TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 339, Number 1, September 1993

REFLECTING BROWNIAN MOTION IN A CUSP

R. DANTE DEBLASSIE AND ELLEN H. TOBY

ABSTRACT. Let C be the cusp $\{(x, y): x \geq 0, \ -x^\beta \leq y \leq x^\beta\}$ where $\beta > 1$. Set $\partial C_1 = \{(x, y): x \geq 0, y = -x^\beta\}$ and $\partial C_2 = \{(x, y): x \geq 0, y = x^\beta\}$. We study the existence and uniqueness in law of reflecting Brownian motion in C. The angle of reflection at $\partial C_j \backslash \{0\}$ (relative to the inward unit normal) is a constant $\theta_j \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and is positive iff the direction of reflection has a negative first component in all sufficiently small neighborhoods of 0. When $\theta_1 + \theta_2 \leq 0$, existence and uniqueness in law hold. When $\theta_1 + \theta_2 > 0$, existence fails. We also obtain results for a large class of asymmetric cusps. We make essential use of results of Warschawski on the differentiability at the boundary of conformal maps.

1. INTRODUCTION AND OUTLINE OF MAIN RESULTS

In recent years there has been considerable interest in reflecting Brownian motion, abbreviated as RBM, in 2 dimensions. This interest was precipitated by the work of Varadhan and Williams [8] in which they consider an RBM in

$$D = \{z = re^{i\theta} \in \mathbb{C}: r \geq 0, \ 0 \leq \theta \leq \xi\}, \qquad \xi \in (0, 2\pi),$$

with constant oblique angle of reflection on each side of D. They gave simple necessary and sufficient conditions for existence and uniqueness. Rogers [3, 4] and Burdzy and Marshall [2] also consider RMB in D but with variable direction of reflection. All of the above mentioned authors explored the conditions under which RBM can reach the origin. Also at issue was extending the process beyond the first time it hits $\{0\}$. All the conditions are of a geometric nature.

Here we study the existence and uniqueness of reflecting Brownian motion in the cusp

$$(1.1) \qquad C = \{(x, y): x \geq 0, \ -x^\beta \leq y \leq x^\beta\}, \qquad \beta > 1,$$

with constant direction of reflection on each of the sides, relative to the inward normals. Because $\partial C \backslash \{0\}$ is smooth, standard results give existence and uniqueness when the process starts away from 0, up to the first hitting time of 0.

Received by the editors June 10, 1991.

1991 *Mathematics Subject Classification*. Primary 60J65; Secondary 60J50.

Key words and phrases. Brownian motion with skew reflection, cusp, conformal transformation, differentiability at the boundary.

Supported in part by grants DMS-8902418 and DMS-8921369 from the National Science Foundation.

Also supported in part by grant DMS-8657483 of Professor R. J. Williams during a visit in August of 1990 to the University of California at San Diego.

©1993 American Mathematical Society
0002-9947/93 $1.00 + $.25 per page

Thus the problem reduces to deciding whether or not RBM in C starting away from 0 ever hits 0, and once there can it escape in a unique manner. Hence, in studying existence and uniqueness, the focus of attention should be on the behavior near zero.

Roughly speaking, RBM behaves like Brownian motion inside of C, reflects instantly in an oblique direction off the sides and spends zero time at the origin. More precisely, we pose the problem as a question of existence and uniqueness of a solution to a submartingale problem in the style of Stroock and Varadhan [7]. Let Ω_C be the set of continuous functions from $[0, \infty)$ into C. For $t \geq 0$ let $\mathscr{M}_t$ be the σ-algebra of subsets of Ω_C generated by the coordinate maps $Z_s(\omega) = \omega(s)$ for $0 \leq s \leq t$. We use $\mathscr{M}$ to denote $\sigma\{Z_t : 0 \leq t < \infty\}$. Let $C_b^2(C)$ be the set of real-valued continuous functions that are defined and twice continuously differentiable on some domain containing C and that together with their first and second partial derivatives are bounded on C.

Let $\partial C_1 = \{(x, y) \in C : x \geq 0,\ y = -x^\beta\}$ and $\partial C_2 = \{(x, y) \in C : x \geq 0,\ y = x^\beta\}$. For $i = 1, 2$, let $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, for $z \in \partial C_i \backslash \{0\}$ let $n_i(z)$ be the inward unit normal to ∂C_i and let $v_i(z)$ make constant angle θ_i with $n_i(z)$. We take $\theta_i > 0$ iff the first component of $v_i(z)$ is negative in small neighborhoods of the origin. We also make the normalizations $v_i(z) \cdot n_i(z) = 1$, $z \in \partial C_i \backslash \{0\}$.

The Laplacian and gradient will be denoted by Δ and ∇ as usual. A *solution to the submartingale problem on* C *starting from* $z \in C$ is a probability measure P_z on $(\Omega_C, \mathscr{M})$ such that

$$P_z(\omega(0) = z) = 1; \tag{1.2}$$

for each $f \in C_b^2(C)$,

$$f(\omega(t)) - \frac{1}{2} \int_0^t \Delta f(\omega(s))\, ds \tag{1.3}$$

is a P_z-submartingale on $(\Omega_C, M, \{\mathscr{M}_t\})$ whenever f is constant in a neighborhood of 0 and $v_i \cdot \nabla f \geq 0$ on ∂C_i for $i = 1, 2$;

$$E^{P_z}\left[\int_0^\infty I_{\{0\}}(\omega(s))\, ds\right] = 0. \tag{1.4}$$

A family $\{P_z : z \in C\}$ is a *solution of the submartingale problem* on C if for each $z \in C$, P_z is a solution to the submartingale problem on C starting from z. In this case, we say $Z(\cdot)$ together with $\{P_z : z \in C\}$ is an RBM in C. The process Z under P_z is an RBM in C starting from z. Any continuous process having the same law as Z under P_z is also called an RBM in C starting from z.

Let us briefly describe our method. Let $S = \{(x, y) : y \geq 0\}$ be the upper half plane, ∂S_1 the positive x-axis, ∂S_2 the negative x-axis, and let $\{G(t) : t \geq 0\}$ be RBM in S with constant angles of reflection θ_1 and θ_2 on ∂S_1 and ∂S_2 respectively. Here $\theta_1, \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The sign convention in this case is distinct from that described for C: now the angle of reflection is measured from the inward unit normal and is positive if and only if the associated direction points toward the origin. The process G is shown to exist for all values of θ_1 and θ_2 in Varadhan and Williams [8]. We will be more precise later. Let $\widetilde{\mathscr{F}} : S \to C$

be one-to-one, onto, continuous, and conformal on $S\backslash\{0\}$ with $\widetilde{\mathscr{F}}(0)=0$. It is reasonable to conjecture that if

$$(1.5) \qquad A(t)=\int_0^t |\widetilde{\mathscr{F}}'(G_u)|^2 I(G_u\neq 0)\,du<\infty \quad \text{a.s.}$$

then $\widetilde{Z}(\cdot)=\widetilde{\mathscr{F}}(G(A^{-1}(\cdot)))$ ought to have the law of an RBM in C with the desired reflection angles. Hence existence of RBM in C basically comes down to verifying (1.5). Our main result is the following theorem.

Theorem 1.1. *There is a unique solution of the submartingale problem on C if $\theta_1+\theta_2\le 0$. If $\theta_1+\theta_2>0$, there is no solution of the submartingale problem on C starting from any $z\in C$.*

The reason for nonexistence is that for $\theta_1+\theta_2>0$, the origin is hit with positive probability and then the process is forced to absorb there, contradicting (1.4). When $\theta_1+\theta_2\le 0$, $G(\cdot)$ started away from 0 will never hit 0, and hence once $\widetilde{Z}(\cdot)$ is away from 0 it never hits it again. As a by-product of our method we obtain a similar theorem for a large class of asymmetric cusps.

Theorem 1.2. *Consider the asymmetric cusp $C=\{(x,y)\colon x\ge 0,\ -x^\delta\le y\le x^\beta\}$ where $\beta>1$ and $\delta>2\beta-1$. There is a unique solution of the submartingale problem on C if $\theta_1+\theta_2\le 0$. If $\theta_1+\theta_2>0$, there is no solution of the submartingale problem on C starting from any $z\in C$.*

The question of existence and uniqueness is still open for the case $\delta\in(\beta, 2\beta-1]$. Unfortunately our method fails in this case. Indeed, it is possible to show that in this case the crucial Theorem 6.2 below is false, by the results of Warschawski [9].

The paper is organized as follows. In §2 we introduce the conformal transformation $\mathscr{F}$ and various properties are given with the proofs deferred to §6. In §3 we develop some preliminaries. Section 4 considers the existence and uniqueness when $\theta_1+\theta_2\le 0$ and in §5 we prove nonexistence for $\theta_1+\theta_2>0$. Section 7 is devoted to proving Theorem 6.2, a key technical result used in §6. We prove Theorems 1.1 and 1.2 simultaneously. Hence for the rest of this paper we will assume

$$C=\{(x,y)\colon x\ge 0,\ -x^\delta\le y\le x^\beta\},\qquad \beta>1,\ \delta=\beta \text{ or } \delta>2\beta-1,$$
$$\partial C_1=\{(x,y)\colon x\ge 0,\ y=-x^\delta\},\quad \partial C_2=\{(x,y)\colon x\ge 0,\ y=x^\beta\}.$$

Acknowledgment. We are indebted to Professor Ruth Williams for suggesting this problem to us. Secondly, she told us how to prove Theorem 3.2 and why we needed it in our proof of uniqueness in §4. Our original proof of nonexistence was rather lengthy and she suggested the method of Varadhan-Williams (1985) should work with much less work. Last but not least we are grateful to her for providing two weeks of support at UCSD for DeBlassie during August of 1990. We are grateful to the referee for the careful review and detailed suggestions for an improved presentation.

2. Properties of the conformal transformation

Since $\partial C\backslash\{0\}$ is smooth, it suffices to construct an RBM in C starting at 0 up to the first time it leaves a small neighborhood of 0 in C. Thus we only really care about C in a neighborhood of 0. Hence rather than map S onto C, we map a neighborhood of 0 in S onto a neighborhood of 0 in C. The next theorem describes properties of a particular choice of such a map. We defer the proof to §6.

For any $\varepsilon > 0$ and $x \in \mathbb{R}^2$ let $B_\varepsilon(x) = \{y \in \mathbb{R}^2 \colon |x-y| < \varepsilon\}$. For $B \subseteq \mathbb{R}^2$ and $n \geq 0$ an integer, let $C^n(B)$ denote the set of real-valued continuous functions that are defined and n times continuously differentiable on some domain containing B.

Theorem 2.1. *There exists an* $\varepsilon \in (0, 1)$, *a closed set* $H \subseteq B_1(0) \cap C$ *and a homeomorphism* $\mathscr{F} \colon S \cap \overline{B_\varepsilon(0)} \to H$ *that is conformal on* $S \cap \overline{B_\varepsilon(0)} \setminus \{0\}$ *and has inverse* F *that is conformal on* $H \setminus \{0\}$ *such that for some finite constant* K_1,

(i) $F(0) = 0$,
(ii) $0 \in \partial H$,
(iii) $H \cap B_s(0) = C \cap B_s(0)$ *for* s *sufficiently small,*
(iv) $|\mathscr{F}'(\zeta)| \leq K_1 |\zeta|^{-1}(-\ln|\zeta|)^{-\beta/(\beta-1)}$, $\zeta \in S \cap \overline{B_\varepsilon(0)} \setminus \{0\}$,
(v) $|F'(z)| \leq K_1$, $z \in H \setminus \{0\}$,
(vi) $|\frac{F'(z)}{F(z)}|(-\ln|F(z)|)^{-\beta/(\beta-1)} \leq K_1$, $z \in H \setminus \{0\}$.

Moreover, the components of $\mathscr{F}$ *are in* $C^2(S \cap \overline{B_\varepsilon(0)} \setminus \{0\})$.

