THE NEF VALUE AND DEFECT OF HOMOGENEOUS LINE BUNDLES

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ABSTRACT. Formulas for the nef value of a homogeneous line bundle are derived and applied to the classification of homogeneous spaces with positive defect and to the classification of complete homogeneous real hypersurfaces of projective space.

Let X be a smooth projective variety imbedded in $\mathbb{P}^N$ by the sections of some very ample line bundle L . If the canonical bundle K_X is not numerically effective (nef), then there is a smallest rational number $\tau = \tau(X, L)$ called the nef value of (X, L) such that $K_X \otimes L^\tau$ is nef. The map $\psi: X \rightarrow Y$ defined by the sections of some power of $K_X \otimes L^\tau$ is called the nef value morphism. In this paper a general formula is derived for the nef value of L when X is a homogeneous space equivariantly imbedded in $\mathbb{P}^N$ by the sections of L , see Theorem 2.2. It is then an easy matter to tabulate the exact values for $\tau(X, L)$ when $\text{Pic}(X) \cong \mathbb{Z}$, see Corollary 2.4.

As is shown in [2, 3], there is a connection between the nef value, $\tau(X, L)$, and the codimension of the variety $X' \subset \mathbb{P}^N$ of hyperplanes tangent to X , known as the dual or discriminant variety of X . The defect of (X, L) is defined to be $\text{def}(X, L) = \text{codim } X' - 1$. Most smooth varieties have defect 0. If $\text{def}(X, L) > 0$, then the defect is determined by the nef value [2],

$$\text{def}(X, L) = 2(\tau(X, L) - 1) - \dim X.$$

Moreover, if Z is a general fiber of the nef value morphism $\psi: X \rightarrow Y$, then $\text{def}(X, L) = \text{def}(Z, L_Z) - \dim Y$ and $\text{Pic}(Z) \cong \mathbb{Z}$. If the defect of X is greater than 1, then a smooth hyperplane section of X also has positive defect, see [7]. Up to such hyperplane sections and fibrations, the only known examples of smooth varieties with positive defect k are linear projective spaces, $\mathbb{P}^n$, $k = n$, the Plücker imbedding of the Grassmann variety, $\text{Gr}(2, 2m+1)$, $k = 2$, and the 10-dimensional spinor variety S_4 in $\mathbb{P}^{15}$, $k = 4$. These last examples are all homogeneous spaces. In fact, they are the only homogeneous projective varieties with $\text{def}(X, L) > 0$, along with products $X_1 \times X_2$ built from them satisfying $\text{def}(X_1) - \dim X_2 > 0$, see [11]. The proof of the classification given in [11] is difficult and proceeds through many cases based on the type of the group. In §4 a simple proof is given based on the above relationship between the defect and the nef value. Only a few special cases arise which are handled

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by determining the fiber dimension of the duality map $\phi: \mathbb{P}(N_X^*(1)) \rightarrow \mathbb{P}^N$. A general method for calculating such fiber dimensions is presented in Theorem 3.7 which reduces the problem for these cases to computing the rank of a matrix constructed from structure constants and weights.

Finally, a list of the self-dual homogeneous spaces is derived, see Corollary 4.3. Up to Hartshorne's Conjecture, this is the same list obtained without the assumption of homogeneity, see [6]. Curiously, these spaces also appear in the classification of homogeneous real hypersurfaces in $\mathbb{P}^N$, see [1, 15]. This connection is explained in Corollary 4.4.

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1. PRELIMINARIES

In this section we organize some facts about homogeneous spaces. General references are [4, 9]. We assume throughout that we are working over the complex numbers $\mathbb{C}$.

Let G be a simply-connected semisimple complex Lie group. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let Λ denote the character group of T which we write additively and refer to as the weights of G . Let $\Phi \subset \Lambda$ denote the roots of G relative to T . To every root $\alpha \in \Phi$ there corresponds a 1-dimensional unipotent subgroup $U_\alpha \cong \mathbb{C}$ called the root group of α . We define the negative roots, Φ^- , to be those α such that $U_\alpha \subset B$. In particular, B is generated by T and the negative root groups U_α , $\alpha \in \Phi^-$. The positive roots are $\Phi^+ = \Phi \setminus \Phi^- = -\Phi^-$. Let $\Delta = \{\alpha_1, \dots, \alpha_l\} \subset \Phi^+$ denote the subset of simple roots, $l = \dim T = \text{rank } G$, so that each root α can be expressed uniquely as an integral combination $\alpha = \sum_i n_i(\alpha) \alpha_i$ with either all $n_i(\alpha) \geq 0$ ($\alpha \in \Phi^+$) or all $n_i(\alpha) \leq 0$ ($\alpha \in \Phi^-$).

The weights are generated as a $\mathbb{Z}$ -module by the fundamental weights $\lambda_1, \dots, \lambda_l$ dual to the simple roots under the Killing form: $(\lambda_i, \alpha_j) = \delta_{ij}(\alpha_j, \alpha_j)/2$. A weight $\lambda = \sum_i m_i(\lambda) \lambda_i$ is dominant if all $m_i(\lambda) \geq 0$ (equivalently, $(\lambda, \alpha) \geq 0$ for all positive roots $\alpha \in \Phi^+$). We denote the dominant weights by Λ^+ . A weight λ is regular if $(\lambda, \alpha) \neq 0$ for all positive roots $\alpha \in \Phi^+$. There is also a partial ordering on Λ : $\lambda > \mu$ iff $\lambda - \mu$ is 0 or a sum of positive roots.

A parabolic subgroup P of G is a subgroup that contains a Borel subgroup. Up to conjugacy, we may assume that P contains B and therefore is determined by a subset of simple roots $\Psi \subset \Delta$. Let $\langle \Psi \rangle$ denote the positive roots of G that are linear combinations of the roots in Ψ . The subgroup P is generated by B and all the root groups U_α for $\alpha \in \langle \Psi \rangle$. If $X = G/P$, then we refer to the roots of G that are not roots of P as the roots of X , denoted R_X . Clearly, $R_X = \langle \Delta \rangle \setminus \langle \Psi \rangle$. We denote by $\Delta_X = \Delta \setminus \Psi$ the set of simple roots of X . A parabolic subgroup P is maximal if it is generated by a maximal subset of simple roots $\Psi = \Delta \setminus \{\alpha_i\}$ so that $\Delta_X = \{\alpha_i\}$.

