HOMOLOGY AND COHOMOLOGY OF Π -ALGEBRAS

W. G. DWYER AND D. M. KAN

ABSTRACT. We study a type of homological algebra associated to the collection of *all* homotopy groups of a space (just as the theory of group homology is associated to the fundamental group).

1. INTRODUCTION

1.1. Summary. In [3] we started an investigation of Π -algebras (i.e., (≥ 1) -graded groups, together with an action of the primary homotopy operations) by constructing an *enveloping ring* of a Π -algebra, which generalized the integral group ring of a group as well as the enveloping algebra of a connected graded rational Lie algebra. In the present paper we

- (i) use the enveloping ring construction to define the *homology and cohomology of Π -algebras* in a manner which generalizes the usual homology and cohomology of groups and connected graded rational Lie algebras,
- (ii) obtain *Serre spectral sequences* which, for a short exact sequence $* \rightarrow U \rightarrow W \rightarrow V \rightarrow *$ of Π -algebras, relate the homology and cohomology groups of U and V to those of W , and
- (iii) discuss a slightly different, *Quillen-like approach* to the homology and cohomology of Π -algebras and note that (except in the bottom dimension) the resulting groups differ from the earlier ones (see (i)) by only a shift in dimension.

1.2. A more detailed outline of the paper. (i) *Homology of Π -algebras.* After fixing some notation and terminology (in §1.3) and a brief review of Π -algebras (in §2) and the enveloping ring functor E (in §3), we recall (in §4) some results of Quillen on simplicial modules over a simplicial ring R (with a slight change in notation: we write $\mathrm{Tor}_i^R(-, -)$ for Quillen's functors $\pi_i(- \otimes_R -)$). In §5 we then define the homology groups $H_i(X; M)$ of a Π -algebra X with coefficients in a right EX -module M , by

$$H_i(X; M) = \mathrm{Tor}^{EF \cdot X}(M, Z), \quad i \geq 0$$

(where $F \cdot X$ denotes the standard free simplicial resolution (2.5) of X), and we obtain, for a short exact sequence $* \rightarrow U \rightarrow W \rightarrow V \rightarrow *$ of Π -algebras and a right EW -module M , a Serre spectral sequence with

$$E_{p,q}^2 = H_p(V; H_q(U; M)) \Rightarrow H_{p+q}(W; M).$$

Received by the editors August 9, 1991 and, in revised form, January 24, 1992.

1991 *Mathematics Subject Classification.* Primary 55Q05; Secondary 18G15.

Research partly supported by the National Science Foundation.

(ii) *Cohomology of Π -algebras.* This requires “dualizing” the results of §§4 and 5. To do this we construct (in §6) functors $\text{Ext}_R^i(-, -)$, which involve in the second variable rather curious “cosimplicial modules over the simplicial ring R ”. In §7 we then define the cohomology groups $H^i(X; N)$ of a Π -algebra X with coefficients in a left EX -module N , by

$$H^i(X, N) = \text{Ext}_{EF, X}^i(Z, N), \quad i \geq 0,$$

and we obtain, for a short exact sequence $* \rightarrow U \rightarrow W \rightarrow V \rightarrow *$ of Π -algebras and a left EW -module N , a Serre spectral sequence with

$$E_2^{p, q} = H^p(V; H^q(U; N)) \Rightarrow H^{p+q}(W; N).$$

(iii) *Quillen homology and cohomology.* The last two sections (§§8 and 9) are devoted to a Quillen-like approach to the homology and cohomology of a Π -algebra X . The key notion here is that of a strongly abelian group object over X , i.e., a diagram of Π -algebras

$$* \rightarrow B \xrightarrow{i} Y' \xrightleftharpoons[j]{p} X \rightarrow *$$

in which $pj = \text{id}$, the right-pointing arrows form an exact sequence and B is strongly abelian (i.e., (2.2) just a (≥ 1) -graded abelian group). We show that these objects are in a natural 1-1 correspondence with the (≥ 0) -graded left EX -modules and that, for an object $(Y \rightarrow X)$ over X , the EX -module so corresponding to the strong abelianization of $(Y \rightarrow X)$ is the module $EX \otimes_{EY} IY$ (where $IY \subset EY$ denotes the augmentation ideal). As a result the Quillen homology groups $H_i^Q(X; M)$ and cohomology groups $H_i^Q(X; N)$ can be described by

$$H_i^Q(X; M) = \text{Tor}_i^{EF, X}(M, IF \cdot X),$$

$$H_i^Q(X; N) = \text{Ext}_{EF, X}^i(IF \cdot X, N), \quad i \geq 0,$$

and the short exact sequence $0 \rightarrow IF \cdot X \rightarrow EF \cdot X \rightarrow Z \rightarrow 0$ therefore readily yields natural isomorphisms

$$H_i^Q(X; M) \approx H_{i+1}(X; M), \quad H_i^Q(X; N) \approx H^{i+1}(X; N), \quad i \geq 1,$$

and natural exact sequences

$$0 \rightarrow H_1(X; M) \rightarrow H_0^Q(X; M) \rightarrow M \rightarrow H_0(X; M) \rightarrow 0,$$

$$0 \rightarrow H^0(X; N) \rightarrow N \rightarrow H_0^Q(X; N) \rightarrow H^1(X; N) \rightarrow 0.$$

1.3. Notation, terminology, etc. We will freely use notation, terminology and results of [3] and [7, Chapter II]. In particular:

(i) *Rings and modules.* All rings will have an identity and will be associative (but not necessarily commutative) and augmented over Z (the integers). Moreover, they will be (≥ 0) -graded, except that, for a simplicial ring R , its homotopy ring $\pi_* R$ will be bi- (≥ 0) -graded. All modules over bigraded rings will be bi-graded, and all other modules will be graded. This also applies to Z -modules, i.e., abelian groups.

(ii) *Whitehead products.* For a pointed connected CW complex L and elements $a \in \pi_{p+1} L$ and $b \in \pi_{q-1} L$ ($p, q \geq 1$), we denote by $[a, b]$ the

Whitehead product which differs by $(-1)^p$ from the usual one [10, Chapter X]. As a result

$$[a, b] + (-1)^{pq}[b \cdot a] = 0$$

and if $c \in \pi_{r+1}L$ ($r \geq 1$), then the Jacobi identity becomes

$$(-1)^{pr}[a, [b, c]] + (-1)^{pq}[b, [c, a]] + (-1)^{qr}[c, [a, b]] = 0.$$

The authors would like to thank the referee for helpful comments.

