COHOPFICITY OF SEIFERT-BUNDLE GROUPS

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ABSTRACT. A group G is *cohopfian*, if every monomorphism $G \rightarrow G$ is an automorphism. In this paper, we answer the cohopficity question for the fundamental groups of compact Seifert fiber spaces (or *Seifert bundles*, in the current vernacular). If M is a closed Seifert bundle, then the following are equivalent: (a) $\pi_1 M$ is cohopfian; (b) M does not cover itself nontrivially; (c) M admits a geometric structure modeled on S^3 or on $\widetilde{\mathrm{SL}}_2\mathbf{R}$. If M is a compact Seifert bundle with nonempty boundary, then $\pi_1 M$ is not cohopfian.

An object $\mathbb{C}$ of a category is *hopfian* if any epimorphism from $\mathbb{C}$ to itself is an automorphism. Dually, $\mathbb{C}$ is *cohopfian*, if any monomorphism from $\mathbb{C}$ to itself is an automorphism. Thus, a group is cohopfian if and only if it cannot be properly imbedded in itself. A group is *complete*, if its automorphisms are all inner and its center is trivial. Finite groups and certain two-generator complete hopfian groups [MS] are cohopfian, while infinite f.g. abelian groups and free products of nontrivial groups are not cohopfian, for example.

In [GW], the following two results concerning the cohopficity of 3-manifold groups were obtained.

Theorem. *Let M be a Haken manifold, different from a collar, whose boundary is a nonempty union of incompressible tori. Then $\pi_1 M$ is cohopfian if and only if the collection of those components of the characteristic submanifold meeting ∂M is a disjoint union of collars.*

Corollary. *The group of a nontrivial knot K is cohopfian if and only if K is not a torus knot, a cable knot, or a composite knot.*

Since, as we have remarked, free products and Z are not cohopfian while finite groups are, the cohopficity question for the fundamental groups of *closed* 3-manifolds is reduced to the following.

Which closed irreducible 3-manifolds M^3 with infinite π_1 have a cohopfian fundamental group?

A sufficient (but not necessary) condition is that M^3 be either Haken with positive Gromov invariant or hyperbolic. Moreover, no subgroup G of infinite index in $\pi_1 M^3$ is isomorphic to $\pi_1 M^3$ when M^3 is also P^2 -irreducible, since $H_3(\pi_1 M^3; \mathbb{Z}_2) \neq 0$ while $H_3(G; \mathbb{Z}_2) = 0$ [GG].

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In this paper, we settle the cohopficity question for the class of fundamental groups of compact Seifert fiber spaces. To do this, it is convenient to consider a compact 3-manifold as a *Seifert “bundle”* if and only if it admits a foliation by circles. We then divide the collection of *closed* 3-manifolds that can be foliated by circles into six classes with each manifold in a class admitting the same geometric structure. These classes do not overlap, and all the Seifert-bundle structures of a closed manifold admit the same geometric structure (cf. [Sc, §4]). This geometric subdivision allows us to prove our main result, Corollary 5.

Corollary 5. *Let M be a closed Seifert fiber space with base orbifold B . The following statements are equivalent.*

- (a) $\pi_1 M$ is cohopfian.
- (b) M does not cover itself nontrivially.
- (c) M admits a geometric structure modeled on S^3 or on $\widetilde{\mathrm{SL}_2\mathbf{R}}$.
- (d) $e(M) \cdot \bar{\chi}(B) = 0$.

The paper concludes with the following two theorems.

Corollary 6. *The fundamental group of a homology 3-sphere admitting a Seifert fibration is cohopfian.*

Theorem 7. *The fundamental group of a compact Seifert fiber space with non-empty boundary is not cohopfian.*

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1. INTRODUCTION

Following Scott [Sc, §3], we call a 3-manifold M a *Seifert bundle* (or a *Seifert fiber space*), if M admits a decomposition into disjoint circles (*fibers*) each having a regular neighborhood that is either a fibered solid torus or a fibered solid Klein bottle. With this definition, a compact 3-manifold admits a Seifert fibration if and only if it can be foliated by circles [Ep].

Now, let M be a Seifert fiber space. An exceptional fiber of M is either isolated, in which case it is the core of a fibered solid torus covered by a trivially fibered solid torus, or it belongs to a fibered solid Klein bottle K . The critical fibers of K together form a 1-sided annulus, and so the collection of all exceptional fibers of M is a set of isolated fibers along with 1-sided annuli, tori, and Klein bottles. It follows that, if M is connected, the union of all regular fibers in M forms a circle bundle over a surface.

We consider the base space B of M as a 2-dimensional orbifold that may contain cone points and reflector curves, but no corner reflectors. If M is closed, then B is an orbifold without boundary, where any boundary components of the underlying space $|B|$ of B serve as reflector circles for B .

Associated with each isolated exceptional fiber of M are the fiber's *Seifert invariants*, (α, β) , where α and β are relatively prime; we assume β to be normalized so that $0 < \beta < \alpha$, except where noted in the proof of Lemma 3. If

$(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ are the Seifert invariants of M , then a natural invariant of the orbifold structure of B is its *orbifold Euler characteristic*

$$\bar{\chi}(B) = \chi(B) - \sum_{i=1}^r \left(1 - \frac{1}{\alpha_i}\right);$$

this equation is sometimes called the Riemann-Hurwitz formula. If $\tilde{B} \rightarrow B$ is an orbifold covering of degree m , then $\bar{\chi}(\tilde{B}) = m\bar{\chi}(B)$.

