UNIVERSAL COVER OF SALVETTI'S COMPLEX AND TOPOLOGY OF SIMPLICIAL ARRANGEMENTS OF HYPERPLANES

LUIS PARIS

ABSTRACT. Let V be a real vector space. An *arrangement of hyperplanes* in V is a finite set $\mathcal{A}$ of hyperplanes through the origin. A *chamber* of $\mathcal{A}$ is a connected component of $V - (\bigcup_{H \in \mathcal{A}} H)$. The arrangement $\mathcal{A}$ is called *simplicial* if $\bigcap_{H \in \mathcal{A}} H = \{0\}$ and every chamber of $\mathcal{A}$ is a simplicial cone. For an arrangement $\mathcal{A}$ of hyperplanes in V , we set

$$M(\mathcal{A}) = V_{\mathbb{C}} - \left(\bigcup_{H \in \mathcal{A}} H_{\mathbb{C}} \right),$$

where $V_{\mathbb{C}} = \mathbb{C} \otimes V$ is the complexification of V , and, for $H \in \mathcal{A}$, $H_{\mathbb{C}}$ is the complex hyperplane of $V_{\mathbb{C}}$ spanned by H .

Let $\mathcal{A}$ be an arrangement of hyperplanes of V . Salvetti constructed a simplicial complex $\text{Sal}(\mathcal{A})$ and proved that $\text{Sal}(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$. In this paper we give a new short proof of this fact. Afterwards, we define a new simplicial complex $\widehat{\text{Sal}}(\mathcal{A})$ and prove that there is a natural map $p: \widehat{\text{Sal}}(\mathcal{A}) \rightarrow \text{Sal}(\mathcal{A})$ which is the universal cover of $\text{Sal}(\mathcal{A})$. At the end, we use $\widehat{\text{Sal}}(\mathcal{A})$ to give a new proof of Deligne's result: "if $\mathcal{A}$ is a simplicial arrangement of hyperplanes, then $M(\mathcal{A})$ is a $K(\pi, 1)$ space." Namely, we prove that $\widehat{\text{Sal}}(\mathcal{A})$ is contractible if $\mathcal{A}$ is a simplicial arrangement.

1. INTRODUCTION

Let V be a real vector space. An *arrangement of hyperplanes* in V is a finite set $\mathcal{A}$ of hyperplanes through the origin. We say that $\mathcal{A}$ is *essential* if $\bigcap_{H \in \mathcal{A}} H = \{0\}$. A *chamber* of $\mathcal{A}$ is a connected component of $V - \bigcup_{H \in \mathcal{A}} H$. The arrangement $\mathcal{A}$ is called *simplicial* if $\mathcal{A}$ is essential and every chamber of $\mathcal{A}$ is an open simplicial cone.

Let $V_{\mathbb{C}} = \mathbb{C} \otimes V$ be the *complexification* of V . Every element z of $V_{\mathbb{C}}$ can be written in a unique way $z = x + iy$, where $x, y \in 1 \otimes V = V$. We say that x is the *real part* of z and that y is its *imaginary part*. For two subsets $X, Y \subseteq V$, we write

$$X + iY = \{(x + iy) \in V_{\mathbb{C}} \mid x \in X \text{ and } y \in Y\}.$$

Let H be a hyperplane of V . The *complexification* $H_{\mathbb{C}}$ of H is the hyperplane of $V_{\mathbb{C}}$ spanned by H ; $H_{\mathbb{C}} = H + iH$.

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Let $\mathcal{A}$ be an arrangement of hyperplanes in a real vector space V . We set

$$M(\mathcal{A}) = V_{\mathbb{C}} - \left(\bigcup_{H \in \mathcal{A}} H_{\mathbb{C}} \right).$$

This space is an open connected submanifold of $V_{\mathbb{C}}$.

The study of the topology of $M(\mathcal{A})$ can be easily reduced to the case of an essential arrangement of hyperplanes. Thus we assume throughout all the arrangements to be essential.

In [Sa1], Salvetti associates with a real arrangement $\mathcal{A}$ of hyperplanes a simplicial complex $\text{Sal}(\mathcal{A})$, and proves that $\text{Sal}(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$.

An *oriented system* is a pair $(\Gamma, \sim)$ where Γ is an oriented graph and $\sim$ is an equivalence relation on the set of paths of Γ with some properties described in §2. In [Pa1], the author shows that there is a natural notion of *universal cover* $\rho: (\hat{\Gamma}, \sim) \rightarrow (\Gamma, \sim)$ of an oriented system $(\Gamma, \sim)$, associates an oriented system $(\Gamma(\mathcal{A}), \sim)$ with a real arrangement $\mathcal{A}$ of hyperplanes, and constructs, from the universal cover of $(\Gamma(\mathcal{A}), \sim)$, the universal cover of $M(\mathcal{A})$.

Note that, in [Sa1], Salvetti uses this same oriented system $(\Gamma(\mathcal{A}), \sim)$ to compute the fundamental group of $M(\mathcal{A})$. He also proves that $\Gamma(\mathcal{A})$ is the 1-skeleton of $\text{Sal}(\mathcal{A})$ provided with an orientation, and that $\sim$ is the homotopy relation on the paths of $\Gamma(\mathcal{A})$. Nevertheless, both works, [Sa1] and [Pa1], are completely independent.

In this paper we give a new short proof of Salvetti's result: $\text{Sal}(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$ (Theorem 3.3). Afterwards, using techniques introduced in [Pa1], we define another simplicial complex $\hat{\text{Sal}}(\mathcal{A})$, and prove that there is a natural map $p: \hat{\text{Sal}}(\mathcal{A}) \rightarrow \text{Sal}(\mathcal{A})$ which is the universal cover of $\text{Sal}(\mathcal{A})$ (Theorem 3.7). In particular, $\hat{\text{Sal}}(\mathcal{A})$ has the same homotopy type as the universal cover of $M(\mathcal{A})$. At the end, we use $\hat{\text{Sal}}(\mathcal{A})$ to give a new proof of the following result of Deligne.

Theorem 1.1 (Deligne [De]). *If $\mathcal{A}$ is a simplicial arrangement of hyperplanes, then $M(\mathcal{A})$ is a $K(\pi, 1)$ space (i.e., the universal cover of $M(\mathcal{A})$ is contractible).*

Namely, we prove that $\hat{\text{Sal}}(\mathcal{A})$ is contractible if $\mathcal{A}$ is a simplicial arrangement.

We say that a real arrangement $\mathcal{A}$ of hyperplanes is a $K(\pi, 1)$ arrangement if $M(\mathcal{A})$ is a $K(\pi, 1)$ space. Most of the already known $K(\pi, 1)$ arrangements are supersolvable (see [Te2]) or simplicial. To find a general criterion for an arrangement to be $K(\pi, 1)$ remains an open problem. In particular, Saito's conjecture that free arrangements are $K(\pi, 1)$ is unsolved (see [Te1]). Supersolvable arrangements are free (see [JT]) but simplicial arrangements are not always free (see [Te1]). We refer to [FR] for a good exposition on $K(\pi, 1)$ arrangements.

The interest of our simplicial complex $\hat{\text{Sal}}(\mathcal{A})$ is, in order to prove that a real arrangement $\mathcal{A}$ is $K(\pi, 1)$, it suffices to prove that $\hat{\text{Sal}}(\mathcal{A})$ is contractible.

Our proof of Theorem 1.1 is more simple than Deligne's one and, more-

over, we isolate the part of the proof where the hypothesis “ $\mathcal{A}$ is a simplicial arrangement of hyperplanes” is essential.

For a simplicial arrangement $\mathcal{A}$ of hyperplanes, Deligne constructs in [De] a cover $q: \widehat{M} \rightarrow M(\mathcal{A})$, defines a simplicial complex $\text{Del}(\mathcal{A})$, and proves that $\text{Del}(\mathcal{A})$ is contractible and has the same homotopy type as $\widehat{M}$. In particular, $\widehat{M}$ is the universal cover of $M(\mathcal{A})$.

Our first innovation was in [Pa1] to introduce a new combinatorial tool: the *oriented system*. Deligne's combinatorial tool, the groupoid Gal , is, in some sense, equivalent to our oriented system, but it cannot be defined for any real arrangement of hyperplanes ((i) and (ii) of Proposition 1.19 of [De] are needed to define Gal and their proof strongly uses the fact that $\mathcal{A}$ is simplicial), and, moreover, unlike in oriented systems, several preliminary results are needed to define it (in particular, Ore's criterion for a semigroup to be embedded in a group (see [Lj, p. 400]) must be generalized to “semigroupoids” and groupoids).

Another important innovation is to substitute the simplicial complex $\text{Del}(\mathcal{A})$ for $\widehat{\text{Sal}}(\mathcal{A})$. Using oriented systems, the complex $\text{Del}(\mathcal{A})$ can be generalized to any real arrangement of hyperplanes (in the general case, $\text{Del}(\mathcal{A})$ is a CW complex), but it does not always have the same homotopy type as the universal cover $\widehat{M}$ of $M(\mathcal{A})$ (see [Pa2]). An advantage of our complex $\widehat{\text{Sal}}(\mathcal{A})$ over $\text{Del}(\mathcal{A})$ is, in order to prove that $\text{Del}(\mathcal{A})$ has the same homotopy type as $\widehat{M}$, several considerations on some kind of “subgroupoids” of Gal are needed (see (1.5), (1.6), (1.9), (1.24), (1.28), (1.29), (1.30), (1.31), (1.32) of [De]) which are not necessary in our case.

To prove that either $\widehat{\text{Sal}}(\mathcal{A})$ or $\text{Del}(\mathcal{A})$ are contractible, a strong property of simplicial arrangements is needed: the *property D* (see Theorem 4.1). In our proof of Theorem 1.1 the hypothesis “ $\mathcal{A}$ is a simplicial arrangement of hyperplanes” is only used to show that $\mathcal{A}$ has the property D. In fact, for an essential arrangement $\mathcal{A}$ of hyperplanes, having the property D is equivalent to being simplicial. I actually proved it some time after the first version of this paper (see [Pa3]).

The only place where our proof of Theorem 1.1 coincides with Deligne's one is in our Lemma 4.4. It is a preliminary result to the proof of Theorem 4.1.

Since I submitted for publication the first version of my paper, Raul Cordevil has informed me he has independently of my paper generalized Deligne's theorem for simplicial oriented matroids (see [Co]). On the other hand, Mario Salvetti has informed me he has also proved independently Theorem 4.1 and Theorem 4.6 (see [Sa2]).

Our work is organized as follows.

Section 2 is a summary of [Pa1]. Its aim is to introduce our main combinatorial tool, the oriented systems, and to give the construction of the universal cover $\widehat{M} \rightarrow M(\mathcal{A})$ of $M(\mathcal{A})$.

In §3 we define the simplicial complexes $\text{Sal}(\mathcal{A})$ and $\widehat{\text{Sal}}(\mathcal{A})$, and prove that $\text{Sal}(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$, and that there is a natural map $p: \widehat{\text{Sal}}(\mathcal{A}) \rightarrow \text{Sal}(\mathcal{A})$ which is the universal cover of $\text{Sal}(\mathcal{A})$.

The goal of §4 is to prove Theorem 1.1.

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2. THE UNIVERSAL COVER OF $M(\mathcal{A})$

This section is divided into three subsections. In the first one we introduce our main combinatorial tool: the *oriented system*. In the second subsection we define the oriented system $(\Gamma(\mathcal{A}), \sim)$ associated with a real arrangement $\mathcal{A}$ of hyperplanes. In the third subsection, from the universal cover $\rho: (\widehat{\Gamma}, \sim) \rightarrow (\Gamma(\mathcal{A}), \sim)$ of the oriented system $(\Gamma(\mathcal{A}), \sim)$, we give the construction of the universal cover $\widehat{M} \rightarrow M(\mathcal{A})$ of $M(\mathcal{A})$.

All the results stated in this section are derived from [Pa1], so we will not give any proofs.

2A. Oriented systems. An *oriented graph* Γ is the following data:

- (1) a set $V(\Gamma)$ of *vertices*,
- (2) a subset $A(\Gamma) \subseteq (V(\Gamma) \times V(\Gamma)) - \{(v, v) | v \in V(\Gamma)\}$ of *arrows*.

The *begin* of an arrow $a = (v, w)$ is v and its *end* is w . An oriented graph Γ is *locally finite* if every vertex $v \in V(\Gamma)$ is the begin or the end of only a finite number of arrows.

A *path* of an oriented graph Γ is an expression

$$f = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n},$$

where $a_i \in A(\Gamma)$ and $\varepsilon_i \in \{\pm 1\}$ (for $i = 1, \dots, n$), such that there exists a sequence $v_0, v_1, \dots, v_n$ of vertices of Γ with

$$\begin{aligned} a_i &= (v_{i-1}, v_i) \quad \text{if } \varepsilon_i = 1, \\ a_i &= (v_i, v_{i-1}) \quad \text{if } \varepsilon_i = -1. \end{aligned}$$

We say that v_0 is the *begin* of f and that v_n is its *end*. The integer n is its *length* and $\sum_{i=1}^n \varepsilon_i$ is its *weight*. Every vertex of Γ is assumed to be a path of length 0 and of weight 0. For a path $f = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$, we write $f^{-1} = a_n^{-\varepsilon_n} \cdots a_1^{-\varepsilon_1}$. For two paths $f = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ and $g = b_1^{\mu_1} \cdots b_m^{\mu_m}$ with $\text{end}(f) = \text{begin}(g)$, we write $fg = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} b_1^{\mu_1} \cdots b_m^{\mu_m}$.

An oriented graph Γ is *connected* if, for every pair (v, w) of vertices of Γ , there exists a path of Γ which begins at v and ends in w .

We assume throughout all the oriented graphs to be locally finite and connected.

Let Γ be an oriented graph. An *identification* of Γ is an equivalence relation $\sim$ in the set of paths of Γ with the following properties:

- (1) $f \sim g \Rightarrow \text{begin}(f) = \text{begin}(g)$, $\text{end}(f) = \text{end}(g)$, and $\text{weight}(f) = \text{weight}(g)$,
- (2) $ff^{-1} \sim \text{begin}(f)$, for every path f ,
- (3) $f \sim g \Rightarrow f^{-1} \sim g^{-1}$,
- (4) $f \sim g \Rightarrow h_1 f h_2 \sim h_1 g h_2$ for any two paths h_1 and h_2 such that $\text{end}(h_1) = \text{begin}(f) = \text{begin}(g)$ and $\text{begin}(h_2) = \text{end}(f) = \text{end}(g)$.

An *oriented system* is a pair $(\Gamma, \sim)$, where Γ is an oriented graph and $\sim$ is an identification of Γ .

Let $\rho: \Theta \rightarrow \Gamma$ be a morphism of oriented graphs. We say that ρ is a *cover* of Γ if, for every vertex v of Θ and every path f of Γ beginning at $\rho(v)$, there exists a unique path $\hat{f}$ of Θ such that $\text{begin}(\hat{f}) = v$ and $\rho(\hat{f}) = f$.

Let $\rho: (\Theta, \sim) \rightarrow (\Gamma, \sim)$ be a morphism of oriented systems (i.e., $\hat{f} \sim \hat{g} \Rightarrow \rho(\hat{f}) \sim \rho(\hat{g})$). We say that ρ is a *cover* of $(\Gamma, \sim)$ if it has the following two properties:

- (1) $\rho: \Theta \rightarrow \Gamma$ is a cover of Γ .
- (2) Let $v \in V(\Theta)$, let f and g be two paths of Γ which both begin at $\rho(v)$, and let $\hat{f}$ and $\hat{g}$ be the lifts of f and g respectively into Θ beginning at v . If $f \sim g$ ($\Rightarrow \text{end}(f) = \text{end}(g)$), then $\hat{f} \sim \hat{g}$ ($\Rightarrow \text{end}(\hat{f}) = \text{end}(\hat{g})$).