2. Π -ALGEBRAS

We start with recalling from [3] some facts about

2.1. Π -algebras. A Π -algebra consists [3, 2.3] of a (≥ 1) -graded group $\{X_p\}_{p=1}^\infty$, with X_p abelian for $p > 1$, together with three kinds of operations:

- (i) a *Whitehead product* homomorphism $[-, -]: X_p \otimes X_q \rightarrow X_{p+q-1}$ for all $p, q > 1$, and
- (ii) a *composition* function $(-\circ\alpha): X_p \rightarrow X_r$, for every element $\alpha \in \pi_r S^p$ with $r > p > 1$ (which need not be a homomorphism, but which is right additive, i.e., $(x \circ \alpha_1) + (x \circ \alpha_2) = (x \circ (\alpha_1 + \alpha_2))$ for all $x \in X_p$ and $\alpha_1, \alpha_2 \in \pi_r S^p$),

which satisfy all the relations that hold for the Whitehead product (1.3) and composition operations on the higher homotopy groups of pointed connected CW complexes, and

- (iii) a *left action* of X_1 on the X_p ($p > 1$) which commutes with the Whitehead product and composition operations (we will write τ_{xy} for the result of this left action by an element $x \in X_1$ on an element $y \in X_p$ ($p > 1$)).

Thus a Π -algebra X is completely determined by its universal covering $\tilde{X}$ (i.e., the sub- Π -algebra $\tilde{X} \subset X$ consisting of the elements of degree > 1) and the left action of the group X_1 on this Π -algebra $\tilde{X}$.

The category of Π -algebras will be denoted by $\Pi\text{-al}$.

2.2. Examples. (i) The *homotopy Π -algebra* of a pointed connected CW complex. Let L be a pointed connected CW complex. Then the graded group $\{\pi_p L\}_{p=1}^\infty$, together with the usual Whitehead product (1.3) and composition operations and fundamental group action, is clearly a Π -algebra, which will be denoted by $\pi_* L$.

(ii) *Aspherical Π -algebras*. These are Π -algebras X such that (2.1) $\tilde{X}$ is trivial. Clearly such an aspherical Π -algebra X is completely determined by the group X_1 . Aspherical Π -algebras thus are just *groups*.

(iii) *Simply connected rational Π -algebras*. These are Π -algebras X such that $X_1 = 1$ and each X_p ($p > 1$) is a rational vector space. Such Π -algebras are [3, 2.5] essentially *connected graded rational Lie algebras*; they are completely determined by their “underlying” connected graded rational Lie algebra, i.e., the Lie algebra obtained by lowering the degrees by 1 and taking the Whitehead product as Lie product.

(iv) *Strongly abelian Π -algebras*. These are Π -algebras X in which all Whitehead product and composition operations are trivial and in which X_1 is abelian and acts trivially on the X_p ($p > 1$). Strongly abelian Π -algebras thus are just (≥ 1) -graded abelian groups.

2.3. Free Π -algebras. Another important class of Π -algebras is that of the *free Π -algebras*, i.e., Π -algebras which are isomorphic to homotopy Π -algebras of wedges of spheres. If $M = \bigvee_{j \in J} S^{p_j}$ ($p_j \geq 1$), then $\pi_* M$ is the free Π -algebra on the obvious generators $i_{p_j} \in \pi_{p_j} S^{p_j} \subset \pi_{p_j} M$ ($j \in J$). Moreover the universal covering of $\pi_* M$ is the homotopy Π -algebra of the universal covering of M , i.e., the free Π -algebra on the elements $\tau_x i_{p_j}$ with $x \in \pi_1 M$, $j \in J$, and $p_j > 1$, and $\pi_1 M$ acts on this Π -algebra from the left by permuting these generators in the obvious manner.

Clearly a sub- Π -algebra of a free Π -algebra need not be free. However, if U and V are both free Π -algebras, then the kernel of the projection $U \amalg V \rightarrow V$ is also free. To prove this one first notes: If $U_1 = V_1 = 1$ and $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are sets of free generators for U and V respectively, then [3, 2.5(v); 6, Theorem 3] the kernel of the projection map $U \amalg V \rightarrow V$ is freely generated by the elements $[\cdots [a_i, b_{j_1}], \dots, b_{j_n}]$ with $i \in I$, $n \geq 0$, and $j_1, \dots, j_n \in J$.

2.4. Simplicial Π -algebras. The category $\mathbf{s}\Pi\text{-al}$ of simplicial Π -algebras admits [7, II, §4] a closed simplicial model category structure in which the simplicial structure is the obvious one and in which a map is a fibration or a weak equivalence whenever the underlying map of simplicial sets is so. The cofibrant objects are the retracts of the free ones, where a simplicial Π -algebra Y is called *free* if there exists a subset $B \subset Y$, closed under the degeneracy operators, such that, for each $i \geq 0$, the Π -algebra of Y in dimension i is freely generated by the elements of B in dimension i . Moreover all simplicial Π -algebras are fibrant and hence [7, II, §2] *every weak equivalence $f: Y \rightarrow Y' \in \mathbf{s}\Pi\text{-al}$ between cofibrant objects is actually a homotopy equivalence*; i.e., there exists a map $f': Y' \rightarrow Y \in \mathbf{s}\Pi\text{-al}$ such that the compositions $f'f$ and ff' are simplicially homotopic to the identity maps of Y and Y' respectively.

2.5. The standard resolution of a Π -algebra. The *standard resolution* of a Π -algebra X consists of the free Π -algebra $F.X$ and weak equivalence $F.X \rightarrow X \in \mathbf{s}\Pi\text{-al}$, in which each $(F.X)_n$ ($n \geq 0$) consists of the Π -algebra $F^{n+1}X$ obtained by $(n+1)$ -fold application of the free Π -algebra functor F (i.e., the forgetful functor from $\Pi\text{-al}$ to (≥ 1) -graded pointed sets) followed by its left adjoint) and in which the face and degeneracy operators and the map $F.X \rightarrow X$ are the obvious ones. Clearly this construction is functorial.

