_t$. $\square$

For our purposes, the significance of the interior ball condition lies in the following facts established in [T1]. If an obstacle L satisfies an interior ball condition of radius R in $\Omega' \Subset \Omega$, then

$$\psi_0(L, B(x, r)) \leq n r^n / R \tag{3.16}$$

whenever $B(x, r) \subset \Omega'$. Now, by considering complements, we see that if M satisfies an interior ball condition of radius R in Ω', then

$$\psi_1(R^n - M, B(x, r)) \leq n r^n / R \tag{3.17}$$

whenever $B(x, r) \subset \Omega'$.

Theorem 3.5. *If E_t is a minimizer of* (3.11), *then ∂E_t is a $C^{1,1/2}$-hypersurface in a neighborhood of each point in $\Omega \cap \partial E_t \cap (\partial L_t \cup \partial M_t)$.*

Thus, one can represent ∂E_t as the graph of a function f in a neighborhood of some point $x \in \Omega \cap (\partial L_t \cup (\partial M_t)$, and for all small $r > 0$, the oscillation of $|\nabla f|$ in $B(x, r)$ is less than $Cr^{1/2}$ where $C = C(R)$. In [SWZ2] the regularity in the above theorem is improved by showing that ∂E_t is, in fact, of class $C^{1,1}$. Reference to Figure 4 below shows that this result is essentially optimal.

Proof. Let E_t be a solution of (3.11). Then from Lemma 3.1, E_t satisfies the double obstacle problem

$$L_t \subset E_t \subset \overline{\Omega} - (M_t)^i$$

and

$$P(E_t, U) \leq P(F, U)$$

for all U open, $U \Subset \Omega$ and all F such that $F \Delta E \Subset U$ and $L_t \subset F \subset \overline{\Omega} - (M_t)^i$. We claim

$$\psi(E_t, U) \leq \psi_0(L_t, U) + \psi_1(\overline{\Omega} - (M_t)^i, U) \tag{3.18}$$

whenever $U \Subset \Omega$. To this end, consider $F \Delta E_t \Subset U$. Define $G = (F \cup L_t) \cap (\overline{\Omega} - (M_t)^i)$. By appealing to (2.9) we have

$$\begin{aligned} P((F \cup L_t) \cap (\overline{\Omega} - (M_t)^i), U) + P((F \cup L_t) \cup (\overline{\Omega} - (M_t)^i), U) \\ \leq P(F \cup L_t, U) + P(\overline{\Omega} - (M_t)^i, U) \end{aligned}$$

and

$$P(F \cup L_t, U) + P(F \cap L_t, U) \leq P(F, U) + P(L_t, U).$$

Thus, since G is a competitor,

$$P(E_t, U) \leq P(G, U) = P((F \cup L_t) \cap (\overline{\Omega} - (M_t)^i), U).$$

Hence,

$$\begin{aligned} P(E_t, U) - P(F, U) &\leq P(L_t, U) - P(F \cap L_t, U) \\ &\quad + P(\overline{\Omega} - (M_t)^i, U) - P(F \cup (\overline{\Omega} - (M_t)^i), U) \\ &\leq \psi_0(L_t, U) + \psi_1(\overline{\Omega} - (M_t)^i, U) \end{aligned}$$

which establishes our claim (3.18). An easy modification of the argument of Tamanini [T1] shows that $\partial E_t \in C^1$ in a neighborhood of $\partial E_t \cap (\partial L_t \cup \partial M_t)$. Now reference to (3.16) and (3.17) allows us to apply Theorem 3.3 to establish the theorem. □

In summary we have the following result.

Theorem 3.6. *If E_t is a minimizer of* (3.11), *then*

(i) *∂E_t is real analytic in a neighborhood of each point in $\Omega \cap \partial E_t - (\operatorname{sing} \partial E_t \cup L_t \cup M_t)$ and $\operatorname{sing} \partial E_t \subset \Omega \cap \partial E_t - (L_t \cup M_t)$.*

(ii) *∂E_t is $C^{1,1/2}$-regular in a neighborhood of each point of $\Omega \cap \partial E_t \cap (\partial L_t \cup \partial M_t)$.*

Remark 3.7. We conclude this section by observing that there is a set E that attains the supremum in (3.12). Indeed, if $\{E_i\}$ is a sequence of admissible sets in (3.12) such that $\lim_{i\to\infty} |E_i| = \mu_t$, then Theorem 2.1 yields a set E of finite perimeter in Ω and a subsequence such that $\chi_{E_i} \to \chi_E$ almost everywhere. Letting $E_0 = E \cup (L_t \cap \overline{\Omega} - E)$, we have $|L_t \cap \overline{\Omega} - E| = 0$, $E_0 \supset L_t \cap \overline{\Omega}$ and $\overline{E_0} \cap (M_t)^i = \varnothing$. Therefore

$$\lambda_t = \liminf_{i \to \infty} \|\nabla \chi_{E_i}\|(\Omega) \geq \|\nabla \chi_{E_0}\|(\Omega) \geq \lambda_t .$$

Thus, there is a set E_0 that is an extremal element of both (3.11) and (3.12) and therefore enjoys the regularity properties of the preceding theorem. Henceforth, we will denote the closure of this set by E_t.

4. Analysis of the level sets

Now that the structure of the sets E_t is known from the previous section, we proceed to establish the separation of the sets ∂E_t for different values of t. This will be a crucial ingredient in the construction of our solution to (1.2). Recall our convention regarding the definition of E_t in Remark 3.7.

Lemma 4.1. *If $a \leq s < t \leq b$, where $[a, b]$ is defined in* (3.3), *then $E_s \supset E_t$.*

Proof. Clearly $E_s \cap E_t \supset L_t$ and $\overline{(E_s \cap E_t)} \cap (F_t)^i = \varnothing$. Hence

$$P(E_s \cap E_t, \Omega) \geq P(E_t, \Omega). \tag{4.1}$$

In view of (2.9), this implies that

$$P(E_s \cup E_t, \Omega) \leq P(E_s, \Omega). \tag{4.2}$$

Since $E_s \cup E_t$ is admissible for λ_s in (3.11), the inequality in (4.2) cannot be strict. On the other hand, if equality holds in (4.2), then $|E_s \cup E_t| \leq |E_s|$ because of the maximality of $|E_s|$, (see Remark 3.7). Hence, $|E_t - E_s| = 0$. In light of (2.11), we conclude that $E_t \subset E_s$. □

A function $u \in C^1(W)$ is called a *weak subsolution* (*supersolution*) *of the minimal surface equation in W* if

$$Mu(\varphi) = \int_W \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}}\, dx \leq 0 \qquad (\geq 0)$$

whenever $\varphi \in C_0^1(W)$, $\varphi \geq 0$.

The following result will be stated in the context of R^{n-1} because of its applications in the subsequent development.

Lemma 4.2. *Suppose W is an open subset of R^{n-1}. If $u_1, u_2 \in C^1(W)$ are respectively weak super and subsolutions of the minimal surface equation in W and if $u_1(x_0) = u_2(x_0)$ for some $x_0 \in W$ while $u_1(x) \geq u_2(x)$ for all $x \in W$, then $u_1(x) = u_2(x)$ for all x in some closed ball contained in W centered at x_0.*

Proof. Define

$$\begin{aligned} u_t &= t u_1 + (1-t) u_2 \quad \text{for } t \in [0, 1], \\ w &= u_1 - u_2, \\ a^{ij}(x) &= \int_0^1 D_{p_j}\left(\frac{D_{x_i} u_t(x)}{\sqrt{1 + |\nabla u_t(x)|^2}} \right) dt \\ &= \int_0^1 \frac{\sqrt{1 + |\nabla u_t(x)|^2}\delta_{ij} - D_{x_i} u_t(x) D_{x_j} u_t(x)}{(1 + |\nabla u_t(x)^2|)^{3/2}}\, dt. \end{aligned}$$

Since both u_1 and u_2 are continuously differentiable in W, for each open set $V \Subset W$ containing x_0 there exists $K > 0$ such that $|\nabla u_t(x)| \leq K$ for all $x \in V$ and all $t \in [0, 1]$. Hence,

$$a^{ij}(x)\xi_i\xi_j \geq \frac{1}{(1+K^2)^{3/2}}|\xi|^2, \quad \text{for all } \xi \in R^{n-1}, \ x \in V,$$
$$\sum_{i,j} a^{ij}(x)^2 \leq \Lambda, \quad \text{for all } x \in V$$

for some $\Lambda = \Lambda(K) < \infty$. For $\varphi \in C_0^1(W)$, $\varphi \geq 0$, we have

$$\begin{aligned} 0 &\leq Mu_1(\varphi) - Mu_2(\varphi) \\ &= \int_W \int_0^1 \frac{d}{dt}\left(\frac{\nabla u_t(x) \cdot \nabla \varphi(x)}{\sqrt{1+|\nabla u_t(x)|^2}}\right) dt\, dx \\ &= \int_W a^{ij}(x) D_j w(x) D_i \varphi(x)\, dx. \end{aligned}$$

Thus, w is a weak supersolution of the equation $D_i(a^{ij}D_j w) = 0$ and since $w \geq 0$, the weak Harnack inequality [GT, Theorem 8.18] yields

$$\left(r^{-n}\int_{B(x_0, 2r)} |w(x)|^p\, dx\right)^{1/p} \leq C \inf_{B(x_0, r)} w = 0$$

whenever $1 \leq p < p/(n-2)$ and $B(x_0, 4r) \subset W$. □

Since our objective is to construct a Lipschitz solution u with Lipschitz constant 1, we will need to prove that

$$d_\Omega(\partial E_s, \partial E_t) \geq |t - s|. \tag{4.3}$$

For this purpose, we introduce the following notation. For fixed s and t, let

$$\begin{aligned} X &= (\overline{\Omega \cap \partial E_s}) \times (\overline{\Omega \cap \partial E_t}), \\ \Lambda &= \inf\{d_\Omega(x, y) : (x, y) \in X\}, \\ N &= X \cap \{(x, y) : d_\Omega(x, y) = \Lambda\}. \end{aligned} \tag{4.4}$$

We also need to define the following 5 sets. In the definitions of S_4 and S_5, the following notation will be employed. If $x \in \Omega \cap \partial E_s \cap (\partial L_s \cup \partial M_s)$, then by Theorem 3.6, ∂E_s is a $C^{1,1/2}$-hypersurface in $B(x, r)$ for some $r > 0$. Therefore, if we let π denote the orthogonal projection $\pi: R^n \to T_{\partial E_s}(x)$ where $T_{\partial E_s}(x)$ is the tangent hyperplane to ∂E_s at x, then the restriction of π to $B(x, r) \cap \partial E_s(x)$ is univalent. We will denote the inverse of this mapping by $\overline{u}_s$.