Example. Let Γ be the oriented graph shown in Figure 1a. Let $\sim$ be the smallest identification of Γ such that $ab \sim dc$. The identification $\sim$ can be viewed as a “homotopy relation”, namely, to identify ab with dc is like to “add” a 2-cell to Γ having $abc^{-1}d^{-1}$ as border. Let $\hat{\Gamma}$ be the oriented graph shown in Figure 1b. The morphism $\pi: \hat{\Gamma} \rightarrow \Gamma$ which sends a_i onto a , b_i onto b , c_i onto c , d_i onto d , and e_i onto e (where $i \in \mathbb{Z}$) is obviously a cover of Γ . Let $\sim$ be the smallest identification of $\hat{\Gamma}$ such that $a_i b_i \sim d_i c_i$ for every $i \in \mathbb{Z}$. The map $\pi: (\hat{\Gamma}, \sim) \rightarrow (\Gamma, \sim)$ is a cover of $(\Gamma, \sim)$; in fact, it is the universal cover of $(\Gamma, \sim)$ (see Proposition 2.1).

Proposition 2.1. *Let $(\Gamma, \sim)$ be an oriented system. There exists a unique cover $\pi: (\hat{\Gamma}, \sim) \rightarrow (\Gamma, \sim)$ of $(\Gamma, \sim)$ (up to isomorphism) which has the following universal property. If $\rho: (\Theta, \sim) \rightarrow (\Gamma, \sim)$ is a cover of $(\Gamma, \sim)$, then there exists a unique cover $\pi': (\hat{\Gamma}, \sim) \rightarrow (\Theta, \sim)$ of $(\Theta, \sim)$ (up to isomorphism) such that $\pi = \rho \circ \pi'$.*

We call $\pi: (\hat{\Gamma}, \sim) \rightarrow (\Gamma, \sim)$ the *universal cover* of $(\Gamma, \sim)$.

FIGURE 1a

FIGURE 1b

Proposition 2.2. Let $\pi: (\widehat{\Gamma}, \sim) \rightarrow (\Gamma, \sim)$ be the universal cover of an oriented system $(\Gamma, \sim)$. Two paths $\hat{f}$ and $\hat{g}$ of $\widehat{\Gamma}$ are identified by $\sim$ if and only if $\text{begin}(\hat{f}) = \text{begin}(\hat{g})$ and $\text{end}(\hat{f}) = \text{end}(\hat{g})$.

2B. Definition of $(\Gamma(\mathcal{A}), \sim)$. Let $\mathcal{A}$ be an (essential) arrangement of hyperplanes in a real vector space V . Our goal in this subsection is to associate with $\mathcal{A}$ an oriented system $(\Gamma(\mathcal{A}), \sim)$.

First, recall some definitions. The *lattice* of $\mathcal{A}$ is the poset

$$\mathcal{L}(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\},$$

ordered by reverse inclusion. $V = \bigcap_{H \in \emptyset} H$ is the smallest element of $\mathcal{L}(\mathcal{A})$ and $\{0\} = \bigcap_{H \in \mathcal{A}} H$ is the greatest one. For $X \in \mathcal{L}(\mathcal{A})$, we set

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\}.$$

We refer to [Or] for a good exposition on $\mathcal{L}(\mathcal{A})$ and its properties.

The hyperplanes of $\mathcal{A}$ subdivide V into *facets*. We denote by $\mathcal{F}(\mathcal{A})$ the set of all the facets. The *support* $|F|$ of a facet F is the vector space $|F| \in \mathcal{L}(\mathcal{A})$ spanned by F . Every facet is open in its support. We denote by $\overline{F}$ the closure of F in V . There is a partial order in $\mathcal{F}(\mathcal{A})$ defined by $F \leq G$ if $F \subseteq \overline{G}$.

A *chamber* of $\mathcal{A}$ is a 0 codimension facet. A *face* is a 1 codimension facet. Two chambers C and D are *adjacent* if they have a common face (i.e., a common 1 codimension facet).

Now, let us define the oriented system $(\Gamma, \sim) = (\Gamma(\mathcal{A}), \sim)$ associated with $\mathcal{A}$.

The vertices of Γ are the chambers of $\mathcal{A}$. An arrow of Γ is a pair (C, D) , where C and D are adjacent chambers. Note that, in this oriented graph, if (C, D) is an arrow, then (D, C) is also an arrow.

A *positive path* of an oriented graph Δ is a path $f = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ with $\varepsilon_1 = \cdots = \varepsilon_n = 1$. This positive path is *minimal* if there is no positive path in Δ having the same begin, the same end and a smaller length than f .

The relation $\sim$ is the smallest identification of Γ such that if f and g are both positive minimal paths with the same begin and the same end, then $f \sim g$.

Remark. A *gallery* of $\mathcal{A}$ is a sequence $(C_0, C_1, \dots, C_n)$ of chambers of $\mathcal{A}$ such that C_{i-1} and C_i are adjacent for $i = 1, \dots, n$ (here we assume $C_{i-1} \neq C_i$ for $i = 1, \dots, n$). Any positive path $f = a_1 \cdots a_n$ of $\Gamma(\mathcal{A})$ can be viewed as the gallery $G = (C_0, C_1, \dots, C_n)$, where $C_i = \text{end}(a_1 \cdots a_i)$ for $i = 0, 1, \dots, n$. In particular, if $f = a_1 \cdots a_n$ is a positive minimal path of $\Gamma(\mathcal{A})$, then $G = (C_0, C_1, \dots, C_n)$ is a minimal gallery (i.e., a gallery of minimal length among the galleries of $\mathcal{A}$ connecting C_0 with C_n). Thus we “identify” two minimal galleries which join the same chambers.

Example. Let $\mathcal{A}$ be the arrangement of two lines in $\mathbb{R}^2$. The oriented graph shown in Figure 2a is $\Gamma(\mathcal{A})$. The relation $\sim$ is the smallest identification of $\Gamma(\mathcal{A})$ such that $ab \sim a'd'$, $bc \sim b'a'$, $cd \sim c'b'$, and $da \sim d'c'$. Let $(\widehat{\Gamma}, \sim) \rightarrow (\Gamma(\mathcal{A}), \sim)$ be the universal cover of $(\Gamma(\mathcal{A}), \sim)$. The oriented graph shown in Figure 2b is $\widehat{\Gamma}$.

FIGURE 2a

FIGURE 2b

2C. Universal cover of $M(\mathcal{A})$. Let $\mathcal{A}$ be an (essential) arrangement of hyperplanes in a real vector space V . We set

$$M = M(\mathcal{A}) = V_{\mathbb{C}} - \left(\bigcup_{H \in \mathcal{A}} H_{\mathbb{C}} \right).$$

Our goal in this subsection is to explain the construction of the universal cover $q: \widehat{M} \rightarrow M$ of $M(\mathcal{A})$.

Let C be a chamber of $\mathcal{A}$. For a facet $F \in \mathcal{F}(\mathcal{A})$, we denote by C_F the

unique chamber of $\mathcal{A}_{|F|}$ containing C . We write

$$M(C) = \bigcup_{F \in \mathcal{F}(\mathcal{A})} (F + iC_F) \subseteq (V + iV) = V_{\mathbb{C}}.$$

Note that this union is disjoint.

Lemma 2.3. *The set $\{M(C) \mid C \in V(\Gamma)\}$ is a covering of $M(\mathcal{A})$ by open subsets.*

Proof. First, let us prove that $M(C) \subseteq M(\mathcal{A})$ for every chamber C of $\mathcal{A}$.

Fix a chamber C of $\mathcal{A}$. Pick $F \in \mathcal{F}(\mathcal{A})$. Let H be a hyperplane of $\mathcal{A}$. If $H \supseteq F$, then $C_F \cap H = \emptyset$ (since C_F is a chamber of $\mathcal{A}_{|F|}$), thus $(F + iC_F) \cap H_{\mathbb{C}} = \emptyset$. If $H \not\supseteq F$, then $F \cap H = \emptyset$, thus $(F + iC_F) \cap H_{\mathbb{C}} = \emptyset$. Therefore $(F + iC_F) \subseteq M(\mathcal{A})$ for every $F \in \mathcal{F}(\mathcal{A})$, thus $M(C) \subseteq M(\mathcal{A})$.

Now, let us prove that $M(C)$ is an open subset of $V_{\mathbb{C}}$ for every chamber C of $\mathcal{A}$.

Fix a chamber C of $\mathcal{A}$. Pick $z = (x + iy) \in M(C)$. Let F be the facet of $\mathcal{A}$ such that $x \in F$. Then, by the definition of $M(C)$, we have $y \in C_F$. If G is a facet of $\mathcal{A}$ with $G \geq F$, then $\mathcal{A}_{|F|} \supseteq \mathcal{A}_{|G|}$, thus $C_F \subseteq C_G$. We write $U(z) = (\bigcup_{G \geq F} G) + iC_F$. The set $U(z)$ is clearly an open neighbourhood of z and, by the above considerations, $U(z) \subseteq M(C)$.

Now, let us prove that $M(\mathcal{A}) \subseteq \bigcup_{C \in V(\Gamma)} M(C)$.

Pick $z = (x + iy) \in M(\mathcal{A})$. Let F be the facet of $\mathcal{A}$ such that $x \in F$. Then there is a chamber D of $\mathcal{A}_{|F|}$ such that $y \in D$. Choose a chamber C of $\mathcal{A}$ such that $C_F = D$. Then $z = (x + iy) \in (F + iC_F) \subseteq M(C)$. $\square$

Now, consider the universal cover $\rho: (\widehat{\Gamma}, \sim) \rightarrow (\Gamma(\mathcal{A}), \sim)$ of $(\Gamma(\mathcal{A}), \sim)$. For every vertex v of $\widehat{\Gamma}$, write

$$M(v) = M(\rho(v)).$$

Set

$$M' = \bigsqcup_{v \in V(\widehat{\Gamma})} M(v),$$

and let $q': M' \rightarrow M$ be the natural projection.

It is easy to see that, if two chambers C and D are adjacent, then there is only one hyperplane $H \in \mathcal{A}$ which separates C and D ; it is the support of their common face. For a chamber C of $\mathcal{A}$ and a hyperplane $H \in \mathcal{A}$, we denote by H_C^+ the open half-space of V bordered by H and containing C .

Let $\mathcal{R}$ be the smallest equivalence relation on M' such that if $a = (v, w) \in A(\widehat{\Gamma})$, $z \in M(v)$, $z' \in M(w)$, and

$$q'(z) = q'(z') \in M(v) \cap M(w) \cap (H_{\rho(w)}^+ + iV),$$

where H is the unique hyperplane of $\mathcal{A}$ which separates $\rho(v)$ and $\rho(w)$, then $z \mathcal{R} z'$. The space $\widehat{M}$ is the quotient $\widehat{M} = M'/\mathcal{R}$, and $q: \widehat{M} \rightarrow M$ is the map induced by q' .

Theorem 2.4. *The map $q: \widehat{M} \rightarrow M$ is the universal cover of $M(\mathcal{A})$.*

Example. Let $V = \mathbb{R}$ and let $\mathcal{A} = \{0\}$. The chambers of $\mathcal{A}$ are $C = \{x \in \mathbb{R} \mid x > 0\}$ and $D = \{x \in \mathbb{R} \mid x < 0\}$. The oriented graph shown in Figure 3a is $\Gamma(\mathcal{A})$ and the oriented graph shown in Figure 3b is $\widehat{\Gamma}$. The subsets of $\mathbb{C}$

shown in Figure 3c are $M(C)$ and $M(D)$. We have $M(C) = \mathbb{C} - \{iy | y \leq 0\}$ and $M(D) = \mathbb{C} - \{iy | y \geq 0\}$. Let (v, w) be an arrow of $\widehat{\Gamma}$ with $\rho(v) = C$ and $\rho(w) = D$. We have

$$\begin{aligned} M(v) \cap M(w) \cap (H_{\rho(w)}^+ + iV) \\ = (\mathbb{C} - \{iy | y \leq 0\}) \cap (\mathbb{C} - \{iy | y \geq 0\}) \cap \{(x + iy) \in \mathbb{C} | x < 0\} \\ = \{(x + iy) \in \mathbb{C} | x < 0\}. \end{aligned}$$

The space shown in Figure 3d is $(M(v) \amalg M(w)) / \mathcal{R}$. If we extend this construction to all $\widehat{\Gamma}$, then we clearly obtain the universal cover of $M(\mathcal{A}) = \mathbb{C} - \{0\}$.

Lemmas 2.5, 2.6 and 2.7 are in [Pa1] preliminary results to the proof of Theorem 2.4; nevertheless, we state them since they will be used later in the paper.

Fix a vertex $v \in V(\widehat{\Gamma})$. Write $C = \rho(v)$. For every chamber D of $\mathcal{A}$, we choose a positive minimal path f_D of $\Gamma(\mathcal{A})$ beginning at C and ending in D . We denote by $\hat{f}_D$ the lift of f_D into $\widehat{\Gamma}$ beginning at v . Note that the end of $\hat{f}_D$ does not depend on the choice of f_D (see the definition of the identification $\sim$ of Γ). We set

$$\Sigma(v) = \{\text{end}(\hat{f}_D) | D \in V(\Gamma)\}.$$

FIGURE 3a

FIGURE 3b

FIGURE 3c

FIGURE 3d

The restriction of ρ to $\Sigma(v)$ is clearly a bijection $\Sigma(v) \rightarrow V(\Gamma)$.

Let v and w be two vertices of $\widehat{\Gamma}$. We write

$$\overline{Z}(v, w) = \bigcup_u \overline{\rho}(u),$$

where the union is over all the vertices $u \in \Sigma(v) \cap \Sigma(w)$ and, for $u \in \Sigma(v) \cap \Sigma(w)$, the set $\overline{\rho}(u)$ is the closure of $\rho(u)$ in V . We denote by $Z(v, w)$ the interior of $\overline{Z}(v, w)$. Note that $Z(v, w)$ is a union of facets of $\mathcal{A}$.

Consider the natural projection

$$p: M' = \bigsqcup_{v \in V(\widehat{\Gamma})} M(v) \rightarrow \widehat{M}.$$

For every $v \in V(\widehat{\Gamma})$, we write $\widehat{M}(v) = p(M(v))$. Since $q': M' \rightarrow M$ sends $M(v)$ homeomorphically onto $M(v)$ and $q' = q \circ p$, the map $q: \widehat{M} \rightarrow M$ sends $\widehat{M}(v)$ homeomorphically onto $M(v)$. Moreover, since q is a cover, $\widehat{M}(v)$ is an open subset of $\widehat{M}$.

Lemma 2.5. *Let v and w be two vertices of $\widehat{\Gamma}$. The border of $Z(v, w)$ is included in the union of the hyperplanes $H \in \mathcal{A}$ which separate $\rho(v)$ and $\rho(w)$.*

Lemma 2.6. *Let v and w be two vertices of $\widehat{\Gamma}$. Then*

$$q(\widehat{M}(v) \cap \widehat{M}(w)) = M(v) \cap M(w) \cap (Z(v, w) + iV).$$

Corollary. *Let v, w be two vertices of $\widehat{\Gamma}$. If $\Sigma(v) \cap \Sigma(w) = \emptyset$, then $\widehat{M}(v) \cap \widehat{M}(w) = \emptyset$.*

Lemma 2.7. *For every chamber C of $\mathcal{A}$, we have*

$$q^{-1}(M(C)) = \bigcup_{v \in \rho^{-1}(C)} \widehat{M}(v),$$

and this union is disjoint.

3. UNIVERSAL COVER OF SALVETTI'S COMPLEX

We assume throughout this section $\mathcal{A}$ to be an (essential) arrangement of hyperplanes in a real vector space V of dimension l .