If P acts linearly on a vector space E , then we define the twisted product $G \times_P E$ to be the quotient of $G \times E$ by the diagonal action of P : $p \cdot (g, z) = (gp^{-1}, p \cdot z)$. We represent an equivalence class in $G \times_P E$ by a pair $[g, z]$, remembering that $[gp, z] = [g, p \cdot z]$ for $p \in P$. Projection onto the first coordinate induces a map $\pi: G \times_P E \rightarrow X = G/P$, $\pi([g, z]) = gP$, which realizes the twisted product as a vector bundle on X with fiber E . Any vector

bundle on X which is invariant under G can be realized in this way. Since any line bundle L on X is invariant under G , L is isomorphic to $G \times_P \mathbb{C}$ where P acts on $\mathbb{C}$ by a character $\lambda: P \rightarrow \mathbb{C}^*$. The character λ also defines by restriction a character on a maximal torus T and therefore a weight of G . To be a character on P the weight λ must be orthogonal to the positive roots of P . Therefore, λ must be an integral combination of the fundamental weights $\mu_1, \dots, \mu_t$ dual to the simple roots in Δ_X . We call $\mu_1, \dots, \mu_t$ the fundamental weights of X and the sublattice $\Lambda_X \subset \Lambda$ they generate the weights of X .

Every dominant weight $\lambda \in \Lambda^+$ defines a character on B and therefore determines a line bundle $L = G \times_B \mathbb{C}$ on $X = G/B$. The vector space of sections $V = H^0(X, L)$ is an irreducible representation of G and every irreducible representation of G is obtained in this way. If $\Psi_\lambda = \{\alpha \in \Delta | (\lambda, \alpha) = 0\}$ then Ψ_λ defines a parabolic subgroup P_λ to which λ can be extended and the corresponding line bundle L on G/P_λ is very ample.

The following proposition summarizes some well-known facts about line bundles on homogeneous spaces.

1.1 Proposition. *Let $X = G/P$ where G is a semisimple complex Lie group and P is a parabolic subgroup. Let $\mu_1, \dots, \mu_t$ be the fundamental weights of X and let L be a line bundle on X defined by $\lambda = \sum_{j=1}^t m_j(\lambda) \mu_j \in \Lambda_X$. Then*

(1) $X = X_1 \times \dots \times X_s$ where $X_i = G_i/P_i$, G_i is a simple complex Lie group, and P_i is a parabolic subgroup of G_i , $i = 1, \dots, s$.

(2) $L = \text{pr}_1^* L_1 \otimes \dots \otimes \text{pr}_s^* L_s$ where L_i is a line bundle on X_i , $i = 1, \dots, s$.

(3) $\text{Pic}(X) \cong \Lambda_X$. In particular, $\text{Pic}(X) \cong \mathbb{Z}$ iff P is a maximal parabolic subgroup of G .

(4) L is numerically effective (nef) iff λ is dominant, i.e., $m_j(\lambda) \geq 0$ for all $1 \leq j \leq t$.

(5) L is ample iff it is very ample iff λ is a regular dominant weight, i.e., $m_j(\lambda) > 0$ for all $1 \leq j \leq t$.

2. THE NEF VALUE

Let X be a smooth projective variety.

2.1 Definition [2, 0.10]. If the canonical bundle K_X is not nef, then for any very ample line bundle L on X there is a smallest rational number $\tau = \tau(X, L)$ called the nef value of (X, L) such that $K_X \otimes L^\tau$ is nef. Moreover, τ is determined by the condition that $K_X \otimes L^\tau$ is nef but not ample. The map $\psi: X \rightarrow Y$ defined by the sections of some power of $K_X \otimes L^\tau$ is called the nef value morphism.

We first show how to compute the nef value of an ample line bundle on a homogeneous space $X = G/P$ using root data.

2.2 Theorem. *Let $X = G/P$ where G is a semisimple complex Lie group and P is a parabolic subgroup defined by $\Psi \subset \Delta$. Let λ be the weight of the anticanonical bundle $K_X^* = \bigwedge^n T_X$, $n = \dim X = |R_X|$. Then λ is regular and dominant and given by $\lambda = \sum_{\alpha \in R_X} \alpha$. In particular, K_X is not nef. The nef value of an ample line bundle L on X defined by a weight $\mu = \sum_{j=1}^t m_j(\mu) \mu_j \in \Lambda_X^+$ is given by*

$$\tau(X, L) = \max_{1 \leq j \leq t} \frac{m_j(\lambda)}{m_j(\mu)} = \max_{\beta \in \Delta_X} \frac{(\lambda, \beta)}{(\mu, \beta)}.$$

The nef value morphism $\psi: X \rightarrow Y$ is a homogeneous fiber bundle $G/P \rightarrow G/P_1$ where P_1 is the parabolic subgroup defined by $\Psi \cup \{\alpha_j | j \in J\}$ and J is the set of indexes for which the above maximum occurs.

Proof. The tangent space at the identity coset in X is isomorphic to the quotient of Lie algebras $\mathfrak{g}/\mathfrak{p}$ on which P acts via the adjoint representation on $\mathfrak{g}$ projected to the quotient. The weights of this representation therefore consist of the positive roots of G which are not in the subgroup P , namely R_X . The n th exterior power of this representation reduces to a 1-dimensional space whose weight λ is the sum of these roots.

Let $\tau = \tau(X, L)$. The weight of $K_X \otimes L^\tau$ is

$$\tau\mu - \lambda = \sum_{j=1}^t (\tau m_j(\mu) - m_j(\lambda))\mu_j.$$

Now, $K_X \otimes L^\tau$ is nef but not ample iff $\tau m_j(\mu) - m_j(\lambda) \geq 0$ for all $1 \leq j \leq t$ and at least one of these coefficients $= 0$, see Proposition 1.1. This gives the first formula for τ . The second follows from $m_j(\nu) = 2(\nu, \beta_j)/(\beta_j, \beta_j)$ for any weight ν and simple root β_j .