(4.5)

$S_1 = N \cap \{(x, y)$: a geodesic from x to y intersects $\partial\Omega\}$,

$S_2 = N \cap \{(x, y) : x \neq y$, there exists $q_1 \in \partial\Omega$ such that $x \in \partial B(q_1, |g(q_1) - s|)$ and $x = q_1 + \tau(y - q_1)$ for some $\tau \in (0, 1)\}$,

$S_3 = N \cap \{(x, y) : x \neq y$, there exists $q_2 \in \partial\Omega$ such that $y \in \partial B(q_2, |g(q_2) - t|)$ and $y = q_2 + \tau(x - q_2)$ for some $\tau \in (0, 1)\}$.

The next result is the final bit of preparation needed for the main results to follow. Let E_t be a solution of (3.11) and let $x \in \Omega \cap \partial E_t - M_t$ be a point at which ∂E_t is $C^{1,1/2}$-regular. Then, with the notation introduced in the paragraph preceding (4.5), ∂E_t can be represented as the graph of a function u_t where

$$u_t(w) = \nu(x, E_t) \cdot (\overline{u}_t(w) - x)$$

for all $w \in B(x, r) \cap T_{\partial E_t}(x)$ and where $\nu(x, E_t)$ is the unit exterior normal to E_t at x. Thus, since u_t is at least of class C^1 near x,

$$H^{n-1}(B(x, r) \cap \partial E_t) = \int_{\pi(B(x,r)\cap\partial E_t)} \sqrt{1 + |\nabla u_t(w)|^2}\, dH^{n-1}(w)$$

for all small $r > 0$. Let W_x be an open set relative to $T_{\partial E_t(x)}$ such that $W_x \subset \pi(B(x, r) \cap \partial E_t)$. Let $\varphi \in C_0^1(W_x)$. Because $x \notin M_t$ by assumption, for all sufficiently small $r > 0$ the set

$$E_{t,\varepsilon} = (E_t - B(x, r)) \cup (G_{t,\varepsilon} \cap B(x, r)),$$

is admissible in (3.11), where $G_{t,\varepsilon}$ is the set under the graph of $u_t + \varepsilon\varphi$, where "under the graph" is defined relative to the coordinate system induced by $T_{\partial E_t(x)}$ and $\nu(x, E_t)$. Therefore, with $f(\varepsilon)$ defined by

$$\begin{aligned} 0 \le f(\varepsilon) &= P(E_{t,\varepsilon}, \Omega) - P(E_t, \Omega) \\ &= \int_{W_x} \sqrt{1 + |\nabla u_t(w) + \varepsilon\nabla\varphi(w)|^2} - \sqrt{1 + |\nabla u_t(w)|^2}\, dH^{n-1}(w), \end{aligned}$$

it follows that

$$0 \le f'(0) = \int_{W_x} \frac{\nabla u_t(w) \cdot \nabla\varphi(w)}{\sqrt{1 + |\nabla u_t(w)|^2}}\, dH^{n-1}(w)$$

for all test functions φ such that $\varphi \in C^1(W_x)$, $\operatorname{spt}\varphi \subset W_x$, and $\varphi \ge 0$. That is, u_t is a weak supersolution of the minimal surface equation in W_x, relative to the coordinate system defined in terms of $T_{\partial E_t(x)}$ and $\nu(x, E_t)$.

Similarly, if $x \in \Omega \cap \partial E_t - L_t$, then u_t is a weak supersolution of the minimal surface equation relative to the coordinate frame determined by $T_{\partial E_t(x)}$ and $-\nu(x, \partial E_t)$. In summary, we have the following.

Lemma 4.3. *If $x \in \Omega \cap \partial E_t - M_t$ and ∂E_t is $C^{1,1/2}$-regular at x, then in some neighborhood of x, u_t is a weak supersolution of the minimal surface equation relative to the coordinate frame determined by $T_{\partial E_t(x)}$ and $\nu(x, E_t)$. Also, if $x \in \Omega \cap \partial E_t - L_t$ and ∂E_t is $C^{1,1/2}$-regular at x, then in some neighborhood of x, u_t is a weak subsolution of the minimal surface equation relative to the coordinate frame determined by $T_{\partial E_t(x)}$ and $\nu(x, E_t)$.*

We now are able to establish a result fundamental to this paper.

Theorem 4.4. *With the notation of* (4.4) *and* (4.5)

$$N \cap \left(\bigcup_{i=1}^{3} S_i \right) \neq \varnothing. \tag{4.6}$$

Proof. Define

$$\begin{aligned} C_s &= \{x \in \overline{\Omega \cap \partial E_s} \colon (x, y) \in N \text{ for some } y \in \overline{\Omega \cap \partial E_t}\}, \\ C_t &= \{y \in \overline{\Omega \cap \partial E_t} \colon (x, y) \in N \text{ for some } x \in \overline{\Omega \cap \partial E_s}\} \end{aligned}$$

and observe that both sets are closed in R^n. The proof is divided into four parts and is by contradiction. Thus we assume that

$$N \cap \left(\bigcup_{i=1}^{3} S_i \right) = \varnothing. \tag{4.7}$$

With this assumption, we will prove that C_s and C_t are relatively open in $\overline{\Omega \cap \partial E_s}$ and $\overline{\Omega \cap \partial E_t}$ respectively. We will assume without loss of generality, that $s < t$.

Part 1. Under assumption (4.7), C_s is relatively open in ∂E_s. That is, for each point $x_0 \in C_s$ there exists an open ball $B(x_0, r)$ such that

$$B(x_0, r) \cap \partial E_s \subset C_s. \tag{4.8}$$

We begin by selecting $x_0 \in C_s$. Then there exists $y_0 \in C_t$ such that $|x_0 - y_0| = \Lambda$; that is $(x_0, y_0) \in N$. We now examine the various possibilities that can occur when $(x_0, y_0) \in N$. There are 7 possible configurations and we examine each one in light of assumption (4.7).

Case 1. Assume $x_0 \notin \partial L_s \cup \partial M_s$, $y_0 \notin \partial L_t \cup \partial M_t$ and $\Lambda > 0$. Since $S_1 \cap N = \varnothing$ by assumption, the geodesic joining x_0 to y_0 is a straight line segment. Corollary 3.2 states that ∂E_s is area-minimizing in $\Omega - (L_s \cup M_s)$ and similarly for ∂E_t. Consequently, $x_0 \in \operatorname{reg} \partial E_s$ and $y_0 \in \operatorname{reg} \partial E_t$ by Lemma 2.3. Moreover, the respective tangent hyperplanes at x_0 and y_0 must be parallel. Thus, the functions u_s and u_t whose graphs locally describe ∂E_s and ∂E_t are solutions to the minimal surface equation as described in Lemma 4.3 and may be assumed to share the same domain of definition. We may now invoke Lemma 4.2 to establish the claim (4.8) in this case.

Case 2. Assume $x_0 \notin \partial L_s \cup \partial M_s$, $y_0 \notin \partial L_t \cup \partial M_t$ and $\Lambda = 0$. Refer to Lemma 4.1 to conclude that $E_t \subset E_s$ and then use Theorem 2.2 to establish (4.8).

Case 3. Assume $x_0 \in \Omega \cap (\partial L_s \cup \partial M_s)$, so that there exists $q_1 \in \partial\Omega$ such that $d_\Omega(x_0, q_1) = |g(q_1) - s|$ and $\Lambda > 0$. By virtue of (3.9) and (3.10), there exists $q^* \in \partial\Omega$ such that $|x_0 - q^*| = |g(q^*) - s|$. Therefore, we may take $q^* = q_1$ and assume $|x_0 - q_1| = |g(q_1) - s|$. Note that $x_0 \notin \partial M_s$ for otherwise the assumption $N \cap (S_1 \cup S_2) = \varnothing$ would imply $|y_0 - q_1| \le |x_0 - q_1| \le s - g(q_1)$. Hence, $y_0 \in M_s$, an impossibility since $y_0 \in E_t$ and $E_t \cap M_s = \varnothing$. Thus, $x_0 \in \Omega \cap \partial E_s \cap \partial L_s - M_s$ and Theorem 3.6 along with Lemma 4.3 yield that u_s, whose graph locally describes ∂E_s in a coordinate system defined by $T_{\partial E_s(x_0)}$ and $(y_0 - x_0)/|y_0 - x_0| = -\nu(x_0, E_s)$ is a weak subsolution of the minimal surface equation. Furthermore, the assumption $N \cap S_2 = \varnothing$ implies that the points q_1, y_0 and x_0 are colinear and are aligned in the order stated. We now claim that ∂E_t is either unconstrained at y_0 or $y_0 \in \partial M_t$. To this end, we assume by contradiction that $y_0 \in \partial L_t$. In view of Theorem 3.6, there exists a tangent plane to ∂E_t at y_0. Because $N \cap S_1 = \varnothing$ by assumption, we have that the geodesic joining x_0 to y_0 is a line segment. Hence, the tangent planes at the points in question are parallel. By the assumption $N \cap S_3 = \varnothing$ and Lemma 3.4 there exists a point $p \in \partial\Omega$ such that