This section is divided in two subsections. In the first one we define an (abstract) simplicial complex $\text{Sal}(\mathcal{A})$ (Salvetti's complex) and prove that $\text{Sal}(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$. Our complex $\text{Sal}(\mathcal{A})$ is essentially the same complex as this defined in [Sal] but our proof that $\text{Sal}(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$ is completely new. In the second subsection we define another simplicial complex $\widehat{\text{Sal}}$ and prove that there is a natural map $p: \widehat{\text{Sal}} \rightarrow \text{Sal}(\mathcal{A})$ which is the universal cover of $\text{Sal}(\mathcal{A})$ (Theorem 3.7). In particular, $\widehat{\text{Sal}}$ has the same homotopy type as $\widehat{M}$, where $\widehat{M}$ is the universal cover of $M(\mathcal{A})$.

3A. Salvetti's complex. We provide V with an arbitrary scalar product. Let $S^{l-1} = \{x \in V \mid |x| = 1\}$ be the unit sphere. The arrangement $\mathcal{A}$ determines a cellular decomposition of S^{l-1} . With a facet $F \neq \{0\}$ of $\mathcal{A}$ corresponds the (open) cell $\Delta(F) = F \cap S^{l-1}$, and every open cell of this decomposition of S^{l-1} has that form.

This cellular decomposition of S^{l-1} determines a simplicial decomposition of S^{l-1} (called *barycentric subdivision*). For every facet $F \neq \{0\}$ of $\mathcal{A}$ we fix a point $x(F) \in \Delta(F)$. A chain $\{0\} \neq F_0 < F_1 < \dots < F_r$ of facets of $\mathcal{A}$ determines a simplex $\phi = x(F_0) \vee x(F_1) \vee \dots \vee x(F_r)$ having $x(F_0), x(F_1), \dots, x(F_r)$ as vertices, and every simplex of this simplicial decomposition of S^{l-1} has that form. From now on, we always assume S^{l-1} to be provided with the simplicial decomposition described above.

Let $\mathbb{B}^l = \{x \in V \mid |x| \leq 1\}$ be the unit disc. The simplicial decomposition of S^{l-1} determines a simplicial decomposition of $\mathbb{B}^l$ (called the *cone* over S^{l-1}). We add the vertex $x(\{0\}) = 0$ to the set of vertices of S^{l-1} . A chain $F_0 < F_1 < \dots < F_r$ of facets of $\mathcal{A}$ determines a simplex $\phi = x(F_0) \vee x(F_1) \vee \dots \vee x(F_r)$ having $x(F_0), x(F_1), \dots, x(F_r)$ as vertices (recall that $\{0\}$ is a facet of $\mathcal{A}$), and every simplex of this simplicial decomposition of $\mathbb{B}^l$ has that form. Note that, if $F_0 \neq \{0\}$, then $\phi = x(F_0) \vee x(F_1) \vee \dots \vee x(F_r) \subseteq S^{l-1}$. From now on, we always assume $\mathbb{B}^l$ to be provided with the simplicial decomposition described above.

Now we are going to define the (abstract) simplicial complex $\text{Sal}(\mathcal{A})$. For every $X \in \mathcal{L}(\mathcal{A})$ and every chamber D of $\mathcal{A}_X$, we fix a point $y(D) \in D$. For every facet F of $\mathcal{A}$ and every chamber C , we set

$$z(F, C) = x(F) + iy(C_F)$$

(recall that C_F is the unique chamber of $\mathcal{A}_{[F]}$ containing C). Note that $z(F, C) \in (F + iC_F) \subseteq M(C)$. We have $z(F_1, C_1) = z(F_2, C_2)$ if and only if $F_1 = F_2$ and $(C_1)_{F_1} = (C_2)_{F_2}$.

Let $V(\text{Sal})$ be an abstract set in bijection with the set of points of $M(\mathcal{A})$ having the form $z(F, C)$ with $F \in \mathcal{F}(\mathcal{A})$ and C a chamber of $\mathcal{A}$. We denote by $\omega(F, C)$ the element of $V(\text{Sal})$ corresponding with $z(F, C)$. We have $\omega(F_1, C_1) = \omega(F_2, C_2)$ if and only if $F_1 = F_2$ and $(C_1)_{F_1} = (C_2)_{F_2}$. The set $V(\text{Sal})$ will be the set of vertices of $\text{Sal}(\mathcal{A})$.

Let F_1 and F_2 be two facets of $\mathcal{A}$ and let C be a chamber. We set $\omega(F_1, C) < \omega(F_2, C)$ if $F_1 < F_2$.

Lemma 3.1. *The relation “ $<$ ” is a partial order in $V(\text{Sal})$.*

Proof. Pick $\omega_1, \omega_2, \omega_3 \in V(\text{Sal})$ such that $\omega_1 < \omega_2$ and $\omega_2 < \omega_3$, and let us prove that $\omega_1 < \omega_3$.

Since $\omega_1 < \omega_2$, there exist two facets F_1 and F_2 of $\mathcal{A}$ and a chamber C such that $F_1 < F_2$, $\omega_1 = \omega(F_1, C)$, and $\omega_2 = \omega(F_2, C)$. Since $\omega_2 < \omega_3$, there exist two facets F'_2 and F_3 of $\mathcal{A}$ and a chamber D such that $F'_2 < F_3$, $\omega_2 = \omega(F'_2, D)$, and $\omega_3 = \omega(F_3, D)$. We have $\omega_2 = \omega(F_2, C) = \omega(F'_2, D)$, thus $F_2 = F'_2$ and $C_{F_2} = D_{F_2}$. The inequality $F_3 > F'_2 = F_2$ implies $\mathcal{A}_{|F_3|} \subseteq \mathcal{A}_{|F_2|}$, thus $C_{F_3} = D_{F_3}$ (since $C_{F_2} = D_{F_2}$), therefore $\omega_3 = \omega(F_3, D) = \omega(F_3, C)$. It follows that $\omega_1 = \omega(F_1, C) < \omega_3 = \omega(F_3, C)$ (since $F_1 < F_2 = F'_2 < F_3$). $\square$

An r -simplex Φ of $\text{Sal}(\mathcal{A})$ is a $(r+1)$ chain $\omega_0 < \omega_1 < \dots < \omega_r$ in $V(\text{Sal})$. We write $\Phi = \omega_0 \vee \omega_1 \vee \dots \vee \omega_r$. A subset of a chain is clearly still a chain, so $\text{Sal}(\mathcal{A})$ is well defined.

Let $\phi = x(F_0) \vee \dots \vee x(F_r)$ be a simplex of $\mathbb{B}^l$ and let C be a chamber of $\mathcal{A}$. Then ϕ and C determine a simplex $\Phi(\phi, C)$ of $\text{Sal}(\mathcal{A})$ defined by

$$\Phi(\phi, C) = \omega(F_0, C) \vee \dots \vee \omega(F_r, C).$$

Lemma 3.2. *Let Φ be a simplex of $\text{Sal}(\mathcal{A})$. Then there exist a simplex ϕ of $\mathbb{B}^l$ and a chamber C of $\mathcal{A}$ such that $\Phi = \Phi(\phi, C)$.*

Proof. Write $\Phi = \omega_0 \vee \omega_1 \vee \dots \vee \omega_r$, with $\omega_0 < \omega_1 < \dots < \omega_r$. Fix a facet F_0 of $\mathcal{A}$ and a chamber C such that $\omega_0 = \omega(F_0, C)$. Let us prove, by induction on $i = 1, \dots, r$, that there exists a facet F_i of $\mathcal{A}$ such that $\omega_i = \omega(F_i, C)$ and $F_i > F_{i-1}$. It follows that $\Phi = \Phi(\phi, C)$, where $\phi = x(F_0) \vee x(F_1) \vee \dots \vee x(F_r)$.

Assume there exists a facet F_{i-1} of $\mathcal{A}$ such that $\omega_{i-1} = \omega(F_{i-1}, C)$. Since $\omega_{i-1} < \omega_i$, there exist two facets F'_{i-1} and F_i of $\mathcal{A}$ and a chamber D such that $F'_{i-1} < F_i$, $\omega_{i-1} = \omega(F'_{i-1}, D)$ and $\omega_i = \omega(F_i, D)$. We have $\omega_{i-1} = \omega(F_{i-1}, C) = \omega(F'_{i-1}, D)$, thus $F_{i-1} = F'_{i-1}$ and $C_{F_{i-1}} = D_{F_{i-1}}$. The inequality $F_i > F'_{i-1} = F_{i-1}$ implies $\mathcal{A}_{|F_i|} \subseteq \mathcal{A}_{|F_{i-1}|}$, thus $C_{F_i} = D_{F_i}$ (since $C_{F_{i-1}} = D_{F_{i-1}}$). It follows that $\omega_i = \omega(F_i, D) = \omega(F_i, C)$. $\square$

Note that, if $\Phi(\phi_1, C_1) = \Phi(\phi_2, C_2)$, then $\phi_1 = \phi_2$. The map $\pi: \text{Sal}(\mathcal{A}) \rightarrow \mathbb{B}^l$ which sends $\Phi(\phi, C)$ onto ϕ , for every simplex ϕ of $\mathbb{B}^l$ and every chamber C of $\mathcal{A}$, is clearly a well-defined simplicial map, and sends every simplex of $\text{Sal}(\mathcal{A})$ onto a simplex of $\mathbb{B}^l$ having the same dimension.

For a chamber C of $\mathcal{A}$, we denote by $\mathbb{B}^l(C)$ the subcomplex of $\text{Sal}(\mathcal{A})$ generated by the vertices of $\text{Sal}(\mathcal{A})$ having the form $\omega(F, C)$ with $F \in \mathcal{F}(\mathcal{A})$. We have

$$\mathbb{B}^l(C) = \bigcup_{\phi} \Phi(\phi, C),$$

where the union is over all the simplexes ϕ of $\mathbb{B}^l$. The restriction $\pi_C: \mathbb{B}^l(C) \rightarrow \mathbb{B}^l$ of π to $\mathbb{B}^l(C)$ is clearly an isomorphism of simplicial complexes. Moreover,

$$\text{Sal}(\mathcal{A}) = \bigcup_C \mathbb{B}^l(C),$$

where the union is over all chambers C of $\mathcal{A}$.

Theorem 3.3. $\text{Sal}(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$.

Proof. With every vertex ω of $\text{Sal}(\mathcal{A})$ we will associate an open convex subset $U(\omega)$ of $M(\mathcal{A})$. We will prove that $\mathcal{U} = \{U(\omega) | \omega \in V(\text{Sal})\}$ is a covering of $M(\mathcal{A})$ having $\text{Sal}(\mathcal{A})$ as nerve. Since $U(\omega)$ will be convex for every vertex ω of $\text{Sal}(\mathcal{A})$, any nonempty intersection of elements of $\mathcal{U}$ will be convex (thus contractible). This implies, by [We], that $\text{Sal}(\mathcal{A})$ has the same homotopy type as $M(\mathcal{A})$.

For a simplex ϕ of $\mathbb{S}^{l-1}$, we write

$$K(\phi) = \{\lambda x | x \in \phi \text{ and } \lambda > 0\}.$$

Note that, if $\phi = x(F_0) \vee \cdots \vee x(F_r)$ with $\{0\} \neq F_0 < F_1 < \cdots < F_r$, then $K(\phi) \subseteq F_r$. Furthermore, the family $\{K(\phi) | \phi \text{ a simplex of } \mathbb{S}^{l-1}\}$ is a partition of $V - \{0\}$.

Let F be a facet of $\mathcal{A}$ and let C be a chamber. If $F = \{0\}$, then we set

$$U(\omega(F, C)) = (V + iC).$$

If $F \neq \{0\}$, then we set

$$U(\omega(F, C)) = \left(\bigcup_{\phi} K(\phi) \right) + iC_F,$$

where the union is over all the simplexes ϕ of $\mathbb{S}^{l-1}$ having $x(F)$ as vertex. We obviously have $U(\omega(F_1, C_1)) = U(\omega(F_2, C_2))$ if $\omega(F_1, C_1) = \omega(F_2, C_2)$, thus $U(\omega)$ is well defined for every vertex ω of $\text{Sal}(\mathcal{A})$. Moreover, $U(\omega)$ is clearly an open convex subset of $V_{\mathbb{C}}$.

Now, we are going to prove successively the following four assertions.

- (1) $U(\omega) \subseteq M(\mathcal{A})$ for every vertex ω of $\text{Sal}(\mathcal{A})$.
- (2) $M(\mathcal{A}) \subseteq \bigcup_{\omega} U(\omega)$, where the union is over all the vertices ω of $\text{Sal}(\mathcal{A})$.
- (3) Let $\omega_0, \omega_1, \dots, \omega_r$ be $(r+1)$ distinct vertices of $\text{Sal}(\mathcal{A})$. If $U(\omega_0) \cap U(\omega_1) \cap \cdots \cap U(\omega_r) \neq \emptyset$, then $\omega_0, \omega_1, \dots, \omega_r$ are the vertices of a simplex Φ of $\text{Sal}(\mathcal{A})$.
- (4) If $\omega_0, \omega_1, \dots, \omega_r$ are the vertices of a simplex Φ of $\text{Sal}(\mathcal{A})$, then $U(\omega_0) \cap U(\omega_1) \cap \cdots \cap U(\omega_r) \neq \emptyset$.

Assertions (1)–(4) obviously prove that $\mathcal{U} = \{U(\omega) | \omega \in V(\text{Sal})\}$ is a covering of $M(\mathcal{A})$ having $\text{Sal}(\mathcal{A})$ as nerve.

(1) Let F be a facet of $\mathcal{A}$ and let C be a chamber. If $F = \{0\}$, then $U(\omega(F, C)) = (V + iC)$ is obviously included in $M(\mathcal{A})$. Now, assume $F \neq \{0\}$. Pick $z = (x + iy) \in U(\omega(F, C))$. There is a simplex ϕ of $\mathbb{S}^{l-1}$ having $x(F)$ as vertex and such that $x \in K(\phi)$. We write $\phi = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_r)$, with $\{0\} \neq F_0 < F_1 < \cdots < F_r$. We have

$$K(\phi) \subseteq F_r \Rightarrow x \in F_r,$$

and

$$F \leq F_r \Rightarrow C_F \subseteq C_{F_r} \Rightarrow y \in C_{F_r}.$$

Therefore $z = (x + iy) \in (F_r + iC_{F_r}) \subseteq M(\mathcal{A})$.

(2) Pick $z = (x + iy) \in M(\mathcal{A})$. If $x = 0$, then $x \in H$ for every $H \in \mathcal{A}$, thus $y \notin H$ for every $H \in \mathcal{A}$, thus there exists a chamber C of $\mathcal{A}$ such that $y \in C$. Therefore $z = (x + iy) \in (V + iC) = U(\omega(\{0\}, C))$.

Now, assume $x \neq 0$. There exists a simplex ϕ of $\mathbb{S}^{l-1}$ such that $x \in K(\phi)$. We write $\phi = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_r)$ with $\{0\} \neq F_0 < F_1 < \cdots < F_r$. Since $x \in K(\phi) \subseteq F_r$, there is no hyperplane $H \in \mathcal{A}$ containing F_r which contains y , thus there is a chamber D of $\mathcal{A}_{|F_r|}$ such that $y \in D$. Pick a chamber C of $\mathcal{A}$ such that $C_{F_r} = D$. Then

$$z = (x + iy) \in (K(\phi) + iC_{F_r}) \subseteq U(\omega(F_r, C)).$$

(3) Let $\omega_0, \omega_1, \dots, \omega_r$ be $(r+1)$ distinct vertices of $\text{Sal}(\mathcal{A})$ such that $U(\omega_0) \cap U(\omega_1) \cap \cdots \cap U(\omega_r) \neq \emptyset$. Write $\omega_i = \omega(F_i, C_i)$, where $F_i \in \mathcal{F}(\mathcal{A})$ and C_i is a chamber of $\mathcal{A}$, for $i = 0, 1, \dots, r$. Pick $z = (x + iy) \in \bigcap_{i=0}^r U(\omega_i)$.