3. THE ENVELOPING RING OF A Π -ALGEBRA

Next we briefly discuss the notion of

3.1. The enveloping of a Π -algebra. The *enveloping ring* of a Π -algebra X is [3, 4.1] the ring (1.3) EX which has, for every integer $p \geq 1$ and element $x \in X_p$, a generator ex in degree $p-1$, and which has the following relations:

$$\begin{aligned} e(x+y) &= ex + ey, & x, y \in \tilde{X}, \\ e[x, y] &= (ex)(ey) - (-1)^{pq}(ey)(ex), & x \in \tilde{X}_{p+1}, y \in \tilde{X}_{q+1}, \\ e(xy) &= (ex)(ey), & x, y \in X_1, \\ e(\tau_x y) &= (ex)(ey)(ex^{-1}), & x \in X_1, y \in \tilde{X}, \\ e(x \circ \alpha) &= H(\alpha)(ex)^2, & \alpha \in \pi_{2p-1} S^p, p \text{ even}, \\ &= 0, & \text{otherwise,} \end{aligned}$$

where $H(\alpha)$ denotes the Hopf invariant [3, 1.3(iv)] of α .

This definition immediately implies

3.2. Proposition. *Let $X \in \Pi\text{-al}$. Then EX is the “semitensor product” of $E\tilde{X}$ and ZX_1 (the integral group ring of X_1), in the sense that*

- (i) *additively EX is actually isomorphic to $E\tilde{X} \otimes ZX_1$,*
- (ii) *the multiplication in EX is given by*

$$(ey \otimes ex)(ey' \otimes ex') = (ey)(e\tau_x y') \otimes (ex)(ex')$$

for all $x, x' \in X_1$ and $y, y' \in \tilde{X}$.

3.3. Classical examples. (i) If $X \in \Pi\text{-al}$ is *aspherical* (2.2), then $EX = ZX_1$, the integral group ring of X_1 , and the function $X_1 \rightarrow EX = ZX_1$ given by $x \rightarrow ex$ for all $x \in X_1$, is the usual map which sends the elements of X_1 to the corresponding (additive) generators of ZX_1 .

(ii) If $X \in \Pi\text{-al}$ is *simply connected rational* (2.2), then EX is the *enveloping algebra of the underlying Lie algebra of X* (2.2), and the function $X \rightarrow EX$ given by $x \rightarrow ex$ for all $x \in X$, is the usual inclusion of one in the other.

Another easy consequence of Definition 3.1 is

3.4. Proposition. *Let X be a simply connected (i.e., $X_1 = 1$) free Π -algebra (2.3) and let $\{b_j\}_{j \in J}$ be a set of free generators for X . Then EX is the free (tensor) algebra on the elements eb_j ($j \in J$). In particular, EX is additively just the free abelian group on the elements $(eb_{j_1}) \cdots (eb_{j_n})$ with $n \geq 0$ and $j_1, \dots, j_n \in J$.*

3.5. Corollary (2.3 and 3.2). *Let X be a free Π -algebra and let $\{b_j\}_{j \in J}$ be a set of free generators for X . Then EX is additively the free abelian group on the elements $(e\tau_{x_1} b_{j_1}) \cdots (e\tau_{x_n} b_{j_n})(ex)$ with $n \geq 0$, $x, x_1, \dots, x_n \in X_1$, $j_1, \dots, j_n \in J$, and $b_{j_1}, \dots, b_{j_n} \in \tilde{X}$.*

A straightforward calculation now yields

3.6. Proposition. *Let X and $\{b_j\}_{j \in J}$ be as in 3.5. Then the augmentation ideal $IX \subset EX$ is a free left EX -module on the elements eb_j with $b_j \in \tilde{X}$ and the elements $1 - eb_j$ with $b_j \in X_1$.*

We will also need

3.7. Proposition. *Let U and V be free Π -algebras with generators $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ respectively and let X be the kernel of the projection $U \amalg V \rightarrow V$. Then $E(U \amalg V)$ is a free right EX -module on the elements*

$$(e\tau_{v_1} b_{j_1}) \cdots (e\tau_{v_n} b_{j_n})(ev)$$

with $n \geq 0$, $v, v_1, \dots, v_n \in V_1$, $j_1, \dots, j_n \in J$, and $b_{j_1}, \dots, b_{j_n} \in \tilde{V}$.

Proof. If U and V are simply connected (i.e., $U_1 = V_1 = 1$), then one proves this by a lengthy but essentially straightforward calculation using 2.3. The general case now follows readily using 3.2 and 3.5.

We end with considering

3.8. E-flat Π -algebras. A Π -algebra X is [3, 5.2] called *E-flat* if (2.5) $\pi_i EF.X = 0$ for $i > 0$. Some examples of *E-flat Π -algebras* are, in view of [3, 5.4(i) and (ii)] and a slight variation on [3, 5.4(iii)]

- (i) *aspherical* (2.2) Π -algebras,
- (ii) *simply connected rational* (2.2) Π -algebras, and
- (iii) *free* (2.3) Π -algebras.

4. THE FUNCTORS Tor_i^R FOR A SIMPLICIAL RING R

In preparation for the definition (in §5) of the homology of a Π -algebra, we recall from [7, II, §6] some facts on simplicial modules over a simplicial ring R . In a slight change in notation we will write Tor_i^R for the functors which were denoted there by $\pi_i(- \otimes_R^L -)$.

4.1. Simplicial modules over a simplicial ring. Given a simplicial ring R (1.3), a *left simplicial R -module* consists of a simplicial abelian group B , together with a map of simplicial abelian groups $R \otimes B \rightarrow B$ which turns each B_i ($i \geq 0$) into a left R -module. The resulting category of left simplicial R -modules will be denoted by $\mathbf{M}_R$.

Of course, *right simplicial R -modules* are just left simplicial R^{op} -modules (where R^{op} denotes the simplicial ring which in each dimension $i \geq 0$ consists of the opposite of the ring R_i) and the category of right simplicial R -modules is therefore denoted by $\mathbf{M}_{R^{\text{op}}}$.

