WEIGHTED NORM INEQUALITIES FOR VILENKIN-FOURIER SERIES

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ABSTRACT. Let $S_n f$ be the n th partial sum of the Vilenkin-Fourier series of $f \in L^1$. For $1 < p < \infty$, we characterize all weight functions w such that if $f \in L^p(w)$, $S_n f$ converges to f in $L^p(w)$. We also determine all weight functions w such that $\{S_n\}$ is uniformly of weak type $(1, 1)$ with respect to w .

1. INTRODUCTION

The Vilenkin-Fourier series is a generalization of the Walsh-Fourier series. While the Walsh functions are characters of the countable direct product of groups of order 2, the Vilenkin system consists of characters of $G = \prod_{i=0}^{\infty} Z_{p_i}$, a direct product of cyclic groups of order p_i , where p_i is any integer ≥ 2 . In this paper, there is no restriction on the orders $\{p_i\}$.

Let μ be the Haar measure on G normalized by $\mu(G) = 1$. We identify G with the unit interval $(0, 1)$ in the following manner. Let $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \dots$. We associate with each $\{x_i\} \in G$, $0 \leq x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0, 1)$. If we disregard the countable set of p_i -rationals, this mapping is one-one, onto and measure-preserving.

The characters of G are the finite products of an orthonormal system $\{\phi_k\}$ of functions on G , defined by $\phi_k(x) = \exp(2\pi i x_k / p_k)$, $x = \{x_k\} \in G$, $k = 0, 1, \dots$. To enumerate the finite products of $\{\phi_k\}$, we write each nonnegative integer n as a finite sum, $n = \sum_{k=0}^{\infty} \alpha_k m_k$, with $0 \leq \alpha_k < p_k$, and define $\chi_n = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of G , and they form a complete orthonormal system on G .

For $f \in L^1$, let

$$S_n f(x) = \int_G f(t) \sum_{j=0}^{n-1} \chi_j(x-t) d\mu(t), \quad n = 1, 2, \dots,$$

be the n th partial sum of the Vilenkin-Fourier series of f . In [10], we showed that for $f \in L^p$, $1 < p < \infty$,

$$\lim_{n \rightarrow \infty} \int_G |S_n f - f|^p d\mu = 0.$$

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In this paper, we identify, for $1 < p < \infty$, all weight functions w such that for any $f \in L^p(w)$, we have $f \in L^1$ and

$$\lim_{n \rightarrow \infty} \int_G |S_n f - f|^p w \, d\mu = 0.$$

(Here $L^p(w)$ denotes the space of measurable functions on G such that $\|f\|_{p,w} \equiv (\int_G |f|^p w \, d\mu)^{1/p} < \infty$.)

In order to characterize these weight functions, we use the following notation. Let $\{G_k\}$ be the sequence of subgroups of G defined by

$$G_0 = G, \quad G_k = \prod_{i=0}^{k-1} \{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \quad k = 1, 2, \dots$$

On the interval $(0, 1)$, cosets of G_k are intervals of the form $(jm_k^{-1}, (j+1)m_k^{-1})$, $j = 0, 1, \dots, m_k - 1$. For $k = 0, 1, \dots$, $j = 1, \dots, p_k$, let I_{jk} be the set in G_k that corresponds to the interval $(0, jm_{k+1}^{-1})$. The collection of all translates of I_{jk} in G , for all $j = 1, \dots, p_k$, $k = 0, 1, \dots$, is denoted by $\mathcal{J}$. In other words, a set I belongs to $\mathcal{J}$ if

- (i) for some $x \in G$, $k = 0, 1, \dots$, $I \subset x + G_k$,
- (ii) I is a union of cosets of G_{k+1} , and
- (iii) if we consider $x + G_k$ as a circle, I is an interval.

We shall call the sets in $\mathcal{J}$ generalized intervals.

We say that w is a weight function on G if w is measurable and $0 \leq w(x) \leq \infty$. We shall exclude the trivial cases $w \equiv 0$ a.e. and $w \equiv \infty$ a.e.

Our main results are as follows:

Theorem 1.1. *Let w be a weight function on G and $1 < p < \infty$. The following statements are equivalent:*

(i) *w satisfies the $A_p(G)$ condition: there is a constant C such that for every $I \in \mathcal{J}$,*

$$(1.1) \quad \left(\frac{1}{\mu(I)} \int_I w \, d\mu \right) \left(\frac{1}{\mu(I)} \int_I w^{-1/(p-1)} \, d\mu \right)^{p-1} \leq C.$$

(ii) *There is a constant C , depending only on w and p , such that for every $f \in L^p(w)$, we have $f \in L^1$ and*

$$(1.2) \quad \int_G |S_n f|^p w \, d\mu \leq C \int_G |f|^p w \, d\mu, \quad n = 1, 2, \dots$$

(iii) *For every $f \in L^p(w)$, we have $f \in L^1$ and*

$$(1.3) \quad \lim_{n \rightarrow \infty} \int_G |S_n f - f|^p w \, d\mu = 0.$$

For the case where the orders of cyclic groups are bounded, i.e., $\sup_i p_i < \infty$, J. Gosselin [5] defined the A_p condition, $1 < p < \infty$, as the one where (1.1) holds for all I that are cosets of G_k , $k = 0, 1, \dots$. For this case, our $A_p(G)$ condition and his A_p condition are equivalent. This can be seen as follows. Suppose w satisfies the A_p condition in [5], and $I \in \mathcal{J}$. Then, for some $x \in G$, $k = 0, 1, \dots$, $I \subset x + G_k$, and I is a union of cosets of G_{k+1} . It

follows that $\mu(I) \geq \mu(G_{k+1}) = p_k^{-1} \mu(G_k)$. Hence

$$\begin{aligned} & \left(\frac{1}{\mu(I)} \int_I w \, d\mu \right) \left(\frac{1}{\mu(I)} \int_I w^{-1/(p-1)} \, d\mu \right)^{p-1} \\ & \leq p_k^p \left(\frac{1}{\mu(G_k)} \int_{x+G_k} w \, d\mu \right) \left(\frac{1}{\mu(G_k)} \int_{x+G_k} w^{-1/(p-1)} \, d\mu \right)^{p-1} \\ & \leq C, \quad \text{if } \sup_i p_i < \infty. \end{aligned}$$

For the bounded order case, the implication (i) $\Rightarrow$ (ii) in Theorem 1.1 is a consequence of the result in [5]. For another proof, see [8].

We also have the following theorem for functions in $L^1(w)$.

Theorem 1.2. *Let w be a weight function on G . The following statements are equivalent:*

(i) *w satisfies the $A_1(G)$ condition: there is a constant C such that for every $I \in \mathcal{I}$,*

$$(1.4) \quad \frac{1}{\mu(I)} \int_I w \, d\mu \leq C \operatorname{ess\,inf}_I w,$$

where $\operatorname{ess\,inf}_I w = \inf\{t > 0: \mu\{x \in I: w(x) < t\} > 0\}$.