The final invariant of the Seifert-bundle structure of M that we shall need is the *rational Euler number*, $e(M)$. This is the obstruction to the existence of a “multifold section” in M —a surface F in M that is orthogonal to the (Seifert) fibers of M and for which the restriction, $p|_F: F \rightarrow B$, is an orbifold covering; here, B is the base space of M , and $p: M \rightarrow B$ is the natural projection map inducing the orbifold structure of B . When M is closed and orientable and has normalized Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$, then

$$e(M) = -b - \sum_{i=1}^r \frac{\beta_i}{\alpha_i},$$

where b is an integral invariant of the circle bundle obtained by removing the exceptional fibers of M . When M is nonorientable, then $e(M) = 0$ and $b \in \mathbb{Z}_2$.

Now, if the base orbifold B of M is closed, there is a nice relationship between $e(M)$ and the rational Euler number, $e(\tilde{M})$, of any finite covering, $\tilde{M}$, of M . If $\tilde{B}$ is the base orbifold of $\tilde{M}$, then a covering $\Psi: \tilde{M} \rightarrow M$ induces an orbifold covering $\psi: \tilde{B} \rightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\Psi} & M \\ \tilde{p} \downarrow & & \downarrow p \\ \tilde{B} & \xrightarrow{\psi} & B \end{array}$$

In this diagram, $\tilde{p}$ and p denote the natural projections. Suppose that the degree of $\psi: \tilde{B} \rightarrow B$ is m and the degree with which a regular fiber of $\tilde{M}$ covers a regular fiber of M is n . Then the degree of $\Psi: \tilde{M} \rightarrow M$ is $m \cdot n$ and $e(\tilde{M}) = \frac{m}{n}e(M)$. See [NR, Theorem 1.2] for a sketch of the proof.

Scott has shown [Sc, Lemma 3.7] that if M is compact, then $e(M) = 0$ if and only if there exists a surface F and a finite, fiber-preserving covering, $F \times S^1 \rightarrow M$, whose group of covering transformations respects the product structure of $F \times S^1$. Thus, in the cases of interest to us with M closed and $e(M) = 0$, we see that M has the structure of a bundle over a 1-dimensional orbifold that is either a circle with no singular points or a closed interval with two reflector points.

As discussed by Thurston [Th] and shown in detail by Scott [Sc, §4], each closed Seifert fiber space M admits precisely one of six geometric structures. The geometric structure of M does not change with different Seifert fibrations of the underlying space of M , and the geometry that M admits depends only upon $e(M)$ and $\bar{\chi}(B)$. The six possibilities are shown in the following table from [Sc].

	$\bar{\chi} > 0$	$\bar{\chi} = 0$	$\bar{\chi} < 0$
$e = 0$	$S^2 \times \mathbf{R}$	E^3	$H^2 \times \mathbf{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}_2\mathbf{R}}$

For details on the ideas touched upon in this introduction, we refer the reader to [Sc, S, BS, Th], and [O].

2. COHOPFICITY RESULTS

Theorem 1. *Let M be a closed Seifert fiber space with rational Euler number $e(M) = 0$. Then $\pi_1 M$ is not cohopfian. In fact, there exists a nontrivial covering $M \rightarrow M$.*

Proof. It suffices to show that M covers itself nontrivially. First notice that M has a geometric structure modeled on exactly one of $S^2 \times \mathbf{R}$, E^3 , and $H^2 \times \mathbf{R}$.

If M is closed and admits a geometric structure modeled on $S^2 \times \mathbf{R}$, then M is homeomorphic to one of four possible manifolds, $P^2 \times S^1$, $S^2 \times S^1$, $P^3 \# P^3$, and $S^2 \tilde{\times} S^1$. Now, each of $P^2 \times S^1$, $S^2 \times S^1$, and $S^2 \tilde{\times} S^1$ fibers over S^1 , and $P^3 \# P^3$ fibers over a closed interval with two reflector points. The three spaces that fiber over S^1 have periodic gluing maps (with period either 1 or 2), and so one easily constructs a nontrivial covering of each by itself.

Tollefson [To] has shown that $P^3 \# P^3$ is an n -fold cover of itself for every positive integer n . To see, for that example, that $P^3 \# P^3$ double-covers itself, think of $P^3 \# P^3$ as $P^3 \# S^3 \# P^3$. There is then a two-fold covering, $\varphi: P^3 \# S^3 \# P^3 \rightarrow P^3 \# P^3$, that takes each P^3 -summand of $P^3 \# S^3 \# P^3$ homeomorphically onto the left P^3 -summand of $P^3 \# P^3$ and takes the S^3 -summand onto the right P^3 -summand of $P^3 \# P^3$ as the usual double covering, $S^2 \times I \rightarrow P^2 \tilde{\times} I$.

In the remaining case, when M is geometrically modeled on either E^3 or $H^2 \times \mathbf{R}$, we prefer an algebraic approach, which in fact can be adopted to the case when M is modeled on $S^2 \times \mathbf{R}$.