Case a. Assume $F_0 = \{0\}$.

Suppose there exists an $i \in \{1, \dots, r\}$ such that $F_i = \{0\}$. Then

$$\begin{aligned} z = (x + iy) &\in U(\omega_0) \cap U(\omega_i) = (V + iC_0) \cap (V + iC_i) \\ &\Rightarrow y \in C_0 \cap C_i \\ &\Rightarrow C_0 \cap C_i \neq \emptyset \\ &\Rightarrow C_0 = C_i \\ &\Rightarrow \omega_0 = \omega(\{0\}, C_0) = \omega_i = \omega(\{0\}, C_i). \end{aligned}$$

This contradicts the fact that $\omega_0 \neq \omega_i$. Therefore $F_i \neq \{0\}$ for $i = 1, \dots, r$.

There is a simplex ϕ_i of $\mathbb{S}^{l-1}$ having $x(F_i)$ as vertex and such that $x \in K(\phi_i)$, for $i = 1, \dots, r$. Since $\{K(\phi) | \phi \text{ a simplex of } \mathbb{S}^{l-1}\}$ is a partition of $V - \{0\}$, we have $\phi_1 = \cdots = \phi_r$. Therefore $x(F_1), \dots, x(F_r)$ are vertices of a same simplex ϕ of $\mathbb{S}^{l-1}$, thus $\{F_1, \dots, F_r\}$ is a chain. Assume $\{0\} = F_0 < F_1 \leq \cdots \leq F_r$. For $i = 1, \dots, r$ we have

$$\begin{aligned} y \in C_0 \quad \text{and} \quad y \in (C_i)_{F_i} &\Rightarrow y \in (C_0)_{F_i} \quad \text{and} \quad y \in (C_i)_{F_i} \\ &\Rightarrow (C_0)_{F_i} \cap (C_i)_{F_i} \neq \emptyset \\ &\Rightarrow (C_0)_{F_i} = (C_i)_{F_i} \\ &\Rightarrow \omega_i = \omega(F_i, C_i) = \omega(F_i, C_0). \end{aligned}$$

Moreover, we have $F_{i-1} \neq F_i$ for $i = 1, \dots, r$; otherwise $\omega_{i-1} = \omega(F_{i-1}, C_0) = \omega_i = \omega(F_i, C_0)$. It follows that $\omega_0, \omega_1, \dots, \omega_r$ are the vertices of the simplex

$$\Phi = \omega(F_0, C_0) \vee \omega(F_1, C_0) \vee \cdots \vee \omega(F_r, C_0)$$

of $\text{Sal}(\mathcal{A})$.

Case b. Assume $F_i \neq \{0\}$ for $i = 0, 1, \dots, r$.

There is a simplex ϕ_i of $\mathbb{S}^{l-1}$ having $x(F_i)$ as vertex and such that $x \in K(\phi_i)$, for $i = 0, 1, \dots, r$. Since $\{K(\phi) | \phi \text{ a simplex of } \mathbb{S}^{l-1}\}$ is a partition of $V - \{0\}$, we have $\phi_0 = \phi_1 = \cdots = \phi_r$. Therefore $\{F_0, F_1, \dots, F_r\}$ is a chain. Assume $\{0\} \neq F_0 \leq F_1 \leq \cdots \leq F_r$. For $i = 1, \dots, r$ we have

$$\begin{aligned} y \in (C_0)_{F_0}, \quad F_i \geq F_0 \quad \text{and} \quad y \in (C_i)_{F_i} &\Rightarrow y \in (C_0)_{F_i} \quad \text{and} \quad y \in (C_i)_{F_i} \\ &\Rightarrow (C_0)_{F_i} \cap (C_i)_{F_i} \neq \emptyset \\ &\Rightarrow (C_0)_{F_i} = (C_i)_{F_i} \\ &\Rightarrow \omega_i = \omega(F_i, C_i) = \omega(F_i, C_0). \end{aligned}$$

Moreover, we have $F_{i-1} \neq F_i$ for $i = 1, \dots, r$; otherwise $\omega_{i-1} = \omega(F_{i-1}, C_0) = \omega_i = \omega(F_i, C_0)$. It follows that $\omega_0, \omega_1, \dots, \omega_r$ are the vertices of the

simplex

$$\Phi = \omega(F_0, C_0) \vee \omega(F_1, C_0) \vee \cdots \vee \omega(F_r, C_0)$$

of $\text{Sal}(\mathcal{A})$.

(4) Let $\Phi = \Phi(\phi, C)$ be a simplex of $\text{Sal}(\mathcal{A})$. We write $\phi = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_r)$ with $F_0 < F_1 < \cdots < F_r$. The vertices of Φ are $\omega(F_0, C)$, $\omega(F_1, C)$, $\dots$, $\omega(F_r, C)$. Let us prove that $\bigcap_{i=0}^r U(\omega(F_i, C)) = \emptyset$.

Case a. Assume $F_0 = \{0\}$.

Consider the simplex $\phi' = x(F_1) \vee \cdots \vee x(F_r)$ of $\mathbb{S}^{l-1}$. Pick $x \in K(\phi')$ and $y \in C$, and write $z = x + iy$. We obviously have $z \in (V + iC) = U(\omega(F_0, C))$. The simplex ϕ' has $x(F_i)$ as vertex, $x \in K(\phi')$ and $y \in C \subseteq C_{F_i}$, thus $z = (x + iy) \in U(\omega(F_i, C))$ for $i = 1, \dots, r$. It follows that $z \in \bigcap_{i=0}^r U(\omega(F_i, C))$.

Case b. Assume $F_0 \neq \{0\}$.

Then ϕ is a simplex of $\mathbb{S}^{l-1}$. Pick $x \in K(\phi)$ and $y \in C$, and write $z = x + iy$. The simplex ϕ has $x(F_i)$ as vertex, $x \in K(\phi)$ and $y \in C \subseteq C_{F_i}$, thus $z = (x + iy) \in U(\omega(F_i, C))$ for $i = 0, 1, \dots, r$. It follows that $z \in \bigcap_{i=0}^r U(\omega(F_i, C))$. $\square$

3B. Universal cover of Salvetti's complex. Now, we are going to define the (abstract) simplicial complex $\widehat{\text{Sal}}$.

Throughout this subsection, we denote by $\rho: (\widehat{\Gamma}, \sim) \rightarrow (\Gamma(\mathcal{A}), \sim)$ the universal cover of $(\Gamma(\mathcal{A}), \sim)$, and by $q: \widehat{M} \rightarrow M(\mathcal{A})$ the universal cover of $M(\mathcal{A})$, as defined in §2.

Let C be a chamber of $\mathcal{A}$. For every facet $F \in \mathcal{F}(\mathcal{A})$ we have

$$z(F, C) = (x(F) + iy(C_F)) \in (F + iC_F) \subseteq M(C).$$

By Lemma 2.7,

$$q^{-1}(M(C)) = \bigcup_{v \in \rho^{-1}(C)} \widehat{M}(v),$$

and this union is disjoint. Moreover, recall that q sends $\widehat{M}(v)$ homeomorphically onto $M(v) = M(C)$ for every vertex $v \in \rho^{-1}(C)$. This implies that, for every facet $F \in \mathcal{F}(\mathcal{A})$ and every vertex $v \in V(\widehat{\Gamma})$, there exists a unique point $e(F, v) \in \widehat{M}(v)$ such that $q(e(F, v)) = z(F, \rho(v))$ (i.e., $e(F, v)$ is the lift of $z(F, C)$ into $\widehat{M}(v)$, where $C = \rho(v)$).

Let $V(\widehat{\text{Sal}})$ be an abstract set in bijection with the set of points of $\widehat{M}$ having the form $e(F, v)$ with $F \in \mathcal{F}(\mathcal{A})$ and $v \in V(\widehat{\Gamma})$. We denote by $\widehat{\omega}(F, v)$ the element of $V(\widehat{\text{Sal}})$ corresponding to $e(F, v)$. The set $V(\widehat{\text{Sal}})$ will be the set of vertices of $\widehat{\text{Sal}}$.

Lemma 3.4. *Let F_1, F_2 be two facets of $\mathcal{A}$, and let v_1, v_2 be two vertices of $\widehat{\Gamma}$. Then*

$$\widehat{\omega}(F_1, v_1) = \widehat{\omega}(F_2, v_2)$$

if and only if $F_1 = F_2 \subseteq Z(v_1, v_2)$ and $\rho(v_1)_{F_1} = \rho(v_2)_{F_2}$.

Proof. Assume $\widehat{\omega}(F_1, v_1) = \widehat{\omega}(F_2, v_2)$ (thus $e(F_1, v_1) = e(F_2, v_2)$). Since $e(F_1, v_1) \in \widehat{M}(v_1)$ and $e(F_2, v_2) \in \widehat{M}(v_2)$, by Lemma 2.6,

$$\begin{aligned} q(e(F_1, v_1)) &= z(F_1, \rho(v_1)) = q(e(F_2, v_2)) \\ &= z(F_2, \rho(v_2)) \in M(v_1) \cap M(v_2) \cap (Z(v_1, v_2) + iV). \end{aligned}$$

The equality $z(F_1, \rho(v_1)) = z(F_2, \rho(v_2))$ implies $F_1 = F_2$ and $\rho(v_1)_{F_1} = \rho(v_2)_{F_2}$. On the other hand, $x(F_1) \in Z(v_1, v_2)$, the set $Z(v_1, v_2)$ is a union of facets of $\mathcal{A}$ and $x(F_1) \in F_1$, thus $F_1 \subseteq Z(v_1, v_2)$.

Now, assume $F_1 = F_2 \subseteq Z(v_1, v_2)$ and $\rho(v_1)_{F_1} = \rho(v_2)_{F_2}$. This implies

$$\begin{aligned} z(F_1, \rho(v_1)) &= z(F_2, \rho(v_2)) \in (F_1 + i\rho(v_1)_{F_1}) \cap (F_2 + i\rho(v_2)_{F_2}) \\ &\cap (Z(v_1, v_2) + iV) \subseteq M(v_1) \cap M(v_2) \cap (Z(v_1, v_2) + iV). \end{aligned}$$

It follows, by Lemma 2.6, that

$$z(F_1, \rho(v_1)) = z(F_2, \rho(v_2)) \in q(\widehat{M}(v_1) \cap \widehat{M}(v_2)).$$

The map q sends $\widehat{M}(v_1)$ homeomorphically onto $M(v_1)$, the point $e(F_1, v_1)$ is the lift of $z(F_1, \rho(v_1))$ into $\widehat{M}(v_1)$, and $z(F_1, \rho(v_1)) \in q(\widehat{M}(v_1) \cap \widehat{M}(v_2))$, thus $e(F_1, v_1) \in \widehat{M}(v_1) \cap \widehat{M}(v_2)$. It follows that $e(F_1, v_1)$ is the unique lift of $z(F_2, \rho(v_2)) = z(F_1, \rho(v_1))$ into $\widehat{M}(v_2)$, thus $e(F_1, v_1) = e(F_2, v_2)$, therefore $\hat{\omega}(F_1, v_1) = \hat{\omega}(F_2, v_2)$. $\square$

Let F_1 and F_2 be two facets of $\mathcal{A}$ and let v be a vertex of $\widehat{\Gamma}$. We set $\hat{\omega}(F_1, v) < \hat{\omega}(F_2, v)$ if $F_1 < F_2$.

Lemma 3.5. *The relation “ $<$ ” is a partial order in $V(\widehat{\text{Sal}})$.*

Proof. Pick $\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3 \in V(\widehat{\text{Sal}})$ such that $\hat{\omega}_1 < \hat{\omega}_2$ and $\hat{\omega}_2 < \hat{\omega}_3$, and let us prove that $\hat{\omega}_1 < \hat{\omega}_3$.

Since $\hat{\omega}_1 < \hat{\omega}_2$, there exist two facets F_1 and F_2 of $\mathcal{A}$ and a vertex v of $\widehat{\Gamma}$ such that $F_1 < F_2$, $\hat{\omega}_1 = \hat{\omega}(F_1, v)$, and $\hat{\omega}_2 = \hat{\omega}(F_2, v)$. Since $\hat{\omega}_2 < \hat{\omega}_3$, there exist two facets F'_2 and F_3 of $\mathcal{A}$ and a vertex w of $\widehat{\Gamma}$ such that $F'_2 < F_3$, $\hat{\omega}_2 = \hat{\omega}(F'_2, w)$, and $\hat{\omega}_3 = \hat{\omega}(F_3, w)$. We have $\hat{\omega}_2 = \hat{\omega}(F_2, v) = \hat{\omega}(F'_2, w)$, thus, by Lemma 3.4, $F_2 = F'_2 \subseteq Z(v, w)$ and $\rho(v)_{F_2} = \rho(w)_{F_2}$. Note that $Z(v, w)$ is a union of facets of $\mathcal{A}$ and is an open subset of V , thus, if F and G are two facets of $\mathcal{A}$ such that $G \geq F$ and $F \subseteq Z(v, w)$, then $G \subseteq Z(v, w)$. Since $F_3 > F'_2 = F_2$ and $F_2 \subseteq Z(v, w)$, we have $F_3 \subseteq Z(v, w)$. Furthermore, $\mathcal{A}_{|F_2|} \supseteq \mathcal{A}_{|F_3|}$ and $\rho(v)_{F_2} = \rho(w)_{F_2}$, thus $\rho(v)_{F_3} = \rho(w)_{F_3}$. It follows, by Lemma 3.4, that $\hat{\omega}_3 = \hat{\omega}(F_3, w) = \hat{\omega}(F_3, v)$. Therefore $\hat{\omega}_1 = \hat{\omega}(F_1, v) < \hat{\omega}_3 = \hat{\omega}(F_3, v)$ (since $F_1 < F_2 = F'_2 < F_3$). $\square$

An r -simplex Φ of $\widehat{\text{Sal}}$ is a $(r+1)$ chain $\hat{\omega}_0 < \hat{\omega}_1 < \dots < \hat{\omega}_r$ in $V(\widehat{\text{Sal}})$. We write $\Phi = \hat{\omega}_0 \vee \hat{\omega}_1 \vee \dots \vee \hat{\omega}_r$. A subset of a chain is clearly still a chain, so $\widehat{\text{Sal}}$ is well defined.

Let $\phi = x(F_0) \vee x(F_1) \vee \dots \vee x(F_r)$ be a simplex of $\mathbb{B}^l$ and let v be a vertex of $\widehat{\Gamma}$. Then ϕ and v determine a simplex $\Phi(\phi, v)$ of $\widehat{\text{Sal}}$ defined by

$$\Phi(\phi, v) = \hat{\omega}(F_0, v) \vee \hat{\omega}(F_1, v) \vee \dots \vee \hat{\omega}(F_r, v).$$

Lemma 3.6. *Let Φ be a simplex of $\widehat{\text{Sal}}$. Then there exist a simplex ϕ of $\mathbb{B}^l$ and a vertex v of $\widehat{\Gamma}$ such that $\Phi = \Phi(\phi, v)$.*

Proof. Write $\Phi = \hat{\omega}_0 \vee \hat{\omega}_1 \vee \dots \vee \hat{\omega}_r$ with $\hat{\omega}_0 < \hat{\omega}_1 < \dots < \hat{\omega}_r$. Fix a facet F_0 of $\mathcal{A}$ and a vertex v of $\widehat{\Gamma}$ such that $\hat{\omega}_0 = \hat{\omega}(F_0, v)$. Let us prove, by induction on $i = 1, \dots, r$, that there exists a facet F_i of $\mathcal{A}$ such that $\hat{\omega}_i = \hat{\omega}(F_i, v)$ and $F_i > F_{i-1}$. It follows that $\Phi = \Phi(\phi, v)$, where $\phi = x(F_0) \vee x(F_1) \vee \dots \vee x(F_r)$.