(ii) *There is a constant C , depending only on w , such that for every $f \in L^1(w)$, we have $f \in L^1$ and*

$$(1.5) \quad \int_{\{|S_n f| > y\}} w \, d\mu \leq C y^{-1} \int_G |f| w \, d\mu, \quad n = 1, 2, \dots, y > 0.$$

Our proofs of Theorems 1.1 and 1.2 consist of adapting the methods of Hunt, Muckenhoupt, and Wheeden [6] and Cordoba and Fefferman [2] to Vilenkin-Fourier series. We recall some properties of Vilenkin-Fourier series. Let $S_n^* f = \bar{\chi}_n S_n(f \chi_n)$ be the n th modified partial sum, $n = 1, 2, \dots$. It is shown in [8] that if $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$, then

$$(1.6) \quad S_n^* f = \sum_{k=0}^{\infty} S_{\alpha_k m_k}^* f$$

and

$$(1.7) \quad S_{\alpha_k m_k}^* f(x) = \frac{1}{\mu(G_k)} \int_{x+G_k} f(t) \phi_k^{-\alpha_k}(x-t) \left(\sum_{j=0}^{\alpha_k-1} \phi_k^j(x-t) \right) d\mu(t).$$

(If $\alpha_k = 0$, we interpret the sum on the right as zero.) $S_{\alpha_k m_k}^* f$ can be expressed in terms of certain conjugate functions of f and $f \phi_k^{\alpha_k}$, which are defined as follows. For $f \in L^1$ and $x = \{x_k\} \in G$, let the k th conjugate function of f be

$$(1.8) \quad H_k f(x) = \frac{1}{2} \frac{1}{\mu(G_k)} \int_{(x+G_k) \cap \{x_k \neq t_k\}} f(t) \cot(\pi(x_k - t_k)/p_k) d\mu(t),$$

$k = 0, 1, \dots$. Then

$$\begin{aligned}
 S_{\alpha_k m_k}^* f(x) &= \frac{\alpha_k}{\mu(G_k)} \int_{(x+G_k) \cap \{x_k = t_k\}} f(t) d\mu(t) \\
 &+ \frac{1}{2} \phi_k^{-\alpha_k}(x) \frac{1}{\mu(G_k)} \int_{(x+G_k) \cap \{x_k \neq t_k\}} f(t) \phi_k^{\alpha_k}(t) d\mu(t) \\
 &- \frac{1}{2} \frac{1}{\mu(G_k)} \int_{(x+G_k) \cap \{x_k \neq t_k\}} f(t) d\mu(t) \\
 &+ i \phi_k^{-\alpha_k}(x) H_k(f \phi_k^{\alpha_k})(x) - i H_k f(x).
 \end{aligned}
 \tag{1.9}$$

To estimate $S_n^* f$, we have to deal simultaneously with the conjugate functions of f and $f \phi_k^{\alpha_k}$, $k = 0, 1, \dots$. In order to do this, we make the following changes in the proofs of [6 and 2]. In the proof of the weak type $(1, 1)$ result, we modify the function $b(t)$ in the Calderón-Zygmund decomposition to satisfy both $\int_I b d\mu = 0$ and $\int_I b \phi_k^{\alpha_k} d\mu = 0$, where α_k depends on I and n . This modification of the Calderón-Zygmund decomposition was introduced in [10]. In the proof of the strong type (p, p) result, the sharp function is modified as follows: the mean value of f over I is replaced by a function $f_{I,n} = A + B \phi_k^{-\alpha_k}$ (α_k depending on I and n) that satisfies $\int_I (f - f_{I,n}) d\mu = 0$ and $\int_I (f - f_{I,n}) \phi_k^{\alpha_k} d\mu = 0$. These modified sharp functions will be discussed in §3.

Section 2 contains some properties of $A_p(G)$. We also show that the $A_p(G)$ condition is necessary and sufficient for the $L^p(w)$ -boundedness of a Hardy-Littlewood type maximal function that is appropriate for the study of Vilenkin-Fourier series.

In §3 we introduce the modified sharp functions. We obtain estimates for the modified sharp function of $S_n^* f$ in terms of the maximal function of f . We also prove weighted norm inequalities relating the maximal function and sharp functions.

The results in §§2 and 3 will be used in the proof of (i) $\Rightarrow$ (ii) of Theorem 1.1. This implication together with the implications (ii) $\Rightarrow$ (i) and (ii) $\Leftrightarrow$ (iii) are proved in §4.

Theorem 1.2 is proved in §5, with the implication (i) $\Rightarrow$ (ii) based on a modified Calderón-Zygmund decomposition lemma and Theorem 1.1.

In what follows, C will denote an absolute constant which may vary from line to line.

2. THE CONDITION $A_p(G)$ AND THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

The following notation will be used throughout this paper. We decompose the collection $\mathcal{J}$ of generalized intervals into disjoint subcollections in the following manner. Let $\mathcal{J}_{-1} = \{G\}$. For $k = 0, 1, \dots$, let $\mathcal{J}_k$ be the collection of all $I \in \mathcal{J}$ such that I is a proper subset of a coset of G_k , and is a union of cosets of G_{k+1} . The collections $\mathcal{J}_k$ are disjoint, and $\mathcal{J} = \bigcup_{k=-1}^{\infty} \mathcal{J}_k$.

For $I \in \mathcal{J}$, we define the set $3I \in \mathcal{J}$ as follows. If $I = G$, let $3I = G$. For $I \in \mathcal{J}_k$, $k = 0, 1, \dots$, there is $x \in G$ such that $I \subset x + G_k$. If $\mu(I) \geq \mu(G_k)/3$, let $3I = x + G_k$. If $\mu(I) < \mu(G_k)/3$, consider $x + G_k$ as a circle. Then I is an interval in this circle. Define $3I \in \mathcal{J}_k$ to be the interval in this circle which contains I at its center and has measure $\mu(3I) = 3\mu(I)$. In all cases, for $I \in \mathcal{J}$, $\mu(3I) \leq 3\mu(I)$.

The properties of weights in $A_p(G)$ are analogous to the properties of those that satisfy Muckenhoupt's A_p condition. (See [1 and 4].)

If $w \in A_p(G)$, $1 < p < \infty$, then w and $w^{-1/(p-1)} \in L^1$. Hence $0 < w < \infty$ a.e. and $L^p(w) \subset L^1$. Similarly, if $w \in A_1(G)$, then $w \in L^1$ and $\text{ess inf}_G w > 0$. Hence $0 < w < \infty$ a.e. and $L^1(w) \subset L^1$.

If $w \in A_p(G)$, $1 < p < \infty$, then Hölder's inequality implies that there is a constant C such that for any $I \in \mathcal{I}$,

$$(2.1) \quad \int_{3I} w \, d\mu \leq C \int_I w \, d\mu.$$

A basic property of $A_p(G)$ weights is the reverse Hölder inequality.

Lemma 2.1. *Let $w \in A_p(G)$, $1 < p < \infty$. Then there exist $s > 1$ and a constant C such that for any $I \in \mathcal{I}$,*

$$\left(\frac{1}{\mu(I)} \int_I w^s \, d\mu \right)^{1/s} \leq \frac{C}{\mu(I)} \int_I w \, d\mu.$$

The following two lemmas are consequences of the reverse Hölder inequality.

Lemma 2.2. *Suppose $w \in A_p(G)$, $1 < p < \infty$. Then there is $1 < q < p$ such that $w \in A_q(G)$.*

We also observe that if $w \in A_p(G)$, $1 \leq p < \infty$, then $w \in A_q(G)$, $p < q < \infty$.

Lemma 2.3. *Suppose $w \in A_p(G)$, $1 < p < \infty$. Given any $\varepsilon > 0$, there is $\delta > 0$ such that for any $I \in \mathcal{I}$, and any measurable $E \subset I$ with $\mu(E) \leq \delta \mu(I)$, we have $\int_E w \, d\mu \leq \varepsilon \int_I w \, d\mu$.*

The proofs of Lemmas 2.1–2.3 are similar to those given in [1]. (See also [4].)

We now define the Hardy-Littlewood maximal function that is appropriate for the study of Vilenkin-Fourier series. For $f \in L^1$, let

$$Mf(x) = \sup_{\substack{x \in I \\ I \in \mathcal{I}}} \frac{1}{\mu(I)} \int_I |f| \, d\mu.$$

This maximal function was first introduced by P. Simon in [9]. He also showed that the maximal operator is bounded in L^p , $1 < p \leq \infty$, and is of weak type $(1, 1)$. Using the above properties of $A_p(G)$ weights, we obtain the following analogue of Muckenhoupt's theorem [7].