$$|p - y_0| = g(p) - t \tag{4.9}$$

and such that y_0, x_0 and p lie in this order on a line segment joining y_0 to p. We then have the colinear ordering q_1, y_0, x_0, and p. By (4.9), it follows that the entire line segment joining y_0 to p lies in L_t. However, $x_0 \in \partial L_s$ implies the line segment joining x_0 to p has nonempty intersection with $R^n - L_s$. This contradicts the containment $L_t \subset L_s$. We have thus established our claim that $y_0 \notin \partial L_t$. Regardless of whether ∂E_t is unconstrained at y_0 or $y_0 \in \partial M_t$, the set ∂E_t is locally representable near y_0 as a graph, u_t, of a $C^{1,1/2}$ function through an appeal to Lemma 2.3 in the first case and Theorem 3.6 in the second. In either case, u_t will be a weak supersolution of the minimal surface equation relative to the coordinates determined by $T_{\partial E_t(y_0)}$ and $(y_0 - x_0)/|y_0 - x_0| = -\nu(y_0, E_t)$. Since the respective hyperplanes at x_0 and y_0 are parallel, we may redefine u_t so that u_s and u_t are defined on some open set of $T_{\partial E_s}$, with $u_s(x_0) = u_t(x_0) = 0$, and $u_t \geq u_s$ near x_0. Now apply Lemma 4.2 to conclude that the two functions are equal near x_0, thus showing that for each $x \in \overline{\Omega} \cap \partial E_s$ near x_0, there exists $y \in \overline{\Omega} \cap \partial E_t$ such that $|x - y| = \Lambda$.

Case 4. Assume that $y_0 \in \Omega \cap (\partial L_t \cup \partial M_t)$. The treatment here is similar to the previous case in view of the assumption $N \cap S_3 = \varnothing$, which implies that $x_0 = \tau q_2 + (1 - \tau) y_0$ for some $\tau \in (0, 1)$.

Case 5. Assume $x_0 = y_0$ where ∂E_s is constrained at x_0 while ∂E_t is not constrained. Thus there exists $q_1 \in \partial\Omega$ such that $d_\Omega(x_0, q_1) = |g(q_1) - s|$, with $\Lambda = 0$. By (3.9) and (3.10), we may as well assume that $|x_0 - q_1| = |g(q_1) - s|$. If $g(q_1) < s$, then $|x_0 - q_1| < t - g(q_1)$. This implies that $x \in (M_t)^i$, but $(M_t)^i \cap E_t = \varnothing$, thus precluding $x_0 \in \partial E_t$. Thus, we must have $g(q_1) > s$, so that $x_0 \in \partial L_s$. Therefore $\nu(x_0, E_s) = (x_0 - q_1)/|x_0 - q_1|$. Note that $x_0 \notin M_s$ for otherwise $x_0 \in B(q_1, s - g(q_1))$, a contradiction since $x_0 = y_0 \in E_t$, and $E_t \cap M_s = \varnothing$. Hence, $x_0 \in \Omega \cap \partial E_s - M_s$, which implies that u_s is a weak supersolution of the minimal surface equation relative to the coordinates induced by $T_{\partial E_s(x_0)}$ and $(x_0 - q_1)/|x_0 - q_1| = \nu(x_0, \partial E_s)$, by Lemma 4.3. Now we employ the hypothesis that $x_0 \notin \partial L_t \cup \partial M_t$ to show that ∂E_t is also regular at x_0. For this purpose, first observe that ∂E_s has a tangent hyperplane at x_0. This implies that for each $\varepsilon > 0$, there exists $r > 0$ such that

$$E_s \cap B(x_0, r) \subset \left\{ x \colon \frac{(x - x_0)}{|x - x_0|} \cdot \frac{(q_1 - x_0)}{|q_1 - x_0|} > -\varepsilon \right\} \cap B(x_0, r). \tag{4.10}$$

However, if $y_0 \in \operatorname{sing} \partial E_t$, then any tangent cone of ∂E_t at y_0 could not be contained in

$$\overline{H} = \left\{ x \colon \frac{(x - x_0)}{|x - x_0|} \cdot \frac{(q_1 - x_0)}{|q_1 - x_0|} \geq 0 \right\},$$

by (2.14). Hence, there is an element x^* of a tangent cone such that $x^* \notin \overline{H}$. Taking $x_0 = 0$ for simplicity of notation, this implies the existence of a sequence $\{x_i\} \to x^*$, where $x_i \notin \overline{H}$, $x_i = y_i / r_i$, $r_i \to 0^+$, and $y_i \in E_t$. Hence,

$$\lim_{i \to \infty} \frac{x_i}{|x_i|} \cdot \frac{q_1}{|q_1|} = \frac{x^*}{|x^*|} \cdot \frac{q_1}{|q_1|}.$$

If we set

$$-2\varepsilon = \frac{x^*}{|x^*|} \cdot \frac{q_1}{|q_1|}$$

then

$$\limsup_{i\to\infty} \frac{y_i}{|y_i|} \cdot \frac{q_1}{|q_1|} < -\varepsilon .$$

But $E_t \subset E_s$ (Lemma 4.1) and (4.10) imply that

$$\liminf_{i\to\infty} \frac{y_i}{|y_i|} \cdot \frac{q_1}{|q_1|} > -\varepsilon ,$$

a contradiction. Thus, $y_0 \in \operatorname{reg} \partial E_t$ and therefore u_t is a weak solution (in fact, a strong solution) of the minimal surface equation near y_0. Now the containment $E_t \subset E_s$ implies $u_t \le u_s$ near x_0 and therefore $u_t = u_s$ there by Lemma 4.2.

Case 6. Assume $x_0 = y_0$ where ∂E_t is constrained at x_0 while ∂E_s is not constrained. This case is completely analogous to the previous one, and its proof is omitted.

Case 7. Assume that $x_0 = y_0$ where both ∂E_s and ∂E_t are constrained. Then there exist $q_1, q_2 \in \partial\Omega$ such that $d_\Omega(x_0, q_1) = |g(q_1)-s|$, $|x_0-q_2| = |g(q_2)-t|$, and $\Lambda = 0$. As in Case 5, we conclude that $x_0 \in \Omega\cap\partial E_s - M_s$ so that $x_0 \in \partial L_s$. Therefore, ∂E_s is $C^{1,1/2}$-regular at x_0 and u_s is a weak subsolution of the minimal surface equation relative to the coordinates determined by $T_{\partial E_s(x_0)}$, and $(q_1 - x_0)/|q_1 - x_0| = -\nu(x_0, E_s)$ (Lemma 4.3). Similarly, $y_0 \in \Omega \cap \partial E_t - L_t$ and therefore u_t is a weak supersolution of the minimal surface equation relative to the coordinates described above. Moreover, the containment $E_t \subset E_s$ implies that $u_t \ge u_s$ near x_0 with $u_t(x_0) = u_s(x_0)$. Hence, $u_t = u_s$ near $x_0 = y_0$ by Lemma 4.2.

This concludes Part 1 of the proof, as we have shown that C_s is relatively open in $\overline{\Omega \cap \partial E_s}$. Let C be a component of C_s. In our attempt to contradict (4.7), we have thus far shown through the analysis of the sets in (4.5), that $C - \operatorname{sing} \partial E_s$ is $C^{1,1/2}$-regular at each of its points. Thus, except for $\operatorname{sing} \partial E_s$, C would be a compact $(n-1)$-manifold (without boundary). If it were true that $\operatorname{sing} \partial E_s = \varnothing$, we would be able to conclude that C is the boundary of a set contained in either E_s or its complement, from which it would be an easy matter to reach a contradiction, thus finally proving (4.6).

Our next step then is to show that this argument is essentially correct, even in the presence of $\operatorname{sing} \partial E_s$.

Part 2. There exists an open set $V \subset R^n$ such that $\overline{V} \subset \Omega$ and $\partial V \supset C$.

For this purpose we first find an open set U such that

(4.11)
- (i) ∂U is an $(n-1)$-manifold with finitely many components,
- (ii) $\partial U \cap \partial E_s = \varnothing$,
- (iii) $U \cap \partial E_s = C$,
- (iv) $\overline{U} \subset \Omega$,
- (v) U is connected.

To find such a set consider a smooth approximation, $\overline{d}$, to the function $d(x) = \operatorname{dist}(x, C)$. That is, let $\overline{d} \in C_0^\infty(R_n - C)$ be such that $K^{-1} d(x) \le \overline{d}(x) \le Kd(x)$ for all $x \in R^n$, cf. [Z, Lemma 3.6.1]. Since C is relatively open in $\overline{\Omega \cap \partial E_s}$, it follows that $\partial\{x : \overline{d}(x) < t\} \cap \partial E_s = \varnothing$ for all small values of t.

Moreover, by Sard's theorem and the Implicit Function Theorem, $\overline{d}^{-1}(t)$ is a smooth $(n-1)$-manifold for almost all values of t and for each t that is not a critical value of $\overline{d}$, $\overline{d}^{-1}(t) = \partial\{x : \overline{d}(x) < t\}$. For any such value of t, let U be that component of $\{x : \overline{d}(x) < t\}$ that contains C to produce a set satisfying all conditions of (4.11) except possibly (iv). By choosing t sufficiently small, this too will be satisfied because $C_s \cap \partial\Omega = \varnothing$.

Using only the fact that ∂U is a compact $(n-1)$-manifold, we invoke Alexander Duality of algebraic topology to conclude that $R^n - \partial U$ consists of finitely many components, one more than the number of components of ∂U, [GH, Theorem 27.10]. Moreover, each component of ∂U is the boundary of precisely one bounded, open set. Note that ∂U_∞ is connected where U_∞ denotes the unbounded component of $R^n - \partial U$. Indeed, since U is connected, it is one of the components of $R^n - \partial U$. Thus, there is a one-to-one correspondence between the bounded components of $R^n - \partial U$ and the components of ∂U, which implies that ∂U_∞ is connected.