The set of simple roots orthogonal to $\tau\mu - \lambda$ is clearly $\Psi \cup \{\alpha_j | j \in J\}$. The character $\tau\mu - \lambda$ therefore extends to the parabolic subgroup P_1 and the corresponding map given by sections is $G/P \rightarrow G/P_1$. $\square$

When P is a maximal parabolic subgroup of G the formula of Theorem 2.2 can be refined. Recall that we write a positive root α as a linear combination of simple roots, $\alpha = \sum_{j=1}^l n_j(\alpha)\alpha_j$.

2.3 Corollary. Let $X = G/P$ where G is a simple complex Lie group and $P = P_i$ is a maximal parabolic subgroup, so that $\Delta_X = \{\alpha_i\}$. Let L be the ample generator of the line bundles on X defined by the dual fundamental weight λ_i . Then $R_X = \{\alpha \in \Phi^+ | n_i(\alpha) > 0\}$ and

$$\tau(X, L) = \frac{(\alpha_i, \alpha_i)}{2(\lambda_i, \lambda_i)} \sum_{\alpha \in R_X} n_i(\alpha).$$

Proof. The roots of X are those α that are not linear combinations of simple roots α_j , $j \neq i$. Thus, R_X consists of those roots whose α_i -coefficient is positive. If λ is the weight of K_X^* then clearly $\lambda = \tau(X, L)\lambda_i$. Since $\lambda = \sum_{\alpha \in R_X} \alpha$, we can use the Killing form to extract $\tau(X, L)$: $(\lambda_i, \lambda_i)\tau(X, L) = \sum_{\alpha \in R_X} (\alpha, \lambda_i) = \sum_{\alpha \in R_X} \sum_j n_j(\alpha)(\alpha_j, \lambda_i) = \sum_{\alpha \in R_X} n_i(\alpha)(\alpha_i, \alpha_i)/2$. $\square$

For each type of simple Lie group G , it is straightforward to list the positive roots and for each maximal parabolic subgroup P to determine the roots of G/P . Plugging these lists into the above formula yields the following table of nef values. We use the notation A_l , B_l , etc., to denote a simply-connected simple complex Lie group of type a_l , b_l , etc., and rank l . We assume the simple roots are ordered according to [16] and denote by P_i the maximal parabolic subgroup generated by $\Psi = \Delta \setminus \{\alpha_i\}$, $1 \leq i \leq l$.

2.4 Corollary. Let $X = G/P$ where G is a simple complex Lie group and $P = P_i$ is a maximal parabolic subgroup. Let $n = \dim X$ and let L be the ample generator for the line bundles on X . Then the nef value $\tau = \tau(X, L)$ is as given below.

(1) $G = A_l :$

i	$1, \dots, l$
n	$i(l+1-i)$
τ	$l+1$

(2) $G = B_l :$

i	$1, \dots, l-1$	l
n	$i(4l+1-3i)/2$	$l(l+1)/2$
τ	$2l-i$	$2l$

(3) $G = C_l :$

i	$1, \dots, l-1$
n	$i(4l+1-3i)/2$
τ	$2l-i+1$

(4) $G = D_l :$

i	$1, \dots, l-2$	$l-1, l$
n	$i(4l-1-3i)/2$	$l(l-1)/2$
τ	$2l-i-1$	$2l-2$

(5) $G = E_6 :$

i	1, 5	2, 4	3	6
n	16	25	29	21
τ	12	9	7	11

(6) $G = E_7 :$

i	1	2	3	4	5	6	7
n	27	42	50	53	47	33	42
τ	18	13	10	8	11	17	14

(7) $G = E_8 :$

i	1	2	3	4	5	6	7	8
n	57	83	97	104	106	98	78	92
τ	29	19	14	11	9	13	23	17

(8) $G = F_4 :$

i	1	2	3	4
n	15	20	20	15
τ	8	5	7	11

(9) $G = G_2 :$

i	1	2
n	5	5
τ	5	3

3. THE DEFECT

References for this material can be found in [2, 6, 7, 10]. The dual variety of a projective variety X is defined as follows. Let $L = \mathcal{O}_X(1)$ be a very ample line bundle on X and let $V = H^0(X, L)$ so that X is naturally imbedded in $\mathbb{P}^N = \mathbb{P}(V^*)$. Let N_X be the normal bundle of X defined by

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}^N}|_X \rightarrow N_X \rightarrow 0.$$

Tensoring this sequence by $\mathcal{O}_X(-1)$ we find that $N_X(-1)$ is generated by sections from $H^0(X, T_{\mathbb{P}^N}|_X(-1)) \cong V^*$. Therefore, we obtain an imbedding $\mathbb{P}(N_X^*(1)) \subset X \times \mathbb{P}(V)$ and projection onto the second factor defines the duality map

$$\phi: \mathbb{P}(N_X^*(1)) \rightarrow \mathbb{P}^N$$

from the conormal variety $\mathbb{P}(N_X^*(1))$ to the dual projective space $\mathbb{P}^N = \mathbb{P}(V)$. The image of ϕ is defined to be the dual variety of X : $X' = \phi(X) \subset \mathbb{P}^N$. This construction is a true duality in the sense that $(X')' \cong X$. The dual variety is also known as the discriminant variety since it is isomorphic to the variety of singular (tangent) hyperplane sections. A simple dimension count shows that $\dim \mathbb{P}(N_X^*(1)) = N - 1$. For most smooth projective varieties the duality map is generically one-to-one and X' has codimension 1. The difference from this norm is called the defect.

3.1 Definition. The defect of a subvariety $X \subset \mathbb{P}^N$ is defined to be

$$\text{def } X = N - \dim X' - 1.$$

In particular, $\text{def } X$ is the dimension of a general fiber of the duality map ϕ . When the imbedding is defined by the sections of a particular very ample line bundle L we refer to the defect of X as the defect of (X, L) or of L and denote it by $\text{def}(X, L)$.

If X is contained in a hyperplane H then X' is a cone over the dual variety of $X \subset H$ with vertex equal to the point dual to H . Conversely, if X' is a cone, then X is contained in the hyperplane dual to the vertex of the cone, see [6]. The defect remains the same whether we consider X as a subvariety of $\mathbb{P}^N$ or of $H \cong \mathbb{P}^{N-1}$. Projective space $\mathbb{P}^n$ with $L = \mathcal{O}_{\mathbb{P}^n}(1)$ is a special case. Since there are no singular hyperplane sections, the dual variety is empty. In order

to be compatible with later formulas, we adopt the convention that $\text{def } \mathbb{P}^n = n$ which by the above definition is the same as assigning the dimension -1 to the empty set.