4.2. A model category structure for $\mathbf{M}_R$. The category $\mathbf{M}_R$ admits [7, II, §6] a closed simplicial model category structure in which the simplicial structure is the obvious one and in which a map is a fibration or a weak equivalence whenever the underlying map of simplicial sets is so. The cofibrant objects are the retracts of the free ones, where an object $B \in \mathbf{M}_R$ is called *free* if there exists a subset $X \subset B$, closed under the degeneracy operators, such that each B_i ($i \geq 0$) is the free left R_i -module on $X \cap B_i$. Moreover, all objects of $\mathbf{M}_R$ are fibrant and it follows [7, II, §2] that every weak equivalence $f: B \rightarrow B' \in \mathbf{M}_R$ *between cofibrant objects is actually a homotopy equivalence*, i.e., there exists a map $f': B' \rightarrow B \in \mathbf{M}_R$ such that the compositions $f'f$ and ff' are simplicially homotopic to the identity maps of B and B' respectively.

4.3. Cofibrant resolutions. Given $B \in \mathbf{M}_R$, a *cofibrant resolution* of B (over R) consists of a cofibrant object $B_{\#} \in \mathbf{M}_R$, together with a trivial fibration $j: B_{\#} \rightarrow B \in \mathbf{M}_R$ (i.e., a weak equivalence which is onto). Clearly (4.2) such cofibrant resolutions always exist. A convenient and functorial one is the *standard resolution* $R_{\#}B \rightarrow B \in \mathbf{M}_R$ in which, for each $i \geq 0$, $(R_{\#}B)_i = (R^{i+1}B)_i$, where $R^{i+1}B$ is obtained from B by $(i+1)$ -fold application of the free left simplicial R -module functor R (i.e., the forgetful functor from $\mathbf{M}_R$ to (**pointed simplicial sets**), followed by its left adjoint) and in which the face and degeneracy maps and the map $R_{\#}B \rightarrow B \in \mathbf{M}_R$ are the obvious ones.

4.4. The functor $\otimes_R$ for a simplicial ring R . Given (4.1) simplicial modules $A \in \mathbf{M}_{R^{\text{op}}}$ and $B \in \mathbf{M}_R$, the simplicial abelian group $A \otimes_R B$ (given by $(A \otimes_R B)_i = A_i \otimes_{R_i} B_i$ for all $i \geq 0$) *has homotopy meaning whenever B is cofibrant*, as a diagonal argument using [4] readily yields

4.5. Proposition. *Let $A \in \mathbf{M}_{R^{\text{op}}}$ and let $f: B \rightarrow B' \in \mathbf{M}_R$ be a weak equivalence between cofibrant objects. Then the map $A \otimes_R B \rightarrow A \otimes_R B'$ induces isomorphisms $\pi_i(A \otimes_R B) \approx \pi_i(A \otimes_R B')$ ($i \geq 0$).*

Remark. It is also useful to observe that if $f: A \rightarrow A' \in \mathbf{M}_{R^{\text{op}}}$ is a weak equivalence and $B \in \mathbf{M}_R$ is a cofibrant object, then the map $f \otimes_R B: A \otimes_R B \rightarrow A' \otimes_R B$ induces isomorphisms $\pi_i(A \otimes_R B) \rightarrow \pi_i(A' \otimes_R B)$ ($i \geq 0$). This follows from the analogue of 4.5 if A and A' are in the evident sense cofibrant objects of $\mathbf{M}_{R^{\text{op}}}$, from the Corollary on page 6.10 of [7, II] if $f: A \rightarrow A'$ is a cofibrant resolution of A' , and then in general by the technique of comparing f to the induced map of standard (functorial) cofibrant resolutions.

4.6. The functors Tor_i^R for a simplicial ring R . These functors will be somewhat different from the ones denoted by Tor_i^R in [7, II, §6]. Let $A \in \mathbf{M}_{R^{\text{op}}}$ and $B \in \mathbf{M}_R$ and let $j: B_{\#} \rightarrow B$ be a cofibrant resolution (4.3) of B . Then the abelian groups $\pi_i(A \otimes_R B_{\#})$ do not depend on the choice of this cofibrant resolution and will be denoted by $\text{Tor}_i^R(A, B)$ (instead of by $\pi_i(A \otimes_R^L B)$ as in [7, II, §6]).

Similarly, for maps $a: A \rightarrow A' \in \mathbf{M}_{R^{\text{op}}}$ and $b: B \rightarrow B' \in \mathbf{M}_R$, cofibrant resolutions $j: B_{\#} \rightarrow B$ and $j': B'_{\#} \rightarrow B'$ and a map $b_{\#}: B_{\#} \rightarrow B'_{\#} \in \mathbf{M}_R$ such that $j'b_{\#} = bj$ (which always exists), the resulting maps

$$\pi_i(a \otimes_R b_{\#}): \pi_i(A \otimes_R B_{\#}) = \text{Tor}_i^R(A, B) \rightarrow \pi_i(A' \otimes_R B'_{\#}) = \text{Tor}_i^R(A', B')$$

($i > 0$) do not depend on the choices of j, j' and $b_{\#}$ and will be denoted by $\text{Tor}_i^R(a, b)$.

The functor Tor_i^R so defined are clearly functors from $\mathbf{M}_{R^{\text{op}}} \times \mathbf{M}$ to (abelian groups) and one readily verifies

4.7. Proposition. *Let $* \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow *$ and $* \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow *$ be short exact sequences in $\mathbf{M}_{R^{\text{op}}}$ and $\mathbf{M}_R$ respectively. Then there are natural long exact sequences*

$$\begin{aligned} \cdots \rightarrow \text{Tor}_i^R(A', B) &\rightarrow \text{Tor}_i^R(A, B) \rightarrow \text{Tor}_i^R(A'', B) \\ &\rightarrow \text{Tor}_{i-1}^R(A', B) \rightarrow \cdots \rightarrow \text{Tor}_0^R(A'', B), \\ \cdots \rightarrow \text{Tor}_i^R(A, B') &\rightarrow \text{Tor}_i^R(A, B) \rightarrow \text{Tor}_i^R(A, B'') \\ &\rightarrow \text{Tor}_{i-1}^R(A, B') \rightarrow \cdots \rightarrow \text{Tor}_0^R(A, B''). \end{aligned}$$

To get a further hold on the groups $\text{Tor}_i^R(A, B)$, one notes [7, II, §6] that the multiplication on R turns the (≥ 0) -graded abelian group $\pi_* R$ into a ring, that the R -module structures on A and B turn the (≥ 0) -graded abelian groups $\pi_* A$ and $\pi_* B$ into right and left $\pi_* R$ -modules respectively and that the resulting groups $\text{Tor}_i^{\pi_* R}(\pi_* A, \pi_* B)$ are related to the groups $\text{Tor}_i^R(A, B)$ by [7, II, §6].