Since ∂U_∞ is connected, either $\partial U_\infty \subset (E_s)^i$ or $\partial U_\infty \subset (E_s)^e$ because $\partial U_\infty \cap \partial E_s = \varnothing$. In case $\partial U_\infty \cap (E_s)^i = \varnothing$, define V by $V = (U_\infty)^e \cap (E_s)^i$. Since $(U_\infty)^e \supset U$ and $U \cap \partial E_s = C$, it follows that

$$\partial V = (\partial U_\infty \cap (E_s)^i) \cup ((U_\infty)^e \cap \partial E_s) \supset C.$$

Similarly, if $(E_s)^e \cap \partial U_\infty = \varnothing$, define V by $V = (U_\infty)^e \cap (E_s)^e$, so that

$$\partial V = (\partial U_\infty \cap (E_s)^e) \cup ((U_\infty)^e \cap \partial E_s) \supset C.$$

Thus, we have established the existence of an open set V that is either a subset of $(E_s)^i$ or a subset of $(E_s)^e$ and satisfies $\overline{V} \subset \Omega$, $\partial V \supset C$. To finish the proof of (4.6), we will now show that this leads to a contradiction.

Part 3. If $V \subset (E_s)^i$, then E_s is not a minimizer of (3.11).

We claim that $V \cap L_s = \varnothing$. If not, there exists $x \in V \cap L_s$ and $q \in \partial\Omega$ such that $x \in \overline{B}(q, |g(q) - s|)$. Since $\overline{V} \subset \Omega$, there must exist $t \in (0, 1)$ such that $tq + (1-t)x \in \partial V \subset \partial E_s$. Thus, $x \in (L_s)^i \cap \partial E_s$, which is a contradiction to the containment $E_s \supset L_s$. Thus, $V \cap L_s = \varnothing$ and this implies that the closed set F_s defined by $F_s = \overline{E_s - \overline{V}}$ is admissible in the minimization problem (3.11). If we can show that

$$H^{n-1}(\Omega \cap \partial E_s) \geq H^{n-1}(\Omega \cap \partial F_s) + H^{n-1}(\partial V) \tag{4.12}$$

the desired conclusion is reached since then, $H^{n-1}(\Omega \cap \partial E_s) > H^{n-1}(\Omega \cap \partial F_s)$, contradicting the minimality of $H^{n-1}(\Omega \cap \partial E_s)$. To establish (4.12) it is sufficient to prove

$$\partial F_s \cap \partial V = \varnothing, \tag{4.13}$$

since

$$\Omega \cap \partial E_s \supset [(\Omega \cap \partial F_s) - (\partial F_s) \cap (\partial V)] \cup [\partial V].$$

Because $\partial V = (U_\infty)^e \cap \partial E_s$ it follows that for all sufficiently small $r > 0$,

$$B(x, r) \cap \partial V = B(x, r) \cap \partial E_s.$$

Furthermore, for all small $r > 0$, $B(x, r) \cap \overline{V} = B(x, r) \cap E_s$. It follows immediately that $x \notin \partial F_s$. Thus $(\partial F_s) \cap (\partial V) = \varnothing$ and therefore (4.13) is established.

Part 4. If $V \subset (E_s)^e$, then E_s is not a minimizer of (3.11). An argument similar to the one above shows that $\overline{V} \cap (M_s)^i = \varnothing$, thus allowing $G_s = E_s \cup \overline{V}$ as an admissible competitor in (3.11). Now repeat the argument of Part 3 with F_s replaced by G_s to contradict the minimality of ∂E_s. □

We now pursue the consequences of (4.6).

Theorem 4.5. *Suppose* $s, t \in [a, b]$ *with* $s < t$ *where* $[a, b]$ *is defined by* (3.3). *Then*

$$d_\Omega(\overline{\Omega \cap \partial E_s}, \overline{\Omega \cap \partial E_t}) \geq t - s.$$

Proof. With the notation of (4.4) and (4.5), select $(x, y) \in N$ and employ Theorem 4.4 to analyze the three cases corresponding to

$$N \cap \left(\bigcup_{i=1}^{3} S_i \right) \neq \varnothing.$$

Case 1. $N \cap S_1 \neq \varnothing$. With $(x, y) \in N$ and either x or y in $\partial\Omega$, it follows from the construction that $d_\Omega(x, y) \geq t - s$. For example, if $x \in \partial\Omega$, then $g(x) = s$ because otherwise we would have $x \in (L_s)^i \cup (M_s)^i$, thus precluding $x \in \partial E_s$. If neither x nor y are in $\partial\Omega$, then there exists a point $q \in \partial\Omega$ lying on a geodesic joining x to y such that

$$d_\Omega(x, y) = d_\Omega(x, q) + d_\Omega(q, y) \geq |s - g(q)| + |t - g(q)| \geq |t - s|.$$

Case 2. $N \cap S_2 \neq \varnothing$. First, consider the possibility $g(q_1) < s$. We may take $d_\Omega(x, y) = |x - y|$ since otherwise Case 1 applies. Then

$$|x - y| = |q_1 - y| - |q_1 - x| \geq |g(q_1) - t| + |s - g(q_1)| \geq |t - s|.$$

We will show that the other possibility, $g(q_i) > s$, leads to a contradiction. Under this assumption, $\{x' : d_\Omega(x', q_1) < g(q_1) - s\} \subset E_s$, and therefore

$$\nu(x, E_s) = \frac{y - x}{|y - x|}. \tag{4.14}$$

(Note that $\nu(x, E_s)$ exists because ∂E_s is regular at x, Theorem 3.6.) Lemma 4.1 implies that $y \in E_s$. Now $y \notin \partial E_s$ for otherwise $\operatorname{dist}(\overline{\Omega \cap \partial E_s}, \overline{\Omega \cap \partial E_t})$ $= 0 < \Lambda$, contradicting the fact that $(x, y) \in N$. On the other hand, if $y \in (E_s)^i$, then (4.14) implies that there exists $\tau \in (0, 1)$ such that $x\tau + (1 - \tau)y \in \partial E_s$, again contradicting that $(x, y) \in N$.

Case 3. $N \cap S_3 \neq \varnothing$. If $g(q_2) > t$, then

$$|x - y| \geq |x - q_2| - |q_2 - y| \geq (g(q_2) - s) - (g(q_2) - t) = t - s.$$

As in the previous case, we will show that the other possibility, $g(q_2) < t$, leads to a contradiction. In this situation, $\{x' : d_\Omega(x', q_2) < t - g(q_2)\} \cap E_t = \varnothing$, thus implying that $\nu(y, E_t) = (y - x)/|y - x|$. Because of this and the regularity of ∂E_t at y, it follows that

$$H^1[(0, 1) \cap \{\tau : \tau x + (1 - \tau)y \in E_t\}] > 0.$$

On the other hand note that

$$H^1[(0, 1) \cap \{\tau : \tau x + (1 - \tau)y \in E_t\}] < 1$$

for otherwise, since $E_t \subset E_s$ (Lemma 4.1), $x \in \partial E_t$ and this would contradict $(x, y) \in N$. Hence, there exists $\tau \in (0, 1)$ such that $z = \tau x + (1-\tau)y \in \partial E_t$. But then, $|x - z| < \Lambda$, a contradiction. □

5. A simpler proof using the Euclidean metric

In this section we present a simpler proof that level sets are sufficiently separated provided the Dirichlet data g in (1.2) satisfies a Lipschitz condition $|g(p) - g(q)| \leq |p - q|$ rather than $|g(p) - g(q)| \leq d_\Omega(p, q)$ for all $p, q \in \partial\Omega$. While utilizing the same notation for the obstacles as in (3.4) and (3.5), in this section only we define them in terms of the Euclidean metric

$$L_t = \left\{ \bigcup \overline{B}(p, g(p) - t) : p \in \partial\Omega, \ g(p) \geq t \right\}, \tag{5.1}$$

$$M_t = \left\{ \bigcup \overline{B}(p, t - g(p)) : p \in \partial\Omega, \ g(p) \leq t \right\}. \tag{5.2}$$

We will first consider the following alternative formulation of problems (3.11) and (3.12)

$$\inf\{P(E, R^n) : L_t \subset E, \ \overline{E} \cap (M_t)^i = \varnothing, \ E - \Omega = L_t - \Omega\} \tag{5.3}$$

and

$$\sup\{|E| : E \text{ a solution of the above}\}. \tag{5.4}$$

The equivalence of this formulation with (3.11) follows from the fact that if E is admissible in (5.3), then

$$P(E, R^n) = H^{n-1}(\partial^* E) = H^{n-1}(\partial^* E \cap \Omega) + H^{n-1}(\partial^* L_t - \Omega).$$

We shall denote the solution of (5.4) by $\mathscr{E}_t$. It is thus related to E_t, the solution to (3.12), by $\mathscr{E}_t \cap \overline{\Omega} = E_t$.

Lemma 5.1. *If $a \leq s < t \leq b$ where $[a, b]$ is defined in* (3.3) *and $\eta \in R^n$ with $|\eta| \leq t - s$, then $\mathscr{E}_t + \eta \subset \mathscr{E}_s$ where $\mathscr{E}_t + \eta = \{x + \eta : x \in \mathscr{E}_t\}$.*

Proof. Define $L_t' = L_t + \eta$, $M_t' = M_t + \eta$, $\mathscr{E}_t' = \mathscr{E}_t + \eta$, and $\Omega' = \Omega + \eta$. Then, $\mathscr{E}_t'$ is a solution to

$$\inf\{P(E, R^n) : L_t' \subset E, \ \overline{E} \cap (M_t')^i = \varnothing, \ E - \Omega' = L_t' - \Omega'\} \tag{5.5}$$

and further maximizes $|E|$ among all such minimizers. Note that $L_t' \subset L_s$, $M_s \subset M_t'$, and $(M_s)^i \subset (M_t')^i$. For example, to prove the first inclusion, note that if $x \in L_t'$ then $x = y + \eta$, $y \in L_t$. This implies that $y \in \overline{B}(p, g(p) - t)$ for some $p \in \partial\Omega$, $g(p) \geq t \geq s$. Thus, $x \in \overline{B}(p, g(p) - t + |\eta|)$, $g(p) \geq s$, $p \in \partial\Omega$. That is, $x \in \overline{B}(p, g(p) - s)$, $g(p) \geq s$, $p \in \partial\Omega$, or $x \in L_s$.