Assume there exists a facet F_{i-1} of $\mathcal{A}$ such that $\hat{\omega}_{i-1} = \hat{\omega}(F_{i-1}, v)$. Since $\hat{\omega}_{i-1} < \hat{\omega}_i$, there exist two facets F'_{i-1} and F_i of $\mathcal{A}$ and a vertex w of $\hat{\Gamma}$ such that $F'_{i-1} < F_i$, $\hat{\omega}_{i-1} = \hat{\omega}(F'_{i-1}, w)$ and $\hat{\omega}_i = \hat{\omega}(F_i, w)$. We have $\hat{\omega}_{i-1} = \hat{\omega}(F_{i-1}, v) = \hat{\omega}(F'_{i-1}, w)$, thus, by Lemma 3.4, $F_{i-1} = F'_{i-1} \subseteq Z(v, w)$ and $\rho(v)_{F_{i-1}} = \rho(w)_{F_{i-1}}$. Since $F_i > F'_{i-1} = F_{i-1}$ and $F_{i-1} \subseteq Z(v, w)$, we have $F_i \subseteq Z(v, w)$. Since $F_i > F_{i-1}$ and $\rho(v)_{F_{i-1}} = \rho(w)_{F_{i-1}}$, we have $\rho(v)_{F_i} = \rho(w)_{F_i}$. It follows, by Lemma 3.4, that $\hat{\omega}_i = \hat{\omega}(F_i, w) = \hat{\omega}(F_i, v)$. $\square$

Note that, if $\Phi(\phi_1, v_1) = \Phi(\phi_2, v_2)$, then $\phi_1 = \phi_2$. The map $\hat{\pi}: \hat{\text{Sal}} \rightarrow \mathbb{B}'$ which sends $\Phi(\phi, v)$ onto ϕ , for every simplex ϕ of $\mathbb{B}'$ and every vertex v of $\hat{\Gamma}$, is clearly a well-defined simplicial map and sends every simplex of $\hat{\text{Sal}}$ onto a simplex of $\mathbb{B}'$ having the same dimension.

For a vertex v of $\hat{\Gamma}$, we denote by $\mathbb{B}'(v)$ the subcomplex of $\hat{\text{Sal}}$ generated by the vertices of $\hat{\text{Sal}}$ having the form $\hat{\omega}(F, v)$ with $F \in \mathcal{F}(\mathcal{A})$. We have

$$\mathbb{B}'(v) = \bigcup_{\phi} \Phi(\phi, v),$$

where the union is over all the simplexes ϕ of $\mathbb{B}'$. The restriction $\hat{\pi}_v: \mathbb{B}'(v) \rightarrow \mathbb{B}'$ of $\hat{\pi}$ to $\mathbb{B}'(v)$ is clearly an isomorphism of simplicial complexes. Moreover,

$$\hat{\text{Sal}} = \bigcup_{v \in V(\hat{\Gamma})} \mathbb{B}'(v).$$

Consider the map $p: \hat{\text{Sal}} \rightarrow \text{Sal}(\mathcal{A})$ which sends $\Phi(\phi, v)$ onto $\Phi(\phi, \rho(v))$ for every simplex ϕ of $\mathbb{B}'$ and every vertex v of $\hat{\Gamma}$. It is clearly a simplicial map and sends every simplex of $\hat{\text{Sal}}$ onto a simplex of $\text{Sal}(\mathcal{A})$ having the same dimension.

Theorem 3.7. *The map $p: \hat{\text{Sal}} \rightarrow \text{Sal}(\mathcal{A})$ is the universal cover of $\text{Sal}(\mathcal{A})$. In particular, $\hat{\text{Sal}}$ has the same homotopy type as $\widehat{M}$.*

Remark. Using the same ideas as in the proof of Theorem 3.3, one can show directly that $\hat{\text{Sal}}$ has the same homotopy type as $\widehat{M}$. Namely, for every facet F of $\mathcal{A}$ and every chamber C , we have $U(\omega(F, C)) \subseteq M(C)$. Thus, if $F \in \mathcal{F}(\mathcal{A})$ and $v \in V(\hat{\Gamma})$, then there exists a unique open “convex” set $U(\hat{\omega}(F, v)) \subseteq \widehat{M}(v)$ such that $q(U(\hat{\omega}(F, v))) = U(\omega(F, \rho(v)))$. One can show that $U(\hat{\omega}(F_1, v_1)) = U(\hat{\omega}(F_2, v_2))$ if $\hat{\omega}(F_1, v_1) = \hat{\omega}(F_2, v_2)$, the set $\mathcal{U} = \{U(\hat{\omega}) | \hat{\omega} \in V(\hat{\text{Sal}})\}$ is a covering of $\widehat{M}$ having $\hat{\text{Sal}}$ as nerve, and every nonempty intersection of elements of $\mathcal{U}$ is “convex”. Nevertheless, we present here a different proof which may be more complicated but which carries more information; for example, we will give explicitly a homotopy equivalence between $\hat{\text{Sal}}$ and $\widehat{M}$.

The following Lemma 3.8 is a preliminary result to the proof of Theorem 3.7.

For $(r+1)$ distinct points $p_0, p_1, \dots, p_r$ of a real vector space, we set

$$\Delta(p_0, p_1, \dots, p_r) = \left\{ \sum_{i=0}^r t_i p_i \mid 0 < t_i \leq 1 \text{ and } \sum_{i=0}^r t_i = 1 \right\}.$$

Consider the map $\iota: \text{Sal}(\mathcal{A}) \rightarrow V_{\mathbb{C}}$ which sends a simplex

$$\Phi = \omega(F_0, C) \vee \omega(F_1, C) \vee \cdots \vee \omega(F_r, C)$$

of $\text{Sal}(\mathcal{A})$ onto $\Delta(z(F_0, C), z(F_1, C), \dots, z(F_r, C))$. This map sends $\omega(F, C)$ onto $z(F, C)$ for every facet F of $\mathcal{A}$ and every chamber C . It is obviously well defined.

Lemma 3.8. (i) Let ϕ be a simplex of $\mathbb{B}^l$, and let C be a chamber of $\mathcal{A}$. Write $\phi = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_r)$ with $F_0 < F_1 < \cdots < F_r$. Then $\iota(\Phi(\phi, C)) \subseteq (F_r + iC_{F_r})$. In particular, $\iota(\text{Sal}(\mathcal{A})) \subseteq M(\mathcal{A})$.

(ii) The map ι is injective.

(iii) The map $\iota: \text{Sal}(\mathcal{A}) \rightarrow M(\mathcal{A})$ is a homotopy equivalence.

Proof. (i) Let ϕ be a simplex of $\mathbb{B}^l$, and let C be a chamber of $\mathcal{A}$. Write $\phi = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_r)$ with $F_0 < F_1 < \cdots < F_r$. Pick $z = (x + iy) = \sum_{i=0}^r \lambda_i z(F_i, C) \in \iota(\Phi(\phi, C))$. We have $\Delta(x(F_0), x(F_1), \dots, x(F_r)) \subseteq F_r$, thus $x = \sum_{i=0}^r \lambda_i x(F_i) \in F_r$. On the other hand, $y(C_{F_i}) \in C_{F_i} \subseteq C_{F_r}$ for $i = 0, 1, \dots, r$, thus $y = \sum_{i=0}^r \lambda_i y(C_{F_i}) \in C_{F_r}$ (since C_{F_r} is convex).

(ii) Let ϕ and ψ be two simplexes of $\mathbb{B}^l$, and let C and D be two chambers of $\mathcal{A}$. Write $\phi = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_r)$ with $F_0 < F_1 < \cdots < F_r$, and $\psi = x(G_0) \vee x(G_1) \vee \cdots \vee x(G_s)$ with $G_0 < G_1 < \cdots < G_s$. Let us prove, by induction on r , that, if $\iota(\Phi(\phi, C)) \cap \iota(\Phi(\psi, D)) \neq \emptyset$, then $\Phi(\phi, C) = \Phi(\psi, D)$.

Pick $z = (x + iy) \in \iota(\Phi(\phi, C)) \cap \iota(\Phi(\psi, D))$. Write $z = \sum_{i=0}^r t_i z(F_i, C) = \sum_{j=0}^s t'_j z(G_j, D)$, where $0 < t_i, t'_j \leq 1$ and $\sum_{i=0}^r t_i = \sum_{j=0}^s t'_j = 1$. Recall that $\{x(F) | F \in \mathcal{F}(\mathcal{A})\}$ is the set of vertices of a triangulation of $\mathbb{B}^l$. Since $x = \sum_{i=0}^r t_i x(F_i) = \sum_{j=0}^s t'_j x(G_j)$, we have $r = s$, $F_i = G_i$ (for $i = 0, 1, \dots, r$) and $t_i = t'_i$ (for $i = 0, \dots, r$). In particular, $\phi = \psi$. Moreover,

$$\emptyset \neq \iota(\Phi(\phi, C)) \cap \iota(\Phi(\phi, D)) \subseteq (F_r + iC_{F_r}) \cap (F_r + iD_{F_r}),$$

thus $C_{F_r} \cap D_{F_r} \neq \emptyset$, therefore $C_{F_r} = D_{F_r}$. It follows that

$$\begin{aligned} z' &= \left(\sum_{i=0}^{r-1} t_i \right)^{-1} (z - t_r z(F_r, C)) \\ &= \left(\sum_{i=0}^{r-1} t_i \right)^{-1} (z - t_r z(F_r, D)) \in \iota(\Phi(\phi', C)) \cap \iota(\Phi(\phi', D)), \end{aligned}$$

where $\phi' = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_{r-1})$. By the inductive hypothesis, we have $\Phi(\phi', C) = \Phi(\phi', D)$, therefore $\Phi(\phi, C) = \Phi(\phi, D) = \Phi(\psi, D)$.

(iii) Let $(f_\omega)_{\omega \in V(\text{Sal})}$ be a partition of the unity subordinated to the covering $\{U(\omega) | \omega \in V(\text{Sal})\}$ of $M(\mathcal{A})$ (see the proof of Theorem 3.3). Namely, $(f_\omega)_{\omega \in V(\text{Sal})}: M(\mathcal{A}) \rightarrow [0, 1]$ is a collection of maps with

(1) $f_\omega(z) > 0$ if $z \in U(\omega)$ and $f_\omega(z) = 0$ if $z \notin U(\omega)$, for all $z \in M(\mathcal{A})$ and all $\omega \in V(\text{Sal})$,

(2) $\sum_{\omega \in V(\text{Sal})} f_\omega(z) = 1$ for all $z \in M(\mathcal{A})$.

Let $\kappa: M(\mathcal{A}) \rightarrow \text{Sal}(\mathcal{A})$ be the map defined by $\kappa(z) = \sum_{\omega \in V(\text{Sal})} f_\omega(z) \omega$, for all $z \in M(\mathcal{A})$. Since the covering $\{U(\omega) | \omega \in V(\text{Sal})\}$ has $\text{Sal}(\mathcal{A})$ as nerve, κ is well defined. Moreover, by [We, p. 143], κ is a homotopy equivalence.

We are going to prove that $\text{id}_{M(\mathcal{A})}$ and $\iota\kappa$ are homotopic. This implies that ι is a homotopy inverse of κ , and thus is a homotopy equivalence.

Consider the homotopy $(\theta_t)_{0 \leq t \leq 1}: M(\mathcal{A}) \rightarrow V_C$ defined by

$$\theta_t(z) = tz + (1-t)(\iota\kappa)(z).$$

Let us prove that $\theta_t(z) \in M(\mathcal{A})$ for all $z \in M(\mathcal{A})$ and all $t \in [0, 1]$. This shows that $(\theta_t)_{0 \leq t \leq 1}: M(\mathcal{A}) \rightarrow M(\mathcal{A})$ is a homotopy connecting $\text{id}_{M(\mathcal{A})}$ with $\iota\kappa$.

Pick $z = (x + iy) \in M(\mathcal{A})$.

Case a. Assume $x = 0$.

Let C be the chamber of $\mathcal{A}$ such that $y \in C$. The only vertex ω of Sal such that $z \in U(\omega)$ is $\omega = \omega(\{0\}, C)$. We have $z = (x + iy) \in (\{0\} + iC)$ and $\iota\kappa(z) = z(\{0\}, C) \in (\{0\} + iC)$, thus $\theta_t(z) \in (\{0\} + iC) \subseteq M(\mathcal{A})$ (since $(\{0\} + iC)$ is convex) for all $t \in [0, 1]$.

Case b. Assume $x \neq 0$.

There is a unique simplex ϕ of $\mathbb{S}^{l-1}$ such that $x \in K(\phi)$. Write $\phi = x(F_1) \vee \dots \vee x(F_r)$ with $\{0\} \neq F_1 < \dots < F_r$. Set $F_0 = \{0\}$. Since $K(\phi) \subseteq F_r$, there is a chamber D_r of $\mathcal{A}_{|F_r|}$ with $y \in D_r$. On the other hand, if there is a chamber D_{i-1} of $\mathcal{A}_{|F_{i-1}|}$ with $y \in D_{i-1}$, then there is a chamber D_i of $\mathcal{A}_{|F_i|}$ with $y \in D_i$ and $D_{i-1} \subseteq D_i$. It follows that there exists a $j \in \{0, 1, \dots, r\}$ such that

- (1) there is a chamber D_i of $\mathcal{A}_{|F_i|}$ containing y for all $i = j, \dots, r$,
- (2) there is no chamber of $\mathcal{A}_{|F_i|}$ containing y for any $i = 0, \dots, j-1$,
- (3) $D_j \subseteq D_{j+1} \subseteq \dots \subseteq D_r$.

Choose a chamber C of $\mathcal{A}$ such that $C_{F_j} = D_j$. We obviously have $C_{F_i} = D_i$ for $i = j, \dots, r$. The set of vertices ω of Sal such that $z \in U(\omega)$ is $\{\omega(F_j, C), \omega(F_{j+1}, C), \dots, \omega(F_r, C)\}$. Write $\Phi = \omega(F_j, C) \vee \dots \vee \omega(F_r, C)$. We have $z = (x + iy) \in (F_r + iC_{F_r})$ and $\iota\kappa(z) \in \iota(\Phi) \subseteq (F_r + iC_{F_r})$, thus $\theta_t(z) \in (F_r + iC_{F_r}) \subseteq M(\mathcal{A})$ (since $(F_r + iC_{F_r})$ is convex) for all $t \in [0, 1]$. $\square$

Proof of Theorem 3.7. Recall that, for every chamber C of $\mathcal{A}$,

$$q^{-1}(M(C)) = \bigcup_{v \in \rho^{-1}(C)} \widehat{M}(v),$$

this union is disjoint (Lemma 2.7), and q sends $\widehat{M}(v)$ homeomorphically onto $M(C) = M(v)$ for every vertex $v \in \rho^{-1}(C)$. Furthermore, for every simplex ϕ of $\mathbb{B}^l$ and every chamber C of $\mathcal{A}$, we have $\iota(\Phi(\phi, C)) \subseteq M(C)$ (see Lemma 4.8(i)).