4.8. A Künneth spectral sequence. *There is a natural first quadrant spectral sequence with*

$$E_{p,q}^2 = (\text{Tor}_p^{\pi_* R}(\pi_* A, \pi_* B))_q \Rightarrow \text{Tor}_{p+q}^R(A, B).$$

One also has [7, II, §6]

4.9. A partial Künneth spectral sequence. *There is a natural first quadrant spectral sequence with*

$$E_{p,q}^2 = \operatorname{Tor}_p^R(\pi_q A, B) \Rightarrow \operatorname{Tor}_{p+q}^R(A, B)$$

where R acts on each $\pi_q A$ ($q \geq 0$) through the projection $R \rightarrow \pi_0 R$.

5. HOMOLOGY OF Π -ALGEBRAS

In this section we

(i) define, for a Π -algebra X and a right module M over the enveloping ring EX of X , homology groups $H_i(X; M)$ of X with coefficients in M , in such a manner that, for aspherical Π -algebras (i.e., groups) and for simply connected rational Π -algebras (which are essentially connected rational Lie algebras), this definition of homology coincides with the usual one, and

(ii) note that a short exact sequence of coefficient modules, as usual, gives rise to a *long exact sequence* of homology groups, while a short exact sequence $* \rightarrow U \rightarrow W \rightarrow V \rightarrow *$ of Π -algebras yields a *Serre spectral sequence* relating the homology groups of U and V to those of W .

We thus start with defining the

5.1. Homology of Π -algebras. Let X be a Π -algebra (2.1), let EX be its enveloping ring (3.1) and let M be a right EX -module. Then the *homology groups* $H_i(X; M)$ of X with coefficients in M are defined by (2.5 and 4.6)

$$H_i(X; M) = \operatorname{Tor}_i^{EF.X}(M, Z), \quad i \geq 0,$$

where the simplicial ring $EF.X$ acts on M through the canonical map $EF.X \rightarrow EX$.

In view of 4.8 this definition implies

5.2. Proposition. *There is a natural first quadrant spectral sequence with*

$$E_{p,q}^2 = (\operatorname{Tor}_p^{\pi_* EF.X}(M, Z))_q \Rightarrow H_{p+q}(X; M).$$

5.3. Corollary. *If X is E -flat (3.8), then there are natural isomorphisms*

$$\operatorname{Tor}_i^{EX}(M, Z) \approx H_i(X; M), \quad i \geq 0.$$

Thus (3.8) for aspherical and for simply connected rational Π -algebras, the above definition of homology reduces to the usual one [5] for groups and for connected graded rational Lie algebras respectively.

Another immediate consequence of Definition 5.1 is (4.7):

5.4. Proposition. *Let $* \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow *$ be a short exact sequence of right EX -modules. Then there is a natural long exact sequence*

$$\begin{aligned} \cdots \rightarrow H_i(X; M') \rightarrow H_i(X; M) \rightarrow H_i(X; M'') \\ \rightarrow H_{i-1}(X; M') \rightarrow \cdots \rightarrow H_0(X; M'') \rightarrow 0. \end{aligned}$$

Less trivial is the existence of

5.5. A Serre spectral sequence. *Let $* \rightarrow U \rightarrow W \rightarrow V \rightarrow *$ be a short exact sequence of Π -algebras. Then there are, for every right EW -module M ,*

(i) *a natural right EV -action on each $H_i(U; M)$ ($i \geq 0$), and*

(ii) a natural first quadrant spectral sequence with

$$E_{p,q}^2 = H_p(V; H_q(U; M)) \Rightarrow H_{p+q}(W; M).$$

Proof. Let $F'U$ be the kernel of the induced map $F.W \rightarrow F.V \in \mathbf{s}\Pi\text{-al}$ of standard resolutions (2.5). The arguments of 2.3 then readily imply that $F'U$ is a free simplicial Π -algebra. It follows (2.4) that the inclusion $F.U \rightarrow F'U \in \mathbf{s}\Pi\text{-al}$ is a homotopy equivalence and therefore [4] induces an isomorphism $\pi_* EF.U \approx \pi_* EF'U$. The desired result now follows readily from 3.4, 3.7, 4.8, and

5.6. **Lemma.** Let $R \rightarrow S \rightarrow T$ be a sequence of simplicial rings (1.3) such that

- (a) T is a free simplicial Z -module,
- (b) S is free as a right simplicial R -module, and
- (c) the composition $R \rightarrow T$ factors through the augmentation $R \rightarrow Z$ and the induced map $S \otimes_R Z \rightarrow T$ is an isomorphism of left simplicial S -modules.

Then there are, for every right simplicial S -module M ,

- (i) a natural right action of $\pi_0 T$ (and hence T) on each $\mathrm{Tor}_i^R(M, Z)$ ($i \geq 0$),
- (ii) a natural first quadrant spectral sequence with

$$E_{p,q}^2 = \mathrm{Tor}_p^T(\mathrm{Tor}_q^R(M, Z), Z) \Rightarrow \mathrm{Tor}_{p+q}^S(M, Z).$$

Proof. In view of (a) the canonical map (4.3) $(S \otimes_Z T^{\mathrm{op}})_\# T \rightarrow T$ is a cofibrant resolution of T over S and hence (4.8) the induced map $(S \otimes_Z T^{\mathrm{op}})_\# T \otimes_T T_\# Z \rightarrow Z$ is a cofibrant resolution of Z over S , so that, for all $i \geq 0$, $\pi_i(M \otimes_S (S \otimes_Z T^{\mathrm{op}})_\# T \otimes_T T_\# Z) \approx \mathrm{Tor}_i^S(M, Z)$. Furthermore (b), (c), and §4.3 imply that the composition

$$S \otimes_R R_\# Z = R_\# Z \otimes_{R^{\mathrm{op}}} S \rightarrow Z \otimes_{R^{\mathrm{op}}} S = S \otimes_R Z \approx T$$

is also a cofibrant resolution of T over S . It follows (4.2) that there are canonical isomorphisms

$$\begin{aligned} \mathrm{Tor}_i^R(M, Z) &= \pi_i(M \otimes_R R_\# Z) \approx \pi_i(M \otimes_S S \otimes_R R_\# Z) \\ &\approx \pi_i(M \otimes_S (S \otimes_Z T^{\mathrm{op}})_\# T) \end{aligned}$$

($i \geq 0$), which yield a natural right $\pi_0 T$ -action on each $\mathrm{Tor}_i^R(M, Z)$ and the lemma now becomes an immediate consequence of §4.9.