3. Preliminaries

Let P_z be a solution of the submartingale problem on C starting from z and let Z be the canonical process on Ω_C. For $0 < s < \varepsilon$, define

$$\sigma_s = \inf\{t \geq 0 \colon Z(t) \in \{x \in H \colon |F(x)| = s\}\}. \tag{3.1}$$

Lemma 3.1. *If* $0 < |F(z)| < \varepsilon$ *and* $\theta_1 + \theta_2 \leq 0$ *then*

$$P_z(\sigma_0 \wedge \sigma_\varepsilon < \infty) = 1.$$

If $\theta_1 + \theta_2 > 0$ *then there exists* $\delta_1 \in (0, \varepsilon]$ *such that for* $0 < |F(z)| < \delta_1$,

$$P_z(\sigma_0 \wedge \sigma_{\delta_1} < \infty) = 1.$$

Proof. Define $v_j(0) = \lim_{\zeta \to 0} v_j(\zeta)$ along $\partial C_j \setminus \{0\}$, $j = 1, 2$. The angle $v_1(\zeta)$ makes with the positive x-axis (the angle measured as positive when taken in the counterclockwise sense) decreases continuously from $\frac{\pi}{2} + \theta_1$ towards θ_1 as ζ moves from $\{0\}$ outward along ∂C_1. Similarly, the angle $v_2(\zeta)$ makes with the positive x-axis increases continuously from $-(\frac{\pi}{2} + \theta_2)$ towards $-\theta_2$ as ζ moves from $\{0\}$ outward along ∂C_2.

First consider $\theta_1 + \theta_2 \leq 0$. If we regard $v_j(\zeta)$ ($\zeta \in \partial C_j$) as a vector starting at $\{0\}$, then

$$\{v_1(\zeta) \colon \zeta \in \partial C_1\} \subseteq \bigcup_{r \geq 0} \left\{re^{i\theta} \colon \theta_1 \leq \theta \leq \frac{\pi}{2} + \theta_1\right\} =: A_1,$$

$$\{v_2(\zeta) \colon \zeta \in \partial C_2\} \subseteq \bigcup_{r \geq 0} \left\{re^{i\theta} \colon -\left(\frac{\pi}{2} + \theta_2\right) \leq \theta \leq -\theta_2\right\} =: A_2.$$

Notice $z_j \in A_j \Rightarrow v_j(\zeta) \cdot z_j \geq 0$ for $\zeta \in \partial C_j$, $j = 1, 2$. Since $\theta_1 + \theta_2 \leq 0$, $A_1 \cap A_2 \setminus \{0\} \neq \varnothing$. Thus we can choose a unit vector $v \in A_1 \cap A_2 \setminus \{0\}$ satisfying

$$v \cdot v_j(\zeta) \geq 0, \qquad \zeta \in \partial C_j, \ j = 1, 2. \tag{3.2}$$

Next consider $\theta_1+\theta_2>0$. Then the smallest angle between $v_1(0)$ and $v_2(0)$ is strictly less than π. Let v be the unit vector bisecting the angle between $v_1(0)$ and $v_2(0)$. Then $v\cdot v_j(0)>0$ for $j=1,2$. By continuity, for some neighborhood N of 0 in $\mathbb{R}^2$ we have $v\cdot v_j(\zeta)>0$ for $\zeta\in N\cap\partial C_j$, $j=1,2$. Hence we can choose $\delta_1\in(0,\varepsilon]$ so small that

$$v\cdot v_j(\zeta)\geq 0,\qquad \zeta\in\{w\colon |F(w)|\leq\delta_1\}\cap\partial C_j,\ j=1,2. \tag{3.3}$$

This is possible because $\mathscr{F}$ is a homeomorphism.

Now we modify the proof of Lemma 2.1 on p. 411 of Varadhan and Williams [8] as follows. Since $\partial C\backslash\{0\}$ is smooth, for $z\neq 0$ the process $Z(\cdot\wedge\sigma_0)$ has a decomposition under P_z:

$$Z(t\wedge\sigma_0)=\begin{cases} B(t)+\int_0^t v_1(Z(s))dY_1(s)+\int_0^t v_2(Z(s))dY_2(s), & 0\leq t<\sigma_0,\\ 0, & t\geq\sigma_0\end{cases}$$

where $Z(t\wedge\sigma_0)\in C$ for all $t\geq 0$, B, Y_1, Y_2 are adapted to Z, $B(\cdot\wedge\sigma_0)$ is a two-dimensional martingale having mutual variation $\langle B_i(\cdot\wedge\sigma_0), B_j(\cdot\wedge\sigma_0)\rangle_t=\delta_{ij}(t\wedge\sigma_0)$ for $i,j=1,2$, and $B(0)=z$ P_z-a.s. Y_1 and Y_2 are continuous increasing processes with $Y_1(0)=Y_2(0)=0$, and $Y_1(t)$, $Y_2(t)$ are finite for all $t<\sigma_0$. Moreover, for each $j=1,2$, Y_j increases only when Z is on $\partial C_j\backslash\{0\}$.

Choose v as in (3.2) or (3.3) according to $\theta_1+\theta_2\leq 0$ or $\theta_1+\theta_2>0$, and observe that on $\{\sigma_0=\infty\}$ we have for

$$a=\begin{cases}\varepsilon, & \text{if } \theta_1+\theta_2\leq 0,\\ \delta_1, & \text{if } \theta_1+\theta_2>0\end{cases}\qquad \text{and}\quad 0<|F(z)|<a,$$

$$\sup_t|Z(t\wedge\sigma_a)|\geq\sup_t[Z(t\wedge\sigma_a)\cdot v]\geq\sup_t[B(t\wedge\sigma_a)\cdot v].$$

Since $\{\zeta\in C\colon |F(\zeta)|\leq a\}\subseteq\{\zeta\in C\colon |F(\zeta)|\leq\varepsilon\}\subseteq H\subseteq B_1(0)\cap C$ this yields

$$1\geq\sup_t[B(t\wedge\sigma_a)\cdot v],\qquad \text{a.s.}$$

By (null) recurrence of the one-dimensional Brownian motion $B(\cdot)\cdot v$,

$$\sup_t[B(t)\cdot v]=+\infty\quad \text{almost surely.}$$

Hence $\sigma_a<\infty$ a.s. on $\{\sigma_0=\infty\}$. $\square$

Theorem 3.2. *If* $0<|F(z)|<\varepsilon$ *and* $\theta_1+\theta_2\leq 0$ *then* $P_z(\sigma_0<\sigma_\varepsilon)=0$. *For* δ_1 *as in Lemma* 3.1, *if* $\theta_1+\theta_2>0$ *then there exists* $\delta_2\in(0,\delta_1)$ *such that*

$$\inf\{P_z(\sigma_0<\sigma_{\delta_1})\colon 0<|F(z)|<\delta_2\}>0.$$

Proof. Let $\alpha=(\theta_1+\theta_2)/\pi$ and in polar coordinates $\zeta=re^{i\theta}$, set

$$\Phi(r,\theta)=\begin{cases} r^\alpha\cos(\alpha\theta-\theta_1), & \alpha\neq 0,\\ \log r+\theta\tan\theta_1, & \alpha=0.\end{cases}$$

Then for some constant K_2 (see Varadhan and Williams [8]—here V_i is the direction of reflection on ∂S_i making angle θ_i relative to the inward unit normal N_i to ∂S_i with the sign convention on θ_i described in §1 where S was introduced)

$$\begin{aligned}
&\Phi \in C^2(S\backslash\{0\}), \\
&\Delta\Phi = 0 \quad \text{in } S\backslash\{0\}, \\
&V_i \cdot \nabla\Phi = 0 \quad \text{on } \partial S_i\backslash\{0\}, \; i = 1, 2, \\
&\Phi(\zeta) \geq K_2 r^\alpha \quad \text{if } \alpha \neq 0.
\end{aligned}$$

So if we define $\Psi(\zeta) = \Phi \circ F(\zeta)$, $\zeta \in H$, we see (cf. proof of Proposition 4.1 below) for some positive constants K_3 and K_4

$$\begin{aligned}
&\Psi \in C^2(H\backslash\{0\}), \\
&\Delta\Psi = 0 \quad \text{in } H\backslash\{0\}, \\
&v_i \cdot \nabla\Psi = 0 \quad \text{on } \partial C_i \cap H\backslash\{0\}, \; i = 1, 2, \\
&\Psi(\zeta) \geq K_3|F(\zeta)|^\alpha, \qquad \zeta \in H\backslash\{0\} \quad \text{if } \alpha \neq 0, \\
&\Psi(\zeta) \leq \log|F(\zeta)| + K_4 \qquad \zeta \in H\backslash\{0\} \quad \text{if } \alpha = 0.
\end{aligned}$$

Thus as in the proof of Theorem 2.2, p. 411 of Varadhan and Williams [8], for $0 < s \leq |F(z)| \leq \varepsilon$, by Lemma 3.1, optional stopping and dominated convergence, (recall $\alpha \leq 0$),

$$E_z[\Psi(Z(\sigma_s \wedge \sigma_\varepsilon))] = \Psi(z). \tag{3.4}$$

Hence for $\alpha < 0$ we get

$$\begin{aligned}
K_3 s^\alpha P_z(\sigma_s < \sigma_\varepsilon) &\leq E_z[\Psi(Z(\sigma_s))I(\sigma_s < \sigma_\varepsilon)] \\
&\leq E_z[\Psi(Z(\sigma_s \wedge \sigma_\varepsilon))] = \Psi(z)
\end{aligned}$$

or

$$P_z(\sigma_s < \sigma_\varepsilon) \leq K_3^{-1} s^{-\alpha}\Psi(z). \tag{3.5}$$

By Lemma 3.1, $\sigma_0 \wedge \sigma_\varepsilon < \infty$ a.s. so by continuity of paths, $\{\sigma_0 < \sigma_\varepsilon\} = \bigcap_{s>0}\{\sigma_s < \sigma_\varepsilon\}$ a.s. Hence letting $s \downarrow 0$ in (3.5), $P_z(\sigma_0 < \sigma_\varepsilon) = 0$ (since $\alpha < 0$) as desired.

When $\alpha = 0$ and s is small, from (3.4)

$$\Psi(z) = E_z[\Psi(Z(\sigma_s \wedge \sigma_\varepsilon))] \leq (\log s)P_z(\sigma_s < \sigma_\varepsilon) + \log\varepsilon + 2K_4$$

or (for $s < 1$)

$$P_z(\sigma_s < \sigma_\varepsilon) \leq (\log s)^{-1}[\Psi(z) - \log\varepsilon - 2K_4]$$

and upon letting $s \downarrow 0$, $P_z(\sigma_0 < \sigma_\varepsilon) = 0$ once again.

Finally, we consider $\alpha > 0$ (i.e., $\theta_1 + \theta_2 > 0$). Choose $\delta_3 > 0$ such that for $\delta_1 \in (0, \varepsilon]$ as in Lemma 3.1

$$\{\zeta: \Psi(\zeta) \leq \delta_3\} \subseteq \{\zeta: |F(\zeta)| < \delta_1\}. \tag{3.6}$$

This is possible because for some positive k, $\Phi(\zeta) \geq k|\zeta|^\alpha$, $\zeta \in S$. Then choose $\delta_2 > 0$ so that

$$\delta_2 < \min(\delta, (\tfrac{1}{2}\delta_3)^{1/\alpha}) \tag{3.7}$$

and

$$\{\zeta: |F(\zeta)| \leq \delta_2\} \subseteq \{\zeta: \Psi(\zeta) < \delta_3\}. \tag{3.8}$$

The latter is possible because $\alpha > 0$ and $\Psi(\zeta) \le |F(\zeta)|^\alpha$ for $\zeta \in H$.