Theorem 2.4. *Let w be a weight function on G . For $1 < p < \infty$, the following two statements are equivalent:*

- (i) $w \in A_p(G)$.
- (ii) *There is a constant C , depending only on w and p , such that for every $f \in L^p(w)$, we have $f \in L^1$ and*

$$\int_G (Mf)^p w \, d\mu \leq C \int_G |f|^p w \, d\mu.$$

The following two statements are also equivalent:

(iii) $w \in A_1(G)$.

(iv) There is a constant C , depending only on w , such that for every $f \in L^1(w)$, we have $f \in L^1$ and

$$\int_{\{Mf > y\}} w \, d\mu \leq Cy^{-1} \int_G |f|w \, d\mu, \quad y > 0.$$

Proof. The proof of this theorem follows the same lines as those in Muckenhoupt [7] and Coifman and Fefferman [1]. (See also [4].) We shall only discuss one modification which involves the generalized intervals.

To prove that the $A_p(G)$ condition is sufficient, suppose $1 \leq p < \infty$ and $w \in A_p(G)$. It is enough to prove the weak type inequality

$$(2.2) \quad \int_{\{Mf > 10y\}} w \, d\mu \leq Cy^{-p} \int_G |f|^p w \, d\mu, \quad y > 0, \quad f \in L_p(w).$$

The strong type inequality for $1 < p < \infty$ then follows from Lemma 2.2 and the Marcinkiewicz interpolation theorem.

To prove (2.2), we can assume that $y \geq \|f\|_1$. (If $y < \|f\|_1$, Hölder's inequality and the $A_p(G)$ condition imply

$$\int_G w \, d\mu \leq y^{-p} \|f\|_1^p \int_G w \, d\mu \leq Cy^{-p} \int_G |f|^p w \, d\mu.)$$

Apply the decomposition in [10], Lemma 2 to the function f and the value y to obtain a sequence of disjoint generalized intervals $\{I_j\}$ such that

$$(2.3) \quad y < \frac{1}{\mu(I_j)} \int_{I_j} |f| \, d\mu \leq 3y, \quad j = 1, 2, \dots,$$

and

$$(2.4) \quad |f(x)| \leq y \quad \text{for a.e. } x \notin \bigcup_j I_j.$$

We shall show that

$$(2.5) \quad \{Mf > 10y\} \subset \bigcup_j (3I_j).$$

Suppose $x \notin \bigcup_j (3I_j)$. Let $I \in \mathcal{I}$ such that $x \in I$. Then

$$\begin{aligned} \int_I |f| \, d\mu &= \int_{I \cap (\bigcup_j I_j)} |f| \, d\mu + \int_{I \cap (\bigcup_j I_j)^c} |f| \, d\mu \\ &\leq \sum_j \int_{I \cap I_j} |f| \, d\mu + y\mu(I), \end{aligned}$$

by (2.4). To estimate the second last term, suppose $I \cap I_j \neq \emptyset$. From this and the fact that $I \cap (3I_j) \neq \emptyset$ (since it contains x), we have $I_j \subset 3I$. (If I_j and I belong to distinct $\mathcal{I}_k$'s, then, since they are not disjoint, one is a subset of the other. From this it follows that $I_j \subset I \subset 3I$. If I_j and I belong to the same $\mathcal{I}_k$, then they belong to the same coset of G_k , and we obtain $I_j \subset 3I$ from a geometric observation.) Hence, by (2.3),

$$\begin{aligned} \sum_j \int_{I \cap I_j} |f| \, d\mu &\leq \sum_{I_j \subset 3I} \int_{I_j} |f| \, d\mu \leq \sum_{I_j \subset 3I} 3y\mu(I_j) \\ &\leq 3y\mu(3I) \leq 9y\mu(I). \end{aligned}$$

Therefore, for all such I , $\int_I |f| d\mu \leq 10y\mu(I)$, and we have $\{Mf > 10y\} \subset \bigcup_j (3I_j)$.

The rest of the proof of (2.2) then follows as in the above references, using (2.1), (2.3) and the $A_p(G)$ condition. This concludes the proof of Theorem 2.4.

3. THE SHARP FUNCTIONS

In this section, we introduce a sequence of sharp functions $\{f_n^\#\}$, which will be used to prove (i) $\Rightarrow$ (ii) of Theorem 1.1. Our sharp functions are modifications of that in Fefferman and Stein [3]. Before we define these modified sharp functions, we first introduce the functions $f_{I,n}$ which take the place of the mean value of f over I in the original sharp function.

Let n be a nonnegative integer expressed in the form $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$. For $I \in \mathcal{J}$ and $f \in L^1$, define $f_{I,n}$ as follows. If $I = G$, let $f_{I,n} = \int_G f d\mu$. For $I \in \mathcal{J}_k$, $k = 0, 1, \dots$, let $f_{I,n}$ be a function of the form $f_{I,n}(x) = a_{I,n}(f) + b_{I,n}(f)\phi_k^{-\alpha_k}(x)$, where $a_{I,n}(f)$ and $b_{I,n}(f)$ are constants, such that

$$(3.1) \quad \int_I f d\mu = \int_I f_{I,n} d\mu$$

and

$$(3.2) \quad \int_I f \phi_k^{\alpha_k} d\mu = \int_I f_{I,n} \phi_k^{\alpha_k} d\mu.$$

If I is not a coset of G_{k+1} , and if $\alpha_k \neq 0$, then $|(\mu(I))^{-1} \int_I \phi_k^{\alpha_k} d\mu| \neq 1$, and the choice for $a_{I,n}(f)$ and $b_{I,n}(f)$ is unique. (See [10, pp. 315–316].) If I is a coset of G_{k+1} , or if $\alpha_k = 0$, then $\phi_k^{-\alpha_k}(x)$ is a constant in I . In this case, we define $a_{I,n}(f) = (\mu(I))^{-1} \int_I f d\mu$, and $b_{I,n}(f) = 0$. For $n = 0, 1, \dots$, $f \in L^1$, define the n th sharp function as

$$f_n^\#(x) = \sup_{\substack{x \in I \\ I \in \mathcal{J}}} \frac{1}{\mu(I)} \int_I |f - f_{I,n}| d\mu.$$

The functions $f_{I,n}$ and $f_n^\#$ have the following properties.

Lemma 3.1. *There is a constant C such that for any $f \in L^1$, $I \in \mathcal{J}$, $n = 0, 1, \dots$, we have*

$$|f_{I,n}(x)| \leq \frac{C}{\mu(I)} \int_I |f| d\mu, \quad x \in I.$$

Proof. The proof is the same as that for (25) in [10]. See [10, pp. 315–316].

In the following, we shall set $\alpha_{-1} = 0$ for the sake of convenience.

Lemma 3.2. *There is a constant C such that for any constants A and B , we have*

$$(3.3) \quad \int_I |f - f_{I,n}| d\mu \leq C \int_I |f - A - B\phi_k^{-\alpha_k}| d\mu,$$

for all $f \in L^1$, $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$, and $I \in \mathcal{J}_k$, $k = -1, 0, 1, \dots$.

Proof. We observe that if I is a coset of G_k , $k = 0, 1, \dots$, or if $I \in \mathcal{J}_k$, $k = 0, 1, \dots$ and $\alpha_k = 0$, then $\phi_k^{-\alpha_k}(x)$ is constant in I . In these cases, we may assume $B = 0$. (3.3) holds with $C = 2$ since $f_{I,n} = (\mu(I))^{-1} \int_I f d\mu$.