So, let M be a closed Seifert bundle with base orbifold B and rational Euler number $e(M) = 0$. If the underlying space $|B|$ of B is orientable, then $\pi_1 M$ has a presentation with generators

$$a_1, b_1, \dots, a_p, b_p, q_1, \dots, q_n, d_1, \dots, d_m, y_1, \dots, y_m, h$$

and with relations

$$a_i h a_i^{-1} = h^{e_i}, \quad b_i h b_i^{-1} = h^{e'_i}, \quad q_i h q_i^{-1} = h, \quad q_i^{\alpha_i} h^{\beta_i} = 1,$$

$$d_1 \cdots d_m \cdot q_1 \cdots q_n \prod_{i=1}^p [a_i, b_i] = h^b, \quad y_i^2 = h, \quad d_i y_i d_i^{-1} = y_i^{e''_i}.$$

If $|B|$ is nonorientable, then $\pi_1 M$ has a presentation with generators

$$v_1, \dots, v_k, q_1, \dots, q_n, d_1, \dots, d_m, y_1, \dots, y_m, h$$

and with relations

$$v_i h v_i^{-1} = h^{e_i}, \quad q_i h q_i^{-1} = h, \quad q_i^{\alpha_i} h^{\beta_i} = 1,$$

$$d_1 \cdots d_m \cdot q_1 \cdots q_n \prod_{i=1}^p v_i^2 = h^b, \quad y_i^2 = h, \quad d_i y_i d_i^{-1} = y_i^{e''_i};$$

cf. [OVZ]. We shall refer to either of these presentations as presentation $\mathcal{P}$.

Here, the y_i and d_i occur only when B has reflector circles. Also, when $|B|$ is orientable, each ε_i and ε'_i is $+1$ when M is orientable, and each is -1 , otherwise. When $|B|$ is nonorientable, each ε_i is -1 when M is orientable, and ε_1 is always $+1$, otherwise.

Let α denote the least common multiple of the numbers $1, \alpha_1, \dots, \alpha_n$, and let r be an integer such that $r \equiv 1 \pmod{2\alpha}$ ($r \equiv -1 \pmod{2\alpha}$ would also work) and $r > 1$. Let t_i denote the *even* integer $(1-r)\beta_i/\alpha_i$, and set $s = \frac{1}{2}((1-r)b + \sum_{i=1}^n t_i)$.

Now replace the generator q_i by $q_i h^{t_i}$. In case M is nonorientable, replace the generator a_1 by $a_1 h^s$ (if $|B|$ is orientable and $p > 0$) and the generator v_1 by $v_1 h^{-s}$ (if $|B|$ is nonorientable). If $|B|$ is orientable and if $p = 0$ and $m > 0$, also replace the generator d_1 by $h^{-2s} d_1$. Then we see that $\pi_1 M$ has a presentation $\mathcal{P}'$ with the same generators and relations as those of $\mathcal{P}$ above, except that $q_i^{\alpha_i} h^{\beta_i} = 1$ is replaced by $q_i^{\alpha_i} h^{r\beta_i} = 1$, and h^b on the right hand side of the long relation is replaced by h^{rb} .

Next define a homomorphism φ from $\|\mathcal{P}\|$ ($= \pi_1 M$) to $\|\mathcal{P}'\|$ ($= \pi_1 M$) by sending h to h^r , y_i to y'_i , and the remaining generators of $\mathcal{P}$ to the corresponding ones of $\mathcal{P}'$. Recalling that h is carried by a regular fiber of M and generates an *infinite*, cycle, normal subgroup of $\pi_1 M$ [Sc, Lemma 3.2], one sees that φ is a monomorphism, but not an isomorphism, by using the diagram

$$\begin{array}{ccccc} \langle h \rangle & \longrightarrow & \|\mathcal{P}\| & \longrightarrow & \|\mathcal{P}\|/\langle h \rangle \\ \downarrow & & \downarrow \varphi & & \downarrow \\ \langle h \rangle & \longrightarrow & \|\mathcal{P}'\| & \longrightarrow & \|\mathcal{P}'\|/\langle h \rangle \end{array}$$

in which the vertical map on the far right is an isomorphism.

We now have an r -fold covering, $\widetilde{M} \rightarrow M$, induced by the monomorphism $\varphi: \pi_1 M \rightarrow \pi_1 M$, with $\pi_1 \widetilde{M} \approx \pi_1 M$. When M is geometrically modeled on either E^3 or $H^2 \times \mathbf{R}$, then $\widetilde{M}$ is modeled on the same geometry. Hence, in these cases, each of M and $\widetilde{M}$ is P^2 -irreducible (see [Sc, §3], say), and they are also sufficiently large, as is easy to see. It follows that $\widetilde{M} \cong M$ [Wa, He] (cf. [OVZ]). $\square$

Remarks. (1) Of the 10 closed, flat, 3-manifolds, there is only one that is the union of two twisted I -bundles; in that case, each of them is an orientable twisted I -bundle over the Klein bottle. As Daverman [Da] has shown, this union of I -bundles is an n -fold cover of itself for each odd n .

(2) Tollefson [To, p. 108] points out an infinite number of closed Seifert bundles that cover themselves nontrivially and, as it turns out, are all geometrically modeled on $H^2 \times \mathbf{R}$.

Theorem 2. *Let M be a closed Seifert fiber space with $e(M) \neq 0$, and suppose that there exists a nontrivial covering, $\Psi: M \rightarrow M$. Then M admits a geometric structure modeled on Nil.*

Proof. Since $e(M) \neq 0$ and $\Psi: M \rightarrow M$ is nontrivial, M is orientable, and the appropriate geometric structure for M must be either Nil or $\widetilde{\text{SL}}_2 \mathbf{R}$. Hence, the universal covering space of M is homeomorphic to $\mathbf{R}^3$. Moreover, M is

not covered by $S^1 \times S^1 \times S^1$, since if it were, then M would admit a geometric structure modeled on E^3 (see [Sc]), making $e(M) = 0$. Therefore, M admits a unique Seifert-bundle structure (up to isomorphism) [Sc, Theorem 3.8], and so we can assume that $\Psi: M \rightarrow M$ is fiber preserving.