For $r \le \delta_3$ define

$$\mu_r = \inf\{t \ge 0 \colon \Psi(Z_t) = r\}.$$

Then $\mu_0 = \sigma_0$ and, as in (3.4) above, for z satisfying $|F(z)| < \delta_2$ and $\Psi(z) > s$,

$$E_z[\Psi(Z(\mu_s \wedge \mu_{\delta_3}))] = \Psi(z).$$

Thus $sP_z(\mu_s < \mu_{\delta_3}) + \delta_3(1 - P_z(\mu_s < \mu_{\delta_3})) = \Psi(z)$, which yields

$$P_z(\mu_s < \mu_{\delta_3}) = \frac{\Psi(z) - \delta_3}{s - \delta_3}. \tag{3.9}$$

For z satisfying $0 < |F(z)| < \delta_2$ we have

$$P_z(\mu_0 \wedge \mu_{\delta_3} < \infty) \ge P_z(\sigma_0 \wedge \sigma_\delta < \infty) = 1$$

by (3.6) and Lemma 3.1. Then by path continuity,

$$\{\mu_0 < \mu_{\delta_3}\} = \bigcap_{s>0}\{\mu_s < \mu_{\delta_3}\} \quad \text{a.s. } P_z.$$

Hence upon letting $s \downarrow 0$ in (3.9) we get

$$P_z(\mu_0 < \mu_{\delta_3}) = 1 - \frac{\Psi(z)}{\delta_3}, \qquad 0 < |F(z)| < \delta_2.$$

But for such z, by (3.7) and that $\alpha > 0$, $\Psi(z) \le |F(z)|^\alpha < \delta_2^\alpha < \frac{1}{2}\delta_3$. Hence

$$\inf\{P_z(\mu_0 < \mu_{\delta_3}) \colon 0 < |F(z)| < \delta_2\} \ge \tfrac{1}{2}.$$

To finish, just observe for $0 < |F(z)| < \delta_2$ we have by (3.6)

$$P_z(\sigma_0 < \sigma_{\delta_1}) \ge P_z(\mu_0 < \mu_{\delta_3}). \quad \square$$

4. Existence and uniqueness for $\theta_1 + \theta_2 \le 0$

Recall $\beta > 1$ and $\delta = \beta$ or $\delta > 2\beta - 1$. Throughout this section we assume $\theta_1 + \theta_2 \le 0$. By Theorem 3.2 it suffices to consider an RBM in C starting from 0. As in the introduction, let $S = \{(x, y) \colon y \ge 0\}$ be the closed upper half-plane. Set $\partial S_1 = \{(x, 0) \colon x \ge 0\}$ and $\partial S_2 = \{(x, 0) \colon x \le 0\}$. Let N_j be the unit inward normal to ∂S_j, and let V_j make angle $\theta_j \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with N_j, $j = 1, 2$. Here V_j points toward 0 iff $\theta_j > 0$. We take $V_j \cdot N_j = 1$. Let $\{G(t) \colon t \ge 0\}$ be a realization of a reflecting Brownian motion (RBM) in S starting at $0 \in S$ with directions of reflection V_1, V_2 on $\partial S_1 \backslash \{0\}$, $\partial S_2 \backslash \{0\}$, respectively. More precisely, on some filtered space $(\Omega, \mathscr{S}, \{\mathscr{S}_t\}, Q)$ there is defined a continuous $\{\mathscr{S}_t\}$-adapted process G such that

(i) $Q(G_u \in S \ \forall u \ge 0) = 1 = Q(G_0 = 0)$;

(ii) for every $f \in C_b^2(S)$ satisfying $V_j \cdot \nabla f \ge 0$ on ∂S_j and $f \equiv$ constant near 0,

$$f(G_t) - \int_0^t \frac{1}{2}\Delta f(G_u)\,du \quad \text{is an } \{\mathscr{S}_t\}\text{-submartingale};$$

(iii) $E^Q[\int_0^\infty I(G_u = 0)\,du] = 0$.

Varadhan and Williams [8] have proved existence and uniqueness in law of G on the space of continuous paths in S. For any number $s > 0$ set

$$\eta_s = \eta_s(G) = \inf\{t > 0\colon |G_t| = s\}$$

and define (here ε is from Theorem 2.1)

$$A^G(t) = \int_0^{t\wedge\eta_\varepsilon} |\mathscr{F}'(G_u)|^2 I(G_u \neq 0)\,du. \tag{4.1}$$

If

$$A^G(\eta_\varepsilon(G)) < \infty \tag{4.2}$$

then A^G is continuous (hence finite) on $[0, \eta_\varepsilon]$. Since $\mathscr{F}$ is conformal away from 0 (so $\mathscr{F}' \neq 0$ there) and G does not spend positive time at 0 a.s. (Q), under (4.2) A^G is strictly increasing on $[0, \eta_\varepsilon]$. Then A^G on $[0, \eta_\varepsilon]$ has a strictly increasing continuous inverse a_t^G defined on $[0, A^G(\eta_\varepsilon)]$. Define

$$Z^G(t) = \mathscr{F}(G(a^G(t \wedge A^G(\eta_\varepsilon)))), \quad \text{for } t \geq 0. \tag{4.3}$$

We will sometimes drop the superscript G in the notation.

Recall that we need only construct an RBM starting from 0 until it leaves a small neighborhood of 0 in C. Thus for existence we need only verify (1.2)–(1.4) for the law of Z^G on $(\Omega_C, \mathscr{M})$ with t replaced by $t \wedge \sigma_\varepsilon$ where

$$\sigma_\varepsilon(\omega) = \sigma_\varepsilon = \inf\{t > 0\colon |F(\omega_t)| = \varepsilon\}.$$

If this is true, we say $Z^G(\cdot)$ is *an RBM in* C, *stopped at time* σ_ε, *starting at* 0. We will usually not include the statement "starting at 0".

Proposition 4.1. *If* $A(\eta_\varepsilon) < \infty$ *then* $Z^G(\cdot)$ *given by* (4.3) *is an RBM in* C *stopped at time* σ_ε.

Proof. Since $F(0) = 0$,

$$E^Q[I(Z_0^G = 0)] = E^Q[I(G_0 = 0)] = 1.$$

Also, since $F(H) = S \cap \overline{B_\varepsilon(0)}$, $\sigma_\varepsilon = A(\eta_\varepsilon)$ and so

$$\begin{aligned} E^Q\left[\int_0^{\sigma_\varepsilon} I(Z_s^G = 0)\,ds\right] &= E^Q\left[\int_0^{\sigma_\varepsilon} I(\mathscr{F}(G(a_s)) = 0)\,ds\right] \\ &= E^Q\left[\int_0^{\sigma_\varepsilon} I(G(a_s) = 0)\,ds\right] \\ &= 0 \end{aligned}$$

since $a(\cdot)$ is strictly increasing and $E^Q[\int_0^\infty I(G_s = 0)\,ds] = 0$.

To show (1.3) suppose $f \in C_b^2(C)$ is constant in a neighborhood of 0 and $v_i \cdot \nabla f \geq 0$ on ∂C_i, $i = 1, 2$. Let $h(\zeta) = f \circ \mathscr{F}(\zeta)$, $\zeta \in S \cap \overline{B_\varepsilon(0)}$. Then h is constant in a neighborhood of 0 and $h \in C_b^2(S \cap \overline{B_\varepsilon(0)})$.

Let

$$J(z) = \begin{pmatrix} \dfrac{\partial F_1}{\partial z_1} & \dfrac{\partial F_1}{\partial z_2} \\[2ex] \dfrac{\partial F_2}{\partial z_1} & \dfrac{\partial F_2}{\partial z_2} \end{pmatrix}, \qquad z = (z_1, z_2),$$

be the Jacobian of F. By conformality, for some positive $c_j(z)$,

$$V_j = c_j(z)J(z)v_j(z), \qquad z \in (\partial C_j \backslash \{0\}) \cap H, \ j = 1, 2.$$

Also, for $\zeta = F(z)$ and $\mathscr{J}(\zeta)$ the Jacobian of $\mathscr{F}$ at ζ,

$$\nabla h(\zeta) = \nabla(f \circ \mathscr{F})(\zeta) = \mathscr{J}(\zeta)^*(\nabla f)(\mathscr{F}(\zeta)) = (J^{-1}(z))^*(\nabla f)(z).$$

Thus for $\zeta \in (\partial S_j \backslash \{0\}) \cap \overline{B_\varepsilon(0)}$, we have $z = \mathscr{F}(\zeta) \in (\partial C_j \backslash \{0\}) \cap H$ and

$$\begin{aligned}\nabla h(\zeta) \cdot V_j &= (\nabla h(\zeta))^* V_j \quad \text{(as matrices)} \\ &= [((\nabla f)(z))^* J^{-1}(z)] c_j(z) J(z) v_j(z) \\ &= c_j(z) \nabla f(z) \cdot v_j(z) \geq 0.\end{aligned}$$

Thus

$$\begin{aligned}&f(Z^G_{t\wedge\sigma_\varepsilon}) - \frac{1}{2}\int_0^{t\wedge\sigma_\varepsilon} \Delta f(Z^G_s)\, ds \\ &= h(G_{a(t\wedge\sigma_\varepsilon)}) - \frac{1}{2}\int_0^{t\wedge\sigma_\varepsilon} \Delta(h \circ F)(\mathscr{F}(G_{a(s)}))\, ds \\ &= h(G_{a(t\wedge\sigma_\varepsilon)}) - \frac{1}{2}\int_0^{t\wedge\sigma_\varepsilon} \Delta h(G_{a(s)})|F'(\mathscr{F}(G_{a(s)}))|^2\, ds \\ &= h(G_{a(t\wedge\sigma_\varepsilon)}) - \frac{1}{2}\int_0^{a(t\wedge\sigma_\varepsilon)} \Delta h(G_u) I(G_u \neq 0)\, du \\ &= h(G_{a(t\wedge\sigma_\varepsilon)}) - \frac{1}{2}\int_0^{a(t\wedge\sigma_\varepsilon)} \Delta h(G_s)\, ds\end{aligned}$$

is a submartingale. □

We can reverse the preceding procedure. Indeed, given a realization $Z(\cdot)$ of RBM in C stopped at time $\sigma_\varepsilon(Z)$ starting from 0, since $\theta_1 + \theta_2 \leq 0$, Lemma 3.1 and Theorem 3.2 imply $\sigma_\varepsilon(Z) < \infty$ a.s. Set

$$V^Z(t) = \int_0^{t\wedge\sigma_\varepsilon(Z)} |F'(Z_u)|^2 I(Z_u \neq 0)\, du, \qquad t \geq 0. \tag{4.4}$$

By Theorem 2.1, for some constant K, $V^Z(t) \leq Kt$ for all $t \geq 0$ and V^Z is continuous and strictly increasing on $[0, \sigma_\varepsilon(Z)]$. Hence it has a continuous strictly increasing inverse $v^Z(t)$ on $[0, V^Z(\sigma_\varepsilon(Z))]$. Define

$$G^Z(t) = F(Z(v^Z(t \wedge V^Z(\sigma_\varepsilon(Z))))), \qquad t \geq 0. \tag{4.5}$$

Similar to Proposition 4.1, G^Z is an RBM in S stopped at $V^Z(\sigma_\varepsilon(Z))$.

We now use the above to show existence and uniqueness of RBM in C stopped at $\sigma_\varepsilon(Z)$.

Theorem 4.2. *Let G be an RBM in S starting at 0. If $\theta_1 + \theta_2 \leq 0$ then the condition*

$$\int_0^{\eta_\varepsilon(G)} |G_u|^{-2} |\ln|G_u||^{-\frac{2\beta}{\beta-1}} I(G_u \neq 0)\, du < \infty \tag{4.6}$$

is sufficient for existence and uniqueness in law of RBM Z in C stopped at time $\sigma_\varepsilon(Z)$.

Proof. Assuming (4.6) holds, then by Theorem 2.1(iv), $A^G(\eta_\varepsilon(G)) < \infty$. Hence existence follows from Proposition 4.1.