3.2 Proposition [6, 7, 10]. *Let $X \subset \mathbb{P}^N$ be a smooth nonlinear projective variety. Let $n = \dim X$, $k = \text{def } X$, and $n' = \dim X' = N - k - 1$.*

- (1) *If Y is a smooth hyperplane section of X , then $\text{def } Y = \max\{0, k - 1\}$.*
- (2) *The general fiber $F \subset \mathbb{P}(N_X^*(1))$ of ϕ over a point corresponding to a tangent hyperplane H is the singular locus of $X \cap H$ and is isomorphic to some linear projective subspace of dimension k : $F \cong \text{Sing}(X \cap H) \cong \mathbb{P}^k$.*
- (3) *If X is a curve, a surface, or a complete intersection, then $k = 0$.*
- (4) *If $k > 0$ then $k \equiv n \pmod{2}$.*
- (5) *$n' \geq n$, and if X' is nonsingular then $n' = n$.*
- (6) *If $k \geq n/2$, then X is a $\mathbb{P}^m$ bundle over a smooth projective variety where $m = (n + k)/2$ and the fibers are imbedded linearly.*
- (7) *The betti numbers of X satisfy*

$$b_n = b_{n-2}, \quad b_{n-1} = b_{n-3}, \dots, \quad b_{n-k+1} = b_{n-k-1}.$$

The defect is related to the nef value in the following way.

3.3 Proposition [2, 0.12, 1.2], [3, 2.4, 3.1]. *Let X be a smooth projective variety and let L be a very ample line bundle on X . Assume the canonical bundle of X is not nef and let $\phi: X \rightarrow Y$ be the nef value morphism with general fiber Z . If $\text{def}(X, L) > 0$ then*

$$\text{def}(X, L) = \text{def}(Z, L_Z) - \dim Y = 2(\tau(X, L) - 1) - \dim X$$

and $\text{Pic } Z \cong \mathbb{Z}$. Conversely, if $\text{def}(Z, L_Z) > 0$, then $\text{def}(X, L) > 0$.

An important part of the classification of homogeneous spaces with positive defect involves products. We state here a general version of a formula for the defect of a product in terms of its factors. This proposition is a refinement of [2, 1.8].

3.4 Proposition. *Let X_1 and X_2 be smooth projective varieties with very ample line bundles L_1 and L_2 , respectively, and $\dim X_1 \geq \dim X_2$. Let $X = X_1 \times X_2$ and $L = \text{pr}_1^* L_1 \otimes \text{pr}_2^* L_2$. Then $\text{def}(X, L) > 0$ if and only if $\text{def}(X_1, L_1) > \dim X_2$. When this is the case, $\text{def}(X, L) = \text{def}(X_1, L_1) - \dim X_2 > 0$.*

Proof. ($\Rightarrow$): Assume $\text{def}(X, L) > 0$ and let $\tau = \tau(X, L)$. Since

$$K_X \otimes L^\tau = \text{pr}_1^*(K_{X_1} \otimes L_1^\tau) \otimes \text{pr}_2^*(K_{X_2} \otimes L_2^\tau)$$

both $K_{X_i} \otimes L_i^\tau$, $i = 1, 2$, are nef but not both can be ample.

Suppose $K_{X_i} \otimes L_i^\tau$, $i = 1, 2$, are both not ample. Then the nef values of all the varieties are equal, $\tau = \tau(X_1, L_1) = \tau(X_2, L_2)$. By Proposition 3.3, $\tau = (\dim X_1 + \dim X_2 + \text{def } X)/2 + 1$, but since $\tau(X_i, L_i) \leq \dim X_i + 1$, $i = 1, 2$, we get

$$(\dim X_1 + \dim X_2 + \text{def } X)/2 + 1 \leq \dim X_1 + 1, \quad \dim X_2 + 1$$

which implies $\dim X_1 + \text{def } X \leq \dim X_2$ and $\dim X_2 + \text{def } X \leq \dim X_1$ so $\text{def } X \leq 0$, contradicting our hypothesis. Therefore, we may assume that $K_{X_2} \otimes L_2^\tau$ is ample and $K_{X_1} \otimes L_1^\tau$ is not (the other possibility would ultimately contradict $\dim X_1 \geq \dim X_2$). In particular, $\tau = \tau(X_1, L_1)$. It follows that

the nef value morphism $\psi: X \rightarrow Y$ must have the form $\psi = \psi_1 \times \text{id}_{X_2}$ where $\psi_1: X_1 \rightarrow Y_1$ is the nef value morphism for X_1 . Since the general fiber Z of ψ is isomorphic to the general fiber of ψ_1 , and $L_Z = L_{1Z}$, we apply Proposition 3.3 to obtain

$$\text{def } X = \text{def } Z - \dim Y = \text{def } Z - \dim Y_1 - \dim X_2 > 0$$

and $\text{def } X_1 = \text{def } Z - \dim Y_1 > \dim X_2$.

($\Leftarrow$): Assume $\text{def } X_1 > \dim X_2$ and let $\tau_1 = \tau(X_1, L_1)$. By Proposition 3.3, $\tau_1 = (\dim X_1 + \text{def } X_1)/2 + 1$. Since $\dim X_1 \geq \text{def } X_1 > \dim X_2$, we obtain $\tau_1 > \dim X_2 + 1$. It follows that $K_{X_1} \otimes L_1^{\tau_1}$ is nef but not ample, and $K_{X_2} \otimes L_2^{\tau_1}$ is ample. Therefore, $K_X \otimes L^{\tau_1}$ is nef but not ample and $\tau = \tau_1$. As above, we find that $\psi = \psi_1 \times \text{id}_{X_2}$, the general fiber Z of ψ is isomorphic to the general fiber of ψ_1 , $L_Z = L_{1Z}$, and $Y = Y_1 \times X_2$. Again by Proposition 3.3, $\text{def } X = \text{def } Z - \dim Y = \text{def } Z - \dim X_2 - \dim Y_1$, and $\text{def } X_1 = \text{def } Z - \dim Y_1$, hence $\text{def } X = \text{def } X_1 - \dim X_2 > 0$. $\square$

These propositions yield the following statements about the defect of a homogeneous line bundle.