Since $\iota: \text{Sal}(\mathcal{A}) \rightarrow M(\mathcal{A})$ is injective, $\iota(\text{Sal}(\mathcal{A}))$ can be viewed as a geometric realisation of $\text{Sal}(\mathcal{A})$, so $q^{-1}(\iota(\text{Sal}(\mathcal{A})))$ can also be seen as a geometric realization of a simplicial complex (since q is a cover). Let us denote by W the set of vertices of $q^{-1}(\iota(\text{Sal}(\mathcal{A})))$. The application $\hat{i}: V(\widehat{\text{Sal}}) \rightarrow W$ which sends $\hat{\omega}(F, v)$ onto $e(F, v)$, for every $F \in \mathcal{F}(\mathcal{A})$ and every $v \in V(\widehat{\text{Sal}})$, is, by the definition of $V(\widehat{\text{Sal}})$, a bijection. It is easy to see that $\hat{i}$ can be extended to an isomorphism $\hat{i}: \widehat{\text{Sal}} \rightarrow q^{-1}(\iota(\text{Sal}(\mathcal{A})))$ of simplicial complexes which sends $\Phi(\phi, v)$ onto the lift of $\iota(\Phi(\phi, C))$ into $\widehat{M}(v)$, where $C = \rho(v)$, for all simplexes ϕ of $\mathbb{B}^l$ and all vertices v of $\widehat{\Gamma}$.

The following diagram commutes:

$$\begin{array}{ccc} \widehat{\text{Sal}} & \xrightarrow{i} & \widehat{M} \\ p \downarrow & & \downarrow q \\ \text{Sal}(\mathcal{A}) & \xrightarrow{\iota} & M(\mathcal{A}) \end{array},$$

the map q is a cover, ι and $\hat{i}$ are injective, and $q^{-1}(\iota(\text{Sal}(\mathcal{A}))) = \hat{i}(\widehat{\text{Sal}})$, thus p is a cover. Furthermore, $q: \widehat{M} \rightarrow M(\mathcal{A})$ is the universal cover of $M(\mathcal{A})$ and ι is a homotopy equivalence, thus $p: \widehat{\text{Sal}} \rightarrow \text{Sal}(\mathcal{A})$ is the universal cover of $\text{Sal}(\mathcal{A})$ and $\hat{i}$ is a homotopy equivalence. $\square$

4. TOPOLOGY OF SIMPLICIAL ARRANGEMENTS OF HYPERPLANES

Recall that an essential arrangement $\mathcal{A}$ of hyperplanes is called *simplicial* if every chamber of $\mathcal{A}$ is an open simplicial cone. Our goal in this section is to prove that if $\mathcal{A}$ is a simplicial arrangement of hyperplanes, then $M(\mathcal{A})$ is a $K(\pi, 1)$ space.

This section is divided in two subsections. In the first one we define a property on real arrangements of hyperplanes: the *property D*, and we prove that if $\mathcal{A}$ is a simplicial arrangement of hyperplanes, then $\mathcal{A}$ has the property D (Theorem 4.1). We do not know if, for an essential arrangement $\mathcal{A}$ of hyperplanes, to have the property D is equivalent to being simplicial. We will give a simple example of a supersolvable arrangement which does not have the property D. It is well known that, if $\mathcal{A}$ is a supersolvable arrangement of hyperplanes, then $M(\mathcal{A})$ is a $K(\pi, 1)$ space (see [Te2]).

In the second subsection we prove that if $\mathcal{A}$ has the property D, then $\widehat{\text{Sal}}$ is contractible (Theorem 4.6). Since $\widehat{\text{Sal}}$ has the same homotopy type as the universal cover $\widehat{M}$ of $M(\mathcal{A})$ (Theorem 3.7), the space $\widehat{M}$ is contractible if $\mathcal{A}$ is simplicial; thus, in this case, $M(\mathcal{A})$ is a $K(\pi, 1)$ space.

4A. Property D. Throughout this subsection, $\mathcal{A}$ is an arrangement of hyperplanes in a real vector space V , and $(\Gamma(\mathcal{A}), \sim)$ is the oriented system associated with $\mathcal{A}$.

Let A and B be two chambers of $\mathcal{A}$. We say that a chamber C of $\mathcal{A}$ is *between* A and B if there exists a positive minimal path $f = a_1 \cdots a_n$ of $\Gamma(\mathcal{A})$ beginning at A , ending in B and such that $C = \text{end}(a_1 \cdots a_i)$ for some $i = 0, 1, \dots, n$. In other words, C is *between* A and B if there exists a minimal gallery $(A = C_0, C_1, \dots, C_n = B)$ of $\mathcal{A}$ such that $C = C_i$ for some $i = 0, 1, \dots, n$. We denote by $\text{Bet}(A, B)$ the set of chambers of $\mathcal{A}$ between A and B .

From now on, for every pair (A, B) of chambers of $\mathcal{A}$, we fix a positive minimal path $m(A, B)$ of $\Gamma(\mathcal{A})$ beginning at A and ending in B . Note that, by the definition of the identification $\sim$ of $\Gamma(\mathcal{A})$, the equivalence class of $m(A, B)$ with respect to $\sim$ does not depend on the choice of $m(A, B)$. We obviously have $C \in \text{Bet}(A, B)$ if and only if $m(A, C)m(C, B) \sim m(A, B)$.

Let f and g be two positive paths of $\Gamma(\mathcal{A})$ with $\text{end}(f) = \text{end}(g)$. We say that f *ends with* g if there exists a positive path h of $\Gamma(\mathcal{A})$ such that $f \sim hg$.

FIGURE 4

Let f be a positive path of $\Gamma(\mathcal{A})$. Write $B = \text{end}(f)$. We say that f has the *property D* if there exists a chamber A of $\mathcal{A}$ such that f ends with $m(C, B)$ if and only if $C \in \text{Bet}(A, B)$, for every chamber C of $\mathcal{A}$.

We say that $\mathcal{A}$ has the *property D* if every positive path of $\Gamma(\mathcal{A})$ has the property D.

Theorem 4.1. *If $\mathcal{A}$ is a simplicial arrangement of hyperplanes, then $\mathcal{A}$ has the property D.*

Example. Consider the arrangement $\mathcal{A} = \{H_1, H_2, H_3, H_4, H_5\}$ in $\mathbb{R}^3$ shown in Figure 4. Let us show that $\mathcal{A}$ does not have the property D.

Let f be the positive path of $\Gamma(\mathcal{A})$ corresponding with the gallery $(A, A_1, A_2, A_3, A_4, C)$ of $\mathcal{A}$. One can verify that $f \sim g$ and $f \sim h$, where g is the positive path of $\Gamma(\mathcal{A})$ corresponding with the gallery $(A, A_1, A_2, A_3, A_2, C)$ and h is the positive path of $\Gamma(\mathcal{A})$ corresponding with the gallery $(A, B_1, B_2, B_3, B_2, C)$ of $\mathcal{A}$. Thus f ends with $m(A_4, C)$, with $m(A_2, C)$ and with $m(B_2, C)$.

Suppose that f has the property D. Then there exists a chamber D of $\mathcal{A}$ such that f ends with $m(B, C)$ if and only if $B \in \text{Bet}(D, C)$. We have $A_4, A_2, B_2 \in \text{Bet}(D, C)$, thus the hyperplanes H_2, H_3, H_4 separate D and C (see Lemma 4.3), therefore $D = -C$. It is easy to see that f cannot end with $M(-C, C)$ (it does not “cross” H_5), so we have a contradiction.

The following Lemmas 4.2–4.5 are preliminary results to the proof of Theorem 4.1. Lemmas 4.2 and 4.3 are well-known results. A proof of Lemma 4.2 and Lemma 4.3(i) can be found in [Br, p. 14]. (ii) and (iii) of Lemma 4.3 are immediate corollaries of Lemma 4.3(i).

Lemma 4.2. *Let $G = (C_0, C_1, \dots, C_n)$ be a gallery of $\mathcal{A}$. Let H_i be the hyperplane of $\mathcal{A}$ which separates C_{i-1} and C_i for $i = 1, \dots, n$. The gallery G is minimal if and only if $H_i \neq H_j$ for $i \neq j$.*

Corollary. Let $f = a_1 \cdots a_n$ be a positive path of $\Gamma(\mathcal{A})$. Write $C_i = \text{end}(a_1 \cdots a_i)$ for $i = 0, 1, \dots, n$. Let H_i be the hyperplane of $\mathcal{A}$ which separates C_{i-1} and C_i for $i = 1, \dots, n$. The path f is positive minimal if and only if $H_i \neq H_j$ for $i \neq j$.

Lemma 4.3. Let $G = (C_0, C_1, \dots, C_n)$ be a minimal gallery. Let H_i be the hyperplane of $\mathcal{A}$ which separates C_{i-1} and C_i for $i = 1, \dots, n$.

- (i) The hyperplanes of $\mathcal{A}$ which separate C_0 and C_n are exactly $H_1, \dots, H_n$.
- (ii) Let F be a facet of $\mathcal{A}$. If F is common to C_0 and C_n , then H_i contains F for every $i = 1, \dots, n$.
- (iii) Let F be a facet of $\mathcal{A}$. If $(C_0)_F = (C_n)_F$, then H_i does not contain F for any $i = 1, \dots, n$.

Corollary 1. Let $f = a_1 \cdots a_n$ be a positive minimal path. Write

$$C_i = \text{end}(a_1 \cdots a_i) \quad \text{for } i = 0, 1, \dots, n.$$

Let H_i be the hyperplane of $\mathcal{A}$ which separates C_{i-1} and C_i for $i = 1, \dots, n$.

- (i) The hyperplanes of $\mathcal{A}$ which separate C_0 and C_n are exactly $H_1, \dots, H_n$.
- (ii) Let F be a facet of $\mathcal{A}$. If F is common to C_0 and C_n , then H_i contains F for every $i = 1, \dots, n$.
- (iii) Let F be a facet of $\mathcal{A}$. If $(C_0)_F = (C_n)_F$, then H_i does not contain F for any $i = 1, \dots, n$.

Corollary 2. Let A and B be two chambers of $\mathcal{A}$. Then $m(A, B)m(B, -A)$ is a positive minimal path of $\Gamma(\mathcal{A})$.

Let $\mathcal{R}$ be the smallest equivalence relation on the set of positive paths of $\Gamma(\mathcal{A})$ such that:

- (1) if $f \mathcal{R} g$, then $\text{begin}(f) = \text{begin}(g)$ and $\text{end}(f) = \text{end}(g)$.
- (2) if $f \mathcal{R} g$, then $(h_1 f h_2) \mathcal{R} (h_1 g h_2)$ for any two positive paths h_1 and h_2 such that $\text{end}(h_1) = \text{begin}(f) = \text{begin}(g)$ and $\text{begin}(h_2) = \text{end}(f) = \text{end}(g)$.
- (3) if f and g are two positive minimal paths of $\Gamma(\mathcal{A})$ with the same begin and the same end, then $f \mathcal{R} g$.

Note that, if $f \mathcal{R} g$, then $f \sim g$ and $\text{length}(f) = \text{length}(g)$.

Let f and g be two positive paths of $\Gamma(\mathcal{A})$ such that $\text{end}(f) = \text{end}(g)$. We say that f $\mathcal{R}$ -ends with g if there exists a positive path h of $\Gamma(\mathcal{A})$ such that $f \mathcal{R} (hg)$.

For any chamber A of $\mathcal{A}$, we write $h_A = m(A, -A)m(-A, A)$ and $(h_A)^r = h_A h_A \cdots h_A$ (r times).

Lemma 4.4. Let f and g be two positive paths of $\Gamma(\mathcal{A})$ such that $f \sim g$. Write $A = \text{begin}(f) = \text{begin}(g)$. Then there exists an integer $r \geq 0$ such that $(h_A)^r f \mathcal{R} (h_A)^r g$.

Proof. We denote by $(-a) = (-A, -B)$ the opposite arrow of an arrow $a = (A, B)$ of $\Gamma(\mathcal{A})$.

For every arrow $a = (A, B)$ of $\Gamma(\mathcal{A})$ set $p(a) = a$ and

$$p(a^{-1}) = m(B, -B)m(-B, A),$$

and for every path $f = a_1^{e_1} \cdots a_n^{e_n}$ set $p(f) = p(a_1^{e_1}) \cdots p(a_n^{e_n})$. It is clear that $p(f)$ is a positive path of $\Gamma(\mathcal{A})$, that $p(fg) = p(f)p(g)$, and that $p(f) = f$ for every positive path f .

Assertion. Let f and g be two paths of $\Gamma(\mathcal{A})$ such that $f \sim g$. Write $A = \text{begin}(f) = \text{begin}(g)$. Then there exist two integers $r, s \geq 0$ such that $(h_A)^r p(f) \mathcal{R} (h_A)^s p(g)$.

This assertion proves Lemma 4.4; indeed, if f and g are two positive paths of $\Gamma(\mathcal{A})$ with $f \sim g$, then there exist two integers $r, s \geq 0$ such that $((h_A)^r f) \mathcal{R} ((h_A)^s g)$. Since $\text{length}((h_A)^r f) = \text{length}((h_A)^s g)$ and $\text{length}(f) = \text{length}(g)$ (because $\text{weight}(f) = \text{weight}(g)$), we have $r = s$.

Proof of the Assertion. Let $a = (A, B)$ be an arrow of $\Gamma(\mathcal{A})$. We have, by Corollary 2 of Lemma 4.3,

$$\begin{aligned} m(A, -A)m(-A, A)a \mathcal{R} m(A, -A)(-a)m(-B, A)a \\ \mathcal{R} m(A, -A)(-a)m(-B, B) \\ \mathcal{R} am(B, -A)(-a)m(-B, B) \\ \mathcal{R} am(B, -B)m(-B, B). \end{aligned}$$

Thus, if f is a positive path of $\Gamma(\mathcal{A})$ beginning at A and ending in B , then $((h_A)^t f) \mathcal{R} (f(h_B)^t)$ for every integer $t \geq 0$.

Let f_1 and f_2 be two paths of $\Gamma(\mathcal{A})$ beginning at the same chamber A and ending in the same chamber B . Let g and h be two paths of $\Gamma(\mathcal{A})$, g ending in A and h beginning at B . Write $C = \text{begin}(g)$. Assume there exist two integers $r, s \geq 0$ such that $((h_A)^r p(f_1)) \mathcal{R} ((h_A)^s p(f_2))$. Then

$$\begin{aligned} (h_C)^r p(g f_1 h) &= (h_C)^r p(g) p(f_1) p(h) \\ &\mathcal{R} p(g) (h_A)^r p(f_1) p(h) \\ &\mathcal{R} p(g) (h_A)^s p(f_2) p(h) \\ &\mathcal{R} (h_C)^s p(g f_2 h). \end{aligned}$$

It follows that, in order to prove the Assertion, it suffices to consider the following cases:

(a) f and g are positive minimal paths with the same begin and the same end.

(b) $f = f'^{-1}$ and $g = g'^{-1}$, where f' and g' are positive minimal paths with the same begin and the same end.

(c) $f = aa^{-1}$ and $g = A$, where $a = (A, B)$ is an arrow of $\Gamma(\mathcal{A})$.

(d) $f = a^{-1}a$ and $g = B$, where $a = (A, B)$ is an arrow of $\Gamma(\mathcal{A})$.

(a) Is trivial.

(b) Let $f = a_1 \cdots a_n$ be a positive minimal path of $\Gamma(\mathcal{A})$. Write $A = \text{begin}(f)$ and $B = \text{end}(f)$. Let us prove, by induction on the length n of f , that

$$p(f^{-1}) \mathcal{R} (h_B)^{n-1} m(B, -B) m(-B, A).$$

This clearly implies the Assertion in the case b.