Now, suppose $I \in \mathcal{J}_k$, $k = 0, 1, \dots$, I is not a coset of G_{k+1} , and $\alpha_k \neq 0$. Then

$$\begin{aligned} \int_I |f - f_{I,n}| d\mu &\leq \int_I |f - A - B\phi_k^{-\alpha_k}| d\mu \\ &\quad + \int_I |(a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}| d\mu. \end{aligned}$$

By (3.1) and (3.2),

$$\int_I (f - A - B\phi_k^{-\alpha_k}) d\mu = \int_I \{(a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}\} d\mu$$

and

$$\int_I (f - A - B\phi_k^{-\alpha_k}) d\mu = \int_I \{(a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}\} \phi_k^{\alpha_k} d\mu.$$

Therefore, by the uniqueness of the coefficients, we have

$$(f - A - B\phi_k^{-\alpha_k})_{I,n} = (a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}.$$

It then follows from Lemma 3.1 that

$$\begin{aligned} &\int_I |(a_{I,n}(f) - A) + (b_{I,n}(f) - B)\phi_k^{-\alpha_k}| d\mu \\ &= \int_I |(f - A - B\phi_k^{-\alpha_k})_{I,n}| d\mu \\ &\leq C \int_I |f - A - B\phi_k^{-\alpha_k}| d\mu. \end{aligned}$$

This proves Lemma 3.2.

For $f \in L^r$, $1 < r < \infty$, let

$$M_r f(x) = \sup_{\substack{x \in I \\ I \in \mathcal{J}}} \left(\frac{1}{\mu(I)} \int_I |f|^r d\mu \right)^{1/r}.$$

We have the following pointwise estimate concerning the n th sharp function of the n th modified partial sum $S_n^* f$.

Theorem 3.3. *Let $1 < r < \infty$. There is a constant C , depending only on r , such that for any $f \in L^r$, $n = 1, 2, \dots$,*

$$(S_n^* f)_n^\#(x) \leq C M_r f(x), \quad x \in G.$$

Proof. Let $x \in G$, and $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$. From Lemma 3.2, it is sufficient to show that for any $I \in \mathcal{J}_k$, $k = -1, 0, 1, \dots$, with $x \in I$, there exist constants A and B (depending on I, n and f) such that

$$(3.4) \quad \frac{1}{\mu(I)} \int_I |S_n^* f - A - B\phi_k^{-\alpha_k}| d\mu \leq CM_r f(x),$$

where C depends only on r .

Suppose $I \in \mathcal{J}_k$, $k = -1, 0, 1, \dots$, with $x \in I$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{3I}$. Then $S_n^* f = S_n^* f_1 + S_n^* f_2$. By Theorem 1* of [10], we have

$$(3.5) \quad \begin{aligned} \frac{1}{\mu(I)} \int_I |S_n^* f_1| d\mu &\leq \left(\frac{1}{\mu(I)} \int_G |S_n^* f_1|^r d\mu \right)^{1/r} \leq C \left(\frac{1}{\mu(I)} \int_G |f_1|^r d\mu \right)^{1/r} \\ &\leq C \left(\frac{1}{\mu(3I)} \int_{3I} |f|^r d\mu \right)^{1/r} \leq CM_r f(x). \end{aligned}$$

For $I = G$, $f_2 = 0$, and (3.4) follows with $A = B = 0$. From now on, we can assume $k \geq 0$.

To estimate $S_n^* f_2$, let c be a fixed point in I , and $y \in I$. By (1.6) we have

$$S_n^* f_2(y) - S_n^* f_2(c) = \sum_{l=0}^{\infty} \{S_{\alpha_l m_l}^* f_2(y) - S_{\alpha_l m_l}^* f_2(c)\}.$$

Expand each term on the right according to (1.7). For $l \geq k+1$, $y + G_l$ and $c + G_l$ are subsets of I . Therefore f_2 vanishes in these cosets. For $0 \leq l \leq k$, $y + G_l = c + G_l$. Hence

$$\begin{aligned} S_n^* f_2(y) - S_n^* f_2(c) &= \sum_{l=0}^k \frac{1}{\mu(G_l)} \int_{y+G_l} f_2(t) \left\{ \phi_l^{-\alpha_l}(y-t) \left(\sum_{j=0}^{\alpha_l-1} \phi_l^j(y-t) \right) \right. \\ &\quad \left. - \phi_l^{-\alpha_l}(c-t) \left(\sum_{j=0}^{\alpha_l-1} \phi_l^j(c-t) \right) \right\} d\mu(t). \end{aligned}$$

Now, for any $t \in G$, $y-t$ and $c-t$ belong to the same coset of G_l , for any $0 \leq l \leq k$. Since ϕ_l is constant on each coset of G_{l+1} , $l = 0, 1, \dots$, we have $\phi_l(y-t) = \phi_l(c-t)$, $0 \leq l \leq k-1$. Therefore all integrals on the right vanish except for the term with $l = k$. We thus have

$$(3.6) \quad S_n^* f_2(y) - S_n^* f_2(c) = S_{\alpha_k m_k}^* f_2(y) - S_{\alpha_k m_k}^* f_2(c).$$

The constants A and B are defined in terms of the conjugate functions (see (1.8)). Let $A = S_n^* f_2(c) - i\phi_k^{-\alpha_k}(c)H_k(f_2\phi_k^{\alpha_k})(c)$ and $B = iH_k(f_2\phi_k^{\alpha_k})(c)$. For $y \in I$, it follows from (3.6) that

$$(3.7) \quad \begin{aligned} S_n^* f_2(y) - A - B\phi_k^{-\alpha_k}(y) &= S_{\alpha_k m_k}^* f_2(y) - S_{\alpha_k m_k}^* f_2(c) \\ &\quad + i\phi_k^{-\alpha_k}(c)H_k(f_2\phi_k^{\alpha_k})(c) - i\phi_k^{-\alpha_k}(y)H_k(f_2\phi_k^{\alpha_k})(c). \end{aligned}$$

Using (1.9), we express the right side as a sum of the following terms:

$$\begin{aligned}
 E_1 &= \frac{\alpha_k}{\mu(G_k)} \int_{(y+G_k) \cap \{y_k=t_k\}} f_2(t) d\mu(t) \\
 &\quad - \frac{\alpha_k}{\mu(G_k)} \int_{(c+G_k) \cap \{c_k=t_k\}} f_2(t) d\mu(t), \\
 E_2 &= \frac{1}{2} \phi_k^{-\alpha_k}(y) \frac{1}{\mu(G_k)} \int_{(y+G_k) \cap \{y_k \neq t_k\}} f_2(t) \phi_k^{\alpha_k}(t) d\mu(t) \\
 &\quad - \frac{1}{2} \frac{1}{\mu(G_k)} \int_{(y+G_k) \cap \{y_k \neq t_k\}} f_2(t) d\mu(t) \\
 &\quad - \frac{1}{2} \phi_k^{-\alpha_k}(c) \frac{1}{\mu(G_k)} \int_{(c+G_k) \cap \{c_k \neq t_k\}} f_2(t) \phi_k^{\alpha_k}(t) d\mu(t) \\
 &\quad + \frac{1}{2} \frac{1}{\mu(G_k)} \int_{(c+G_k) \cap \{c_k \neq t_k\}} f_2(t) d\mu(t), \\
 E_3 &= i \phi_k^{-\alpha_k}(y) \{H_k(f_2 \phi_k^{\alpha_k})(y) - H_k(f_2 \phi_k^{\alpha_k})(c)\}, \\
 E_4 &= -i \{H_k f_2(y) - H_k f_2(c)\}.
 \end{aligned}
 \tag{3.8}$$

We shall show that $E_1 = 0$, and each of $|E_2|$, $|E_3|$ and $|E_4|$ is dominated by $CMf(x)$, and hence by $CM_r f(x)$. (3.4) will then follow from (3.5), (3.7), (3.8) and these estimates.

Since f_2 vanishes in I , and hence in $(y + G_k) \cap \{y_k = t_k\}$ and $(c + G_k) \cap \{c_k = t_k\}$, we have $E_1 = 0$.