3.5 Corollary. *Let $X = G/P$ and let L be a line bundle on X . Let $X = X_1 \times \cdots \times X_s$ be the decomposition of X given in Proposition 1.1.1.*

(1) *$\text{def}(X, L) > 0$ iff $\text{def}(X_i, L_i) > \text{codim}_X X_i$ for some $1 \leq i \leq s$. In this case $\text{def}(X, L) = \text{def}(X_i, L_i) - \text{codim}_X X_i > 0$.*

(2) *If L is ample, the nef value morphism $\phi: X \rightarrow Y$ associated to L is a homogeneous fibration $G/P \rightarrow G/Q$ with fiber $Z = Q/P$ where Q is a parabolic subgroup of G and P is a maximal subgroup of Q . If $\text{def}(X, L) > 0$, then $\text{def}(X, L) = \text{def}(Z, L_Z) - \dim Y$.*

This corollary shows that, up to products and fibrations, it is sufficient to classify homogeneous spaces with positive defect for the case where G is simple and P is maximal. The nef values computed in Corollary 2.4 show that very few of those spaces meet the numerical criterion of Proposition 3.3 for positive defect.

3.6 Corollary. *Let $X = G/P$ where G is a simple complex Lie group and P is a maximal parabolic subgroup. Let L be the ample generator of the line bundles on X . Let $\tau = \tau(X, L)$, $n = \dim X$, and $k = 2(\tau - 1) - n$. Then $k > 0$ iff X is one of the following:*

- (1) $A_l/P_1 \cong A_l/P_l$ (projective space), $\tau = l + 1$, $n = k = l$,
- (2) $A_l/P_2 \cong A_l/P_{l-1}$ (Grassmann), $n = 2(l - 1)$, $\tau = l + 1$, $k = 2$,
- (3) A_5/P_3 (Grassmann), $n = 9$, $\tau = 6$, $k = 1$,
- (4) B_l/P_1 (quadric), $n = \tau = 2l - 1$, $k = 2l$,
- (5) $B_2/P_2 = D_4/P_4$ (projective space), $n = 3$, $\tau = 4$, $k = 3$,
- (6) $B_3/P_3 = D_4/P_4$ (quadric), $n = \tau = 6$, $k = 4$,
- (7) $B_4/P_4 = D_5/P_5$ (spinor), $n = 10$, $\tau = 8$, $k = 4$,
- (8) $B_5/P_5 = D_6/P_6$ (spinor), $n = 15$, $\tau = 10$, $k = 3$,
- (9) $B_6/P_6 = D_7/P_6$ (spinor), $n = 21$, $\tau = 12$, $k = 1$,
- (10) C_l/P_1 (projective space), $n = 2l - 1$, $\tau = 2l$, $k = 2l - 1$,
- (11) C_l/P_2 , $n = 4l - 5$, $\tau = 2l - 1$, $k = 1$,
- (12) D_l/P_1 (quadric), $n = \tau = 2l - 2$, $k = 2l - 4$,
- (13) $E_6/P_1 \cong E_6/P_5$, $n = 16$, $\tau = 12$, $k = 6$,

- (14) E_7/P_1 , $n = 27$, $\tau = 18$, $k = 7$,
- (15) F_4/P_4 , $n = 15$, $\tau = 11$, $k = 5$,
- (16) G_2/P_1 (quadric), $n = \tau = 5$, $k = 3$.

The listed value of k is the defect only if the defect is positive, which is not the case for most of the entries. For example, quadrics are hypersurfaces and obviously have defect 0 (they are self-dual). The fact that $A_l/P_2 = \text{Gr}(2, l+1)$ has positive defect only when $l+1$ is odd has been known for some time [8, 12]. The 10-dimensional spinor variety $B_4/P_4 = D_5/P_5$ is also known to have defect 4, see e.g., [6, 13].

The space C_l/P_2 is easily seen to have defect 0. Recall that $C_l = Sp(2l)$ is the stabilizer of a generic 2-form in $\wedge^2 \mathbb{C}^{2l}$ and stabilizes a (complementary) hyperplane through a point of the form $[v_1 \wedge v_2] \in \mathbb{P}(\wedge^2 \mathbb{C}^{2l})$. The isotropy subgroup of $[v_1 \wedge v_2]$ is conjugate to P_2 and therefore C_l/P_2 , which has dimension $4l - 5$, is a hyperplane section in $\text{Gr}(2, 2l)$. This Grassmann variety has defect 0 as we just pointed out, and so by Proposition 3.2.1 the defect of C_l/P_2 is also 0.

The exceptional variety E_6/P_1 cannot have positive defect because its betti numbers do not follow the pattern of Proposition 3.2.7. The odd betti numbers of E_6/P_1 are zero, and the pertinent even betti numbers are $b_{16} = 3$, $b_{14} = 2$, and $b_{12} = 2$, see, e.g., [14]. Therefore, the defect of E_6/P_1 is 0.

The remaining cases are the spinor varieties, D_6/P_6 , D_7/P_7 , and the exceptional varieties E_7/P_1 , F_4/P_4 . These spaces do not violate any of the simple criteria for positive defect. We shall show that they have defect 0 by computing the fiber dimension of the duality map $\phi: \mathbb{P}(N_X^*(1)) \rightarrow \mathbb{P}(V)$. This is a special case of the following general situation.