Let B denote the base orbifold of M . Since M is closed, B is a closed orbifold, and we have $e(M) = (m/n)e(B)$, where m is the degree of the covering $\psi: B \rightarrow B$ induced by Ψ . Thus, $m = n$. Since $m \cdot n$ is the degree of Ψ , we have $m \cdot n > 1$, and it follows that $m > 1$. But since $\bar{\chi}(B) = m\bar{\chi}(B)$, we have $\bar{\chi}(B) = 0$. Hence, M is geometrically modeled on Nil. $\square$

Lemma 3. *If M is a closed Seifert fiber space admitting a geometric structure modeled on Nil, then $\pi_1 M$ is not cohopfian.*

Proof. Since M is geometrically modeled on Nil, we have $\bar{\chi}(B) = 0$, where B is the base orbifold of M . Hence, B is a torus or a Klein bottle with no cone points, S^2 with cone points of multiplicities $(2, 2, 2, 2)$, $(2, 2, 4)$, $(2, 3, 6)$, or $(3, 3, 3)$, or B is P^2 with two cone points of multiplicities $(2, 2)$ (see [Th] or [Sc]). Notice that M is orientable, since $e(M) \neq 0$, and so B can have no reflector circles, that is, the underlying space $|B|$ of B must be closed. We prove the lemma case-by-case.

(1) $|B| \cong S^1 \times S^1$.

$$\pi_1 M \approx \langle a_1, b_1, h: a_1 h a_1^{-1} = h, b_1 h b_1^{-1} = h, [a_1, b_1] = h^b \rangle,$$

with $b \in \mathbb{Z}$ and $b \neq 0$. The assignment, $a_1 \mapsto a_1^2$, $b_1 \mapsto b_1$, $h \mapsto h^2$, defines an endomorphism $\varphi: \pi_1 M \rightarrow \pi_1 M$, and we have a commutative diagram.

$$\begin{array}{ccc} \langle h: \rangle & \xrightarrow{h \mapsto h^2} & \langle h: \rangle \\ \downarrow & & \downarrow \\ \pi_1 M & \xrightarrow{\varphi} & \pi_1 M \\ \downarrow & & \downarrow \\ \pi_1 M / \langle h: \rangle & \xrightarrow{\bar{\varphi}} & \pi_1 M / \langle h: \rangle \end{array}$$

Since $\bar{\varphi}$, which is induced by φ , is clearly a monomorphism (as is easily checked) as is $h \mapsto h^2$, so is φ . Also, φ is not an epimorphism.

(2) $|B| \cong$ Klein bottle.

$$\pi_1 M \approx \langle v_1, v_2, h: v_1 h v_1^{-1} = h^{-1}, v_2 h v_2^{-1} = h^{-1}, v_1^2 v_2^2 = h^b \rangle,$$

with $b \in \mathbb{Z}$ and $b \neq 0$. The assignment, $v_1 \mapsto v_1^3$, $v_2 \mapsto v_2^3$, $h \mapsto h^3$, defines an endomorphism $\varphi: \pi_1 M \rightarrow \pi_1 M$, and one can show by the same method as in case (1) that φ is a monomorphism but not an epimorphism.

(3) $|B| \cong S^2$ or P^2 .

We shall use the following notation throughout the remainder of the proof. We let B_0 denote $|B|$ punctured at the cone points. The number of exceptional fibers of M is k ($= 2, 3$, or 4), and α_i and β_i have their usual meaning except that we take $b = 0$ and, thus, do not assume that $0 < \beta_i < \alpha_i$.

The notation of various groups of interest is as follows.

$$\pi_1(B_0) = \begin{cases} \langle q_1, \dots, q_k: q_1 \cdots q_k = 1 \rangle & \text{if } |B| \cong S^2, \\ \langle v, q_1, q_2: v^2 q_1 q_2 = 1 \rangle & \text{if } |B| \cong P^2. \end{cases}$$

As in cases (1) and (2), we let h denote the element of $\pi_1 M$ represented by a regular fiber so that

$$\pi_1 M = \pi_1 B_0 \ltimes \langle h: \rangle / \langle q_i^{\alpha_i} h^{\beta_i} \ (i = 1, \dots, h) \rangle;$$

the action of $\pi_1 B_0$ on $\langle h: \rangle$ is $q_i h q_i^{-1} = h$ ($i = 1, \dots, k$) and $vhv^{-1} = h^{-1}$ (if present). The orbifold fundamental group of B is

$$\bar{\pi}_1 B = \pi_1 M / \langle h: \rangle = \pi_1 B_0 / \langle q_i^{\alpha_i} \ (i = 1, \dots, k) \rangle.$$

An orbifold covering $\tau: B \rightarrow B$ of degree n will be called *good*, if each cone point is covered by one cone point of the same multiplicity and some regular points. (Note that there are bad coverings: in the case $(3, 3, 3)$, for instance, take a cyclic 3-fold covering of S^2 branched over two of its cone points.) A good covering corresponds to a transitive representation $\rho: \pi_1 B_0 \rightarrow \Sigma_n$ such that each $\rho(q_i)$ has one fixed point and some number m_i of cycles of length α_i ; hence, $n = m_i \alpha_i + 1$. We show first that good coverings exist in each case.
(2, 2, 2, 2) For any $n \equiv 1 \pmod{2}$,

$$\begin{aligned} q_1, q_2 &\mapsto (12)(34) \cdots (n-2, n-1), \\ q_3, q_4 &\mapsto (23)(45) \cdots (n-1, n). \end{aligned}$$

(2, 4, 4) For $n = 5$,

$$q_1 \mapsto (12)(34), \quad q_2 \mapsto (1345), \quad \text{and} \quad q_3 \mapsto (1532).$$