Next, the absolute value of each of the terms in E_2 is bounded by

$$(2\mu(G_k))^{-1} \int_{x+G_k} |f| d\mu.$$

Therefore $|E_2| \leq 2Mf(x)$.

To estimate E_3 , we observe that for $y \in I$,

$$H_k(f_2 \phi_k^{\alpha_k})(y) = \frac{1}{2} \frac{1}{\mu(G_k)} \int_{(x+G_k) \cap {}^c(3I)} f(t) \phi_k^{\alpha_k}(t) \cot(\pi(y_k - t_k)/p_k) d\mu(t),$$

since $f_2 = 0$ in $3I$ and $(y + G_k) \cap \{y_k \neq t_k\} \supset (x + G_k) \cap {}^c(3I)$. A similar expression holds for $H_k(f_2 \phi_k^{\alpha_k})(c)$. Hence

$$\begin{aligned}
 |E_3| &= |H_k(f_2 \phi_k^{\alpha_k})(y) - H_k(f_2 \phi_k^{\alpha_k})(c)| \\
 &\leq \frac{1}{2} \frac{1}{\mu(G_k)} \int_{(x+G_k) \cap {}^c(3I)} |f(t)| \left| \cot\left(\frac{\pi(y_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(c_k - t_k)}{p_k}\right) \right| d\mu(t).
 \end{aligned}
 \tag{3.9}$$

Let $3^{j+1}I = 3(3^jI)$, $j = 1, 2, \dots$. If $3I \neq x + G_k$, write $(x + G_k) \cap {}^c(3I) = \bigcup_{j=1}^{J-1} (x + G_k) \cap (3^{j+1}I \setminus 3^jI)$, where $J = \min\{j \geq 1: 3^jI = x + G_k\}$. Now, for $1 \leq j \leq J-1$ and $t \in (x + G_k) \cap (3^{j+1}I \setminus 3^jI)$,

$$\begin{aligned}
 &\left| \cot\left(\frac{\pi(y_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(c_k - t_k)}{p_k}\right) \right| \\
 &= \left| \sin\left(\frac{\pi(y_k - c_k)}{p_k}\right) / \left\{ \sin\left(\frac{\pi(y_k - t_k)}{p_k}\right) \sin\left(\frac{\pi(c_k - t_k)}{p_k}\right) \right\} \right| \\
 &\leq C\mu(I)\mu(G_k)/(\mu(3^{j-1}I))^2 \leq C3^{-j}\mu(G_k)/\mu(3^{j+1}I).
 \end{aligned}$$

Substituting these into (3.9), we obtain

$$\begin{aligned} |E_3| &\leq C \sum_{j=1}^{J-1} 3^{-j} \frac{1}{\mu(3^{j+1}I)} \int_{3^{j+1}I} |f(t)| d\mu(t) \\ &\leq C \sum_{j=1}^{J-1} 3^{-j} Mf(x) \leq CMf(x). \end{aligned}$$

The same argument shows that $|E_4| \leq CMf(x)$. This completes the proof of Theorem 3.3.

We have the following weighted norm inequality relating the maximal function and the sharp functions.

Theorem 3.4. *Let $1 < p < \infty$, $w \in A_p(G)$. There is a constant C , depending only on p and w , such that for any $f \in L^p(w)$, $n = 0, 1, \dots$,*

$$(3.10) \quad \int_G (Mf)^p w d\mu \leq C \int_G (f_n^*)^p w d\mu + C \left(\int_G |f| d\mu \right)^p \int_G w d\mu.$$

Proof. Let $f \in L^p(w)$. We observe that $w \in A_p(G)$ implies that $f \in L^1$. Let $y_0 = \|f\|_1$ and n be a nonnegative integer. For any $y > y_0$, apply the decomposition in [10, Lemma 2] to the function f and the value y to obtain a sequence of disjoint generalized intervals $\{I_j\}_{j \geq 1}$ such that

$$(3.11) \quad y < \frac{1}{\mu(I_j)} \int_{I_j} |f| d\mu \leq 3y, \quad j = 1, 2, \dots,$$

and

$$|f(x)| \leq y \quad \text{for a.e. } x \notin \bigcup_j I_j.$$

From Lemma 3.1, there is a constant $A > 1$, independent of f , such that for any $I \in \mathcal{J}$, $n = 0, 1, \dots$,

$$(3.12) \quad |f_{I,n}(x)| \leq \frac{A}{\mu(I)} \int_I |f| d\mu, \quad x \in I.$$

Since $y < 4Ay$, the above decomposition applied to f and the value $4Ay$ gives a sequence of disjoint generalized intervals $\{I_{ij}\}_{i,j \geq 1}$ such that $\bigcup_i I_{ij} \subset I_j$, $j = 1, 2, \dots$,

$$(3.13) \quad 4Ay < \frac{1}{\mu(I_{ij})} \int_{I_{ij}} |f| d\mu \leq 12Ay, \quad i, j = 1, 2, \dots,$$

and

$$|f(x)| \leq 4Ay \quad \text{for a.e. } x \notin \bigcup_{i,j} I_{ij}.$$

Let $\delta > 0$ be a constant whose value is to be chosen later. For each I_j , there are two possibilities.

Case 1. $I_j \subset \{f_n^* > \delta Ay\}$. Then

$$\sum_i \int_{I_{ij}} w d\mu \leq \int_{\{f_n^* > \delta Ay\} \cap I_j} w d\mu.$$

Case 2. $I_j \notin \{f_n^* > \delta Ay\}$. Then

$$\frac{1}{\mu(I_j)} \int_{I_j} |f - f_{I_j, n}| d\mu \leq \delta Ay.$$

Hence

$$(3.14) \quad \delta Ay \mu(I_j) \geq \sum_i \left(\int_{I_{ij}} |f| d\mu - \int_{I_{ij}} |f_{I_j, n}| d\mu \right).$$

By (3.13), we have

$$\int_{I_{ij}} |f| d\mu \geq 4Ay \mu(I_{ij}).$$

Also, from (3.12) and (3.11), we have, for $x \in I_{ij} \subset I_j$,

$$|f_{I_j, n}(x)| \leq A(\mu(I_j))^{-1} \int_{I_j} |f| d\mu \leq 3Ay.$$

Hence

$$\int_{I_{ij}} |f_{I_j, n}| d\mu \leq 3Ay \mu(I_{ij}).$$

Substituting these estimates into (3.14), we obtain

$$\mu \left(\bigcup_i I_{ij} \right) \leq \delta \mu(I_j).$$

For $\varepsilon > 0$ to be chosen later, let δ be the value that associates with ε in Lemma 2.3. Then

$$\sum_i \int_{I_{ij}} w d\mu \leq \varepsilon \int_{I_j} w d\mu.$$

Summing over all I_j in Cases 1 and 2, we obtain

$$\sum_{i,j} \int_{I_{ij}} w d\mu \leq \int_{\{f_n^* > \delta Ay\}} w d\mu + \varepsilon \int_{\bigcup_j I_j} w d\mu.$$

Now, from (2.5), we have $\{Mf > 40Ay\} \subset \bigcup_{i,j} (3I_{ij})$. Hence, by (2.1), there is a constant B such that

$$\int_{\{Mf > 40Ay\}} w d\mu \leq B \sum_{i,j} \int_{I_{ij}} w d\mu.$$

Also, $\bigcup_j I_j \subset \{Mf > y\}$. Therefore, for $y \geq y_0$, we have

$$\int_{\{Mf > 40Ay\}} w d\mu \leq B \int_{\{f_n^* > \delta Ay\}} w d\mu + B\varepsilon \int_{\{Mf > y\}} w d\mu.$$

Multiply both sides of the above inequality by py^{p-1} and integrate over $y \in [y_0, \infty)$. We obtain

$$\begin{aligned} p \int_{y_0}^{\infty} y^{p-1} \int_{\{Mf > 40Ay\}} w d\mu dy \\ \leq B(\delta A)^{-p} \int_G (f_n^*)^p w d\mu + B\varepsilon \int_G (Mf)^p w d\mu. \end{aligned}$$

Also,

$$p \int_0^{y_0} y^{p-1} \int_{\{Mf > 40Ay\}} w \, d\mu \, dy \leq \left(\int_G |f| \, d\mu \right)^p \int_G w \, d\mu.$$

Therefore, we have

$$\begin{aligned} \int_G (Mf)^p w \, d\mu &\leq B(40/\delta)^p \int_G (f_n^*)^p w \, d\mu \\ &\quad + B\varepsilon(40A)^p \int_G (Mf)^p w \, d\mu \\ &\quad + (40A)^p \left(\int_G |f| \, d\mu \right)^p \int_G w \, d\mu. \end{aligned}$$

Since $f \in L^p(w)$ and $w \in A_p(G)$, we have $Mf \in L^p(w)$, by Theorem 2.4. Choosing $\varepsilon = (2B)^{-1}(40A)^{-p}$, we obtain (3.10). This completes the proof of Theorem 3.4.