Let $E = G \times_P E_0$ be a homogeneous vector bundle over X . If E is spanned by global sections, then the evaluation map of sections $X \times V^* \rightarrow E$ is surjective, where $V^* = H^0(X, E)$. Therefore, the projectivization $\mathbb{P}(E^*)$ imbeds into $X \times \mathbb{P}(V)$ and projection onto the second factor yields an equivariant map $\phi: \mathbb{P}(E) \rightarrow \mathbb{P}(V)$ which imbeds each fiber $\mathbb{P}(E_0)$ linearly into $P(V)$. We let $\Lambda(E_0) \subset \Lambda(V)$ denote the weights (repeated according to their multiplicity) of a maximal torus on $E_0 \subset V$. Let R_X be the roots of X and define

$$\Xi = \Lambda(V) \cap (\Lambda(E_0) + R_X), \quad \Theta = \Lambda(E_0) \cap (\Xi - R_X).$$

Let z_ν be a fixed weight vector in V of weight $\nu \in \Lambda(V)$, and let x_α be a root vector in u for the root $\alpha \in R_X$. Then for $\mu \in \Lambda(V)$, $\alpha \in R_X$ we define constants M_α^μ by the equation $x_\alpha \cdot z_{\mu-\alpha} = M_\alpha^\mu z_\mu$ if $\mu - \alpha \in \Lambda(V)$ and 0 otherwise.

3.7 Theorem. *We retain the notation and assumptions of the previous paragraph. Let c be an arbitrary vector $[c_\nu]_{\nu \in \Theta}$ and let $M(c)$ be the matrix $[M_\alpha^\mu c_{\mu-\alpha}]$ ($\mu \in \Xi$, $\alpha \in R_X$). Then the dimension k of a general fiber of $\phi: \mathbb{P}(E) \rightarrow \mathbb{P}(V)$ satisfies $k \leq \dim X - \text{rank } M(c)$.*

Proof. The fiber dimension of $\phi: \mathbb{P}(E) \rightarrow \mathbb{P}(V)$ is constant on G -orbits, so the dimension of a general fiber can be calculated over a generic point $[z] \in \mathbb{P}(E_0)$. Now

$$\begin{aligned} k &= \dim \phi^{-1}([z]) = \dim\{[g, [w]] \in G \times_P \mathbb{P}(E_0) \mid g \cdot [w] = [z]\} \\ &= \dim\{g \in G \mid g^{-1} \cdot z \in E_0\} - \dim P = \dim\{u \in U \mid u \cdot z \in N_0^*(1)\} \end{aligned}$$

where U is the unipotent subgroup of G generated by the root groups U_α , $\alpha \in R_X$. The last equality comes from the fact the dimension in question can be determined near the identity in G and any $g \in G$ near the identity can be factored uniquely as $g = u \cdot p$ with $u \in U$, $p \in P$.

Passing to the Lie algebra $\mathfrak{u}$ of U , we find that there exist vectors $x_1, \dots, x_k \in \mathfrak{u}$ such that $x_i \cdot z \in E_0$ and are linearly independent, $i = 1, \dots, k$. With respect to the fixed basis write $z = \sum_{\nu \in \Lambda(E_0)} c_\nu z_\nu \in E_0$ and $x = \sum_{\alpha \in R_X} a_\alpha x_\alpha \in \mathfrak{u}$. Then

$$x \cdot z = \sum_{\alpha \in R_X} \sum_{\nu \in \Lambda(E_0)} a_\alpha c_\nu x_\alpha \cdot z_\nu = \sum_{\mu \in \Lambda(V)} \left[\sum_{\substack{\alpha \in R_X, \nu \in \Lambda(E_0) \\ \nu + \alpha = \mu}} M_\alpha^\mu c_\nu a_\alpha \right] x_\mu.$$

Thus, $x \cdot z \in E_0$ if and only if for every $\mu \in \Xi$ we have $\sum_{\alpha \in R_X} M_\alpha^\mu c_{\mu-\alpha} a_\alpha = 0$. Since we have k independent solutions $[a_\alpha]_{\alpha \in R_X}$ to these equations, the rank of the matrix $[M_\alpha^\mu c_{\mu-\alpha}]$ must be $\leq \dim X - k$ as claimed. $\square$

This theorem applied to the duality map can be used to verify the well-known examples of homogeneous spaces with positive defect. We shall only use it to show that the remaining cases from Corollary 3.6 do not have positive defect.

3.8 Proposition. *If X is D_6/P_6 , D_7/P_7 , E_7/P_1 , or F_4/P_4 and L is the generator of the ample line bundles on X , then $\text{def}(X, L) = 0$.*

Proof. Let $G = D_l$ and let $\lambda = \lambda_l$. Let $\varepsilon_1, \dots, \varepsilon_l$ be the standard orthonormal basis for the Lie algebra so that $\lambda = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_l)$ and the roots of X are $R_X = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq l\}$. The representation V is the spinor representation whose weights are $\Lambda(V) = \{\frac{1}{2}(\sigma_1 \varepsilon_1, \dots, \sigma_l \varepsilon_l) \mid \sigma_i = \pm 1, \sigma_1 \cdots \sigma_l = 1\}$. Thus, a weight of V can be identified by the (even number of) coordinates in the ε -basis that are negative. Let $T_0^*(1)$ and $N_0^*(1)$ denote the fibers of $T_X^*(1)$ and $N_X^*(1)$ over the identity coset in X . We find that $\Lambda(T_0^*(1)) = \{\lambda - (\varepsilon_i + \varepsilon_j) \mid 1 \leq i < j \leq l\}$ so that these weights have exactly two negative coordinates and hence $\Lambda(N_0^*(1)) = \Lambda(V) \setminus (\Lambda(T_0^*(1)) \cup \{\lambda\}) = \{\frac{1}{2}(\sigma_1 \varepsilon_1 + \dots + \sigma_l \varepsilon_l)\}$ where the number of $\sigma_i = -1$ is 4, 6, etc. It follows that

$$\begin{aligned} \Xi &= \{\mu_{pq} = \lambda - (\varepsilon_p + \varepsilon_q) \mid 1 \leq p < q \leq l\}, \\ \Theta &= \{\nu_{ijpq} = \lambda - (\varepsilon_i + \varepsilon_j + \varepsilon_p + \varepsilon_q) \mid 1 \leq i < j < p < q \leq l\}. \end{aligned}$$