6. THE FUNCTORS Ext_R^i FOR A SIMPLICIAL RING R

In preparation for the definition (in §7) of the cohomology of a Π -algebra, we now “dualize” the results of §4. We start with defining

6.1. **Cosimplicial modules over a simplicial ring.** Given a simplicial ring R (1.3), a left cosimplicial R -module will consist of a cosimplicial abelian group C [1, Chapter X], together with a map of cosimplicial abelian groups $C \rightarrow \mathrm{hom}(R, C)$ (where $\mathrm{hom}(R, C)$ denotes the cosimplicial abelian group with $\mathrm{hom}(R, C)^i = \mathrm{hom}(R_i, C^i)$ for all $i \geq 0$ and with the obvious cosimplicial operators) which turns each C^i ($i \geq 0$) into a left R_i -module. This is equivalent to requiring that each C^i ($i \geq 0$) comes with a left R -module structure

such that, for every pair of integers (h, i) with $0 \leq h \leq i$, the following diagrams are commutative:

$$\begin{array}{ccccc}
 & & R_{i-1} \otimes C^{i-1} & \rightarrow & C^{i-1} \\
 & \nearrow d_h & & & \downarrow d^h \\
 R_i \otimes C^{i-1} & & & & \\
 & \searrow d^h & & & \\
 & & R_i \otimes C^i & \rightarrow & C^i \\
 \\
 & & R_{i+1} \otimes C^{i+1} & \rightarrow & C^{i+1} \\
 & \nearrow s_h & & & \downarrow s^h \\
 R_i \otimes C^{i+1} & & & & \\
 & \searrow s^h & & & \\
 & & R_i \otimes C^i & \rightarrow & C^i
 \end{array}$$

The category of these left cosimplicial R -module will be denoted by $\mathbf{M}^R$.

6.2. The functor hom_R for a simplicial ring R . Given (4.1 and 6.1) modules $B \in \mathbf{M}_R$ and $C \in \mathbf{M}^R$, the cosimplicial abelian group $\text{hom}_R(B, C)$ (with $\text{hom}_R(B, C)^i = \text{hom}_R(B_i, C^i)$ for all $i \geq 0$ and with the obvious cosimplicial operators) has homotopy meaning when B is cofibrant (4.2), as a diagonal argument using [4] readily yields

6.3. Proposition. *Let $C \in \mathbf{M}^R$ and let $f: B \rightarrow B' \rightarrow \mathbf{M}_R$ be a weak equivalence between cofibrant objects (4.2). Then the map $\text{hom}_R(f, C): \text{hom}_R(B', C) \rightarrow \text{hom}_R(B, C)$ induces an isomorphism of cohomotopy groups [1, Chapter X] $\pi^i \text{hom}_R(B', C) \approx \pi^i \text{hom}_R(B, C)$ ($i \geq 0$).*

Remark. It is useful to observe (see the remark after 4.5) that if $B \in \mathbf{M}_R$ is a cofibrant object and $f: C \rightarrow C' \in \mathbf{M}^R$ is a weak equivalence (i.e., a map inducing isomorphisms $\pi^* C \cong \pi^* C'$, $i \geq 0$), then the map

$$\text{hom}_R(B, f): \text{hom}_R(B, C) \rightarrow \text{hom}_R(B, C')$$

gives isomorphisms $\pi^i \text{hom}_R(B, C) \simeq \pi^i \text{hom}_R(B, C')$ ($i \geq 0$).

6.4. The functors Ext_R^i for a simplicial ring R . Let $C \in \mathbf{M}^R$ and $B \in \mathbf{M}_R$ and let $j: B_\# \rightarrow B$ be a cofibrant resolution of B (4.3). Then the abelian groups $\pi^i \text{hom}_R(B_\#, C)$ ($i \geq 0$) do not depend on the choice of the cofibrant resolution and will be denoted by $\text{Ext}_R^i(B, C)$. Similarly, for maps $c: C' \rightarrow C \in \mathbf{M}^R$ and $b: B \rightarrow B' \in \mathbf{M}_R$, cofibrant resolutions $j: B_\# \rightarrow B$ and $j': B'_\# \rightarrow B'$ and a map $b_\#: B_\# \rightarrow B'_\# \in \mathbf{M}_R$ such that $j b_\# = b j'$ (which always exists) the resulting maps

$$\begin{aligned}
 \pi^i \text{hom}_R(b_\#, c): \pi^i \text{hom}_R(B'_\#, C') \\
 = \text{Ext}_R^i(B', C') \rightarrow \pi^i \text{hom}_R(B_\#, C) = \text{Ext}_R^i(B, C)
 \end{aligned}$$

($i \geq 0$) do not depend on the choices of j , j' , and $b_\#$ and will be denoted by $\text{Ext}_R^i(b, c)$.