4. PROOF OF THEOREM 1.1

We shall prove Theorem 1.1 by showing (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof of (i) $\Rightarrow$ (ii). Suppose $w \in A_p(G)$. We have $L^p(w) \subset L^1$. It is sufficient to prove (1.2) with S_n replaced by S_n^* . Let $f \in L^p(w)$. We observe that $S_n^* f \in L^p(w)$, since for $x \in G$,

$$|S_n^* f(x)| \leq n \int_G |f| \, d\mu \leq n \left(\int_G |f|^p w \, d\mu \right)^{1/p} \left(\int_G w^{-1/(p-1)} \, d\mu \right)^{1-1/p},$$

by Hölder's inequality. Therefore, using the $A_p(G)$ condition, we have

$$\int_G |S_n^* f|^p w \, d\mu \leq Cn^p \int_G |f|^p w \, d\mu < \infty.$$

We now apply Theorem 3.4 to the function $S_n^* f$ to get

$$\begin{aligned} \int_G |S_n^* f|^p w \, d\mu &\leq \int_G (M(S_n^* f))^p w \, d\mu \\ (4.1) \quad &\leq C \int_G ((S_n^* f)_n^*)^p w \, d\mu + C \left(\int_G |S_n^* f| \, d\mu \right)^p \int_G w \, d\mu. \end{aligned}$$

Since $w \in A_p(G)$, Lemma 2.2 shows there is $1 < r < p$ such that $w \in A_{p/r}(G)$. Hence $|f|^r \in L^{p/r}(w) \subset L^1$, and it follows from Theorem 3.3 and Theorem 2.4 that

$$\begin{aligned} \int_G ((S_n^* f)_n^*)^p w \, d\mu &\leq C \int_G (M_r f)^p w \, d\mu \\ (4.2) \quad &= C \int_G (M|f|^r)^{p/r} w \, d\mu \leq C \int_G |f|^p w \, d\mu. \end{aligned}$$

Also, by Theorem 1* of [10],

$$\int_G |S_n^* f| \, d\mu \leq \left(\int_G |S_n^* f|^r \, d\mu \right)^{1/r} \leq C \left(\int_G |f|^r \, d\mu \right)^{1/r}.$$

Therefore, Hölder's inequality gives

$$\begin{aligned}
 & \left(\int_G |S_n^* f| d\mu \right)^p \int_G w d\mu \\
 (4.3) \quad & \leq C \left(\int_G |f|^p w d\mu \right) \left(\int_G w^{-1/(p/r-1)} d\mu \right)^{p/r-1} \left(\int_G w d\mu \right) \\
 & \leq C \int_G |f|^p w d\mu,
 \end{aligned}$$

since $w \in A_{p/r}(G)$. Substituting (4.2) and (4.3) into (4.1), we obtain

$$\int_G |S_n^* f|^p w d\mu \leq C \int_G |f|^p w d\mu.$$

This completes the proof of (i) $\Rightarrow$ (ii).

Proof of (ii) $\Rightarrow$ (i). Suppose w satisfies (ii). As a consequence of $L^p(w) \subset L^1$, we have $w^{-1/(p-1)} \in L^1$. (Otherwise, $w^{-1/p} \notin L^{p'}$, where $1/p + 1/p' = 1$. Hence there exists $g \in L^p$ such that $gw^{-1/p} \notin L^1$, contradicting $gw^{-1/p} \in L^p(w)$.) Therefore $\|w^{-1/(p-1)}\|_{p,w} = \|w^{-1/(p-1)}\|_1 < \infty$.

To prove (1.1) for $I = G$, let $f = w^{-1/(p-1)}$. Then, for $x \in G$, $S_1 f(x) = \|f\|_1 = \|w^{-1/(p-1)}\|_1 > 0$, otherwise we have $w \equiv \infty$ a.e. Hence, for every $0 < y \leq \|f\|_1$, it follows from (1.2) that

$$\int_G w d\mu = \int_{\{S_1 f > y\}} w d\mu \leq Cy^{-p} \int_G |f|^p w d\mu.$$

Thus

$$(4.4) \quad \int_G w d\mu \leq C \left(\int_G f d\mu \right)^{-p} \int_G |f|^p w d\mu,$$

and hence

$$\left(\int_G w d\mu \right) \left(\int_G w^{-1/(p-1)} d\mu \right)^{p-1} \leq C.$$

Next, consider $I \in \mathcal{J}_k$, $k = 0, 1, \dots$. We can assume $\int_I w^{-1/(p-1)} d\mu > 0$, or else there is nothing to prove. First let us assume $\mu(I) \leq \mu(G_k)/2$. Take $\alpha_k = [\mu(G_k)/(2\mu(I))]$, where $[a]$ is the largest integer less than or equal to a . We have $\alpha_k \geq 1$. Let $f = w^{-1/(p-1)} \chi_I$. From (1.7), we have, for $x \in G$,

$$\begin{aligned}
 S_{\alpha_k m_k} f(x) &= \frac{1}{\mu(G_k)} \int_I f(t) \left(\sum_{j=0}^{\alpha_k-1} \phi_k^j(x-t) \right) d\mu(t) \\
 &= \frac{1}{\mu(G_k)} \int_I f(t) \phi_k^{(\alpha_k-1)/2}(x-t) \frac{\sin(\pi \alpha_k (x_k - t_k)/p_k)}{\sin(\pi (x_k - t_k)/p_k)} d\mu(t).
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \phi_k^{-(\alpha_k-1)/2}(x) S_{\alpha_k m_k} (f \phi_k^{(\alpha_k-1)/2})(x) \\
 &= \frac{1}{\mu(G_k)} \int_I f(t) \frac{\sin(\pi \alpha_k (x_k - t_k)/p_k)}{\sin(\pi (x_k - t_k)/p_k)} d\mu(t).
 \end{aligned}$$

Since for $x, t \in I$,

$$\frac{\sin(\pi \alpha_k (x_k - t_k)/p_k)}{\sin(\pi (x_k - t_k)/p_k)} \geq \frac{2\alpha_k}{\pi} \geq \frac{\mu(G_k)}{2\pi\mu(I)},$$

we have, for $x \in I$,

$$\phi_k^{-(\alpha_k-1)/2}(x) S_{\alpha_k m_k}(f \phi_k^{(\alpha_k-1)/2})(x) \geq \frac{1}{2\pi\mu(I)} \int_I f(t) d\mu(t).$$

Therefore, for any $0 < y < (2\pi\mu(I))^{-1} \int_I f d\mu$,

$$I \subset \{\phi_k^{-(\alpha_k-1)/2} S_{\alpha_k m_k}(f \phi_k^{(\alpha_k-1)/2}) > y\}.$$