Now consider $\mathscr{E} = \mathscr{E}_s \cap \mathscr{E}_t'$. Then $L_t' \subset \mathscr{E}$ since $L_t' \subset L_s \subset \mathscr{E}_s$ and $L_t' \subset \mathscr{E}_t'$. Also, $L_t' - \Omega' \subset \mathscr{E} - \Omega' \subset \mathscr{E}_t' - \Omega' = L_t' - \Omega'$ which implies that $\mathscr{E} - \Omega' = L_t' - \Omega'$. Since $\mathscr{E}_t' \cap (M_t')^i = \varnothing$ it follows that $\mathscr{E} \cap (M_t')^i = \varnothing$. Therefore $\mathscr{E}_s \cap \mathscr{E}_t'$ is a competitor for (5.3) and since $\mathscr{E}_t'$ is a minimizer, we have

$$P(\mathscr{E}_s \cap \mathscr{E}_t', R^n) \geq P(\mathscr{E}_t', R^n). \tag{5.6}$$

For the next step of the proof, let $F = \mathscr{E}_s \cup \mathscr{E}_t'$. Then $L_s \subset F$ since $L_s \subset \mathscr{E}_s$. Moreover, since $(M_s)^i \subset (M_t')^i$ and $\mathscr{E}_t' \cap (M_t')^i = \varnothing$, we have $\mathscr{E}_t' \cap (M_s)^i = \varnothing$ and therefore $F \cap (M_s)^i = \varnothing$. We wish to show that $F - \Omega = L_s - \Omega$. Since

$L_s \subset F$ we have that $L_s - \Omega \subset F - \Omega$ and also $F - \Omega = (\mathscr{E}_s - \Omega) \cup (\mathscr{E}_t' - \Omega) = (L_s - \Omega) \cup (\mathscr{E}_t' - \Omega)$. So we need only show that $\mathscr{E}_t' - \Omega \subset L_s - \Omega$. Suppose $x \in \mathscr{E}_t' - \Omega$. Then $x \notin \Omega$ and $x = y + \eta$ with $y \in \mathscr{E}_t$. If $y \notin \Omega$, then $y \in \mathscr{E}_t - \Omega = L_t - \Omega$ and therefore $x \in L_t' \subset L_s$. If $y \in \Omega$, then there exists $y' \in \partial\Omega$ with $y' = y + \gamma\eta$, $0 \leq \gamma \leq 1$ since $x \notin \Omega$. If $g(y') \geq t \geq s$ then $|x - y'| \leq t - s \leq g(y') - s$ which implies that $x \in L_s$, our desired conclusion. On the other hand, if $g(y') \leq t$, then since $y \in \mathscr{E}_t$, $y \notin (M_t)^i$ and thus $|y - y'| \geq t - g(y')$. Consequently, $g(y') \geq t - |y - y'| \geq t - (t - s) = s$ and $|x - y'| \leq (t - s) - |y - y'| \leq g(y') - s$ which implies $x \in L_s$. This shows that $\mathscr{E}_t' - \Omega \subset L_s - \Omega$ and therefore that $F - \Omega = L_s - \Omega$. Hence $\mathscr{E}_s \cup \mathscr{E}_t'$ is a competitor with $\mathscr{E}_s$ in problem (5.3) thus showing that

$$P(\mathscr{E}_s \cup \mathscr{E}_t', R^n) \geq P(\mathscr{E}_s, R^n). \tag{5.7}$$

Now appeal to (2.9) to conclude that equality holds in both (5.6) and (5.7). Because $|\mathscr{E}_s|$ has maximal measure among solutions to (5.3), we have $|\mathscr{E}_s| = |\mathscr{E}_s \cup \mathscr{E}_t'|$. Note that $|(\mathscr{E}_t' - \mathscr{E}_s) \cap \Omega| = 0$ because $\mathscr{E}_s \cup \mathscr{E}_t' - \Omega = \mathscr{E}_s - \Omega = L_s - \Omega$. Due to (2.11), $\mathscr{E}_t' \cap \Omega \subset \mathscr{E}_s \cap \Omega$. We already have seen that

$$\mathscr{E}_t' - \Omega \subset L_s - \Omega = \mathscr{E}_s - \Omega$$

and thus $\mathscr{E}_t' \subset \mathscr{E}_s$. □

Corollary 5.2. *Suppose $a \leq s < t \leq b$ where $[a, b]$ is defined in* (3.3). *Then*

$$\operatorname{dist}(\partial\mathscr{E}_s, \partial\mathscr{E}_t) \geq t - s.$$

Proof. Assume $\operatorname{dist}(\partial\mathscr{E}_s, \partial\mathscr{E}_t) < t - s$. Choose $x \in \partial\mathscr{E}_t$. There exists $y \notin \mathscr{E}_s$ such that $|x - y| = t - s$. Set $\eta = y - x$. Then $y = x + \eta \in \mathscr{E}_t'$. But $y \notin \mathscr{E}_s$ contradicting $\mathscr{E}_t' \subset \mathscr{E}_s$. □

Remark 5.3. If $\partial\Omega$ is taken to be C^1, then the above proof may be modified to show that there is a function $\gamma\colon R^1 \to R^1$ with $\gamma(t) \to 1$ as $t \to 0$ such that $d\gamma(d) \geq t - s$, where $d = d_\Omega(\overline{\Omega \cap \partial E_s}, \overline{\Omega \cap \partial E_t})$. This is a weaker result than Theorem 4.5 but may still be used to prove the result that our solution u, (see (6.1)) is Lipschitz and $|\nabla u| \leq 1$ almost everywhere.

6. Construction of the solution

We now are prepared to define the solution u to our problem. For this purpose we let $A_t = \overline{E_t}$. Then define u by

$$u(x) = \sup\{t\colon x \in A_t\} \tag{6.1}$$

whenever $x \in \overline{\Omega}$. In order to show that u satisfies the desired properties we define

$$B_t = \bigcap_{s<t} A_s, \quad C_t = \bigcup_{s>t} A_s \quad \text{and} \quad D_t = B_t - C_t.$$

Lemma 6.1. *Each point of $\Omega \cap \partial D_t$ is a limit point of $\{\bigcup(\Omega \cap \partial A_s)\colon s \neq t\}$ for $t \in R^1$.*

Proof. For $x \in \Omega \cap \partial D_t$ consider only those $r > 0$ for which $B(x, r) \subset \Omega$. Then each $B(x, r)$ contains $y \in D_t$ and $z \in \Omega - D_t$. This implies that either

$$y \in \bigcap_{s<t} A_s, \quad \text{and} \quad z \in \bigcup_{s<t} (\Omega - A_s)$$

or

$$y \in \bigcap_{s>t}(\Omega - A_s), \quad \text{and} \quad z \in \bigcup_{s>t} A_s .$$

With the help of Lemma 4.1 the first possibility implies that for each $r > 0$, $B(x, r)$ contains an element of ∂A_s for all $s < t$ sufficiently close to t whereas the second implies that a similar conclusion for all $s > t$ sufficiently close to t. □

Lemma 6.2. *For each* $t \in R^1$ *we have the following*:

(i) $u = g$ *on* $\partial\Omega$,

(ii) D_t *is a closed set,*

(iii) $D_t = \overline{\Omega} \cap \{x : u(x) = t\}$,

(iv) $u^{-1}(t) \supset \Omega \cap \partial A_t$,

(v) $A_t \subset \{x : u(x) \geq t\} = B_t$,

(vi) u *is Lipschitz on* $\overline{\Omega}$ *with Lipschitz constant* 1.

Proof. (i) If $q \in \partial\Omega$ with $g(q) = t$, then $q \in L_s \subset A_s$ for each $s < t$. Hence, $u(q) \geq t$. A similar argument shows that $q \notin A_s$ for each $s > t$. Thus, $u(q) = t$.

(ii) Let $x \in \overline{\Omega} \cap \partial D_t$. We wish to prove that $x \in D_t$. First note that x is a limit of points $x_i \in D_t \subset B_t$. Thus $x \in B_t$ since B_t is closed. We now wish to prove that $x \notin C_t$ to establish that $x \in D_t$. We proceed by contradiction and assume $x \in C_t$. Then there exists $s_0 > t$ such that $x \in A_{s_0}$. First consider the possibility that there exists $r > 0$ such that

$$B(x, r) \cap \Omega = B(x, r) \cap D_t . \tag{6.2}$$

Because our convention (2.11) is in force, we have that $|B(x, r) \cap \Omega \cap A_{s_0}| > 0$ for all $r > 0$. But (6.2) implies $B(x, r) \cap \Omega \cap C_t = B(x, r) \cap \Omega \cap (\bigcup_{s>t} A_s) = \varnothing$ for all small $r > 0$, a contradiction. Hence, $x \notin C_t$ and therefore $x \in D_t$.