Next we consider uniqueness. Observe

$$\left\{ \begin{array}{l} V^Z(\sigma_\varepsilon(Z)) = \eta_\varepsilon(G^Z), \\ A^G(\eta_\varepsilon(G)) = \sigma_\varepsilon(Z^G). \end{array} \right. \tag{4.7}$$

Then since the law of RBM on S stopped at η_ε is unique, we need only show that the associations

$$\begin{aligned} &Z(\cdot \wedge \sigma_\varepsilon(Z)) \text{ RBM in } C \text{ stopped at } \sigma_\varepsilon(Z) \\ &\qquad \to G^Z(\cdot \wedge V^Z(\sigma_\varepsilon(Z))) \text{ RBM in } S \text{ stopped at } \eta_\varepsilon(G^Z) \end{aligned}$$

and

$$\begin{aligned} &G(\cdot \wedge \sigma_\varepsilon(G)) \text{ RBM in } S \text{ stopped at } \eta_\varepsilon(Z) \\ &\qquad \to Z^G(\cdot \wedge A^G(\eta_\varepsilon(G))) \text{ RBM in } C \text{ stopped at } \sigma_\varepsilon(Z^G) \end{aligned}$$

form a one-to-one correspondence. For this it suffices to show

$$\sigma_\varepsilon(Z^{(G^Z)}) = \sigma_\varepsilon(Z), \qquad Z^{(G^Z)}(\cdot) = Z(\cdot \wedge \sigma_\varepsilon(Z)), \tag{4.8}$$

and

$$\eta_\varepsilon(G^{(Z^G)}) = \eta_\varepsilon(G), \qquad G^{(Z^G)}(\cdot) = G(\cdot \wedge \eta_\varepsilon(G)). \tag{4.9}$$

We only prove (4.8), the proof of (4.9) being similar.

Observe first that

$$A^{(G^Z)}(V^Z(t)) = t, \qquad t \leq \sigma_\varepsilon(Z). \tag{4.10}$$

Indeed, for such t, $V^Z(t) \leq V^Z(\sigma_\varepsilon(Z)) = \eta_\varepsilon(G^Z)$ (by (4.7)) so the left side is well defined and by (4.1) is

$$\begin{aligned} &\int_0^{V^Z(t)} |\mathscr{F}'(G_u^Z)|^2 I(G_u^Z \neq 0)\, du \\ &\quad = \int_0^{V^Z(t)} |\mathscr{F}' \circ F(Z(v_u^Z))|^2 I(Z(v_u^Z) \neq 0)\, du \quad \text{(by (4.5))} \\ &\quad = \int_0^t |\mathscr{F}' \circ F(Z_s)|^2 I(Z_s \neq 0) |F'(Z_s)|^2\, ds \quad \text{(by (4.4))} \\ &\quad = \int_0^t I(Z_s \neq 0)\, ds \\ &\quad = t. \end{aligned}$$

Applying this, by (4.7),

$$\sigma_\varepsilon(Z^{(G^Z)}) = A^{(G^Z)}(\eta_\varepsilon(G^Z)) = A^{(G^Z)}(V^Z(\sigma_\varepsilon(Z))) = \sigma_\varepsilon(Z),$$

giving the first part of (4.8).

For the latter part, by (4.10), $v^Z(a^{(G^Z)}(t)) = t$ for $t \leq \sigma_\varepsilon(Z)$ and so

$$\begin{aligned} Z^{(G^Z)}(t) &= \mathscr{F}(G^Z\{a^{G^Z}(t \wedge A^{G^Z}(\eta_\varepsilon(G^Z)))\}) \quad \text{by (4.3)} \\ &= \mathscr{F}(G^Z\{a^{G^Z}(t \wedge \sigma_\varepsilon(Z^{G^Z}))\}) \quad \text{by (4.7)} \\ &= \mathscr{F}(G^Z\{a^{G^Z}(t \wedge \sigma_\varepsilon(Z))\}) \quad \text{(first part of (4.8))} \\ &= Z(v^Z\{[a^{G^Z}(t \wedge \sigma_\varepsilon(Z))] \wedge V^Z(\sigma_\varepsilon(Z))\}) \quad \text{by (4.5)} \\ &= Z(t \wedge \sigma_\varepsilon(Z)) \end{aligned}$$

as desired. □

By Theorem 4.2, to prove existence and uniqueness of RBM for $\theta_1+\theta_2\le 0$, it suffices to verify (4.6). For this we need the following theorem.

Theorem 4.3. *Let* $h\in C(S\cap\overline{B_\varepsilon(0)})\cap C^2(S\cap\overline{B_\varepsilon\{0\}}\backslash\{0\})$ *with* $h(0)=0$, $\Delta h\ge 0$ *on* $S\backslash\{0\}$, *and* $V_j\cdot\nabla h\ge 0$ *on* $(\partial S_j)\cap\overline{B_\varepsilon(0)}\backslash\{0\}$, $j=1,2$. *Then*

$$E^Q\left[\int_0^{t\wedge\eta_\varepsilon}\Delta h(G_s)I_{\{h(G_s)>0\}}ds\right]<\infty.$$

Proof. Suppose $g_n\colon\mathbb{R}\to[0,\infty)$ is continuous with $\operatorname{supp}g_n\subseteq[\frac{1}{3n},\frac{1}{n}]$ and $\int g_n(r)\,dr=1$. Let

$$k_n(t)=\int_0^t\int_0^s g_n(r)\,dr\,ds.$$

Then $k_n\in C^2(\mathbb{R})$, k_n vanishes in a neighborhood of 0 and $|k_n(t)-t\vee 0|\le\frac{A}{n}$ for some constant $A>0$. Moreover, $0\le k_n'\le 1$, $k_n''\ge 0$ and as $n\to\infty$, $k_n'(t)\to I_{(0,\infty)}(t)$.

With h as above, it follows that $k_n\circ h\in C_b^2(S\cap\overline{B_\varepsilon(0)})$. Now $k_n\circ h$ can be extended outside of $S\cap\overline{B_\varepsilon(0)}$ so that it is in $C_b^2(S)$ and satisfies $V_j\cdot\nabla(k_n\circ h)\ge 0$ on ∂S_j, $j=1,2$. Such extensions are made in Varadhan and Williams [8]. Therefore by the submartingale property

$$\begin{aligned}&E^Q[k_n\circ h(G_{t\wedge\eta_\varepsilon})-k_n\circ h(z)]\\&\quad\ge\frac{1}{2}E^Q\left[\int_0^{t\wedge\eta_\varepsilon}[\Delta h(G_s)k_n'\circ h(G_s)+|\nabla h(G_s)|^2k_n''\circ h(G_s)]\,ds\right].\end{aligned}$$

Note that since h is bounded on $S\cap\overline{B_\varepsilon(0)}$, as $n\to\infty$ the left hand side converges to $E^Q[0\vee h(G_{t\wedge\eta_\varepsilon})]-0\vee h(z)$. By hypothesis, all integrands on the right are nonnegative. Hence by Fatou's lemma, since k_n' converges to $I_{(0,\infty)}$,

$$E^Q\left[\int_0^{t\wedge\eta_\varepsilon}\Delta h(G_s)I(h(G_s)>0)\,ds\right]<\infty$$

is desired. □

Now we specialize this to verify (4.6). In polar coordinates (r,θ) for $z=re^{i\theta}$ let

$$h(z)=\begin{cases}[-\ln r-\theta\tan\theta_1]^{-\frac{2}{\beta-1}}, & z\in\overline{B_\varepsilon(0)}\cap S\backslash\{0\},\\ 0, & r=0,\end{cases}$$

where β is the same number used to define the cusp C. By making ε from Theorem 2.1 smaller if necessary, for $r\le\varepsilon$ we have $h\ge 0$, $h(0)=0$, $h\in C(S\cap\overline{B_\varepsilon(0)})\cap C^2(S\cap\overline{B_\varepsilon(0)}\backslash\{0\})$ and

$$\Delta h=\frac{2}{\beta-1}r^{-2}[-\ln r-\theta\tan\theta_1]^{-\frac{2\beta}{\beta-1}}\left(\frac{2}{\beta-1}+1\right)(1+\tan^2\theta_1).$$

Then $\Delta h\ge 0$ on $S\cap\overline{B_\varepsilon(0)}\backslash\{0\}$ and $\Delta h\ge Kr^{-2}[-\ln r]^{-\frac{2\beta}{\beta-1}}$ for $0<r\le\varepsilon$ and some constant $K>0$ depending on ε, β and $\tan|\theta_1|$. In polar coordinates $\nabla h=(\frac{\partial h}{\partial r},\frac{1}{r}\frac{\partial h}{\partial\theta})$ so

$$\nabla h=\frac{2}{\beta-1}(-\ln r-\theta\tan\theta_1)^{-\frac{2}{\beta-1}-1}\left(\frac{1}{r},\frac{1}{r}\tan\theta_1\right);$$

also,

$$V_1 = (-\tan\theta_1, 1) \quad \text{and} \quad V_2 = (-\tan\theta_2, -1).$$

It is easily verified that $V_j \cdot \nabla h \geq 0$ on $(\partial S_j \backslash \{0\}) \cap \overline{B_\varepsilon(0)}$, $j = 1, 2$, because $\theta_1 + \theta_2 \leq 0$. Therefore h satisfies the conditions of Theorem 4.3 on $\overline{B_\varepsilon(0)} \cap S$ and

$$\begin{aligned} E^Q &\left[\int_0^{t\wedge\eta_\varepsilon} |G_s|^{-2} |\ln|G_s||^{-\frac{2\beta}{\beta-1}} I_{\{G_s\neq 0\}}\, ds\right] \\ &\leq K^{-1} E^Q \left[\int_0^{t\wedge\eta_\varepsilon} \Delta h(G_s) I_{\{h(G_s)>0\}}\, ds\right] < \infty \end{aligned}$$

as desired.

Remark. If $\theta_1 + \theta_2 \leq 0$ then by Lemma 3.1 and Theorem 3.2 an RBM in C starting away from 0 never hits 0. Of course, the preceding calculations show the process can be *started* from 0.

5. RBM in C does not exist if $\theta_1 + \theta_2 > 0$

Throughout this section we take $\theta_1 + \theta_2 > 0$.

Theorem 5.1. *Suppose $\theta_1 + \theta_2 > 0$. Then for any $z \in C$, there is no solution to the submartingale problem on C starting from z.*

First we prove the following special case.

Theorem 5.2. *Suppose $\theta_1 + \theta_2 > 0$. Then there exists no solution to the submartingale problem on C starting from 0.*

Proof. Suppose there exists a process Z with law P_0 on $(\Omega_C, \mathscr{M})$ satisfying (1.2) and (1.3). We will show below that (1.4) cannot also hold for this process and therefore an RBM in C starting from 0 does not exist when $\theta_1 + \theta_2 > 0$.

Let α, $\Phi(r, \theta)$ and $\Psi(Z)$ be defined as in the proof of Theorem 3.2. Let ε be as in Theorem 2.1 and set $c = \cos\theta_1 \wedge \cos\theta_2$; note $c > 0$. Define $K = c(\varepsilon/2)^\alpha$ and let

$$\mu_K = \inf\{t > 0 \colon \Psi(Z_t) = K\}.$$

We will prove below that

$$P_0[w(t \wedge \mu_K) = 0 \text{ for all } t] = 1. \tag{5.1}$$

Because $\Phi(r, \theta) > 0$ for $(r, \theta) \in S \cap B_\varepsilon(0)\backslash\{0\}$ and $\Phi(0, \theta) = 0$ when $\theta_1 + \theta_2 > 0$, by continuity of Φ, there exists an open neighborhood J of $\{0\}$ in H for which $\Psi(z) < K$ for all $z \in J$. Therefore continuity of paths and (1.2) imply $\mu_K > 0$ a.s. so (5.1) and (1.4) cannot both hold.