Thus, by (1.2),

$$\begin{aligned} \int_I w d\mu &\leq \int_{\{|S_{\alpha_k m_k}(f \phi_k^{(\alpha_k-1)/2})| > y\}} w d\mu \\ &\leq C y^{-p} \int_G |f \phi_k^{(\alpha_k-1)/2}|^p w d\mu \\ &= C y^{-p} \int_I |f|^p w d\mu. \end{aligned}$$

It follows that

$$(4.5) \quad \int_I w d\mu \leq C(2\pi\mu(I))^p \left(\int_I f d\mu \right)^{-p} \int_I |f|^p w d\mu,$$

and hence

$$\left(\int_I w d\mu \right) \left(\int_I w^{-1/(p-1)} d\mu \right)^{p-1} \leq C(\mu(I))^p.$$

We have thus proved (1.1) for $I = G$, and for $I \in \mathcal{J}_k$, $k = 0, 1, \dots$, with $\mu(I) \leq \mu(G_k)/2$. In particular, since for $k = 1, 2, \dots$, every coset of G_k is in $\mathcal{J}_{k-1}$ and $\mu(G_k) \leq \mu(G_{k-1})/2$, (1.1) holds for all cosets of G_k , $k = 0, 1, \dots$.

For the case $I \in \mathcal{J}_k$, $k = 0, 1, \dots$, $\mu(I) > \mu(G_k)/2$, assume $I \subset x + G_k$, $x \in G$. Then

$$\begin{aligned} &\left(\int_I w d\mu \right) \left(\int_I w^{-1/(p-1)} d\mu \right)^{p-1} \\ &\leq \left(\int_{x+G_k} w d\mu \right) \left(\int_{x+G_k} w^{-1/(p-1)} d\mu \right)^{p-1} \\ &\leq C(\mu(G_k))^p \leq C2^p(\mu(I))^p. \end{aligned}$$

This completes the proof of (ii) $\Rightarrow$ (i).

Before we prove (ii) $\Leftrightarrow$ (iii), we introduce the following notation. For $k = 0, 1, \dots$, let $\mathcal{J}_k$ be the σ -algebra generated by the cosets of G_k . We also say that a function f belongs to $\mathcal{J}_k$, or $f \in \mathcal{J}_k$, if f is measurable with respect to $\mathcal{J}_k$.

Proof of (ii) $\Rightarrow$ (iii). Let $f \in L^p(w)$. Since (ii) $\Rightarrow$ (i), we have $w \in L^1$. Therefore, given $\varepsilon > 0$, there is $g \in \mathcal{J}_k$ for some $k = 0, 1, \dots$, such that $\|f - g\|_{p,w} < \varepsilon$. Now, for $n \geq m_k$, $S_n g = g$. Hence, for all such n ,

$$\begin{aligned} \|S_n f - f\|_{p,w} &\leq \|S_n(f - g)\|_{p,w} + \|S_n g - g\|_{p,w} + \|g - f\|_{p,w} \\ &\leq C\|f - g\|_{p,w} < C\varepsilon, \end{aligned}$$

by (1.2). This proves (1.3).

Proof of (iii) $\Rightarrow$ (ii). We first show that $w \in L^1$. Let $f \in L^p(w)$ such that $f \notin \mathcal{F}_k$ for all $k = 0, 1, \dots$ (f can be obtained as follows. If $w \notin \mathcal{F}_k$ for any $k = 0, 1, \dots$, take $f = w^{-1/p}$. Otherwise $w \in \mathcal{F}_k$ for some $k = 0, 1, \dots$. Since $w \neq \infty$ a.e., there is a coset I of G_k such that $w(x) = c < \infty$ for $x \in I$. Let f be any function not in $\mathcal{F}_k$ for all $k = 0, 1, \dots$, $f = 0$ outside I and $\|f\|_p < \infty$.) For this f , $\hat{f}(n) = \int_G f \bar{\chi}_n d\mu \neq 0$ for infinitely many n . But $|\hat{f}(n)| \|w\|_1^{1/p} = \|\hat{f}(n)\chi_n\|_{p,w} = \|S_{n+1}f - S_nf\|_{p,w} \rightarrow 0$ as $n \rightarrow \infty$ by (1.3). This shows $w \in L^1$.

From the proof of (ii) $\Rightarrow$ (i), we know that w satisfies $L^p(w) \subset L^1$ implies that $w^{-1/(p-1)} \in L^1$. Therefore, for any $n \geq 1$, S_n is a bounded operator from $L^p(w)$ into itself, with norm not exceeding $n\|w\|_1^{1/p}\|w^{-1/(p-1)}\|_1^{1-1/p}$. Since, by (1.3), $\sup_n \|S_nf\|_{p,w} < \infty$ for every $f \in L^p(w)$, (1.2) follows from the uniform boundedness principle. The proof of Theorem 1.1 is now complete.

5. PROOF OF THEOREM 1.2

We shall prove the weak type (1, 1) result using Theorem 1.1 and the modified Calderón-Zygmund decomposition lemma obtained in [10].

Proof of (i) $\Rightarrow$ (ii). Suppose $w \in A_1(G)$. We have $L^1(w) \subset L^1$. It is sufficient to prove (1.5) with S_n replaced by S_n^* . Let $f \in L^1(w)$. If $y < \|f\|_1$, then the $A_1(G)$ condition gives

$$\int_{\{|S_n^* f| > y\}} w d\mu \leq \left(\int_G w d\mu \right) y^{-1} \int_G |f| d\mu \leq Cy^{-1} \int_G |f| w d\mu.$$

Therefore we can restrict ourselves to $y \geq \|f\|_1$.

Let $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$. From Lemma 2 of [10], we obtain a sequence of disjoint generalized intervals $\{I_j\}$ such that

$$(5.1) \quad y < \frac{1}{\mu(I_j)} \int_{I_j} |f| d\mu \leq 3y, \quad j = 1, 2, \dots,$$

and

$$(5.2) \quad |f(x)| \leq y \quad \text{for a.e. } x \notin \bigcup_j I_j \equiv \Omega.$$

Moreover, there is a decomposition of $f = g + b$, with $g, b \in L^1$, such that

$$(5.3) \quad g(x) = f(x) \quad \text{for } x \notin \Omega,$$

$$(5.4) \quad |g(x)| \leq \frac{C}{\mu(I_j)} \int_{I_j} |f| d\mu, \quad x \in I_j, \quad j = 1, 2, \dots,$$

$$(5.5) \quad \int_{I_j} b d\mu = 0, \quad j = 1, 2, \dots,$$

and

$$(5.6) \quad \int_{I_j} b \phi_k^{\alpha_k} d\mu = 0 \quad \text{for all } I_j \in \mathcal{F}_k, \quad k = 0, 1, \dots$$

(The constant C in (5.4) is independent of f and y .)

Since

$$\int_{\{|S_n^* f| > y\}} w \, d\mu \leq \int_{\{|S_n^* g| > y/2\}} w \, d\mu + \int_{\{|S_n^* b| > y/2\}} w \, d\mu,$$

we shall show that each term on the right is bounded by $Cy^{-1}\|f\|_{1,w}$.