Write $A_i = \text{end}(a_1 \cdots a_i)$ for $i = 1, 2, \dots, n$.

$$\begin{aligned}
 & m(-A, -B)m(-B, A) \text{ is positive minimal} \\
 \Rightarrow & (-a_1)(-a_2) \cdots (-a_n)m(-B, A) \text{ is positive minimal} \\
 \Rightarrow & (-a_2) \cdots (-a_n)m(-B, A) \text{ is positive minimal} \\
 \Rightarrow & m(-A_1, -B)m(-B, A) \text{ is positive minimal} \\
 \Rightarrow & m(-A_1, -B)m(-B, A) \mathcal{R} m(-A_1, A).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 p(f^{-1}) &= p(a_n^{-1} \cdots a_2^{-1})p(a_1^{-1}) \\
 &\mathcal{R} (h_B)^{n-2}m(B, -B)m(-B, A_1)m(A_1, -A_1)m(-A_1, A) \\
 &\mathcal{R} (h_B)^{n-2}m(B, -B)m(-B, A_1)m(A_1, B)m(B, -A_1) \\
 &\quad m(-A_1, -B)m(-B, A) \\
 &\mathcal{R} (h_B)^{n-2}m(B, -B)m(-B, B)m(B, -B)m(-B, A) \\
 &\mathcal{R} (h_B)^{n-1}m(B, -B)m(-B, A).
 \end{aligned}$$

(c)

$$\begin{aligned}
 p(f) &= am(B, -B)m(-B, A) \\
 &\mathcal{R} am(B, -A)(-a)m(-B, A) \\
 &\mathcal{R} m(A, -A)m(-A, A) = h_A = h_A p(g).
 \end{aligned}$$

(d)

$$\begin{aligned}
 p(f) &= m(B, -B)m(-B, A)a \\
 &\mathcal{R} m(B, -B)m(-B, B) = h_B p(g). \quad \square
 \end{aligned}$$

Lemma 4.5. Assume $\mathcal{A}$ to be simplicial.

(i) Let f_1, f_2 and g be three positive paths of $\Gamma(\mathcal{A})$ such that $\text{end}(g) = \text{begin}(f_1) = \text{begin}(f_2)$. If $(gf_1)\mathcal{R}(gf_2)$, then $f_1\mathcal{R}f_2$.

(ii) Let f_1, f_2 and h be three positive paths of $\Gamma(\mathcal{A})$ with $\text{begin}(h) = \text{end}(f_1) = \text{end}(f_2)$. If $(f_1h)\mathcal{R}(f_2h)$, then $f_1\mathcal{R}f_2$.

(iii) Let f be a positive path of $\Gamma(\mathcal{A})$. Write $B = \text{end}(f)$. There exists a chamber A of $\mathcal{A}$ such that f $\mathcal{R}$ -ends with $m(C, B)$ if and only if $C \in \text{Bet}(A, B)$.

Proof. See [De, Proposition 1.19]. $\square$

Proof of Theorem 4.1. Assume $\mathcal{A}$ to be simplicial. Let f and g be two positive paths of $\Gamma(\mathcal{A})$. If $f\mathcal{R}g$, then obviously $f \sim g$. On the other hand, if $f \sim g$, then there exists an integer $r \geq 0$ such that $((h_A)'f)\mathcal{R}((h_A)'g)$ (Lemma 4.4), where $A = \text{begin}(f) = \text{begin}(g)$, therefore, by Lemma 4.5(i), $f\mathcal{R}g$. Thus $f \sim g$ if and only if $f\mathcal{R}g$. In particular, a positive path f of $\Gamma(\mathcal{A})$ ends with a positive path g if and only if f $\mathcal{R}$ -ends with g . Then Theorem 4.1 easily follows from Lemma 4.5(iii). $\square$

4.B. Property D and the topology of $\widehat{\text{Sal}}$. Throughout this subsection, $\mathcal{A}$ is an essential arrangement of hyperplanes, $\rho: (\widehat{\Gamma}, \sim) \rightarrow (\Gamma(\mathcal{A}), \sim)$ is the universal cover of the oriented system $(\Gamma(\mathcal{A}), \sim)$ associated with $\mathcal{A}$, and $p: \widehat{\text{Sal}} \rightarrow \text{Sal}(\mathcal{A})$ is the universal cover of Salvetti's complex $\text{Sal}(\mathcal{A})$ as defined in §3.

Theorem 4.6. *If $\mathcal{A}$ has the property D , then $\widehat{\text{Sal}}$ is contractible.*

Corollary. *If $\mathcal{A}$ is a simplicial arrangement of hyperplanes, then $M(\mathcal{A})$ is a $K(\pi, 1)$ space.*

The following Lemmas 4.7–4.12 are preliminary results to the proof of Theorem 4.6.

Lemma 4.7. *Let X be a simplicial complex. We denote by $V(X)$ the set of vertices of X . Let $W \subseteq V(X)$ be a subset. Let Y be the subcomplex of X generated by W (i.e., Y is the union of the simplexes of X having their vertices in W), and let Z be the subcomplex of X generated by $V(X) - W$.*

Then Y is a strong deformation retract of $(X - Z)$.

Proof. We have to define a continuous family $(\theta_t)_{0 \leq t \leq 1}: (X - Z) \rightarrow (X - Z)$ of maps such that

- (1) $\theta_0(x) = x$, for all $x \in (X - Z)$,
- (2) $\theta_1(x) \in Y$, for all $x \in (X - Z)$,
- (3) $\theta_t(x) = x$, for all $x \in Y$ and all $t \in [0, 1]$.

Let Φ be a simplex of X included in $(X - Z)$. Let $\omega_0, \omega_1, \dots, \omega_r$ be the vertices of Φ . Via the canonical embedding $\Phi \rightarrow \mathbb{R}^{r+1}$, every element $x \in \Phi$ can be written in a unique way

$$x = \sum_{i=0}^r t_i \omega_i,$$

with $0 < t_i \leq 1$ for $i = 0, 1, \dots, r$, and $\sum_{i=0}^r t_i = 1$. Since $\Phi \subseteq (X - Z)$, there is at least one vertex of Φ included in W . Assume $\omega_0, \omega_1, \dots, \omega_s$ to be the vertices of Φ included in W . The restriction of θ_t to Φ is defined by

$$\theta_t \left(\sum_{i=0}^r t_i \omega_i \right) = t \left(\sum_{i=0}^r t_i \omega_i \right) + (1-t) \left(\sum_{i=0}^s t_i \right)^{-1} \left(\sum_{i=0}^s t_i \omega_i \right).$$

It is clear that θ_t is well defined and satisfies (1), (2), and (3). $\square$

A wall of a chamber A of $\mathcal{A}$ is the support of a face of A (i.e., of a 1 codimension facet of A).

Lemma 4.8. *Let A be a chamber of $\mathcal{A}$, and let $H_1, \dots, H_r$ be r distinct walls of A . Consider the subcomplex Δ of $\mathbb{S}^{l-1}$ generated by the vertices $x(F)$ of $\mathbb{S}^{l-1}$ included in $\bigcup_{i=1}^r (H_i)_A^+$. Then Δ is a strong deformation retract of $\mathbb{B}^l$.*

Proof. Apply Lemma 4.7 to $X = \mathbb{S}^{l-1}$ and W the set of vertices $x(F)$ of $\mathbb{S}^{l-1}$ included in $\bigcup_{i=1}^r (H_i)_A^+$. We have $X - Z = \mathbb{S}^{l-1} \cap (\bigcup_{i=1}^r (H_i)_A^+)$ and $Y = \Delta$. It follows that Δ is a strong deformation retract of $\mathbb{S}^{l-1} \cap (\bigcup_{i=1}^r (H_i)_A^+)$. Since $\mathbb{S}^{l-1} \cap (\bigcup_{i=1}^r (H_i)_A^+) = \mathbb{S}^{l-1} - (\bigcap_{i=1}^r (\overline{H_i})_A^-)$ is contractible (where $(\overline{H_i})_A^-$ is the closed half-space of V bordered by H_i and not containing A), the subcomplex Δ is contractible, thus is a strong deformation retract of $\mathbb{B}^l$ (see [LW, Theorem 3.1, Chapter IV], for example). $\square$

Fix a vertex v_0 of $\widehat{\Gamma}$. We denote by $V(\widehat{\Gamma})_+(v_0) = V(\widehat{\Gamma})_+$ the set of vertices v of $\widehat{\Gamma}$ such that there exists a positive path in $\widehat{\Gamma}$ beginning at v_0 and ending in v . We denote by $V(\widehat{\Gamma})_n(v_0) = V(\widehat{\Gamma})_n$ the set of vertices v of $\widehat{\Gamma}$ such that there

exists a positive path in $\widehat{\Gamma}$ of length $\leq n$ beginning at v_0 and ending in v . Note that, if f and g are both positive paths of $\widehat{\Gamma}$ beginning at v_0 and ending in $v \in V(\widehat{\Gamma})_+$, then $f \sim g$ (Proposition 2.2), thus $\text{length}(f) = \text{length}(g)$. For $v \in V(\widehat{\Gamma})_+$, we denote by $d(v_0, v)$ the length of a positive path of $\widehat{\Gamma}$ beginning at v_0 and ending in v .

We denote by $\widehat{\text{Sal}}_+(v_0) = \widehat{\text{Sal}}_+$ the subcomplex of $\widehat{\text{Sal}}$ generated by the vertices of $\widehat{\text{Sal}}$ having the form $\hat{\omega}(F, v)$ with $F \in \mathcal{F}(\mathcal{A})$ and $v \in V(\widehat{\Gamma})_+$. We have

$$\widehat{\text{Sal}}_+ = \bigcup_{v \in V(\widehat{\Gamma})_+} \mathbb{B}^l(v).$$

We denote by $\widehat{\text{Sal}}_n(v_0) = \widehat{\text{Sal}}_n$ the subcomplex of $\widehat{\text{Sal}}$ generated by the vertices of $\widehat{\text{Sal}}$ having the form $\hat{\omega}(F, v)$ with $F \in \mathcal{F}(\mathcal{A})$ and $v \in V(\widehat{\Gamma})_n$. We have

$$\widehat{\text{Sal}}_n = \bigcup_{v \in V(\widehat{\Gamma})_n} \mathbb{B}^l(v).$$

Lemma 4.9. *Assume $\mathcal{A}$ to have the property D. Fix a vertex v_0 of $\widehat{\Gamma}$. Let $v, w \in V(\widehat{\Gamma})_{n+1} - V(\widehat{\Gamma})_n$ with $v \neq w$. Then*

$$\mathbb{B}^l(v) \cap \mathbb{B}^l(w) \subseteq \widehat{\text{Sal}}_n.$$

Proof. Let $\hat{\omega}$ be a vertex of $\widehat{\text{Sal}}$ included in $\mathbb{B}^l(v) \cap \mathbb{B}^l(w)$. Write $\rho(v) = A$ and $\rho(w) = B$. There exist two facets F and G of $\mathcal{A}$ such that $\hat{\omega} = \hat{\omega}(F, v) = \hat{\omega}(G, w)$. By Lemma 3.4, $F = G \subseteq Z(v, w)$ and $A_F = B_F$.

Let C be the chamber of $\mathcal{A}$ such that F is a facet of C and $C_F = -A_F = -B_F$. The set $Z(v, w)$ is a union of facets of $\mathcal{A}$, it is an open subset of V , $F \subseteq Z(v, w)$, and $C \geq F$, thus $C \subseteq Z(v, w)$. Therefore there exists a vertex $u \in \Sigma(v) \cap \Sigma(w)$ such that $\rho(u) = C$.

Let $m(v, u)$ be a positive path of $\widehat{\Gamma}$ beginning at v and ending in u , and let $m(w, u)$ be a positive path of $\widehat{\Gamma}$ beginning at w and ending in u . Since $u \in \Sigma(v) \cap \Sigma(w)$, one can assume $\rho(m(v, u)) = m(A, C)$ and $\rho(m(w, u)) = m(B, C)$.

Pick a positive path $\hat{f}$ of $\widehat{\Gamma}$ beginning at v_0 and ending in v , and a positive path $\hat{g}$ of $\widehat{\Gamma}$ beginning at v_0 and ending in w . Write

- (i) $f = \rho(\hat{f})$ and $g = \rho(\hat{g})$,
- (ii) $\hat{f}_0 = \hat{f}m(v, u)$ and $\hat{g}_0 = \hat{g}m(w, u)$,
- (iii) $f_0 = \rho(\hat{f}_0) = fm(A, C)$ and $g_0 = \rho(\hat{g}_0) = gm(B, C)$.

Note that $\hat{f}_0$ and $\hat{g}_0$ have the same begin v_0 and the same end u , thus $\hat{f}_0 \sim \hat{g}_0$ (Proposition 2.2), therefore $f_0 \sim g_0$.

Recall that $\mathcal{A}$ has the property D. There exists a chamber C_0 of $\mathcal{A}$ such that f_0 ends with $m(D, C)$ if and only if $D \in \text{Bet}(C_0, C)$. Choose a positive path h of $\Gamma(\mathcal{A})$ such that $f_0 \sim hm(C_0, C)$. Let $\hat{h}$ be the lift of h into $\widehat{\Gamma}$ beginning at v_0 . Write $u_0 = \text{end}(\hat{h})$.

Let us prove that $u_0 \in V(\widehat{\Gamma})_n$ and that $\hat{\omega} = \hat{\omega}(F, u_0)$. This shows that $\hat{\omega} \in \widehat{\text{Sal}}_n$, thus ends the proof of Lemma 4.9.

First, let us prove that $C_0 \neq A$. If not, then

$$\begin{aligned}
 n+1 &= \text{length}(f) \quad (\text{since } v \in V(\widehat{\Gamma})_{n+1} - V(\widehat{\Gamma})_n) \\
 &= \text{length}(f_0) - \text{length}(m(C_0, C)) \quad (\text{since } C_0 = A) \\
 &= \text{length}(f_0) - \text{length}(m(C_0, B)) - \text{length}(m(B, C)) \\
 &\quad (\text{since } B \in \text{Bet}(C_0, C)) \\
 &= \text{length}(g) - \text{length}(m(C_0, B)) \quad (\text{since } f_0 \sim g_0) \\
 &= n+1 - \text{length}(m(C_0, B)) \quad (\text{since } w \in V(\widehat{\Gamma})_{n+1} - V(\widehat{\Gamma})_n).
 \end{aligned}$$

It follows that $A = C_0 = B$, thus $m(A, C) = m(B, C)$, therefore $m(v, u) = m(w, u)$. This contradicts the fact that $v \neq w$.

Now,

$$\begin{aligned}
 d(v_0, u_0) &= \text{length}(\hat{h}) \\
 &= \text{length}(h) \\
 &= \text{length}(f_0) - \text{length}(m(C_0, C)) \quad (\text{since } f_0 \sim hm(C_0, C)) \\
 &= \text{length}(f_0) - \text{length}(m(C_0, A)) - \text{length}(m(A, C)) \\
 &\quad (\text{since } A \in \text{Bet}(C_0, C)) \\
 &= \text{length}(f) - \text{length}(m(C_0, A)) \\
 &= n+1 - \text{length}(m(C_0, A)) \quad (\text{since } v \in V(\widehat{\Gamma})_{n+1} - V(\widehat{\Gamma})_n) \\
 &\leq n \quad (\text{since } C_0 \neq A).
 \end{aligned}$$

This shows that $u_0 \in V(\widehat{\Gamma})_n$.

If $H \supseteq F$, then H separates A and C (since $C_F = -A_F$), thus H does not separate C_0 and A (since $A \in \text{Bet}(C_0, C)$), therefore $(C_0)_F = A_F$. The vertex u is included in $\Sigma(v) \cap \Sigma(u_0)$ (lift $m(C_0, A)m(A, C)$ into $\widehat{\Gamma}$), thus $\rho(u) = C \subseteq Z(v, u_0)$. Since $F \subseteq \overline{C}$, we have $F \subseteq \overline{Z}(v, u_0)$. If H separates $A = \rho(v)$ and $C_0 = \rho(u_0)$, then H does not contain F (since $(C_0)_F = A_F$). It follows that F is not included in the border of $Z(v, u_0)$ (see Lemma 2.5), thus $F \subseteq Z(v, u_0)$. By Lemma 3.4, we have $\hat{\omega} = \hat{\omega}(F, v) = \hat{\omega}(F, u_0)$. $\square$

Lemma 4.10. Assume $\mathcal{A}$ to have the property D . Fix a vertex v_0 of $\widehat{\Gamma}$. Let $v \in V(\widehat{\Gamma})_{n+1} - V(\widehat{\Gamma})_n$. Write $A = \rho(v)$. Then there exists a set $\{H_1, \dots, H_r\}$ of walls of A such that $\mathbb{B}^l(v) \cap \widehat{\text{Sal}}_n$ is the subcomplex of $\mathbb{B}^l(v)$ generated by the vertices of $\mathbb{B}^l(v)$ having the form $\hat{\omega}(F, v)$ with $F \subseteq \bigcup_{i=1}^r (H_i)_A^+$.