If, on the other hand, (6.2) is not true, then for all $r > 0$, there would exist $z \in B(x, r) \cap \Omega - D_t$ and $y \in B(x, r) \cap \Omega \cap D_t$. This implies that either

$$y \in \bigcap_{s<t} A_s, \quad \text{and} \quad z \in \bigcup_{s<t}(\Omega - A_s) \tag{6.3}$$

or

$$y \in \bigcap_{s>t}(\Omega - A_s), \quad \text{and} \quad z \in \bigcup_{s>t} A_s . \tag{6.4}$$

Assuming that $r > 0$ is chosen small enough that $B(x, r) \cap \Omega$ is connected, Lemma 4.1 implies that (6.3) yields $b_s \in B(x, r) \cap \Omega \cap \partial A_s$ for all $s < t$ sufficiently close to t. Thus, there exists a sequence $\{b_{s_i}\} \to x$ with $b_{s_i} \in \partial A_{s_i}$ and $s_i \to t^-$. Now either $x \in (A_{s_0})^i$ or $x \in \partial A_{s_0}$. If $x \in (A_{s_0})^i$ then $x \in (C_t)^i$, making $x \in \partial D_t$ impossible. Finally, if $x \in \partial A_{s_0}$, then

$$d_\Omega(x, b_{s_i}) \geq d_\Omega(\overline{\Omega \cap \partial A_{s_i}}, \overline{\Omega \cap \partial A_{s_0}}) \geq s_0 - s_i > 0$$

by Lemma 4.5, and we reach a contradiction by letting $i \to \infty$. The same contradiction is reached if (6.4) holds rather than (6.3).

(iii) If $x \in D_t$, then $x \in B_t$ and therefore $u(x) \geq t$. Similarly, $x \notin C_t$ implies $u(x) \leq t$ and therefore $u(x) = t$.

Conversely, if $u(x) = t$, then $x \in B_t$. Moreover, $u(x) < s$ for each $s > t$ which implies $x \notin A_s$. Thus, $x \notin C_t$ and therefore $x \in D_t$.

(iv) If $x \in \partial A_t$, then $x \in A_t \subset B_t$. Moreover, $x \notin A_{s_0}$, $s_0 > t$ for otherwise, $x \in \partial A_{s_0}$ and $\operatorname{dist}(\Omega \cap \partial A_t, \Omega \cap \partial A_{s_0}) = 0$, a contradiction to Lemma 4.5.

(v) It follows immediately from the definition of u that $A_t \subset \{x : u(x) \geq t\}$. On the other hand, $u(x) \geq t$ if and only if $x \in \bigcap_{s<t} A_s = B_t$.

(vi) Consider $x, y \in \Omega$ such that $u(x) = s$ and $u(y) = t$. By (iii), $x \in D_s$ and $y \in D_t$. If $x \in (D_s)^i$, then there exists $x' \in \partial D_s$ on a geodesic joining x to y. Similarly, if $y \in (D_t)^i$, then there exists $y' \in \partial D_t$ on this geodesic. By Lemma 6.1 there exists a sequence $\{x_i\} \to x'$ such that $x_i \in \partial A_{s_i}$ with $s_i \to s$. Likewise, there exists a sequence $\{y_i\} \to y'$ such that $y_i \in \partial E_{t_i}$ with $t_i \to t$. By Theorem 4.5, we obtain the desired Lipschitz condition in Ω,

$$d_\Omega(x, y) \geq d_\Omega(x', y') = \lim_{i\to\infty} d_\Omega(x_i, y_i) \geq \lim_{i\to\infty} |t_i - s_i| = |t - s|.$$

Next, observe that u is continuous at each $x \in \partial\Omega$. To see this, suppose $u(x) = t$. Then $g(x) = t$ by (i) and therefore, with $s < t$, $\overline{\Omega} \cap \{y : d_\Omega(x, y) < g(x) - s\} \subset A_s$. This implies

$$\liminf_{\substack{y\to x\\ y\in\Omega}} u(y) \geq u(x). \tag{6.5}$$

On the other hand, $\overline{\Omega} \cap \{y : d_\Omega(x, y) < s - g(x)\} \subset M_s$ for all $s > t = g(x)$, which implies

$$\limsup_{\substack{y\to x\\ y\in\Omega}} u(y) \leq u(x). \tag{6.6}$$

Now it is obvious that u satisfies the desired Lipschitz condition on $\overline{\Omega}$. $\square$

Theorem 6.3. *With u defined by* (6.1), *a solution to the problem*

$$\inf\left\{\int_\Omega |\nabla u|\,dx : u \in C^{0,1}(\overline{\Omega}),\ |\nabla u| \leq 1 \text{ a.e., } u = g \text{ on } \partial\Omega\right\} \tag{6.7}$$

is given by (6.1).

Proof. Lemma 6.2 states that u has Lipschitz constant 1 on $\overline{\Omega}$ (hence $|\nabla u| \leq 1$ a.e. on Ω by Rademacher's theorem) and that $u = g$ on $\partial\Omega$.

The function $f(t) = |A_t|$ is nonincreasing and is therefore continuous for all but countably many $t > 0$. Consequently,

$$|B_t - A_t| = 0 \tag{6.8}$$

for all such t. Let $A_t = \overline{\Omega} \cap \{x : u(x) \geq t\}$ and invoke the co-area formula (1.3) to conclude

$$\int_\Omega |\nabla u|\,du = \int_{-\infty}^{\infty} P(A_t, \Omega)\,dt = \int_a^b P(A_t, \Omega)\,dt$$

where the last equality is obtained from (6.8) and Lemma 6.2(v). If v is any competitor in (6.7), it is clear from the co-area formula that $A'_t = \{x : v(x) \geq t\}$ is a set of finite perimeter for almost all $t \in R^1$. It is readily verified that for all such t, A'_t is a competitor for (3.11). Consequently,

$$\int_\Omega |\nabla u|\,dx = \int_a^b P(A_t, \Omega)\,dt \leq \int_{-\infty}^{\infty} P(A'_t, \Omega)\,dt = \int_\Omega |\nabla v|\,dx,$$

thus showing that u is a minimizer of (6.7). □

We conclude this section with an example in R^2 (easily modified to hold in R^n) which illustrates the possibility of nonuniqueness in (3.11) and thus shows the need for taking the supremum in (3.12). The example also illustrates that a portion of the boundary of the solution of (3.12) does not coincide with the boundary of the obstacles. That is, this portion of the boundary of the solution will satisfy (in R^n) the minimal surface equation. See Figures 1–4. Note that at point P in Figure 4, the level set is not C^2.

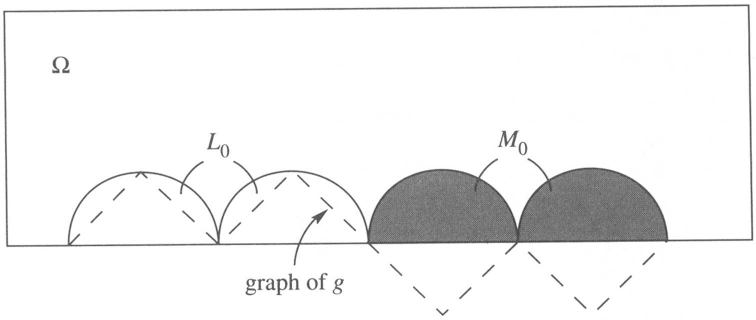

FIGURE 1. The obstacles L_0 and M_0 for the boundary function g which is taken to be saw-toothed where pictured and zero elsewhere on $\partial\Omega$.

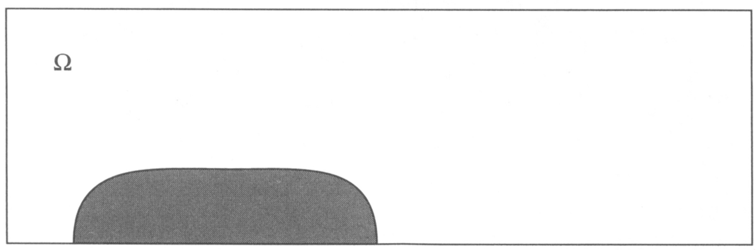

FIGURE 2. The shaded region denotes a solution to (3.11) for $t = 0$, but not the solution to (3.12).

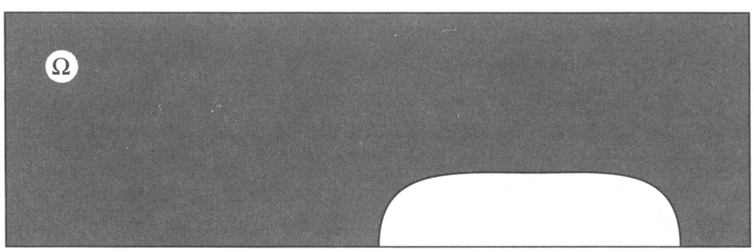

FIGURE 3. The set E_0 that solves (3.12) and corresponds to $\{u(x) \geq 0\}$ in the solution to (1.2).

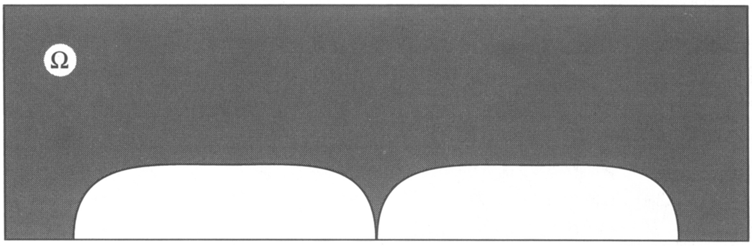

FIGURE 4. The set $D_0 = \{u(x) = 0\}$. In the optimal design setting of [KS1], D_0 corresponds to a hole in the optimal construction.

7. UNIQUENESS

In this section we show that the solution constructed in §6 is the unique solution to (1.2).

Theorem 7.1. *The solution to the constrained least gradient problem* (1.2) *is unique.*

Proof. By considering a component of Ω we may as well assume that Ω is connected. We will proceed by contradiction and assume that there exist solutions u_1 and u_2 and a point x_0 such that

$$u_1(x_0) = \alpha < \beta = u_2(x_0).$$

In light of (1.3), note that any solution u to (1.2) has the property that $\{u \geq t\}$ solves (3.11) and that $|\Omega \cap \{u = t\}| = 0$ for almost all values of t. Thus we may choose s and t with $\alpha < s < t < \beta$ such that $A = \{x\colon u_1(x) \geq s\}$ and $B = \{x\colon u_2(x) \geq t\}$ solve (3.11) with the associated obstacles, while

$$|\Omega \cap \{u_1 = s\}| = 0 = |\Omega \cap \{u_2 = t\}|. \tag{7.1}$$

We will now proceed to show that $\partial A = \partial(A \cup B)$, which will be accomplished by establishing that $\partial A \cap \partial(A \cup B)$ is both open and closed relative to both ∂A and $\partial(A \cup B)$. This readily leads to a contradiction. The argument relies on the fact that A and $A \cup B$ both solve (3.11) at the level s, while B and $A \cap B$ both solve (3.11) at the level t. This is an easy consequence of (2.9).