Let x_{ij} (resp. y_{ij}) be a root vector in $\mathfrak{u}$ for the roots $\varepsilon_i + \varepsilon_j$ (resp. $-(\varepsilon_i + \varepsilon_j)$). We fix a basis for V consisting of the vector v_0 of weight λ , $v_{pq} = y_{pq} v_0$ of weight μ_{ij} , $1 \leq p < q \leq l$, $v_{ijpq} = y_{ij} y_{pq} v_0$ of weight ν_{ijpq} , $1 \leq i < j < p < q \leq l$, etc. Let $\langle ij pq \rangle$ denote the list of indices i, j, p, q rearranged to be in their natural order. If $\mu = \mu_{pq} \in \Xi$ and $\alpha = \varepsilon_i + \varepsilon_j \in R_X$, then we write M_{ij}^{pq} for M_α^μ so that $x_{ij} \cdot v_{\langle ij pq \rangle} = M_{ij}^{pq} v_{pq}$. Then

$$M_{ij}^{pq} = \begin{cases} 0, & i, j, p, q \text{ not pairwise disjoint,} \\ [[x_{ij}, y_{st}], y_{uv}] / y_{pq}, & stuv = \langle ij pq \rangle. \end{cases}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & j & 0 & f & g & h & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & j & 0 & 0 & f & g & 0 & i & b \\ 0 & 0 & 0 & 0 & 0 & j & 0 & 0 & f & 0 & 0 & h & i & 0 & c \\ 0 & 0 & 0 & 0 & j & 0 & f & 0 & 0 & 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & j & 0 & 0 & 0 & 0 & g & 0 & h & i & 0 & 0 & d \\ 0 & 0 & j & 0 & 0 & 0 & g & 0 & 0 & 0 & a & b & 0 & d & 0 \\ 0 & 0 & 0 & f & 0 & g & 0 & h & 0 & i & 0 & 0 & 0 & 0 & e \\ 0 & j & 0 & 0 & 0 & 0 & h & 0 & a & 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & a & 0 & b & 0 & 0 & 0 & e & 0 \\ j & 0 & 0 & 0 & 0 & 0 & i & 0 & b & 0 & c & d & 0 & 0 & 0 \\ 0 & f & 0 & 0 & h & a & 0 & 0 & 0 & c & 0 & 0 & e & 0 & 0 \\ f & g & h & a & i & b & 0 & c & 0 & d & 0 & e & 0 & 0 & 0 \\ g & 0 & i & b & 0 & 0 & 0 & d & 0 & 0 & e & 0 & 0 & 0 & 0 \\ h & i & 0 & c & 0 & d & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & 0 & d & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now choose the vector $[c_\nu]$ so that all but the constants e and j equal 0. The resulting matrix is nonsingular. Moreover, it remains nonsingular if we change each of the j 's and e 's to other nonzero values. This proves that the matrix $[M_\alpha^\mu c_{\mu-\alpha}]$ is nonsingular for this choice of $[c_\nu]$. We therefore conclude that the defect of X is zero.

A similar calculation and argument shows that for the case $X = E_7/P_1$ the matrix $[M_\alpha^\mu c_{\mu-\alpha}]$ has rank $\geq \dim X - 2$, so again the defect of X is 0. $\square$

We can conclude at this point that the only homogeneous spaces $X = G/P$, P maximal parabolic, that have positive defect are the familiar three examples: $\mathbb{P}^n$, $\text{Gr}(2, 2m+1)$, and S_4 .

4. CLASSIFICATIONS

We now make the final conclusions about which homogeneous spaces $X = G/P$ have positive defect and show how the self-dual homogeneous spaces can be used to classify real homogeneous hypersurfaces in $\mathbb{P}^N$. We need one more technical fact.

4.1 Lemma. *Let G be a simple complex Lie group and let $P \subset Q$ be proper parabolic subgroups of G such that $Q/P \cong \mathbb{P}^k$. Then $\dim G/Q > k$.*

Proof. Let $\Psi \subset \Delta$ be the subset of simple roots defining Q . There must be a connected component of the Dynkin diagram of Ψ isomorphic to type a_k or type c_m where $m = (k+1)/2$, since these represent the only two simple groups that act transitively on projective space $Q/P \cong \mathbb{P}^k$. Let Ψ' denote the subset of simple roots corresponding to this connected component. Since G is simple, its Dynkin diagram is connected, and therefore there is a simple root $\beta \in \Delta \setminus \Psi'$ such that $\Phi' = \Psi' \cup \{\beta\}$ forms a set of simple roots for a simple subgroup $G' \subset G$. Let $Q' = G' \cap Q$ and $P' = G' \cap P$. By construction, Q' is a maximal parabolic subgroup of G' , $Q'/P' \cong \mathbb{P}^k$, and $\dim G'/Q' \leq \dim G/Q$. The proof is complete if we show $k < \dim G'/Q'$.

Therefore, we may assume from the outset that Q is a maximal parabolic subgroup of a simple group G defined by a connected subset of simple roots Ψ of type a_k or c_m , $m = (k+1)/2$. In particular, Ψ is either type a_k and G has rank $k+1$, or Ψ is type c_m and $G = C_{m+1}$, $m = (k+1)/2$, or $G = F_4$, $m = 3, k = 5$. It is quite easy to check each of the simple types for these conditions. The smallest possibility for $\dim G/Q$ occurs for the case $G = A_{k+1}$ where $G/Q \cong \mathbb{P}^{k+1}$. $\square$

4.2 Theorem (cf. [11]). *Let $X = G/P \subset \mathbb{P}^N$ be a homogeneous space and let L be an ample line bundle on X . If $k = \text{def}(X, L) > 0$, then X is one of the following:*

- (1) *A linear projective space $\mathbb{P}^n$, $k = n$.*
- (2) *The Plücker imbedding of the Grassmann variety $\text{Gr}(2, 2m+1)$, $k = 2$.*
- (3) *The 10-dimensional spinor variety S_4 in $\mathbb{P}^{15}$, $k = 4$.*
- (4) *$X_1 \times X_2$ where X_1 is one of the varieties in 1–3 and $\text{def } X = \text{def } X_1 - \dim X_2 > 0$.*

Proof. If X is not isomorphic to a product then we may assume G is simple, see Proposition 1.1.1. Also, whenever P is a maximal parabolic subgroup of G and $\text{def}(X, L) > 0$, it follows immediately from [7, Theorem 1.3(b)] that L must be the generator of the ample line bundles on X .