Let $g \colon [0, \infty) \to [0, 1]$ be a twice continuously differentiable function such that

$$g(x) = \begin{cases} 0, & 0 \leq x \leq 1/2, \\ 1, & x \geq 1 \end{cases}$$

and let $M = \max_{x\in[0,\infty)}\{|g'(x)| + |g''(x)|\}$. For each $s > 0$ with $s \leq K$ and $z \in H$ define

$$\begin{aligned} h_s(z) &= g(s^{-1}\Psi(z)); \\ f_s(z) &= \begin{cases} h_s(z)(1 - h_{2K}(z))\Psi(z), & 2K \geq \Psi(z) \geq s/4, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Because F is conformal in $H\backslash\{0\}$, f_s is well defined as an element of $C_b^2(H)$, constant in a neighborhood of $\{0\}$ and $\nabla f_s \cdot v_j = 0$ on $\overline{B_\varepsilon(0)} \cap \partial C_j$. Moreover $\Delta\Psi = ((\Delta\Phi)(F))|F'|^2$ on $H\backslash\{0\}$ and since $\Delta\Phi = 0$ in $S\backslash\{0\}$ it is easy to show for $z \in H$,

$$\Delta f_s(z) = \begin{cases} 0, & 0 \le \Psi \le s/2, \\ (s^{-2}g''\Psi + 2s^{-1}g')|(\nabla\Phi)(F)|^2|F'|^2(z), & s/2 \le \Psi \le s, \\ 0, & s \le \Psi \le K. \end{cases}$$

By Doob's stopping theorem and (1.3), with equality,

$$f_s(w(t \wedge \mu_K)) - \frac{1}{2}\int_0^{t\wedge\mu_K} \Delta f_s(w(u))\,du \tag{5.2}$$

is a P^0 martingale. Also on $\{z \in C\colon 0 \le \Psi(z) \le K\}$, $f_s \to \Psi$ uniformly and $\Delta f_s \to 0$ as $s \downarrow 0$. By (2.31) of Varadhan and Williams [8] there is a constant $b < \infty$ such that $|(\nabla\Phi)(F)|^2 \le b\Psi^{2-2/\alpha}$ on $H\backslash\{0\}$. By Theorem 2.1vi), on $H\backslash\{0\}$,

$$|F'(z)| \le K_1|F(z)|(-\ln|F(z)|)^{\beta/(\beta-1)}.$$

But on $\{z\colon s/2 \le \Psi \le s\}$, $|F(z)| \le (\frac{s}{c})^{1/\alpha}$ so for $|z|$ sufficiently small, since $x(-\ln x)^{\beta/(\beta-1)}$ is an increasing function for small positive x,

$$|F'(z)|^2 \le K_1^2\left(\frac{s}{c}\right)^{2/\alpha}\left(-\frac{1}{\alpha}\ln\left|\frac{s}{c}\right|\right)^{2\beta/(\beta-1)}.$$

Thus on $\{z \in C\colon 0 \le \Psi(z) \le K\}$,

$$\begin{aligned} |\Delta f_s| &\le K_5\{(s^{-2}M)s + 2s^{-1}M\}s^{2-2/\alpha}s^{2/\alpha}\left(-\frac{1}{\alpha}\ln\left|\frac{s}{c}\right|\right)^{2\beta/(\beta-1)} \\ &= K_6 s\left(-\frac{1}{\alpha}\ln\left|\frac{s}{c}\right|\right)^{2\beta/(\beta-1)} \end{aligned}$$

which is bounded uniformly in s on $\{z \in C\colon 0 \le \Psi(z) \le K\}$. Because martingales are preserved under bounded convergence at each time t, by letting $s \downarrow 0$ in (5.2) we find $\Psi(w(t \wedge \mu_K))$ is a P_0 martingale. Hence

$$E_0[\Psi(w(t \wedge \mu_K))] = \Psi(0) = 0$$

for all $t \ge 0$. Since $\Psi > 0$ on $C\backslash\{0\}$ this implies (5.1) and we are done. □

On Ω_C, for $Z(\cdot)$ the coordinate process and $r \ge 0$, define

$$\tau_r = \tau_r(Z) = \inf\{t \ge 0\colon |Z(t)| = r\}.$$

The proof of the following theorem is much like that of Theorem 2.1 on p. 410 of Varadhan-Williams [8].

Theorem 5.3. *For each $z \in C$ there is a unique probability measure P_z^0 on Ω_C such that*

(i) $P_z^0(Z(0) = z) = 1$;

(ii) *for each $f \in C_b^2(C)$ satisfying $v_j \cdot \nabla f \ge 0$ on ∂C_j, $j = 1, 2$,*

$$f(Z(t \wedge \tau_0)) - \frac{1}{2}\int_0^{t\wedge\tau_0} (\Delta f)(Z(s))\,ds$$

is a P_z^0-submartingale;

(iii) $P_z^0(Z(t) = 0\ \forall t \ge \tau_0) = 1$.

Moreover, the family $\{P_z^0 \colon z \in C\}$ *has the strong Markov property.*

In light of Theorem 5.2, Theorem 5.1 is a consequence of the following result.

Theorem 5.4. *For* $z \in C \backslash \{0\}$, $P_z^0(\tau_0 < \infty) > 0$.

Proof. Let δ_1 and δ_2 be from Lemma 3.1 and Theorem 3.2, respectively. Thus $0 < \delta_2 < \delta_1 \leq \varepsilon$. If z satisfies $0 < |F(z)| < \delta_2$, then by Theorem 3.2

$$0 < P_z(\sigma_0 < \sigma_{\delta_1}) = P_z^0(\sigma_0 < \sigma_{\delta_1}) \leq P_z^0(\tau_0 < \infty), \tag{5.3}$$

as desired. Define $B = \{\zeta \in C \colon 0 < |F(\zeta)| < \frac{1}{2}\delta_2\}$ and $\tau_B = \inf\{t \geq 0 \colon Z_t \in B\}$. Then for each $z \in C \backslash \{0\}$, $P_z^0(\tau_B < \infty) > 0$. Indeed, for z in the interior of C, this follows from properties of ordinary Brownian motion in C; for z in the boundary of ∂C excluding the origin, reflecting Brownian motion will enter the interior of C immediately and from there will have positive probability of hitting B. Together with (5.3) and the strong Markov property, this yields the desired conclusion for z satisfying $|F(z)| \geq \delta_2$. □

6. Proof of Theorem 2.1

Our plan of action is to define a function F explicitly enough to prove it has the properties stated in Theorem 2.1. In essence we do the following. Open the cusp via a conformal mapping to locally flatten the boundary near 0 so that the tangent is well defined there. Then conformally map to the unit disc; from there map conformally to the upper half-space S. This seems convoluted because Warschawski [9] does have results on conformal maps from a cusp to the unit disc. The trouble is, the results are too coarse for our purpose. On the other hand, he has results on conformal maps from domains with continuously turning tangents to the unit disc and they *are* good enough for us.

First we open the cusp. Below in (7.1) we give an explicit parametric representation of ∂C_1 and ∂C_2 in polar coordinates. With this in mind, using the principal branch of $\ln z$, set

$$f_\delta(z) = \begin{cases} 0, & z = 0, \\ \exp\left\{-\frac{\pi}{2(\beta-1)} \exp\{-(\beta-1)\ln z\}\right\}, & z \neq 0,\ \delta = \beta, \\ \exp\left\{-\frac{\pi}{(\beta-1)} \exp\{-(\beta-1)\ln z\} - i\frac{\pi}{2}\right\}, & z \neq 0,\ \delta > \beta. \end{cases} \tag{6.1}$$

We will show below ((7.2)–(7.3) and Lemma 7.3) that f_δ does indeed open the cusp. Now for $\gamma \in (0, 1)$ sufficiently small, f is well defined and one-to-one on $N = \{z \colon |z| \leq \gamma,\ \operatorname{Re} z \geq 0\} \cap C$. Moreover, f_δ and f_δ^{-1}, are conformal on $N \backslash \{0\}$ and $f(N \backslash \{0\})$, respectively, with

$$f_\delta^{-1}(w) = \begin{cases} 0, & w = 0, \\ \exp\left\{-\frac{1}{\beta-1} \ln\left(-\frac{2(\beta-1)}{\pi} \ln w\right)\right\}, & w \neq 0,\ \delta = \beta, \\ \exp\left\{-\frac{1}{\beta-1} \ln\left(-\frac{\beta-1}{\pi}\left[\ln w + i\frac{\pi}{2}\right]\right)\right\}, & w \neq 0,\ \delta > \beta. \end{cases} \tag{6.2}$$

Notice

$$f_\delta(z) = e^{-i\pi/2}[f_\beta(z)]^2 \quad \text{for } \delta > \beta. \tag{6.3}$$

In polar coordinates $z = re^{i\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$(6.4)\qquad \begin{cases} f_\beta(z) = \exp\left\{-\frac{\pi}{2(\beta-1)} r^{-(\beta-1)} \cos[(\beta-1)\theta]\right\} e^{i\varphi_\beta}, \\ f_\delta(z) = |f_\beta(z)|^2 e^{i\varphi_\delta}, \qquad \delta > \beta, \end{cases}$$

where

$$(6.5)\qquad \begin{cases} \varphi_\beta = \frac{\pi}{2(\beta-1)} r^{-(\beta-1)} \sin[(\beta-1)\theta], \\ \varphi_\delta = 2\varphi_\beta - \frac{\pi}{2}, \quad \delta > \beta. \end{cases}$$

We leave the proof of the following elementary lemma to the reader.

Lemma 6.1. *The following bounds hold*

$$(6.6)\qquad |f_\delta'(z)| \leq C|f_\delta(z)|\,|z|^{-\beta}, \qquad z \in N\backslash\{0\},$$

$$(6.7)\qquad |(f_\delta^{-1})'(w)| \leq C|w|^{-1}(-\ln|w|)^{-\frac{\beta}{\beta-1}}, \qquad w \in f(N\backslash\{0\}).$$

Consider the set $C \cap \{z: |z| \leq \gamma/2\}$. Smooth out the corners located on $\{z: |z| = \gamma/2\}$ and call the resulting set $\widetilde{H}$. Define $R_\delta = f_\delta(\widetilde{H})$.

Let g_δ be the one-to-one conformal mapping (with conformal inverse g_δ^{-1}) taking $R_\delta\backslash\{0\}$ onto $D\backslash\{0\}$ where $D = \{\zeta: |\zeta - 1| \leq 1\}$ with $g_\delta(0) = 0$. Such a map exists because $\partial R_\delta\backslash\{0\}$ is C^∞. Moreover, g_δ and g_δ^{-1} are continuous on R_δ and D, respectively. The proof of the following theorem about g_δ' is very technical, so we leave it for the next section.

Theorem 6.2. *If $\delta = \beta$ or if $\delta > 2\beta - 1$ then g_δ' on $R_\delta\backslash\{0\}$ has a continuous extension to R_δ and $g_\delta' \neq 0$ on R_δ. In particular, $(g_\delta^{-1})'$ is bounded on D and g_δ' is bounded on R_δ.*

Remark. When $\delta > \beta$ and $\delta \leq 2\beta - 1$, using results of Warschawski [9] it is possible to show Theorem 6.2 is *not* true.