Proof. Choose a positive path $\hat{f}$ of $\widehat{\Gamma}$ beginning at v_0 and ending in v . Write $f = \rho(\hat{f})$. Let $a_1, \dots, a_r$ be all the arrows of $\Gamma(\mathcal{A})$ such that f ends with a_i (for $i = 1, \dots, r$). Write $A_i = \text{begin}(a_i)$ and H_i the hyperplane of $\mathcal{A}$ which separates A_i and A , for $i = 1, \dots, r$. Let us show that $\{H_1, \dots, H_r\}$ is the required set of walls of A .

Let F be a facet of $\mathcal{A}$. Assume there exists an $i \in \{1, \dots, r\}$ such that $F \subseteq (H_i)_A^+$. Since f ends with a_i , there exists a positive path h of $\Gamma(\mathcal{A})$ such that $f \sim ha_i$. Let $\hat{h}$ be the lift of h into $\widehat{\Gamma}$ beginning at v_0 . Write $w_i = \text{end}(\hat{h})$. Let us prove that $w_i \in V(\widehat{\Gamma})_n$ and that $\hat{\omega}(F, v) = \hat{\omega}(F, w_i)$. This shows that $\hat{\omega}(F, v) \in \mathbb{B}^l(v) \cap \widehat{\text{Sal}}_n$.

$$d(v_0, w_i) = \text{length}(\hat{h}) = \text{length}(h) = n,$$

thus $w_i \in V(\widehat{\Gamma})_n$.

Since $F \subseteq (H_i)_A^+$, the facet F is not included in H_i , thus $(A_i)_F = A_F$. Pick a chamber C of $\mathcal{A}$ having F as facet. Since $F \subseteq (H_i)_A^+$ and $F \subseteq \overline{C}$, we have $C \subseteq (H_i)_A^+$, thus H_i does not separate A and C , therefore $A \in \text{Bet}(A_i, C)$ (by Lemmas 4.2 and 4.3(i)). This implies that there exists a vertex $u \in \Sigma(v) \cap \Sigma(w_i)$ such that $\rho(u) = C$ (lift $m(A_i, A)m(A, C)$ into $\widehat{\Gamma}$). It follows that $C \subseteq Z(v, w_i)$, therefore $F \subseteq \overline{Z}(v, w_i)$ (since $F \subseteq \overline{C}$). The equality $(A_i)_F = A_F$ shows that no hyperplane of $\mathcal{A}$ containing F separates $A_i = \rho(w_i)$ and $A = \rho(v)$, thus, by Lemma 2.5, we have $F \subseteq Z(v, w_i)$. It follows, by Lemma 3.4, that $\hat{\omega}(F, v) = \hat{\omega}(F, w_i)$.

Now, let $\hat{\omega}$ be a vertex of $\widehat{\text{Sal}}$ included in $\mathbb{B}^l(v) \cap \widehat{\text{Sal}}_n$. There exist a vertex $w \in V(\widehat{\Gamma})_n$ and two facets F and G of $\mathcal{A}$ such that $\hat{\omega} = \hat{\omega}(F, v) = \hat{\omega}(G, w)$. Write $B = \rho(w)$. By Lemma 3.4, $F = G \subseteq Z(v, w)$ and $A_F = B_F$.

Let C be the chamber of $\mathcal{A}$ having F as facet and such that $C_F = -A_F = -B_F$. The set $Z(v, w)$ is a union of facets of $\mathcal{A}$, it is an open subset of V , $F \subseteq Z(v, w)$, and $C \geq F$, thus $C \subseteq Z(v, w)$. Therefore there exists a vertex $u \in \Sigma(v) \cap \Sigma(w)$ such that $\rho(u) = C$. Let $m(v, u)$ be a positive path of $\widehat{\Gamma}$ beginning at v and ending in u , and let $m(w, u)$ be a positive path of $\widehat{\Gamma}$ beginning at w and ending in u . Since $u \in \Sigma(v) \cap \Sigma(w)$, one can assume $\rho(m(v, u)) = m(A, C)$ and $\rho(m(w, u)) = m(B, C)$.

Pick a positive path $\hat{f}$ of $\widehat{\Gamma}$ beginning at v_0 and ending in v , and a positive path $\hat{g}$ of $\widehat{\Gamma}$ beginning at v_0 and ending in w . Write

- (i) $f = \rho(\hat{f})$ and $g = \rho(\hat{g})$,
- (ii) $\hat{f}_0 = \hat{f}m(v, u)$ and $\hat{g}_0 = \hat{g}m(w, u)$,
- (iii) $f_0 = \rho(\hat{f}_0) = fm(A, C)$ and $g_0 = \rho(\hat{g}_0) = gm(B, C)$.

Note that $\hat{f}_0$ and $\hat{g}_0$ have the same begin v_0 and the same end u , thus $\hat{f}_0 \sim \hat{g}_0$ (Proposition 2.2), therefore $f_0 \sim g_0$.

Recall that $\mathcal{A}$ has the property D. There exists a chamber C_0 of $\mathcal{A}$ such that f_0 ends with $m(D, C)$ if and only if $D \in \text{Bet}(C_0, C)$. Choose a positive path h of $\Gamma(\mathcal{A})$ such that $f_0 \sim hm(C_0, C)$. As in the proof of Lemma 4.9, $f_0 \sim g_0$ and $v \neq w$ imply $C_0 \neq A$. Since

$$f_0 = fm(A, C) \sim hm(C_0, A)m(A, C)$$

indeed, $A \in \text{Bet}(C_0, C)$, the path f ends with $m(C_0, A)$.

Write $m(C_0, A) = b_1 \cdots b_n$. There is an $i \in \{1, \dots, r\}$ such that $b_n = a_i$. Since $A \in \text{Bet}(C_0, C)$ and H_i separates C_0 and A , the hyperplane H_i does not separate A and C (by Lemmas 4.2 and 4.3(i)), thus $C \subseteq (H_i)_A^+$. This implies $F \subseteq (\overline{H_i})_A^+$ (since $F \subseteq \overline{C}$), where $(\overline{H_i})_A^+$ is the closed half-space of V bordered by H_i and containing A . If $H \in \mathcal{A}$ contains F , then H separates A and C (since $C_F = -A_F$), thus H does not separate C_0 and A (since $A \in \text{Bet}(C_0, C)$), therefore $H \neq H_i$. It follows that $F \subseteq (H_i)_A^+$. $\square$

Lemma 4.11. *Assume $\mathcal{A}$ to have the property D. Fix a vertex v_0 of $\widehat{\Gamma}$. Then $\widehat{\text{Sal}}_+(v_0)$ is contractible.*

Proof. Let us show that $\widehat{\text{Sal}}_n$ is a strong deformation retract of $\widehat{\text{Sal}}_{n+1}$ for all $n \geq 0$. Since $\widehat{\text{Sal}}_+ = \varinjlim \widehat{\text{Sal}}_n$ and $\widehat{\text{Sal}}_0 = \mathbb{B}^l(v_0)$ is contractible, this proves Lemma 4.11.

We have to define a continuous family $(\theta_t)_{0 \leq t \leq 1} : \widehat{\text{Sal}}_{n+1} \rightarrow \widehat{\text{Sal}}_{n+1}$ of maps such that

- (1) $\theta_0(x) = x$, for all $x \in \widehat{\text{Sal}}_{n+1}$,
- (2) $\theta_1(x) \in \widehat{\text{Sal}}_n$, for all $x \in \widehat{\text{Sal}}_{n+1}$,
- (3) $\theta_t(x) = x$, for all $x \in \widehat{\text{Sal}}_n$ and all $t \in [0, 1]$.

Let $v \in V(\widehat{\Gamma})_{n+1} - V(\widehat{\Gamma})_n$. Write $A = \rho(v)$. By Lemma 4.10, there exists a set $\{H_1, \dots, H_r\}$ of walls of A such that $\hat{\pi}_v(\mathbb{B}^l(v) \cap \widehat{\text{Sal}}_n)$ is the subcomplex of $\mathbb{B}^l$ generated by the vertices $x(F)$ of $\mathbb{B}^l$ included in $\bigcup_{i=1}^r (H_i)_A^+$. Note that $x(\{0\}) = 0 \notin \bigcup_{i=1}^r (H_i)_A^+$. By Lemma 4.8, $\hat{\pi}_v(\mathbb{B}^l(v) \cap \widehat{\text{Sal}}_n)$ is a strong deformation retract of $\mathbb{B}^l$. It follows that $\mathbb{B}^l(v) \cap \widehat{\text{Sal}}_n$ is a strong deformation retract of $\mathbb{B}^l(v)$ (since $\hat{\pi}_v : \mathbb{B}^l(v) \rightarrow \mathbb{B}^l$ is an isomorphism). Choose a homotopy $(\theta_t^v)_{0 \leq t \leq 1} : \mathbb{B}^l(v) \rightarrow \mathbb{B}^l(v)$ with

- (a) $\theta_0^v(x) = x$, for all $x \in \mathbb{B}^l(v)$,
- (b) $\theta_1^v(x) \in \mathbb{B}^l(v) \cap \widehat{\text{Sal}}_n$, for all $x \in \mathbb{B}^l(v)$,
- (c) $\theta_t^v(x) = x$, for all $x \in \mathbb{B}^l(v) \cap \widehat{\text{Sal}}_n$ and all $t \in [0, 1]$.

Set $\theta_t|_{\mathbb{B}^l(v)} = \theta_t^v$ if $v \in V(\widehat{\Gamma})_{n+1} - V(\widehat{\Gamma})_n$, and $\theta_t|_{\mathbb{B}^l(v)} = \text{id}_{\mathbb{B}^l(v)}$ if $v \in V(\widehat{\Gamma})_n$.

By Lemma 4.9, the homotopy θ_t is well defined. It obviously satisfies (1), (2) and (3). $\square$

Recall that, for a chamber A of $\mathcal{A}$, $h_A = m(A, -A)m(-A, A)$. Write $(h_A)^{-r} = (h_A)^{-1}(h_A)^{-1} \dots (h_A)^{-1}$ (r times) for all $r \geq 0$.

Lemma 4.12. *Let f be a path of $\Gamma(\mathcal{A})$. Write $A = \text{begin}(f)$. Then there exist a positive path g of $\Gamma(\mathcal{A})$ and an integer $r \geq 0$ such that $f \sim (h_A)^{-r}g$.*

Proof. Write $f = a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n}$. Set $r = |\{i \in \{1, \dots, n\} \mid \varepsilon_i = -1\}|$. We are going to prove, by induction on r , that there exists a positive path g of $\Gamma(\mathcal{A})$ such that $f \sim (h_A)^{-r}g$.

Assume $r > 0$. For an arrow $a = (C, D)$ of $\Gamma(\mathcal{A})$, we write $(-a) = (-C, -D)$. Let $A_i = \text{end}(a_1^{\varepsilon_1} \dots a_i^{\varepsilon_i})$ for $i = 1, 2, \dots, n$. There exists a $j \in \{1, \dots, n\}$ such that $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{j-1} = 1$ and $\varepsilon_j = -1$. We have

$$\begin{aligned}
 h_A f &\sim m(A, -A)m(-A, A)a_1 \dots a_{j-1}a_j^{-1}a_{j+1}^{\varepsilon_{j+1}} \dots a_n^{\varepsilon_n} \\
 &\sim m(A, -A)(-a_1)m(-A_1, A)a_1a_2 \dots a_{j-1}a_j^{-1}a_{j+1}^{\varepsilon_{j+1}} \dots a_n^{\varepsilon_n} \\
 &\sim m(A, -A)(-a_1)m(-A_1, A_1)a_2 \dots a_{j-1}a_j^{-1}a_{j+1}^{\varepsilon_{j+1}} \dots a_n^{\varepsilon_n} \\
 &\dots \\
 &\sim m(A, -A)(-a_1)(-a_2) \dots (-a_{j-1})m(-A_{j-1}, A_{j-1})a_j^{-1}a_{j+1}^{\varepsilon_{j+1}} \dots a_n^{\varepsilon_n} \\
 &\sim m(A, -A)(-a_1) \dots (-a_{j-1})m(-A_{j-1}, A_j)a_ja_{j+1}^{-1}a_{j+1}^{\varepsilon_{j+1}} \dots a_n^{\varepsilon_n} \\
 &\sim m(A, -A)(-a_1) \dots (-a_{j-1})m(-A_{j-1}, A_j)a_{j+1}^{\varepsilon_{j+1}} \dots a_n^{\varepsilon_n} \\
 &\sim (h_A)^{1-r}g \quad (\text{inductive hypothesis}),
 \end{aligned}$$

where g is a positive path. Thus $f \sim (h_A)^{-r}g$. $\square$

Proof of Theorem 4.6. Fix a vertex $v_0 \in V(\widehat{\Gamma})$. Write $A = \rho(v_0)$. Let us define, by induction on $r > 0$, a vertex $v_r \in V(\widehat{\Gamma})$ with $\rho(v_r) = A$. Assume v_{r-1} to be defined. Let $\hat{h}_r$ be the lift of h_A into $\widehat{\Gamma}$ ending in v_{r-1} . We set $v_r = \text{begin}(\hat{h}_r)$.

We clearly have $V(\widehat{\Gamma})_+(v_{r-1}) \subseteq V(\widehat{\Gamma})_+(v_r)$, thus $\widehat{\text{Sal}}_+(v_{r-1}) \subseteq \widehat{\text{Sal}}_+(v_r)$, for $r > 0$.

Let us prove that, for every vertex $w \in V(\widehat{\Gamma})$, there exists an integer $r \geq 0$ such that $w \in V(\widehat{\Gamma})_+(v_r)$. This shows that

$$\widehat{\text{Sal}} = \bigcup_{w \in V(\widehat{\Gamma})} \mathbb{B}'(w) = \varinjlim \widehat{\text{Sal}}_+(v_r),$$

thus $\widehat{\text{Sal}}$ is contractible (since, by Lemma 4.11, $\widehat{\text{Sal}}_+(v_r)$ is contractible).

Let $\hat{f}$ be a path of $\widehat{\Gamma}$ beginning at v_0 and ending in w . Write $f = \rho(\hat{f})$. By Lemma 4.12, there exists an integer $r \geq 0$ and a positive path g of $\Gamma(\mathcal{A})$ such that $f \sim (h_A)^{-r}g$. The lift of $(h_A)^{-r}g$ into $\widehat{\Gamma}$ beginning at v_0 has the form $\hat{h}_1^{-1} \cdots \hat{h}_r^{-1} \hat{g}$, where $\hat{g}$ is a positive path of $\widehat{\Gamma}$. By the definition of a cover of an oriented system, we have $\hat{f} \sim \hat{h}_1^{-1} \cdots \hat{h}_r^{-1} \hat{g}$, thus $\hat{g}$ begins at v_r and ends in $w = \text{end}(\hat{f})$, therefore $w \in V(\widehat{\Gamma})_+(v_r)$. $\square$

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SECTION DE MATHÉMATIQUES, UNIVERSITÉ DE GENÈVE, CP 240, CH-1211 GENÈVE 24, SWITZERLAND

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706
 Current address: Département de Mathématiques, Université de Nantes, 2 rue de la Haussinière, 44072 Nantes, France