We first observe that

$$B - A \Subset \Omega. \tag{7.2}$$

Indeed, if $x \in B - A$, then $(u_2 - u_1)(x) \geq t - s > 0$, while $u_2 - u_1 = 0$ on $\partial\Omega$, so that $\operatorname{dist}(x, \partial\Omega) > 0$ follows by the continuity of $u_2 - u_1$.

We now proceed to show that $\partial A \cap \partial(A \cup B)$ is both open and closed relative to both ∂A and $\partial(A \cup B)$. Since ∂A and $\partial(A \cup B)$ are closed sets, it suffices to only prove that $\partial A \cap \partial(A \cup B)$ is open relative to both ∂A and $\partial(A \cup B)$.

First observe that if $x_1 \in \partial A$ then $x_1 \notin (L_s)^i$ and therefore $x_1 \notin L_t$. Now if $x_1 \notin B$, then there exists a neighborhood U of x_1 with $U \cap B = \varnothing$. Hence, $A \cap U = (A \cup B) \cap U$ and therefore $U \cap \partial A = U \cap \partial(A \cup B)$. Thus, in case

$x_1 \notin B$ we arrive at our desired conclusion, so we will therefore assume that $x_1 \in \partial B$. This implies $x_1 \notin (M_t)^i$ and therefore that $x_1 \notin M_s$. Under the assumption that $x_1 \in \partial A \cap \partial B$, the remainder of the proof is divided into 3 cases.

Case 1. $x_1 \notin L_s$.
In this case there is a neighborhood U of x_1 such that A and $A \cup B$ do not intersect $(L_s \cup M_s) \cap U$. Since $A \cup B$ solves the same obstacle problem as does A, it follows that both ∂A and $\partial(A \cup B)$ are area-minimizing in U. In view of the fact that $x_1 \in \partial A \cap \partial(A \cup B)$ we may apply Theorem 2.2 to find that ∂A and $\partial(A \cup B)$ agree in a neighborhood of x_1.

Case 2. $x_1 \in (L_s \cap M_t)$.
Since both A and $A \cup B$ intersect an obstacle, we apply Theorem 3.6 to conclude that both ∂A and $\partial(A \cup B)$ are $C^{1,1/2}$ in a neighborhood of x_1. Similarly, ∂B and $\partial(A \cap B)$ are $C^{1,1/2}$ in a neighborhood of x_1. Now $\partial(A \cap B), \partial A, \partial B$, and $\partial(A \cup B)$ all possess the same tangent plane at x_1 and therefore in a neighborhood of x_1 they all can be represented as graphs of real-valued functions. Moreover, the functions that represent ∂A and $\partial(A \cup B)$ are supersolutions of the minimal surface equation whereas the functions that represent ∂B and $\partial(A \cap B)$ are subsolutions. Recall that ∂B and $\partial(A \cap B)$ solve the same obstacle problem. Referring to Lemma 4.2, we find that the functions that represent $\partial(A \cap B)$ and $\partial(A \cup B)$ agree in a neighborhood of x_1, thus showing that the remaining functions also agree in this neighborhood.

Case 3. $x_1 \in \partial L_s - \partial M_t$.
Observe that ∂A and $\partial(A \cup B)$ are $C^{1,1/2}$-regular in a neighborhood of x_1 since each intersects an obstacle. On the other hand, $\partial B \cap (L_t \cup M_t) = \varnothing$ and $\partial(A \cap B) \cap (L_t \cup M_t) = \varnothing$ in a neighborhood of x_1 and therefore ∂B and $\partial(A \cap B)$ are area-minimizing there. Since $B \subset A \cup B$ it follows that the tangent cone to B at x_1 lies in the half-space determined by the tangent plane to $\partial(A \cup B)$. Hence, ∂B is regular at x_1. Similarly, $\partial(A \cap B)$ is regular at x_1. As in the previous case, $\partial A, \partial B, \partial(A \cap B)$, and $\partial(A \cup B)$ can all be represented as graphs over their common tangent plane. The graph of $\partial(A \cup B)$ is a supersolution while that of ∂B is a solution of the minimal surface equation. Hence, by Lemma 4.2 they agree in a neighborhood of x_1. Also, the graphs of ∂B and $\partial(A \cap B)$ are classical solutions of the minimal surface equation and by appealing once again to Lemma 4.2, they agree near x_1. This implies that $\partial(A \cap B) = \partial(A \cup B)$ near x_1. Finally, the graph of ∂A lies between those of $\partial(A \cap B)$ and $\partial(A \cup B)$, so that $\partial A = \partial(A \cup B)$.

Having established that $\partial A \cap \partial(A \cup B)$ is both open and closed relative to ∂A and $\partial(A \cup B)$ we will now proceed to show that $\partial A = \partial(A \cup B)$. For this, we note that any component of ∂A which intersects $\partial(A \cup B)$ is contained within $\partial(A \cup B)$. Indeed, let C be a component of ∂A such that $C \cap \partial(A \cup B) \neq \varnothing$. There exist open sets U_1 and U_2 such that $U_1 \cap \partial A = C$ and $U_2 \cap \partial A = C \cap \partial(A \cup B)$. Since $C \cap \partial(A \cup B) \neq \varnothing$, it will follow that $C \cap \partial(A \cup B) = C$ if $C \cap \partial(A \cup B)$ is open relative to C. Let $V = U_1 \cap U_2$. We will show that $V \cap C = C \cap \partial(A \cup B)$, thus proving that $C \cap \partial(A \cup B)$ is open relative to C. First, $V \cap C \subset U_2 \cap \partial A = C \cap \partial(A \cup B)$. For the reverse inclusion, note that

$$\begin{aligned} C \cap \partial(A \cup B) &= [U_2 \cap \partial A] \cap [U_1 \cap \partial A \cap \partial(A \cup B)] \\ &\subset V \cap \partial A = U_2 \cap C \\ &= U_2 \cap C \cap U_1 \quad (\text{since } C = C \cap U_1) \\ &= V \cap C. \end{aligned}$$

Thus, $V \cap C = \partial(A \cup B) \cap C$.

We now can show that $\partial A \subset \partial(A \cup B)$. From what has just been proved, it is sufficient to show that each component of ∂A intersects $\partial(A \cup B)$. However, reference to (7.2) shows that it is sufficient to prove that ∂A has no component C such that $C \cap \partial\Omega = \varnothing$. But this follows from the same argument used in Parts 2 and 3 in the proof of Theorem 4.4.

Similarly, to show that $\partial(A \cup B) \subset \partial A$ it is necessary to only show that each component of $\partial(A \cup B)$ intersects ∂A. Since $\partial(A \cup B)$ is also a solution to problem (3.11), we can again apply (7.2) and the aforementioned argument from §6 to arrive at the desired conclusion.

Thus, we now have shown that $\partial A = \partial(A \cup B)$. To finish the proof of the theorem, let S be a component of $B^i - A$. Then $S \Subset \Omega$ and $\partial S \subset \partial A \cup \partial B$. However, it is not possible that $\partial S \subset \partial A$, for if this were true, it would follow that $u_1 = s$ on ∂S and therefore that $u_1 = s$ on S. This would contradict (7.1). Thus, there is a point $x^* \in \partial S \cap (\partial B - \partial A)$ and an open set U containing x^* such that $U \cap A = \varnothing$. This implies $(A \cup B) \cap U = B \cap U$ and therefore $\partial(A \cup B) \cap U = U \cap \partial B$. Hence, $x^* \in \partial(A \cup B)$, which contradicts $\partial A = \partial(A \cup B)$. □

8. A reformulation of the problem

We conclude with another formulation of (1.2), by introducing the following functions, which will serve as constraints

$$\begin{aligned} F(x) &= \inf\{g(p) + d_\Omega(x, p) \colon p \in \partial\Omega\}, \\ f(x) &= \sup\{g(p) - d_\Omega(x, p) \colon p \in \partial\Omega\}. \end{aligned}$$

Theorem 8.1. (i) *If $p \in \partial\Omega$, then $f(p) = g(p) = F(p)$.*

(ii) *Both F and f are Lipschitz functions relative to d_Ω with Lipschitz constant* 1.

(iii) *$M_t = \{x \colon F(x) \le t\}$ and $L_t = \{x \colon f(x) \ge t\}$.*

(iv) *Problem* (1.2) *is equivalent to*

$$\inf\left\{\int_\Omega |Du|\,dx \colon u \in C^{0,1}(\overline{\Omega}),\ f \le u \le F \text{ on } \overline{\Omega}\right\},$$

and so u defined by (6.1) *is a solution to both.*

(v) *The function u is a solution of the problem*

$$\inf\left\{\int_D |Dv|\,dx \colon v \in C^{0,1}(\overline{\Omega}),\ v = u \text{ on } \partial D\right\}$$

where $D = \Omega \cap \{x \colon f(x) < u(x) < F(x)\}$.

Proof. (i) Clearly, $F(p) \le g(p)$ whenever $p \in \partial\Omega$. However,

$$\begin{aligned} g(p) &= g(q) + g(p) - g(q) \\ &\le g(q) + d_\Omega(p, q) \quad \text{for all } q \in \partial\Omega, \end{aligned}$$

which shows that $g(p) \le F(p)$.