In view of Theorem 6.2, if $\delta = \beta$ or $\delta > 2\beta - 1$,

$$(6.8)\qquad |g_\delta^{-1}(\zeta)| \leq c|\zeta|, \qquad \zeta \in D,$$

$$(6.9)\qquad |g_\delta(w)| \leq c|w|, \qquad w \in R_\delta.$$

Next, map $D\backslash\{(2, 0)\}$ onto $S = \{\xi: \operatorname{Im}\xi \geq 0\}$ by the conformal mapping

$$p(\zeta) = \frac{\zeta e^{i\pi/2}}{2-\zeta}, \qquad \zeta \in D\backslash\{(2, 0)\}.$$

Also,

$$p^{-1}(\xi) = \frac{2\xi}{\xi + e^{i\pi/2}}, \qquad \xi \in S,$$

is also conformal and we can choose $\varepsilon > 0$ such that

$$f_\delta^{-1} \circ g_\delta^{-1} \circ p^{-1}(\partial S \cap \overline{B_\varepsilon(0)}) \subseteq \partial C;$$

then

$$(6.10)\qquad (p^{-1})' \text{ is bounded on } \overline{B_\varepsilon(0)} \cap S;$$

$$(6.11)\qquad \frac{1}{c}|\xi| \leq |p^{-1}(\xi)| \leq c|\xi| \quad \text{for } \xi \in \overline{B_\varepsilon(0)} \cap S;$$

(6.12) $$p' \text{ is bounded on } p^{-1}(\overline{B_\varepsilon(0)} \cap S).$$

At last we can define H and F. Set

$$H = f_\delta^{-1} \circ g_\delta^{-1} \circ p^{-1}(\overline{B_\varepsilon(0)} \cap S), \quad F(z) = p \circ g_\delta \circ f_\delta(z), \qquad z \in H.$$

Then F is conformal from $H\backslash\{0\}$ into $\overline{B_\varepsilon(0)} \cap S\backslash\{0\}$ with conformal inverse $\mathscr{F} = f_\delta^{-1} \circ g_\delta^{-1} \circ p^{-1}$ and parts (i)–(iii) of Theorem 2.1 clearly hold.

By (6.9) and (6.11)

(6.13) $$|g_\delta^{-1} \circ p^{-1}(\xi)| \geq c|\xi|, \qquad \xi \in \overline{B_\varepsilon(0)} \cap S.$$

Hence for $\xi \in \overline{B_\varepsilon(0)} \cap S$, by (6.7), Theorem 6.2 and (6.10)

$$\begin{aligned} |\mathscr{F}'(\xi)| &= |(f_\delta^{-1})' \circ g_\delta^{-1} \circ p^{-1}(\xi)| \cdot |(g_\delta^{-1})' \circ p^{-1}(\xi)| \cdot |(p^{-1})'(\xi)| \\ &\leq [c|g_\delta^{-1} \circ p^{-1}(\xi)|^{-1}(-\ln|g_\delta^{-1} \circ p^{-1}(\xi)|)^{-\frac{\beta}{\beta-1}}] \cdot [c] \cdot [c]. \end{aligned}$$

Since $x \to \frac{1}{x}(-\ln x)^{-\beta/(\beta-1)}$ is decreasing for x small and positive, by (6.13) we get

$$|\mathscr{F}'(\xi)| \leq c|\xi|^{-1}(-\ln|\xi|)^{-\frac{\beta}{\beta-1}} \quad \text{for } \xi \in \overline{B_\varepsilon(0)} \cap S.$$

This gives (iv) in Theorem 2.1.

If $z \in H\backslash\{0\}$, then by (6.12), Theorem 6.2 and (6.6)

$$\begin{aligned} |F'(z)| &= |p' \circ g_\delta \circ f_\delta(z)| \cdot |g_\delta' \circ f_\delta(z)| \cdot |f_\delta'(z)| \\ &\leq c \cdot c \cdot c|f_\delta(z)| \cdot |z|^{-\beta} \end{aligned}$$

and by (6.4) the latter is bounded. This gives part (v).

All that remains is part (vi). By (6.11) and (6.8)–(6.9)

$$\frac{1}{c}|f_\delta(z)| \leq |F(z)| \leq c|f_\delta(z)|, \qquad z \in H.$$

Thus for $z \in H\backslash\{0\}$

$$\begin{aligned} \left|\frac{F'(z)}{F(z)}\right|(-\ln|F(z)|)^{-\frac{\beta}{\beta-1}} &\leq \frac{c|f_\delta(z)| \cdot |z|^{-\beta}}{c|f_\delta(z)|}(-\ln c|f_\delta(z)|)^{-\frac{\beta}{\beta-1}} \\ &\leq c|z|^{-\beta}(-\ln|f_\delta(z)|)^{-\frac{\beta}{\beta-1}} \\ &\leq c|z|^{-\beta}[c|z|^{-(\beta-1)}]^{-\frac{\beta}{\beta-1}} \quad \text{(by (6.4))} \\ &= c. \end{aligned}$$

Here too we are using that the neighborhood N is chosen small enough so that $\theta = \arg z$ is sufficiently small that $\cos((\beta-1)\theta) \geq c' > 0$ in (6.4). □

7. Proof of Theorem 6.2

We need the following result of Warschawski [10].

Theorem 7.1 (Warschawski). *Suppose $\mathscr{C}$ is a closed rectifiable Jordan curve with a continuously turning tangent. Furthermore, assume that the tangent angle $\tau(s)$ as a function of arclength s is Dini continuous. If G maps D^0 conformally and injectively onto the interior of $\mathscr{C}$ then G' has a continuous extension to D and $G' \neq 0$ on D.* □

Here $\tau(s)$ Dini continuous means for some nondecreasing continuous function $\kappa(t)$, $t \in [0, \sigma]$, with

$$\int_0^\sigma \frac{\kappa(u)}{u}\, du < \infty$$

we have

$$|\tau(s_2) - \tau(s_1)| \leq \kappa(|s_2 - s_1|), \qquad |s_2 - s_1| \leq \sigma.$$

Theorem 6.2 immediately follows from Theorem 7.1 and the next theorem.

Theorem 7.2. *If $\delta = \beta$ or $\delta > 2\beta - 1$ then ∂R_δ has a continuously turning tangent. The tangent angle $\tau(s)$ is Dini continuous as a function of arclength s.*

Proof. Since $\partial R_\delta \backslash \{0\}$ is C^∞, we need only consider ∂R_δ in a neighborhood of 0. Near 0, ∂R_δ consists of two arcs, call them $\Gamma_1(\delta)$ and $\Gamma_2(\delta)$, separated by 0. We parametrize as follows. Let

$$\text{(7.1)} \qquad \begin{cases} r_1(t) = t(1 + t^{2(\delta-1)})^{1/2}, & \theta_1(t) = -\tan^{-1} t^{\delta-1}, \\ r_2(t) = t(1 + t^{2(\beta-1)})^{1/2}, & \theta_2(t) = \tan^{-1} t^{\beta-1}. \end{cases}$$

These give parametric representations (in polar coordinates) of ∂C_1, ∂C_2, respectively, for $t \geq 0$. Then for some $\delta_1 > 0$, in polar coordinates $\rho e^{i\varphi}$ for $k = 1, 2$,

$$\Gamma_k(\delta)\colon \rho = \rho_k(t), \qquad \varphi = \varphi_k(t), \qquad t \in (0, \delta_1),$$

where (cf. (6.4))

$$\text{(7.2)} \qquad \begin{cases} \rho_k(t) = |f_\delta(r_k(t)e^{i\theta_k(t)})|, \\ \varphi_k(t) = \varphi_\delta(r_k(t)e^{i\theta_k(t)}). \end{cases}$$

Then

$$\text{(7.3)} \qquad w(t) = \begin{cases} w_1(t) = \rho_1(-t)e^{i\varphi_1(-t)}, & t \in (-\delta_1, 0), \\ w_2(t) = \rho_2(t)e^{i\varphi_2(t)}, & t \in (0, \delta_1), \\ 0, & t = 0, \end{cases}$$

is a continuous parametric representation of ∂R_δ near 0.

We need the following expansions.

Lemma 7.3. *For $0 \leq t \leq \delta_1$,*

$$\varphi_1(t) = -\frac{\pi}{2} + t^{\delta-\beta} \sum_{m=0}^{\infty} a_m(\delta) t^{2m(\delta-1)}, \qquad \begin{cases} a_0(\delta) = 0,\ a_1(\delta) > 0 \text{ when } \delta = \beta, \\ a_0(\delta) < 0 \text{ when } \delta > \beta, \end{cases}$$

$$\varphi_2(t) = \frac{\pi}{2} + \sum_{m=1}^{\infty} b_m(\delta) t^{2m(\beta-1)}, \qquad b_1(\delta) < 0;$$

$$\frac{\varphi_1'(t)\rho_1(t)}{\rho_1'(t)} = t^{\delta-1} \sum_{m=0}^{\infty} c_m(\delta) t^{2m(\delta-1)}, \qquad \begin{cases} c_0(\delta) = 0 & \text{if } \delta = \beta, \\ c_0(\delta) \neq 0 & \text{if } \delta > \beta, \end{cases}$$

$$\frac{\varphi_2'(t)\rho_2(t)}{\rho_2'(t)} = t^{3(\beta-1)} \sum_{m=0}^{\infty} d_m(\delta) t^{2m(\beta-1)}, \qquad d_0(\delta) \neq 0.$$

Proof. Since $\tan^{-1} z$ is analytic on $|z| < 1$ and $\sin z$ is entire, $\sin(\gamma \tan^{-1} z)$ has a power series on $|z| < 1$ in odd powers of z. Thus for $\gamma > 0$ and $a > 0$,

$$(1 + u^2)^{-a} \sin(\gamma \tan^{-1} u) = u \sum_{m=0}^{\infty} a_m(a, \gamma) u^{2m}, \qquad -1 < u < 1 \tag{7.4}$$

where

$$a_0(a, \gamma) = \gamma, \qquad a_1(a, \gamma) < 0. \tag{7.5}$$

Hence by (7.1) and (6.5)

$$\begin{aligned} \varphi_\beta(r_1(t)e^{i\theta_1(t)}) &= -\frac{\pi}{2(\beta-1)} t^{-(\beta-1)} t^{\delta-1} \sum_{m=0}^{\infty} a_m\left(\frac{\beta-1}{2}, \beta-1\right) t^{2m(\delta-1)} \\ &= t^{\delta-\beta}\left[-\frac{\pi}{2} - \sum_{m=1}^{\infty} \tilde{a}_m\left(\frac{\beta-1}{2}, \beta-1\right) t^{2m(\delta-1)}\right], \qquad \tilde{a}_1 < 0. \end{aligned}$$

When $\delta = \beta$, by (7.2) this gives

$$\varphi_1(t) = -\frac{\pi}{2} + \sum_{m=1}^{\infty} a_m(\beta) t^{2m(\beta-1)}, \qquad a_1(\beta) > 0;$$

when $\delta > \beta$ we get

$$\begin{aligned} \varphi_1(t) &= 2t^{\delta-\beta}\left[-\frac{\pi}{2} - \sum_{m=1}^{\infty} \tilde{a}_m\left(\frac{\beta-1}{2}, \beta-1\right) t^{2m(\delta-1)}\right] - \frac{\pi}{2} \\ &= -\frac{\pi}{2} + t^{\delta-\beta} \sum_{m=0}^{\infty} a_m(\delta) t^{2m(\delta-1)}, \qquad a_0(\delta) < 0, \end{aligned}$$

as desired.

By (7.1), (7.4) and (6.5)

$$\varphi_\beta(r_2(t)e^{i\theta_2(t)}) = \frac{\pi}{2(\beta-1)} \sum_{m=0}^{\infty} a_m\left(\frac{\beta-1}{2}, \beta-1\right) t^{2m(\beta-1)}.$$

Together with (7.2) this gives for $\delta = \beta$,

$$\varphi_2(t) = \frac{\pi}{2} + \sum_{m=1}^{\infty} b_m(\delta) t^{2m(\beta-1)}, \qquad b_1(\delta) < 0;$$

for $\delta > \beta$,

$$\begin{aligned} \varphi_2(t) &= 2\left[\frac{\pi}{2(\beta-1)} \sum_{m=0}^{\infty} a_m\left(\frac{\beta-1}{2}, \beta-1\right) t^{2m(\beta-1)}\right] - \frac{\pi}{2} \\ &= \frac{\pi}{2} + \sum_{m=1}^{\infty} b_m(\delta) t^{2m(\beta-1)}, \qquad b_1(\delta) < 0, \end{aligned}$$

as desired.