Clearly the functions Ext_R^i so defined are functors from $\mathbf{M}_R^{\text{op}} \times \mathbf{M}^R$ to (abelian groups) and one readily verifies

6.5. Proposition. *Let $* \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow *$ and $* \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow *$ be short exact sequences in $\mathbf{M}_R$ and $\mathbf{M}^R$ respectively. Then there are natural long exact sequences*

$$\begin{aligned} 0 \rightarrow \text{Ext}_R^0(B'', C) \rightarrow \cdots \rightarrow \text{Ext}_R^i(B, C) \rightarrow \text{Ext}_R^i(B, C) \\ \rightarrow \text{Ext}_R^i(B', C) \rightarrow \text{Ext}_R^{i+1}(B'', C) \rightarrow, \\ 0 \rightarrow \text{Ext}_R^0(B, C') \rightarrow \cdots \rightarrow \text{Ext}_R^i(B, C') \rightarrow \text{Ext}_R^i(B, C) \\ \rightarrow \text{Ext}_R^i(B, C'') \rightarrow \text{Ext}_R^{i+1}(B, C') \rightarrow. \end{aligned}$$

To get a hold on the groups $\text{Ext}_R^i(B, C)$ we need

6.6. A natural action of $\pi_* R$ on $\pi^* C$. Given a module $C \in \mathbf{M}^R$, a lengthy but essentially straightforward calculation yields that the maps $R_p \otimes C^n \rightarrow C^{n-p}$ ($0 \leq p \leq n$) which send an element $x \otimes y \in R_p \otimes C^n$ to the element

$$\sum_{(\mu, \nu)} \varepsilon(\mu, \nu) s^\mu(s_\nu x) y \in C^{n-p}$$

(where (μ, ν) runs over all $(p, n-p)$ shuffles, i.e., permutations $(\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_{n-p})$ of $(0, \dots, n-1)$ such that $\mu_1 < \dots < \mu_p$ and $\nu_1 < \dots < \nu_{n-p}$, where $\varepsilon(\mu, \nu)$ denotes the sign of the permutation and where $s^\mu = s^{\mu_1} \dots s^{\mu_p}$ and $s_\nu = s_{\nu_{n-p}} \dots s_{\nu_1}$), gives rise to a *left action of $\pi_* R$ on the (≤ 0) -graded abelian group $\pi^* C$* (i.e., we put $(\text{degree } \pi^* C) = -i$). The resulting groups $\text{Ext}_{\pi_* R}^i(\pi_* B, \pi^* C)$ are then related to the groups $\text{Ext}_R^i(B, C)$ by

6.7. Künneth spectral sequence. *There is a natural third quadrant spectral sequence with*

$$E_2^{p,q} = (\text{Ext}_{\pi_* R}^p(\pi_* B, \pi^* C))_{-q} \Rightarrow \text{Ext}_R^{p+q}(B, C).$$

One also has

6.8. A partial Künneth spectral sequence. *There is a natural third quadrant spectral sequence with*

$$E_2^{p,q} = \text{Ext}_R^p(B, \pi^q C) \Rightarrow \text{Ext}_R^{p+q}(B, C)$$

where R acts on each $\pi^q C$ ($q \geq 0$) through the projection $R \rightarrow \pi_0 R$.

It remains to give

6.9. Proofs of 6.7 and 6.8. The proofs of 6.7 and 6.8 are similar to those of 4.8 and 4.9 [7, II, §6].

To prove 6.7, construct (as in [7, II, §6] or [9, §2]) a “simplicial left simplicial R -module” $V.B$ and a map $V.B \rightarrow B$ to the (discrete) simplicial left simplicial R -module B such that

(i) each left simplicial R -module $V_i B$ ($i \geq 0$) is a free left simplicial R -module such that $\pi_* V_i B$ is a free left $\pi_* R$ -module,

(ii) the induced map $\text{diag } V.B \rightarrow B \in \mathbf{M}_R$ is a cofibrant resolution of B , and

(iii) the induced map $\pi_* V.B \rightarrow \pi_* B$ is a cofibrant (and in fact free) simplicial resolution of $\pi_* B$ over $\pi_* R$.

As $\text{hom}_R(\text{diag } V.B, C) \approx \text{diag } \text{hom}_R(V.B, C)$ and $\pi^* \text{hom}_R(V_i B, C) \approx \text{hom}_{\pi_* R}(\pi_* V_i B, \pi^* C)$ for all $i \geq 0$, the desired spectral sequence now is one of the spectral sequences of the bicosimplicial abelian group $\text{hom}_R(V.B, C)$.

To prove 6.8 one constructs a Postnikov filtration on C , i.e., a filtration

$$0 = P_{-1}C \subset P_0C \subset \cdots \subset P_iC \subset \cdots \subset \bigcup_{n=1}^{\infty} P_nC = C$$

such that $\pi^i(P_iC/P_{i-1}C) = \pi^iC$ for all $i \geq 0$ and $\pi^j(P_iC/P_{i-1}C) = 0$ for $j \neq i$. This can be done, for instance, by constructing a corresponding filtration on the normalization NC of C [2, §3] and then denormalizing. The rest of the proof of 6.8 now is straightforward.

7. COHOMOLOGY OF Π -ALGEBRAS

“Dualizing” the results of §5 we

(i) define, for a Π -algebra X and a left module N over the enveloping ring EX of X , *cohomology groups* $H^i(X; N)$ of X with coefficients in N , in such a manner that for aspherical Π -algebras (i.e., groups) and simply connected rational Π -algebras (which are essentially connected graded rational Lie algebras) this definition of cohomology coincides with the usual one, and

(ii) note that a short exact sequence of coefficient modules, as usual, gives rise to a *long exact sequence* of cohomology groups, while a short exact sequence $* \rightarrow U \rightarrow W \rightarrow V \rightarrow *$ of Π -algebras yields a *Serre spectral sequence* relating the cohomology groups of U and V to those of W .

We thus start with defining the

7.1. Cohomology of Π -algebras. Let X be a Π -algebra (2.1), let EX be its enveloping ring (3.1) and let N be a left EX -module. Then the *cohomology groups* $H^i(X; N)$ of X with coefficients in N are defined by (2.5 and 6.4)

$$H^i(X; N) = \text{Ext}_{EF.X}^i(Z, N)$$

where the simplicial ring $EF.X$ acts on N through the canonical map $EF.X \rightarrow EX$.

In view of 6.7 this definition implies

7.2. Proposition. *There is a natural third quadrant spectral sequence with*

$$E_2^{p,q} = (\text{Ext}_{\pi.EF.X}^p(Z, N))_{-q} \Rightarrow H^{p+q}(X; N).$$

7.3. Corollary. *If X is E -flat (3.8), then there are natural isomorphisms*

$$\text{Ext}_{EX}^i(Z, N) \approx H^i(X; N), \quad i \geq 0.$$

Thus (3.8) for aspherical and for simply connected rational Π -algebras, the above definition of cohomology reduces to the usual ones [5] for groups and for connected graded rational Lie algebras respectively.