(ii) Let $x, y \in \overline{\Omega}$. Then

$$\begin{aligned} F(x) &= \inf\{g(p) + d_\Omega(x, p) : p \in \partial\Omega\} \\ &\le \inf\{g(p) + d_\Omega(y, p) + d_\Omega(x, y) : p \in \partial\Omega\} \\ &= d_\Omega(x, y) + F(y). \end{aligned}$$

That is, $F(x) - F(y) \le d_\Omega(x, y)$ and interchanging x and y gives $|F(x) - F(y)| \le d_\Omega(x, y)$. The proof involving f is similar.

(iii) $$\begin{aligned} F(x) \le t &\Leftrightarrow \text{there exists } p \in \partial\Omega \text{ with } g(p) + d_\Omega(x, p) \le t \\ &\Leftrightarrow x \in M_t. \end{aligned}$$

The proof involving L_t is similar.

(iv) This follows from (iii) and the proof of Theorem 6.3.

(v) This follows from (iv). □

This reformulation of (1.2) allows us to gain further insight into the structure of the solution in the unconstrained region, $D = \Omega \cap \{x : f(x) < u(x) < F(x)\}$. By analogy with the case of gradient constraints for regular elliptic problems such as elastic-plastic torsion problems, one might expect that in D, we would have either $|\nabla u(x)| < 1$ or at least $|u(x) - u(y)| < d_\Omega(x, y)$. However, recall the example of §4. See Figure 5.

We see that there is a region where the surfaces ∂E_t do not coincide with ∂M_t or ∂L_t but are parallel, and the distance from ∂E_t to ∂E_s is $|t - s|$. In this region even though u does not coincide with either f or F, it is differentiable and $|\nabla u(x)| = 1$. Indeed, u is linear in this region. We now show that this situation is typical.

Theorem 8.2. *The set D can be decomposed into $D = D_1 \cup D_2$ such that D_1 is open and*

(i) *If $x \in D_1$ then $|u(x) - u(y)| \le d_\Omega(x, y)$ for all $y \in \overline{\Omega}$.*

(ii) *D_2 is a union of cylinders C. On each cylinder u is a linear function $a \cdot x + b$, $a \in R^n$, $|a| = 1$, and $b \in R^1$. Choosing a coordinate system such that $a = e_n$, where e_n is the nth coordinate vector, we can write $C = U \times I$ with $U \subset R^{n-1}$ open and I an interval.*

(iii) *If $x = (x', y') \in \partial U \times I$, then either $u(x) = F(x)$ or $u(x) = f(x)$, (i.e. $x \in \partial D$). If $u(x) = F(x)$ then there is $p = (p', d') \in \partial\Omega$ such that $F(x) = g(p) + |x - p|$, $p' = x'$ and $F(y) = g(p) + |y - p| = u(y)$ for all $y = (p', \tau)$ with $\tau \in I$. There is an analogous characterization if $u(x) = f(x)$.*

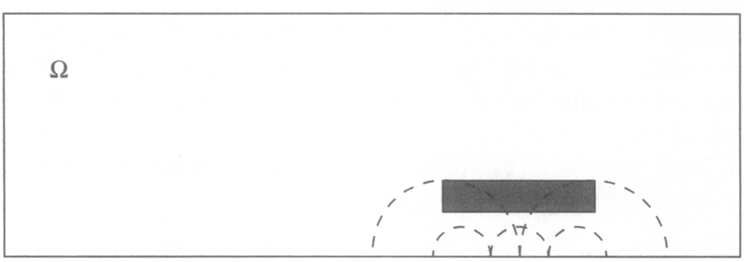

FIGURE 5. The subset of D in which $|\nabla u(x)| = 1$.

Proof. Suppose $x_0 \in D$ and $y_0 \in \overline{\Omega}$ are such that $u(y_0) = u(x_0) + d_\Omega(x_0, y_0)$. Since the difference quotient of u is bounded above by 1, this implies that $|u(x) - u(y)| = d_\Omega(x, y)$ for all x, y on a geodesic joining x_0 and y_0. We show that x_0 must belong to a cylinder of the form given above. First note that if there is a point x' on this geodesic which is also in $\partial\Omega$, then $u(x') = f(x') = F(x')$. But, $f(x_0) \geq f(x') - d_\Omega(x', x_0) = u(x_0)$ which contradicts $x_0 \in D$. Hence, the geodesic lies in Ω and thus is a line segment. Now suppose $u(x_0) = s$ and $u(y_0) = t$ with $s < t$ and set $\eta = x_0 - y_0$ so that $|\eta| = t - s = |x_0 - y_0|$ and $x_0 = y_0 + \eta$. By Lemma 6.1 and the continuity of u, we may as well assume also that $x_0 \in \partial E_s$ and $y_0 \in \partial E_t$.

Now let

$$E = \{x \in \Omega\colon x - \eta \in \Omega,\, u(x) < F(x),\, u(x - \eta) > f(x - \eta),\\ \text{a geodesic from } x \text{ to } x - \eta \text{ lies in } \Omega\}$$

and note that since $x_0 \in D$, we have $x_0 \notin (L_s \cup M_s)$ so that $x_0 \in \Omega$, $x_0 - \eta = y_0 \in \Omega$, and $u(x_0) < F(x_0)$. Furthermore, in light of Theorem 8.1, (ii), we have

$$u(x_0 - \eta) = u(y_0) = u(x_0) + |\eta| > f(x_0) + |\eta| \geq f(x_0 - \eta),$$

so that $x_0 \in \partial E_s \cap (\partial E_t + \eta) \cap E$.

We show that this set is both open and closed in $\partial E_s \cap E$. Since it is obviously closed, we need only show that it is open. Suppose $z \in \partial E_s \cap (\partial E_t + \eta) \cap E$. Since $z \in \partial E_s$ and $z - \eta \in \partial E_t$ are distance $|\eta|$ apart while $\operatorname{dist}(\partial E_s, \partial E_t) \geq t - s = |\eta|$, we have $\operatorname{dist}(z, z - \eta) = \operatorname{dist}(\partial E_s, \partial E_t)$. Now appeal to Lemma 2.3 to conclude that the sets ∂E_s and $\partial E_t + \eta$ are regular at z. Furthermore, since $u(z) < F(z)$ and $u(z - \eta) > f(z - \eta)$, it follows that $\partial E_s \cap \partial M_s \cap N = \varnothing$ and $(\partial E_t + \eta) \cap (\partial L_t + \eta) \cap N = \varnothing$ for some open set N containing z. Thus, near z, ∂E_s may be represented by the graph of a supersolution of the minimal surface equation, u_s, and in the same coordinate system, $\partial E_t + \eta$ may be represented by the graph of a subsolution u_t. Further, since $E_t + \eta \subset E_s$, we have $u_t \leq u_s$ while the graphs coincide at z. Then, by Lemma 4.2, u_t and u_s locally agree, so that ∂E_s and $\partial E_t + \eta$ coincide in a neighborhood of z.

Now let S be the component of $\partial E_s \cap E$ containing x_0. Then S must also be a component of $(\partial E_t + \eta) \cap E$. For each point z in S, the points $z \in \partial E_s$ and $z - \eta \in \partial E_t$ are closest points on the corresponding sets. Thus, S is regular at all of its points z (cf. Lemma 2.3) and the single vector η is normal to S at all of its points. Consequently, S must be the intersection of a hyperplane with normal η and E because S is both open and closed in the intersection of the hyperplane with E.

Let the cylinder $C = \{x - \gamma\eta\colon x \in S,\ 0 \leq \gamma < 1\}$. If $x \in S$, $u(x) = s$ and $u(x - \eta) = t$ with $t - s = |\eta|$. Since, $|u(x) - u(y)| \leq |x - y|$ for all x and y, u must be a linear function on C with $u(x - \gamma\eta) = s + \gamma|\eta|$. We now show that $C \subset D$.

If $z \in C$ and $u(z) = F(z)$, then $z = x - \gamma\eta$ for some $x \in S$ and some γ with $0 \leq \gamma < 1$. Then

$$\begin{aligned} F(x - \eta) &\leq F(z) + |x - \eta - z| = F(z) + (1 - \gamma)|\eta| \\ &= u(z) + (1 - \gamma)|\eta| = u(x - \eta). \end{aligned}$$

Hence, F and u agree at $x-\eta$ and at z. Thus, they are both linear with slope 1 along the line from $x-\eta$ to z and they agree there. Now let $p \in \partial\Omega$ be such that $F(z) = g(p) + d_\Omega(z, p)$. Then,

$$\begin{aligned} F(x-\eta) &\le g(p) + d_\Omega(x-\eta, p) \\ &\le g(p) + d_\Omega(z, p) + |\eta|(1-\gamma) \\ &= F(z) + |\eta|(1-\gamma) = F(x-\eta). \end{aligned}$$

Consequently, z and $x-\eta$ lie on a geodesic from p to $x-\eta$. Thus, x must also be on this geodesic and $u(x) = F(x)$. This contradicts $x \in S \subset E$. Similarly, we cannot have $u(z) = f(z)$, and so $C \subset D$. The union of all such cylinders C constitute the set D_2 thus establishing (ii).

Finally, if $x \in \overline{S} \cap \partial E$, then either $u(x) = F(x)$ or $u(x-\eta) = f(x-\eta)$ or a geodesic from x to $x-\eta$ touches $\partial\Omega$. In the first case, suppose $u(x) = F(x) = s$. Then $u(x-\eta) = t = s + |\eta| = F(x) + |\eta| \ge F(x-\eta)$. Hence, $u(x-\eta) = F(x-\eta)$ and F must increase linearly with slope 1 from x to $x-\eta$, as must u. Therefore, the two functions must coincide along that line segment. Also, if a geodesic touches $\partial\Omega$, then as above $u(x) = F(x)$ or $f(x)$ everywhere along the line from x to $x-\eta$. Furthermore, in either case, as above x must lie along the geodesic from p to $x-\eta$ where $F(x-\eta) = g(p) + d_\Omega(x-\eta, p)$. By taking p as the closest such point on $\partial\Omega$, we may assume the geodesic is a straight line. Thus, equality holds and so $x-p$ and η are colinear, proving (iii). □

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