By the chain rule

$$
\begin{aligned}
\varphi_1'(t) &= (\delta-\beta)t^{\delta-\beta-1}\sum_{m=0}^{\infty} a_m(\delta)t^{2m(\delta-1)} \\
&\quad + t^{\delta-\beta}(\delta-1)t^{\delta-2}\sum_{m=0}^{\infty} a_m(\delta)2mt^{(2m-1)(\delta-1)} \\
&= t^{\delta-\beta-1}\sum_{m=0}^{\infty} e_m(\delta)t^{2m(\delta-1)}, \qquad \begin{cases} e_0(\delta)=0 & \text{if } \delta=\beta, \\ e_0(\delta)\neq 0 & \text{if } \delta>\beta, \end{cases}
\end{aligned} \tag{7.6}
$$

and

$$
\begin{aligned}
\varphi_2'(t) &= t^{\beta-2}\sum_{m=1}^{\infty}(\beta-1)2mb_m(\delta)t^{(2m-1)(\beta-1)} \\
&= t^{-1}\sum_{m=1}^{\infty} f_m(\delta)t^{2m(\beta-1)}, \qquad f_1(\delta)\neq 0.
\end{aligned} \tag{7.7}
$$

An argument similar to that giving (7.4) shows for γ and $a>0$,

$$
(1+u^2)^{-a}\cos[\gamma\tan^{-1}u] = \sum_{m=0}^{\infty} b_m(a,\gamma)u^{2m}, \qquad -1<u<1, \tag{7.8}
$$

where

$$
b_0(a,\gamma)=1. \tag{7.9}
$$

Thus by (6.4) and (7.1)

$$
\begin{aligned}
|f_\beta(r_1(t)e^{i\theta_1(t)})| &= \exp\left\{-\frac{\pi}{2(\beta-1)}t^{-(\beta-1)}\sum_{m=0}^{\infty} b_m\left(\frac{\beta-1}{2},\beta-1\right)t^{2m(\delta-1)}\right\} \\
&= \exp\left\{t^{-(\beta-1)}\sum_{m=0}^{\infty} g_m t^{2m(\delta-1)}\right\}, \qquad g_0<0;
\end{aligned}
$$

moreover, by the chain rule

$$
\begin{aligned}
\frac{\frac{d}{dt}|f_\beta(r_1(t)e^{i\theta_1(t)})|}{|f_\beta(r_1(t)e^{i\theta_1(t)})|} &= -(\beta-1)t^{-\beta}\sum_{m=0}^{\infty} g_m t^{2m(\delta-1)} \\
&\quad + t^{-(\beta-1)}(\delta-1)t^{\delta-2}\sum_{m=0}^{\infty} 2mg_m t^{(2m-1)(\delta-1)} \\
&= t^{-\beta}\sum_{m=0}^{\infty} h_m(\delta)t^{2m(\delta-1)}, \qquad h_0(\delta)\neq 0.
\end{aligned}
$$

Together with (7.2) and (6.4) this yields

$$
\frac{\rho_1(t)}{\rho_1'(t)} = t^\beta\sum_{m=0}^{\infty} j_m(\delta)t^{2m(\delta-1)}, \qquad j_0(\delta)\neq 0. \tag{7.10}
$$

Similarly,

$$
\frac{\rho_2(t)}{\rho_2'(t)} = t^\beta\sum_{m=0}^{\infty} k_m(\delta)t^{2m(\beta-1)}, \qquad k_0(\delta)\neq 0. \tag{7.11}
$$

The formulas (7.6), (7.7), (7.10) and (7.11) yield the desired representation of $\varphi_i' \rho_i / \rho_i'$, $i = 1, 2$. $\square$

We continue with the proof of Theorem 7.2. Write

$$x_k(t) = \operatorname{Re} \rho_k(t) e^{i\varphi_k(t)}, \quad y_k(t) = \operatorname{Im} \rho_k(t) e^{i\varphi_k(t)}.$$

Then

$$\frac{x_k'(t)}{y_k'(t)} = \frac{\cos \varphi_k(t) - \frac{\rho_k(t)}{\rho_k'(t)} \varphi_k'(t) \sin \varphi_k(t)}{\sin \varphi_k(t) + \frac{\rho_k(t)}{\rho_k'(t)} \varphi_k'(t) \cos \varphi_k(t)}. \tag{7.12}$$

By Lemma 7.3, as $t \to 0^+$,

$$\begin{aligned}
\cos \varphi_1(t) &= \sin \left[t^{\delta-\beta} \sum_{m=0}^{\infty} a_m(\delta) t^{2m(\delta-1)} \right] \\
&= \begin{cases} a_1(\delta) t^{2(\beta-1)} + o(t^{2(\beta-1)}) & \text{if } \delta = \beta, \\ a_0(\delta) t^{\delta-\beta} + o(t^{\delta-\beta}) & \text{if } \delta > \beta, \end{cases} \\
\sin \varphi_1(t) &= -1 + o(1), \\
\frac{\rho_1(t)}{\rho_1'(t)} \varphi_1'(t) &= \begin{cases} O(t^{3(\beta-1)}) & \text{if } \delta = \beta, \\ c_0(\delta) t^{\delta-1} + o(t^{\delta-1}) & \text{if } \delta > \beta, \end{cases} \\
&= \begin{cases} o(t^{2(\beta-1)}) & \text{if } \delta = \beta, \\ o(t^{\delta-\beta}) & \text{if } \delta > \beta, \end{cases} \\
\cos \varphi_2(t) &= -\sin \sum_{m=1}^{\infty} b_m(\delta) t^{2m(\beta-1)} = -b_1(\delta) t^{2(\beta-1)} + o(t^{2(\beta-1)}), \\
\sin \varphi_2(t) &= 1 + o(1), \\
\frac{\rho_2(t)}{\rho_2'(t)} \varphi_2'(t) &= d_0(\delta) t^{3(\beta-1)} + o(t^{3(\beta-1)}).
\end{aligned}$$

Then as $t \to 0^+$,

$$\begin{aligned}
\frac{x_1'(t)}{y_1'(t)} &= \begin{cases} \frac{a_1(\delta) t^{2(\beta-1)} + o(t^{2(\beta-1)}) - o(t^{2(\beta-1)})(-1+o(1))}{-1+o(1)+o(1)}, & \delta = \beta, \\ \frac{a_0(\delta) t^{\delta-\beta} + o(t^{\delta-\beta}) - o(t^{\delta-\beta})(-1+o(1))}{-1+o(1)+o(1)}, & \delta > \beta, \end{cases} \\
&\sim \begin{cases} -a_1(\delta) t^{2(\beta-1)} & \text{if } \delta = \beta, \\ -a_0(\delta) t^{\delta-\beta} & \text{if } \delta > \beta, \end{cases} \\
&\to \begin{cases} 0^-, & \delta = \beta, \\ 0^+, & \delta > \beta, \end{cases}
\end{aligned} \tag{7.13}$$

and

$$\begin{aligned}
\frac{x_2'(t)}{y_2'(t)} &= \frac{-b_1(\delta) t^{2(\beta-1)} + o(t^{2(\beta-1)}) - d_0(\delta) t^{3(\beta-1)} + o(t^{3(\beta-1)})}{1 + o(1) + o(1)} \\
&\sim -b_1(\delta) t^{2(\beta-1)} \\
&\to 0^+.
\end{aligned} \tag{7.14}$$

These show that the tangents to $w(t)$ in (7.3) for $t \neq 0$ turn continuously to a vertical line as $t \to 0$. Thus ∂R_δ has a continuously turning tangent.

Let $s(t)$, $t \in (-\delta_1, \delta_1)$, denote arclength; to prove Dini continuity of $\tau(\cdot)$ as a function of s it suffices to find a continuous nondecreasing function κ such that for $\Theta(t) = \tau(s(t))$,

$$\int_{0^+} \frac{\kappa(u)}{u} du < \infty \tag{7.15}$$

and

$$|\Theta(t_1) - \Theta(t_2)| \le \kappa(|s(t_1) - s(t_2)|), \qquad \begin{cases} t_1 \ \& \ t_2 \in [0, \delta_1), \\ \qquad \text{OR} \\ t_1 \ \& \ t_2 \in (-\delta_1, 0]. \end{cases} \tag{7.16}$$

By (7.13)–(7.14) and (7.3)

$$|\Theta(t_1) - \Theta(t_2)| = \begin{cases} \left|\tan^{-1} \frac{y_1'}{x_1'}(-t_1) - \tan^{-1} \frac{y_1'}{x_1'}(-t_2)\right|, & t_1 \ \& \ t_2 \in (-\delta_1, 0], \\ \left|\tan^{-1} \frac{y_2'}{x_2'}(t_1) - \tan^{-1} \frac{y_2'}{x_2'}(t_2)\right|, & t_1 \ \& \ t_2 \in [0, \delta_1). \end{cases}$$

Writing $\tan^{-1} a - \tan^{-1} b = \int_a^b (1 + x^2)^{-1} dx$ for a, b both positive or both negative, we have

$$\left|\tan^{-1} \frac{1}{u} - \tan^{-1} \frac{1}{v}\right| \le |u - v| \quad \text{for } u \ \& \ v > 0 \quad \text{OR} \quad u \ \& \ v < 0.$$

Then to prove (7.16), it suffices to show

$$\left|\frac{x_i'}{y_i'}(t_1) - \frac{x_i'}{y_i'}(t_2)\right| \le \kappa\left(\int_{t_1}^{t_2} \sqrt{x_i'(u)^2 + y_i'(u)^2}\, du\right), \qquad 0 \le t_1 < t_2 < \delta_1. \tag{7.17}$$

For typographical clarity in the next few lines we drop the i subscript. By (7.12) for

$$h(u) = \sin\varphi(u) + \varphi'(u)\frac{\rho(u)}{\rho'(u)} \cos\varphi(u),$$

we have

$$\begin{aligned} &\left|\frac{x'}{y'}(t_1) - \frac{x'}{y'}(t_2)\right| \\ &\quad \le \left\{\left|\left[1 + \varphi'(t_1)\varphi'(t_2)\frac{\rho(t_1)\rho(t_2)}{\rho'(t_1)\rho'(t_2)}\right] \sin(\varphi(t_2) - \varphi(t_1))\right| \right. \\ &\qquad \left. + \left|\left[\varphi'(t_2)\frac{\rho(t_2)}{\rho'(t_2)} - \varphi'(t_1)\frac{\rho(t_1)}{\rho'(t_1)}\right] \cos(\varphi(t_2) - \varphi(t_1))\right|\right\} \cdot |h(t_1)h(t_2)|^{-1}. \end{aligned}$$

By the formulas after (7.12), $|h|$ is bounded below away from 0 for δ_1 small and $1 + \varphi'(t_1)\varphi'(t_2)\rho(t_1)\rho(t_2)/\rho'(t_1)\rho'(t_2)$ is bounded. Therefore

$$\left|\frac{x'}{y'}(t_1) - \frac{x'}{y'}(t_2)\right| \le C\left[|\varphi(t_2) - \varphi(t_1)| + \left|\varphi'(t_2)\frac{\rho(t_2)}{\rho'(t_2)} - \varphi'(t_1)\frac{\rho(t_1)}{\rho'(t_1)}\right|\right]. \tag{7.18}$$

Now we examine each term separately. By Lemma 7.3

$$(7.19)\quad \begin{aligned} |\varphi_1(t_2)-\varphi_1(t_1)| &= \left|\sum_{m=0}^{\infty} a_m(\delta)[t_2^{2m(\delta-1)+\delta-\beta}-t_1^{2m(\delta-1)+\delta-\beta}]\right|, \\ |\varphi_2(t_2)-\varphi_2(t_1)| &= \left|\sum_{m=1}^{\infty} b_m(\delta)[t_2^{2m(\beta-1)}-t_1^{2m(\beta-1)}]\right|, \\ \left|\varphi_1'\frac{\rho_1}{\rho_1'}(t_2)-\varphi_1'\frac{\rho_1}{\rho_1'}(t_1)\right| &= \left|\sum_{m=0}^{\infty} c_m(\delta)[t_2^{(2m+1)(\delta-1)}-t_1^{(2m+1)(\delta-1)}]\right|, \\ \left|\varphi_2'\frac{\rho_2}{\rho_2'}(t_2)-\varphi_2'\frac{\rho_2}{\rho_2'}(t_2)\right| &= \left|\sum_{m=0}^{\infty} d_m(\delta)[t_2^{(2m+3)(\beta-1)}-t_1^{(2m+3)(\beta-1)}]\right|. \end{aligned}$$

Choose $r\in(0,1]$ so small that $\delta-(2\beta-1)-r(\beta-1)>0$ if $\delta>\beta$; otherwise let $r=1$. Here is where we use the hypothesis $2\beta-1<\delta$ whenever $\delta>\beta$. This particular choice of r does not come into play until the end of this section. Define

$$\kappa(u)=(-\ln u)^{-(1+r)}, \qquad 0<u<\tfrac{1}{2}.$$

Then $\kappa(u)$ is increasing and satisfies (7.15). We need the following lemma to examine the right side of (7.17).