(2, 3, 6) For $n = 7$,

$$q_1 \mapsto (12)(34)(56), \quad q_2 \mapsto (136)(275), \quad \text{and} \quad q_3 \mapsto (132657).$$

(3, 3, 3) For $n = 4$,

$$q_1 \mapsto (123), \quad q_2 \mapsto (134), \quad \text{and} \quad q_3 \mapsto (142).$$

For $n = 7$,

$$q_1 \mapsto (123)(456), \quad q_2 \mapsto (247)(365), \quad \text{and} \quad q_3 \mapsto (134)(276).$$

(2, 2) For any $n \equiv 1 \pmod{2}$,

$$\begin{aligned} v &\mapsto (n(n-1) \cdots 1), \\ q_1 &\mapsto (1(n-1))(3(n-3)) \cdots ((n-2)2), \end{aligned}$$

and

$$q_2 \mapsto (3(n-1))(5(n-3)) \cdots (n2).$$

We first consider the case $|B| = S^2$. Let $\tau: B \rightarrow B$ be a good covering. Then τ restricts to a covering $\tilde{B}_0 \rightarrow B_0$.

We can choose generators q_{ip} ($i = 1, \dots, k$; $p = 0, \dots, (n-1)/\alpha_i$ ($= m_i$)) for $\tilde{B}_0$ so that

$$\pi_1 \tilde{B}_0 = \left\langle q_{ip}: \prod_{i=1}^k \prod_{p=0}^{m_i} q_{ip} = 1 \right\rangle$$

and $\tau_*(q_{i0}) = g_i q_i g_i^{-1}$, $\tau_*(q_{ip}) = g_i p q_i^{\alpha_i} g_i p^{-1}$, for $p \geq 1$.

Note that since the relation

$$\prod_{i=1}^k \left[g_i q_i g_i^{-1} \prod_{p=1}^{m_i} g_{ip} q_i^{\alpha_i} g_{ip}^{-1} \right] = 1$$

holds in $\pi_1 B_0$, it certainly holds in $\pi_1 M$. But in $\pi_1 M$, we also have $q_i^{\alpha_i} = h^{-\beta_i}$ ($i = 1, \dots, k$), and so

$$(*) \quad \prod_{i=1}^k g_i q_i g_i^{-1} = h^{\sum_{i=1}^k m_i \beta_i}$$

in $\pi_1 M$.

Now define $\varphi: \pi_1 M \rightarrow \pi_1 M$ by

$$\varphi(q_i) = g_i q_i g_i^{-1} h^{-m_i \beta_i}, \quad \varphi(h) = h^n.$$

We must check that this is well defined.

Certainly $[\varphi(q_i), \varphi(h)] = 1$. Also,

$$\begin{aligned} (\varphi(q_i))^{\alpha_i} \cdot (\varphi(h))^{\beta_i} &= g_i q_i^{\alpha_i} g_i^{-1} h^{-m_i \alpha_i \beta_i} h^{n \beta_i} \\ &= g_i q_i^{\alpha_i} g_i^{-1} h^{\beta_i} \quad (\text{since } n = m_i \alpha_i + 1) \\ &= 1. \end{aligned}$$

Finally,

$$\varphi(q_1) \cdots \varphi(q_k) = \left(\prod_{i=1}^k g_i q_i g_i^{-1} \right) h^{-\sum_{i=1}^k m_i \beta_i} = 1,$$

by relation (*).

We now have a commutative diagram:

$$\begin{array}{ccc} \langle h: \rangle & \xrightarrow{h \mapsto h^n} & \langle h: \rangle \\ \downarrow & & \downarrow \\ \pi_1 M & \xrightarrow{\varphi} & \pi_1 M \\ \downarrow & & \downarrow \\ \overline{\pi}_1 B & \xrightarrow{\tau_*} & \overline{\pi}_1 B \end{array}$$

Since $h \mapsto h^n$ and τ_* are monomorphisms, so is φ . It is also clear that φ is not an epimorphism.

Suppose now that $|B| = P^2$. Let $\tau: B \rightarrow B$ be the good orbifold covering of degree 3 corresponding to the transitive representation $\rho: \pi_1 B_0 \rightarrow \Sigma_3$ given by $\rho(v) = (3 \ 2 \ 1)$, $\rho(q_1) = (1 \ 2)$, $\rho(q_2) = (2 \ 3)$. Much as before, we have

$$\pi_1 \tilde{B}_0 = \langle \tilde{v}, q_{10}, q_{11}, q_{20}, q_{21} : \tilde{v}^2 q_{11} q_{21} q_{10} q_{20} = 1 \rangle$$

and $\tau_*(\tilde{v}) = v^3$, $\tau_*(q_{10}) = q_2^{-1} q_1 q_2$, $\tau_*(q_{20}) = (q_1 q_2)^{-2} q_2 (q_1 q_2)^2$, $\tau_*(q_{11}) = q_1^2$, $\tau_*(q_{21}) = q_2^2$.