Consider the nef value morphism $\phi: X = G/P \rightarrow Y = G/Q$ of Corollary 3.5.2. The fiber $Z = Q/P$ is isomorphic to the quotient of a simple group by a maximal parabolic subgroup. Since $\text{def } Z > 0$, the only possibilities for such quotients are the three listed varieties as was demonstrated in §3. If $Z = \text{Gr}(2, 2m+1)$ then $\dim Y < \text{def } Z = 2$. If $\dim Y = 1$ then $Y \cong \mathbb{P}^1$ and $G = SL(2, \mathbb{C})$ which is too small to provide a nontrivial fibration. Therefore, $\dim Y = 0$ and $X = \text{Gr}(2, 2m+1)$.

Similarly, if Z is S_4 , then $\dim Y < \text{def } Z = 4$. If Y is not a point, then Y must be $\mathbb{P}^r$, $r = 1, 2$, or 3 , the 3-dimensional flag manifold, or a 3-dimensional quadric. The corresponding groups are A_r , $r = 1, 2$, or 3 , and B_2 . None of these groups contains either of the two complex simple Lie groups that acts transitively on the 10-dimensional spinor variety (B_4 or D_5). So again we conclude that the fibration is trivial and $X = S_4$.

The final possibility of a nontrivial fibration with $Z = \mathbb{P}^k$ can also be ruled out because of the condition $\dim Y < \text{def } Z$ and Lemma 4.1. Therefore, if X is not a product, it must be one of the three listed varieties. If X is isomorphic to a product, then Corollary 3.5.1 implies statement 4. $\square$

The following theorem holds for nonlinear smooth projective varieties $X \subset \mathbb{P}^N$ such that $\dim X = \dim X' \leq \frac{2}{3}N$, see [6]. The version we present here for homogeneous spaces is simply a corollary of the above classification. This list also classifies those nonlinear homogeneous spaces with nonsingular dual varieties, see Proposition 3.2.5. If X is a linear projective space $X = \mathbb{P}^n \subset \mathbb{P}^N$ then the tangent hyperplanes are clearly parameterized by a complementary projective space $X' = \mathbb{P}^{N-n-1}$.

4.3 Corollary. *Let $X = G/P$ be a nonlinear homogeneous space imbedded in $\mathbb{P}^N$ by the sections of an ample line bundle L on X . If $\dim X = \dim X'$ then X is one of the following:*

- (1) *A quadric hypersurface in $\mathbb{P}^{n+1}$.*
- (2) *The Segre imbedding of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ in $\mathbb{P}^{2n-1}$.*
- (3) *The Plücker imbedding of $\text{Gr}(2, 5)$ in $\mathbb{P}^9$.*
- (4) *The 10-dimensional spinor variety S_4 in $\mathbb{P}^{15}$.*

Proof. The listed varieties are well known to be self-dual, see e.g., [6, 13]. If $X = \text{Gr}(2, 2m+1)$ then $\dim X = 2(2m-1)$ and $\dim X' = m(2m+1) - 4$ and these are equal only when $m = 2$. By Theorem 4.2 it remains to check the case $X = X_1 \times X_2$. By Proposition 1.1.2, $L = \text{pr}_1^* L_1 \otimes \text{pr}_2^* L_2$, so that $H^0(X, L) \cong H^0(X_1, L_1) \otimes H^0(X_2, L_2)$. Therefore, the imbedding dimension of X satisfies $N+1 = (N_1+1)(N_2+1)$ where N_i is the imbedding dimension of X_i under L_i , $i = 1, 2$. We know that $\text{def } X = N - \dim X - 1 = \text{def } X_1 - \dim X_2$. Hence, $N_1 N_2 + N_1 + N_2 = \dim X_1 + \text{def } X_1 + 1$. Since $N_i \geq \dim X_i$ this equation becomes $\dim X_2(\dim X_1 + 1) \leq \text{def } X_1 + 1$. Now, X_1 must be a projective space, for otherwise $\text{def } X_1 + 1 \leq 5$ and this would imply that $\dim X_1 = 1$. Therefore, $(X_1, L_1) \cong (\mathbb{P}^{n_1}, \mathcal{O}_{\mathbb{P}^{n_1}}(1))$ and the previous equation yields $N_2 = 1$. This implies that $(X_2, L_2) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, and $X = \mathbb{P}^{n-1} \times \mathbb{P}^1$ as claimed. $\square$

Certain real hypersurfaces in complex projective space are tubes over complex submanifolds, see [5]. This fact along with the above classification of self-dual homogeneous spaces can be used to classify homogeneous real hypersurfaces in complex projective space.

4.4 Corollary [1, 15]. *Let M be a homogeneous complete real hypersurface imbedded equivariantly in $\mathbb{P}^N$. Then M is a tube over a linear projective space or one of the 4 self-dual homogeneous spaces $X \subset \mathbb{P}^N$ listed in Corollary 4.3.*

Proof. Let $M = K/L$ where K is a compact Lie group and let ξ denote the normal vector field to M . If J denotes the complex structure operator, then $W = -J\xi$ is a tangent vector field. Because the imbedding is equivariant, W is left invariant under K and therefore its integral curves are given by 1-parameter subgroups of K and are geodesics. By [5], M is a tube over a complex submanifold $X \subset \mathbb{P}^N$ (a focal submanifold). In particular, X is homogeneous, $X = G/P$, and G acts transitively on the normal directions to X . It follows that the conormal variety $\mathbb{P}(N_X^*(1))$ itself is a homogeneous space, G/P_0 , and therefore the image of $\phi: \mathbb{P}(N_X^*(1)) = G/P_0 \rightarrow X'$ must also be homogeneous, $X' = G/P'$. If X is not a linear projective space then $\dim X' = \dim X$, see Proposition 3.2.5, and so X is one of 4 self-dual homogeneous spaces by Corollary 4.3. $\square$

Conversely, the classification of homogeneous real hypersurfaces [15] can also be used to obtain the list of self-dual homogeneous spaces. For, if a homogeneous space $X = G/P$ is self-dual, then by a symmetry argument, the conormal variety must be a homogeneous space under G . This implies that G acts transitively on the normal directions to X and hence a maximal compact subgroup of G must have a hypersurface orbit in the normal bundle of X . The resulting orbit is a homogeneous real hypersurface in $\mathbb{P}^N$ realized as a tube over X and therefore must be on the list given in [15], see also [1].

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