Lemma 7.4. *Assume $a, p, \eta, b>0$. Then for some $b_0, c>0$ independent of η,*

$$\sup_{x>1}(1-x^{-p})^a(\eta-\ln[1-e^{-b(x-1)}])\le \eta\vee cb$$

whenever $b\ge b_0$.

Proof. Call the function to be maximized $k(x)$. If $2>\frac{p+1}{b}$, the function $x^{p+1}e^{-b(x-1)}$ is decreasing for $x\ge 2$. Hence for such b,

$$(7.20)\qquad \sup_{x\ge 2} x^{p+1}e^{-b(x-1)}=2^{p+1}e^{-b}.$$

Consider the function

$$(7.21)\qquad k_1(x)=\frac{1-x^{-p}}{1-e^{-b(x-1)}}, \qquad 1<x\le 2.$$

Since $\lim_{x\to 1^+}k_1(x)=\frac{p}{b}$ and $k_1(2)\le(1-e^{-b})^{-1}$,

$$\sup_{1<x\le 2}k_1(x)=\frac{p}{b}\vee\frac{1}{1-e^{-b}}\vee\sup\{k_1(x_0)\colon 1<x_0<2,\, k_1'(x_0)=0\}.$$

But if $k_1'(x_0)=0$ then

$$e^{-b(x_0-1)}=\frac{px_0^{-p-1}}{px_0^{-p-1}+b(1-x_0^{-p})}$$

and so

$$k_1(x_0)=\frac{px_0^{-p-1}+b(1-x_0^{-p})}{b}<\frac{p+b}{b}.$$

So we get

$$(7.22)\qquad \sup_{1<x\le 2}k_1(x)\le\frac{p+b}{b}\vee\frac{1}{1-e^{-b}}.$$

Using (7.20)–(7.22), for some b_0,

$$\sup_{x>1} \frac{(1-x^{-p})^{a+1}}{1-e^{-b(x-1)}} x^{p+1} e^{-b(x-1)} = c < \infty \quad \text{if } b \geq b_0. \tag{7.23}$$

Since

$$\lim_{x\to 1^+} k(x) = 0, \qquad \lim_{x\to\infty} k(x) = \eta,$$

we have

$$\sup_{x>1} k(x) = \eta \vee \sup\{k(x_0)\colon x_0 > 1 \text{ and } k'(x_0) = 0\}.$$

If $x_0 > 1$ and $k'(x_0) = 0$ then

$$\eta - \ln[1 - e^{-b(x_0-1)}] = \frac{b}{ap} x_0^{p+1} e^{-b(x_0-1)} \frac{1-x_0^{-p}}{1-e^{-b(x_0-1)}}$$

and so

$$\begin{aligned} k(x_0) &= \frac{(1-x_0^{-p})^{a+1}}{1-e^{-b(x_0-1)}} \frac{b}{ap} x_0^{p+1} e^{-b(x_0-1)} \\ &\leq \frac{c}{ap} b \quad \text{provided } b \geq b_0, \end{aligned}$$

by (7.23). Thus

$$\sup_{x>1} k(x) \leq \eta \vee cb, \qquad b \geq b_0 \text{ as desired.} \quad \square$$

By (7.10)–(7.11), (7.2) and (6.4), for some $\gamma_i > 0$, if t is sufficiently small,

$$|\rho_i'(t)| \geq ct^{-\beta} \exp(-\gamma_i t^{-(\beta-1)}),$$

and since

$$\begin{aligned} \sqrt{x_i'(u)^2 + y_i'(u)^2} &= |\rho_i'(u)| \sqrt{1 + \left[\varphi_i' \frac{\rho_i}{\rho_i'}(u)\right]^2} \\ &\geq cu^{-\beta} \exp(-\gamma_i u^{-(\beta-1)}), \end{aligned}$$

it follows that

$$\begin{aligned} \int_{t_1}^{t_2} \sqrt{x_i'(u)^2 + y_i'(u)^2}\, du &\geq c \int_{t_1}^{t_2} u^{-\beta} e^{-\gamma_i u^{-(\beta-1)}}\, du \\ &= c[e^{-\gamma_i t_2^{-(\beta-1)}} - e^{-\gamma_i t_1^{-(\beta-1)}}] \\ &= ce^{-\gamma_i t_2^{-(\beta-1)}} [1 - e^{-\gamma_i [t_1^{-(\beta-1)} - t_2^{-(\beta-1)}]}]. \end{aligned}$$

Thus

$$\begin{aligned} &\kappa\left(\int_{t_1}^{t_2} \sqrt{x_i'(u)^2 + y_i'(u)^2}\, du\right) \\ &\quad \geq (-\ln c + \gamma_i t_2^{-(\beta-1)} - \ln[1 - e^{-\gamma_i [t_1^{-(\beta-1)} - t_2^{-(\beta-1)}]}])^{-(1+r)}. \end{aligned} \tag{7.24}$$

For $q > 0$ (where RHS = Right-hand side), with $0 \leq t_1 < t_2$,

$$\begin{aligned} \frac{t_2^q - t_1^q}{\text{RHS}(7.24)} &= t_2^q \left[\left(1 - \left(\frac{t_1}{t_2}\right)^q\right)^{1/(1+r)} (-\ln c + \gamma_i t_2^{-(\beta-1)} \right. \\ &\qquad \left. - \ln[1 - e^{-\gamma_i t_2^{-(\beta-1)} [(t_1/t_2)^{-(\beta-1)} - 1]}])\right]^{1+r} \\ &\leq t_2^q \{[-\ln c + \gamma_i t_2^{-(\beta-1)}] \vee c\gamma_i t_2^{-(\beta-1)}\}^{1+r} \quad \text{for } t_2 \text{ small}, \end{aligned}$$

by Lemma 7.4 (setting $a = 1/(1+r)$, $\eta = -\ln c + \gamma_i t_2^{-(\beta-1)}$, $b = \gamma_i t_2^{-(\beta-1)}$, $p = q/(\beta-1)$, and $x = (t_1/t_2)^{-(\beta-1)} > 1$). Then for t_2 small,

$$\frac{t_2^q - t_1^q}{\mathrm{RHS}(7.24)} \le C t_2^{q-(1+r)(\beta-1)} \tag{7.25}$$

where $q > 0$ and C is independent of t_1 and t_2.

Finally, we prove

$$\frac{|\varphi_1(t_2) - \varphi_1(t_1)|}{\mathrm{LHS}(7.24)} \le C \quad \text{for } 0 \le t_1 < t_2,\ t_2 \text{ small.} \tag{7.26}$$

Similar arguments prove similar bounds for $|\varphi_1(t_2) - \varphi_1(t_1)|$ replaced by the left hand parts in the 3 remaining equations in (7.19). Then together with (7.18) we see that (7.17) holds with κ replaced by $C\kappa$ and we will be done. So on to (7.26).

Lemma 7.5. *If $p \ge 1$ then $(1-x^p)/(1-x) \le p$ for $0 \le x \le 1$.* □

Corollary 7.6. *If $p \ge 1$ then*

$$|t_2^{pq} - t_1^{pq}| \le p|t_2|^{(p-1)q}|t_2^q - t_1^q|. \quad \square$$

Choose $q > 1$ such that

$$q - (1+r)(\beta-1) > 0. \tag{7.27}$$

Then choose $M \ge 1$ such that

$$2m(\beta-1) - q > 1 \quad \text{for } m \ge M. \tag{7.28}$$

By Corollary 7.6, (7.19) and (7.25)

$$\begin{aligned}
\frac{|\varphi_1(t_2) - \varphi_1(t_1)|}{\mathrm{LHS}(7.24)} &\le \frac{|\varphi_1(t_2) - \varphi_1(t_1)|}{\mathrm{RHS}(7.24)} \\
&\le \frac{\left|\sum_{m=0}^M a_m(\delta)[t_2^{2m(\delta-1)+\delta-\beta} - t_1^{2m(\delta-1)+\delta-\beta}]\right|}{\mathrm{RHS}(7.24)} \\
&\quad + \frac{\sum_{m=M+1}^\infty |a_m(\delta)| t_2^{2m(\delta-1)+\delta-\beta-q} \left[\frac{2m(\delta-1)+\delta-\beta}{q}\right] |t_2^q - t_1^q|}{\mathrm{RHS}(7.24)} \\
&\le c \sum_{m=0}^M |a_m(\delta)| t_2^{2m(\delta-1)+\delta-\beta-(\beta-1)(1+r)} + c t_2^{q-(\beta-1)(1+r)}
\end{aligned}$$

(by (7.25))

where we have used (7.28) and (7.25) to bound the second summation. The last term on the RHS is bounded for t_2 small by (7.27). If $\delta > \beta$ then $2m(\delta-1) + (\delta-\beta) - (1+r)(\beta-1) \ge \delta - (2\beta-1) - r(\beta-1) > 0$ by choice of r. When $\delta = \beta$ then since $a_0 = 0$ we are only concerned with $m \in \{1, 2, \dots, M\}$ in the first summation and then $2m(\delta-1) + (\delta-\beta) - (1+r)(\beta-1) \ge 0$ by choice of r. In any event the first summation is also bounded for t_2 small. This gives (7.26) and we are done. □

References

1. R. D. DeBlassie, *Explicit semimartingale representation of Brownian motion in a wedge*, Stochastic Process. Appl. **34** (1990), 67–97.
2. K. Burdzy and D. Marshall, *Hitting a boundary point with reflected Brownian motion*, (1990), preprint.
3. L. C. G. Rogers, *Brownian motion in a wedge with variable skew reflection*, Trans. Amer. Math. Soc. **326** (1991), 227–236.
4. ——, *Brownian motion in a wedge with variable skew reflection*. II, Proc. Evanston Conf. (1990), (to appear).
5. L. C. G. Rogers and D. Williams, *Diffusions, Markov processes, and martingales*, Vol. 2: *Itô calculus*, Wiley, Chichester, 1987.
6. D. W. Stroock and S. R. S. Varadhan, *Diffusion processes with boundary conditions*, Comm. Pure. Appl. Math. **24** (1971), 147–225.
7. D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*, Springer-Verlag, New York, 1979.
8. S. R. S. Varadhan and R. J. Williams, *Brownian motion in a wedge with oblique reflection*, Comm. Pure Appl. Math. **38** (1985), 405–443.
9. E. Warschawski, *On conformal mapping of infinite strips*, Trans. Amer. Math. Soc. **51** (1942), 280–335.
10. ——, *On differentiability at the boundary of a conformal mapping*, Proc. Amer. Math. Soc. **12** (1961), 614–620.
11. R. J. Williams, *Recurrence classification and invariant measure for reflected Brownian motion in a wedge*, Ann. Probab. **13** (1985), 758–778.

Department of Mathematics, Texas A&M University, College Station, Texas 77843
E-mail address: deblass@math.tamu.edu
E-mail address: toby@math.tamu.edu