Now define $\varphi: \pi_1 M \rightarrow \pi_1 M$ by

$$\begin{aligned} \varphi(v) &= v^3, \quad \varphi(q_1) = q_2^{-1} q_1 q_2 h^{-\beta_1}, \\ \varphi(q_2) &= (q_1 q_2)^{-2} q_2 (q_1 q_2)^2 h^{-\beta_2}, \quad \text{and} \quad \varphi(h) = h^3. \end{aligned}$$

The proof that φ is well defined and is a monomorphism, but not an epimorphism, is exactly as with $|B| = S^2$; note, however, that it is important that conjugating elements are orientation preserving and thus commute with h . We check directly that $(\varphi(v))^2 \cdot \varphi(q_1) \cdot \varphi(q_2) = 1$.

$$\begin{aligned} (\varphi(v))^2 \cdot \varphi(q_1) \cdot \varphi(q_2) &= v^6 q_2^{-1} q_1 q_2 (q_1 q_2)^{-2} q_2 (q_1 q_2)^2 h^{-\beta_1 - \beta_2} \\ &= v^6 q_2^{-2} h^{-\beta_2} q_1^{-1} h^{-\beta_1} q_2 (q_1 q_2)^2 \\ &= v^6 q_1 q_2 (q_1 q_2)^2 = 1. \quad \square \end{aligned}$$

Remark. For M with exceptional fibers as in Lemma 3, one can obtain specific imbeddings, $\pi_1 M \rightarrow \pi_1 M$, by drawing a covering $\tilde{B}_0$ of B_0 and using it to obtain the g_i 's.

Recall that a closed Seifert fiber space M is *small*, if it is geometrically modeled on either $S^2 \times R$ or S^3 , or if it is a circle bundle over a torus or a Klein bottle [O, p. 91]. If M is not small, then it is *large*.

Proposition 4. *Let M be a closed Seifert fiber space. Then $\pi_1 M$ is not cohopfian if and only if there exists a nontrivial covering $M \rightarrow M$.*

Proof. The condition is obviously sufficient, so suppose that $\pi_1 M$ is not cohopfian. If $e(M) = 0$, then M covers itself nontrivially (Theorem 1).

If $e(M) \neq 0$, then M must be geometrically modeled on either Nil or $\widetilde{\text{SL}}_2 \mathbf{R}$; it cannot be modeled on S^3 , since otherwise $\pi_1 M$ would be finite and, therefore, cohopfian. If M admits a geometric structure modeled on Nil, then the base orbifold of M is one of the following seven.

- (1) $|B| \cong S^1 \times S^1$ and B has no cone points;
- (2) $|B| \cong$ Klein bottle and B has no cone points;
- (3) $|B| \cong S^2$ and B has 4 cone points with multiplicities $(2, 2, 2, 2)$;
- (4) $|B| \cong P^2$ and B has 2 cone points with multiplicities $(2, 2)$;
- (5) $|B| \cong S^2$ and B has 3 cone points with multiplicities $(2, 4, 4)$, $(2, 3, 6)$, or $(3, 3, 3)$.

Now M is orientable (since $e(M) \neq 0$), and thus if the base orbifold of M is one of the first four, (1), (2), (3), (4) above, then M must be a Haken manifold (see [J, Theorem VI.15, p. 96], for example). If the base orbifold of M is listed under (3), (4), or (5), then M is large. Thus, $\pi_1 M$ determines M up to homeomorphism among closed, orientable, irreducible 3-manifolds when M is Haken [Wa], or among large Seifert bundles when M is large ([OVZ]; cf. [O, pp. 97, 134]).

Let $\tilde{M}$ denote the covering space of M corresponding to a nontrivial imbedding of $\pi_1 M$ in itself. Since the universal covering space of M is R^3 (and, thus, not $S^2 \times R$), M is P^2 -irreducible (see [Sc, §3], say), and so $\pi_1 M$ cannot be imbedded in itself with infinite index, as we pointed out in the opening remarks of the paper. Hence, the covering $\tilde{M} \rightarrow M$ is finite, and $\tilde{M}$ is a closed, orientable, Seifert bundle geometrically modeled on Nil (and, therefore, irreducible). Thus, $\tilde{M}$ is either large or Haken, and it follows from remarks in the last paragraph that $\tilde{M} \cong M$, since $\pi_1 \tilde{M} \approx \pi_1 M$.

Suppose now that M admits a geometric structure modeled on $\widetilde{\text{SL}}_2 \mathbf{R}$ and that $\pi_1 M$ is not cohopfian. Let $G < \pi_1 M$ be a proper subgroup isomorphic to $\pi_1 M$. As in the case when M was geometrically modeled on Nil, we see that

M is P^2 -irreducible, and so $[\pi_1 M : G] < \infty$. Let $\widetilde{M}$ denote the covering space of M corresponding to G . Then $\widetilde{M}$ is a closed, orientable, Seifert bundle modeled on $\widetilde{\text{SL}_2\mathbf{R}}$. Clearly, both M and $\widetilde{M}$ are large, and so $\widetilde{M} \cong M$, since $\pi_1 \widetilde{M} \approx \pi_1 M$. But no Seifert fiber space admitting a geometric structure modeled on $\widetilde{\text{SL}_2\mathbf{R}}$ can cover itself nontrivially (Theorem 2). Therefore, $\pi_1 M$ is cohopfian. $\square$

Remark. In the above proof, we do not need the powerful result of Scott [Sc₁]: If M and N are closed, orientable, irreducible 3-manifolds with infinite fundamental groups, if N is Seifert fibered, and if $\pi_1 M \approx \pi_1 N$, then $M \cong N$.

Corollary 5. *Let M be a closed Seifert fiber space with base orbifold B . The following statements are equivalent.*

- (a) $\pi_1 M$ is cohopfian.
- (b) M does not cover itself nontrivially.
- (c) M admits a geometric structure modeled on S^3 or on $\widetilde{\text{SL}_2\mathbf{R}}$.
- (d) $e(M) \cdot \overline{\chi}(B) \neq 0$.