Because $g \in L^\infty \subset L^2(w)$ and $w \in A_1(G) \subset A_2(G)$, it follows from Theorem 1.1 that

$$(5.7) \quad \begin{aligned} \int_{\{|S_n^* g| > y/2\}} w \, d\mu &\leq 4y^{-2} \int_G |S_n^* g|^2 w \, d\mu \leq Cy^{-2} \int_G |g|^2 w \, d\mu \\ &= Cy^{-2} \left(\int_{c_\Omega} |g|^2 w \, d\mu + \sum_j \int_{I_j} |g|^2 w \, d\mu \right). \end{aligned}$$

By (5.3) and (5.2),

$$\int_{c_\Omega} |g|^2 w \, d\mu \leq y \int_{c_\Omega} |f| w \, d\mu.$$

Also, by (5.4), (5.1) and the $A_1(G)$ condition,

$$\int_{I_j} |g|^2 w \, d\mu \leq \left(\frac{C}{\mu(I_j)} \int_{I_j} |f| \, d\mu \right)^2 \left(\int_{I_j} w \, d\mu \right) \leq Cy \int_{I_j} |f| w \, d\mu.$$

Substituting these estimates into (5.7), we obtain

$$\int_{\{|S_n^* g| > y/2\}} w \, d\mu \leq Cy^{-1} \int_G |f| w \, d\mu.$$

To estimate $S_n^* b$, let $\Omega^* = \bigcup_j (3I_j)$. From (2.1), (5.1) and the $A_1(G)$ condition, we have

$$\begin{aligned} \int_{\Omega^*} w \, d\mu &\leq C \sum_j \int_{I_j} w \, d\mu \\ &\leq C \sum_j \left(\int_{I_j} w \, d\mu \right) \left(y^{-1} \frac{1}{\mu(I_j)} \int_{I_j} |f| \, d\mu \right) \\ &\leq Cy^{-1} \sum_j \int_{I_j} |f| w \, d\mu \leq Cy^{-1} \int_G |f| w \, d\mu. \end{aligned}$$

Hence, it is sufficient to show that

$$(5.8) \quad \int_{\{x \notin \Omega^* : |S_n^* b(x)| > y/2\}} w \, d\mu \leq Cy^{-1} \int_G |f| w \, d\mu.$$

To do this, we expand $S_n^* b$ as in (1.6) and (1.9). For $x \notin \Omega^*$, it follows from (5.3), (5.5) and (5.6) that the first three terms in (1.9) vanish. (See the explanation in [10, pp. 317–318].) Therefore, for $x \notin \Omega^*$, we have

$$S_n^* b(x) = i \sum_{k=0}^{\infty} \{ \phi_k^{-\alpha_k}(x) H_k(b\phi_k^{\alpha_k})(x) - H_k b(x) \}.$$

(5.8) will be proved if we can show

$$(5.9) \quad \sum_{k=0}^{\infty} \int_{c_{\Omega^*}} |H_k(b\phi_k^{\alpha_k})| w \, d\mu \leq C \int_G |f| w \, d\mu,$$

and

$$(5.10) \quad \sum_{k=0}^{\infty} \int_{c_{\Omega^*}} |H_k b| w \, d\mu \leq C \int_G |f| w \, d\mu.$$

Suppose $x \notin \Omega^*$. It follows from (1.8), (5.3), (5.5) and (5.6) that

$$\begin{aligned} H_k(b\phi_k^{\alpha_k})(x) &= \frac{1}{2} \frac{1}{\mu(G_k)} \sum_{\substack{I_j \subset x+G_k \\ I_j \in \mathcal{J}_k}} \int_{I_j} b(t) \phi_k^{\alpha_k}(t) \\ &\quad \times \left\{ \cot \left(\frac{\pi(x_k - t_k)}{p_k} \right) - \cot \left(\frac{\pi(x_k - t_k^j)}{p_k} \right) \right\} d\mu(t), \end{aligned}$$

where $t^j = \{t_k^j\}_{k \geq 0}$ is any fixed point in I_j . (See [10, p. 318].) Let I be any coset of G_k . Fubini's theorem gives

$$\begin{aligned} \int_{I \cap c_{\Omega^*}} |H_k(b\phi_k^{\alpha_k})(x)| w(x) \, d\mu(x) &\leq \frac{1}{2} \frac{1}{\mu(I)} \sum_{\substack{I_j \subset I \\ I_j \in \mathcal{J}_k}} \int_{I_j} |b(t)| \\ &\quad \cdot \int_{I \cap c(3I_j)} \left| \cot \left(\frac{\pi(x_k - t_k)}{p_k} \right) - \cot \left(\frac{\pi(x_k - t_k^j)}{p_k} \right) \right| w(x) \, d\mu(x) \, d\mu(t). \end{aligned}$$

From the proof of Theorem 3.3 (see the estimate of $|E_3|$ below (3.9)), we have, for any $t \in I_j$, $I_j \subset I$, $I_j \in \mathcal{J}_k$,

$$\frac{1}{\mu(I)} \int_{I \cap c(3I_j)} \left| \cot \left(\frac{\pi(x_k - t_k)}{p_k} \right) - \cot \left(\frac{\pi(x_k - t_k^j)}{p_k} \right) \right| w(x) \, d\mu(x) \leq CMw(t).$$

Moreover, the last term does not exceed $Cw(t)$ for a.e. $t \in I_j$, by the $A_1(G)$ condition. Therefore

$$(5.11) \quad \int_{I \cap c_{\Omega^*}} |H_k(b\phi_k^{\alpha_k})| w \, d\mu \leq C \sum_{\substack{I_j \subset I \\ I_j \in \mathcal{J}_k}} \int_{I_j} |b| w \, d\mu.$$

Writing $b = f - g$, using (5.4) and the $A_1(G)$ condition again, we get

$$\begin{aligned} \int_{I_j} |b| w \, d\mu &\leq \int_{I_j} |f| w \, d\mu + \left(\frac{C}{\mu(I_j)} \int_{I_j} |f| \, d\mu \right) \left(\int_{I_j} w \, d\mu \right) \\ &\leq C \int_{I_j} |f| w \, d\mu. \end{aligned}$$

Substituting this into (5.11), and summing over all cosets I of G_k , and then over all k , we obtain

$$\sum_{k=0}^{\infty} \int_{c_{\Omega^*}} |H_k(b\phi_k^{\alpha_k})| w \, d\mu \leq C \sum_{k=0}^{\infty} \sum_{I_j \in \mathcal{J}_k} \int_{I_j} |f| w \, d\mu \leq C \int_G |f| w \, d\mu.$$

This proves (5.9).

(5.10) can be proved similarly, using (5.5) instead of (5.6). With this we have completed the proof of (i) $\Rightarrow$ (ii).

Proof of (ii) $\Rightarrow$ (i). Suppose w satisfies the conditions in (ii). We first prove (1.4) for $I = G$ and for $I = \mathcal{J}_k$, $k = 0, 1, \dots$, with $\mu(I) \leq \mu(G_k)/2$. Let $z > \text{ess inf}_I w$, and $E_z = \{x \in I: w(x) < z\}$. Then $\mu(E_z) > 0$. Define $f = \chi_{E_z}$. We have $\|f\|_{1,w} \leq z\mu(E_z) < \infty$. Applying the proofs of (4.4) and (4.5) to this f , using (1.5) instead of (1.2), we get

$$\int_I w d\mu \leq C\mu(I) \left(\int_I f d\mu \right)^{-1} \int_I |f|w d\mu \leq C\mu(I)z.$$

Since this holds for any $z > \text{ess inf}_I w$, we have

$$\frac{1}{\mu(I)} \int_I w d\mu \leq C \text{ess inf}_I w.$$

This proves (1.4) for $I = G$, and for $I \in \mathcal{J}_k$, $k = 0, 1, \dots$, with $\mu(I) \leq \mu(G_k)/2$. In particular, (1.4) holds for all cosets of G_k , $k = 0, 1, \dots$.

For $I \in \mathcal{J}_k$, $k = 0, 1, \dots$, with $\mu(I) > \mu(G_k)/2$, let $I \subset x + G_k$, $x \in G$. Then

$$\frac{1}{\mu(I)} \int_I w d\mu \leq \frac{2}{\mu(G_k)} \int_{x+G_k} w d\mu \leq 2C \text{ess inf}_{x+G_k} w \leq 2C \text{ess inf}_I w.$$

This proves (ii) $\Rightarrow$ (i) and concludes the proof of Theorem 1.2.

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