Proof. Statements (a) and (b) are equivalent, by Proposition 4. That (c) $\Rightarrow$ (b) follows from Theorem 2. If (a) holds, then $e(M) \neq 0$, by Theorem 1, but if M were geometrically modeled in Nil, then $\pi_1 M$ would not be cohopfian, by Lemma 3. Therefore, (a) $\Rightarrow$ (c). Finally, the Scott-Thurston chart given at the end of §1 implies that (c) and (d) are equivalent. $\square$

Recall from [S, Theorem 12] that if M is a closed Seifert fiber space that is also a homology 3-sphere different from S^3 , then the base orbifold of M consists of S^2 with r cone points of multiplicities $(\alpha_1, \alpha_2, \dots, \alpha_r)$, where $r \geq 3$ and the multiplicities are pairwise relatively prime. Moreover, there is only one such homology sphere with finite fundamental group; its cone multiplicities are $(2, 3, 5)$ and the space itself is geometrically modeled on S^3 , of course.

Corollary 6. *The fundamental group of a homology 3-sphere admitting a Seifert fibration is cohopfian.*

Proof. Let M denote such a 3-manifold different from S^3 , with base orbifold B . A presentation for $\pi_1 M$ is

$$\langle q_1, \dots, q_r, h : q_1 h q_1^{-1} = h, \dots, q_r h q_r^{-1} = h, \\ q_1^{\alpha_1} h^{\beta_1} = 1, \dots, q_r^{\alpha_r} h^{\beta_r} = 1, q_1 \cdots q_r = h^b \rangle,$$

where $r \geq 3$. The corresponding relation matrix for $H_1(M)$ is square and its determinant is

$$D = b\alpha_1 \cdots \alpha_r + \beta_1 \alpha_2 \cdots \alpha_r + \cdots + \alpha_1 \alpha_2 \cdots \alpha_{r-1} \beta_r;$$

therefore,

$$\frac{D}{\alpha_1 \cdots \alpha_r} = -e(M).$$

Since M is a homology sphere, we have $D = \pm 1$, and so $e(M) \neq 0$.

On the other hand,

$$\overline{\chi}(B) = 2 - \sum_{i=1}^r \left(1 - \frac{1}{\alpha_i}\right) = (2 - r) + \sum_{i=1}^r \frac{1}{\alpha_i},$$

and since $\alpha_1, \dots, \alpha_r$ are pairwise relatively prime, $\sum 1/\alpha_i$ is not an integer. Hence, $\bar{\chi}(B) \neq 0$, and so M is geometrically modeled on either S^3 or $\widetilde{\mathrm{SL}}_2\mathbf{R}$. Since $\pi_1 S^3$ is cohopfian, the result now follows from Corollary 5. $\square$

Our final result settles the cohopficity question for $\pi_1 M$ when M is Seifert fibered and compact, and $\partial M \neq \emptyset$.

Theorem 7. *The fundamental group of a compact Seifert fiber space with non-empty boundary is not cohopfian.*

Proof. Let M denote a compact Seifert fiber space with nonempty boundary. Then each component of ∂M is either a torus or a Klein bottle. We can assume that M is neither a solid torus nor a solid Klein bottle, and so each component of ∂M is incompressible (see [Sc, Corollary 3.3], say). We can also assume that M is not an I -bundle (trivial or not) over a torus or over a Klein bottle, since then $\pi_1 M$ is isomorphic to either $Z \times Z$ or $\langle v, h: v h v^{-1} = h^{-1} \rangle$; the latter imbeds in itself (with index 2) by $v \mapsto v, h \mapsto h^2$.

Before attacking the general case, it is convenient to consider three more special ones. For these, M is obtained from a solid Klein bottle K by removing some (open) fibered solid Klein bottles and perhaps one (open) fibered solid torus from $\mathrm{Int}(K)$, and possibly replacing one regular fiber of K by an exceptional one. We describe the cases in terms of the base orbifold B of M .

Case 1. $|B| = D^2$; $\partial(|B|)$ contains at least two reflector lines; B has no cone points.

Case 2. $|B| = S^1 \times I$; one component of $\partial(|B|)$ contains no singular points while the other contains at least one reflector line; B has no cone points.

Case 3. $|B| = D^2$; $\partial(|B|)$ contains at least one reflector line; B has exactly one cone point.

Let r denote the number of reflector lines in each case. Then, in Cases 1, 2, or 3, $\pi_1 M$ is isomorphic to

$$\begin{aligned} \langle x_1, \dots, x_r: x_1^2 = x_2^2 = \dots = x_r^2 \rangle, \\ \langle x_1, \dots, x_r, t_1: x_1^2 = \dots = x_r^2, t_1 x_r^2 t_1^{-1} = x_r^2 \rangle, \end{aligned}$$

or

$$\langle x_1, \dots, x_r, q_1: x_1^2 = \dots = x_r^2, q_1 x_r^2 q_1^{-1} = x_r^2, q_1^\alpha x_r^{2\beta} = 1 \rangle,$$

respectively. In each case, each x_i^2 is represented by a regular fiber of M and generates an infinite, cyclic, normal subgroup of $\pi_1 M$. In Case 1, a proper imbedding is induced by $x_i \mapsto x_i^3$. In Cases 2 and 3, we let $c_1, c_2, \dots, c_{r+1}$ denote $x_1, x_2, \dots, x_r, t_1$ or $x_1, \dots, x_r, q_1$, respectively, and send $x_1 \mapsto (c_2 c_1)^{-2} x_1 (c_2 c_1)^2$, $x_i \mapsto x_i$ ($i > 1$), and $t_1 \mapsto t_1$ (Case 2) or $q_1 \mapsto q_1$ (Case 3); these are similar to the endomorphism used in the proof of Lemma 2.4 of [GW]; notice that $(c_2 c_1)^2$ commutes with each x_i^2 . Now,

$$\pi_1 M / \langle x_1^2: \rangle \approx Z_2 * \dots * Z_2 * Z, \quad \text{in Case 2,}$$

and

$$\pi_1 M / \langle x_1^2: \rangle \approx Z_2 * \dots * Z_2 * Z_\alpha, \quad \text{in Case 3,}$$

with $\alpha \geq 2$ and with r copies of Z_2 in each case. We then have a commutative diagram

$$\begin{array}{ccccc} \langle x_1^2 \rangle & \hookrightarrow & \pi_1 M & \twoheadrightarrow & \pi_1 M / \langle x_1^2 \rangle \\ \mathrm{id} \downarrow & & \varphi \downarrow & & \downarrow \hat{\varphi} \\ \langle x_1^2 \rangle & \hookrightarrow & \pi_1 M & \twoheadrightarrow & \pi_1 M / \langle x_1^2 \rangle \end{array}$$

where $\hat{\varphi}$ is induced by φ . Since $\hat{\varphi}$ is a monomorphism, but not an automorphism, so is φ .

In the general case, we let $A_1, \dots, A_d$ denote (closed) annular neighborhoods (without cone points) of those boundary components of $|B|$ that are either reflector circles or contain reflector lines of B . Set $|B_0| = \text{cl}(|B| - (A_1 \cup \dots \cup A_d))$, let M_0 denote the Seifert bundle in M over B_0 , and let N_i denote the Seifert bundle over A_i ($i = 1, \dots, d$). There are two cases.

First, suppose that some boundary component of $|B|$ and of A_1 , say, contains reflector lines of B , and let $M_1 = \text{cl}(M - N_1)$. Then

$$\pi_1 M \approx \pi_1 M_1 *_{Z \times Z} \pi_1 N_1,$$

where we are assuming that M is not one of the special cases, 1, 2, or 3, above, so that the inclusion map $M_1 \cap N_1 \rightarrow M_1$ induces a *proper* imbedding, $Z \times Z \approx \pi_1(M_1 \cap N_1) \rightarrow \pi_1 M_1$. Notice that the amalgamating subgroup $Z \times Z$ also properly injects into $\pi_1 N_1$. Now $\pi_1 N_1$ is isomorphic to

$$\langle x_1, \dots, x_r, t_1 : x_1^2 = \dots = x_r^2, t_1 x_r^2 t_1^{-1} = x_r^2 \rangle,$$

as in special Case 2, and the proper imbedding of this group that we gave clearly extends to a proper imbedding $\pi_1 M \rightarrow \pi_1 M$ whose restriction to $\pi_1 M_1$ is the identity. (Here, we take the base point of $\pi_1 M$ in $M_1 \cap N_1$.)

Assume now that no boundary component of $|B|$ contains reflector lines. Then, one component of each A_i is a reflector circle, and $\partial(|B|)$ contains at least one component that is *not* a reflector circle. Moreover, each fiber of the Seifert bundle M_0 over B_0 has a fibered solid torus as a regular neighborhood, and a component of ∂M is either an incompressible torus or Klein bottle. We now show that $\pi_1 M_0$ is not cohopfian.

Let $T_1, \dots, T_m$ be the boundary components of M_0 , and let T_m be a boundary component not projecting to a boundary component of any A_i . Let n be the number of exceptional fibers of M_0 (all isolated now), and let r be the rank of $\pi_1(|B_0|)$. Then $\pi_1 M$ has a presentation with generators t_j ($1 \leq j < m$), q_i ($1 \leq i \leq n$), a_k ($1 \leq k \leq r - m + 1$), and h and with relations $t_j h t_j^{-1} = h^{\varepsilon_j}$, $[q_i, h] = 1$, $a_k h a_k^{-1} = h^{\varepsilon_k}$, and $q_i^{\alpha_i} h^{\beta_i} = 1$, where ε_j and ε_k belong to $\{-1, 1\}$. Notice that these generators and relations take into account all the possible combinations of orientability states for M_0 , $|B_0|$, and the various boundary components of M_0 (see [OVZ, p. 51]).

Let $\{c_1, \dots, c_{r+n}\}$ be the set of generators $\{t_1, \dots, t_{m-1}, q_1, \dots, q_n, a_1, \dots, a_{r-m+1}\}$; we can assume that $r + m \geq 2$. Let $\omega = (c_2 c_1)^2$, and notice that $[\omega, h] = 1$. Define $\varphi: \pi_1 M_0 \rightarrow \pi_1 M_0$ by $\varphi(c_1) = \omega^{-1} c_1 \omega$, $\varphi(c_i) = c_i$ ($i > 1$), and $\varphi(h) = h$. Then

$$\pi_1 M_0 / \langle h : \rangle \approx F_r * Z_{\alpha_1} * \dots * Z_{\alpha_n},$$

and the proof that φ is a monomorphism but not an automorphism is similar to that for special Cases 2 and 3 and for the endomorphism φ in the proof of Lemma 2.4 of [GW].

Finally, if a nonorientable, twisted I -bundle, N_1 say, meets ∂M_0 along the surface T_1 associated with the loop t_1 ($= c_1$) we take the basepoint $*$ of $\pi_1 M$ in T_1 and set $\varphi(x) = \omega^{-1} x \omega$ for each generator x of $\pi_1 N_1 < \pi_1 M$, and we define φ on every other $\pi_1 N_i$ to be the identity; an element of $\pi_1 N_i$ would

here be represented by a path from $*$ to a point p_i on ∂N_i , then a loop in N_i , and back to $*$ along the original path from $*$ to p_i . We define φ on $\pi_1 M_0$ as in the preceding paragraph. Thus, we have a monomorphism φ of $\pi_1 M$ that is not an automorphism. $\square$

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