Another immediate consequence of Definition 7.1 is (6.5)

7.4. Proposition. *Let $* \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow *$ be a short exact sequence of left EX -modules. Then there is a natural long exact sequence*

$$\begin{aligned} 0 \rightarrow H^0(X; N') \rightarrow \cdots \rightarrow H^i(X; N') \rightarrow H^i(X; N) \\ \rightarrow H^i(X; N'') \rightarrow H^{i+1}(X; N') \rightarrow \cdots \end{aligned}$$

Less trivial is again the proof of the existence of

7.5. A Serre spectral sequence. Let $* \rightarrow U \rightarrow W \rightarrow V \rightarrow *$ be a short exact sequence of Π -algebras. Then there are, for every left EW -module N ,

- (i) a natural left EV -action on each $H^i(U; N)$ ($i \geq 0$),
- (ii) a natural third quadrant spectral sequence with

$$E_2^{p,q} = H^p(V; H^q(U; N)) \Rightarrow H^{p+q}(W; N).$$

Proof. This is essentially the same as the proof of 5.5, using (instead of 4.8 and 5.6) 6.7 and

7.6. Lemma. Let $R \rightarrow S \rightarrow T$ be a sequence of simplicial rings such that 5.6(a)–(c) hold. Then there are, for every left cosimplicial S -module N ,

- (i) a natural left action of $\pi_0 T$ (and hence T) on each $\text{Ext}_R^i(Z, N)$ ($i \geq 0$),
- (ii) a natural third quadrant spectral sequence with

$$E_2^{p,q} = \text{Ext}_T^p(Z, \text{Ext}_R^q(Z, N)) \Rightarrow \text{Ext}_S^{p+q}(Z, N).$$

Proof. By the arguments of 5.6 one has

$$\pi^{p+q} \text{hom}_S((S \otimes_Z T^{\text{op}})_\# T \otimes_T T_\# Z, N) \cong \text{Ext}_S^{p+q}(Z, N)$$

and

$$\begin{aligned} \pi^q \text{hom}_S((S \otimes_Z T^{\text{op}})_\# T, N) &\cong \text{Ext}_S^q(T, N) \\ &\cong \pi^q \text{hom}_S(S \otimes_S R_\# Z, N) \cong \text{Ext}_R^q(Z, N). \end{aligned}$$

The result follows from 6.8 with T in place of R , $T_\# Z$ in place of R , and $\text{hom}_S((S \otimes_Z T^{\text{op}})_\# T, N)$ in place of C .

8. QUILLEN HOMOLOGY AND COHOMOLOGY

We end with a Quillen-like approach to the homology and cohomology of Π -algebras and note that (except in the bottom dimension) the resulting groups differ from the ones of §5 and §7 by only a shift in dimension. To do all this we need the notion of

8.1. Strong abelianization over a Π -algebra. Given a Π -algebra X , let $(\Pi\text{-al}/X)$ denote its over category (which has as objects the maps $Y \rightarrow X \in \Pi\text{-al}$ and as maps the obvious commutative triangles) and let $(\Pi\text{-al}/X)_{\text{sab}}$ denote the category of the *abelian group objects* in $(\Pi\text{-al}/X)$ which have a *strongly abelian* (2.2) *kernel*. In other words, the objects of $(\Pi\text{-al}/X)_{\text{sab}}$ are the diagrams

$$* \rightarrow B \xrightarrow{i} Y' \xrightleftharpoons[j]{p} X \rightarrow *$$

in $\Pi\text{-al}$ in which

- (i) B is strongly abelian,
- (ii) the right-pointing arrows form an exact sequence,
- (iii) $pj = \text{id}: X \rightarrow X$.

The *strong abelianization functor*

$$\text{sab}: (\Pi\text{-al}/X) \rightarrow (\Pi\text{-al}/X)_{\text{sab}}$$

then is defined as the left adjoint of the forgetful functor $(\Pi\text{-al}/X)_{\text{sab}} \rightarrow (\Pi\text{-al}/X)$.

To “compute”, for an object $(Y \rightarrow X) \in (\Pi\text{-al}/X)$, the image $\text{sab}(Y \rightarrow X) \in (\Pi\text{-al}/X)_{\text{sab}}$, we need

8.2. Proposition. (i) *Given an object of $(\Pi\text{-al}/X)_{\text{sab}}$, i.e., a diagram*

$$* \rightarrow B \xrightarrow{i} Y' \begin{matrix} \xleftarrow{p} \\ \xleftarrow{j} \end{matrix} X \rightarrow *$$

in $\Pi\text{-al}$ as in 8.1, let $B^{\#}$ denote the (≥ 0) -graded abelian group obtained from B by lowering the degrees by 1, and, for every element $b \in B$, let the same symbol b denote the corresponding element of $B^{\#}$. Then there is a unique left EX -module structure on $B^{\#}$ such that, if we identify the elements $x \in X$ and $b \in B$ with their images in Y' under i and j respectively,

$$\begin{aligned} (ex)b &= xbx^{-1}, & x \in X_1, b \in \mathbf{B}_1, \\ &= \tau_x b, & x \in \mathbf{X}_1, b \in \tilde{B}, \\ &= x - \tau_b x, & x \in \tilde{\mathbf{X}}, b \in \mathbf{B}_1, \\ &= [x, b], & x \in \tilde{X}, b \in \tilde{B}. \end{aligned}$$

(ii) *Moreover, the resulting functor*

$$\Phi: (\Pi\text{-al}/X)_{\text{sab}} \rightarrow ((\geq 0)\text{-graded left } EX\text{-modules})$$

is an equivalence of categories.

Now we can state

8.3. Proposition. *Let $(Y \rightarrow X) \in (\Pi\text{-al}/x)$. Then*

$$\Phi(\text{sab}(Y \rightarrow X)) = EX \otimes_{EY} IY,$$

where $IY \subset EY$ denotes the augmentation ideal. In particular

$$\Phi(\text{sab}(\text{id}: X \rightarrow X)) = IX.$$

As the proofs of these two propositions are rather long, we will postpone them until §9, and proceed here with defining

8.4. The Quillen homology of a Π -algebra. Given a Π -algebra X and a right EX -module M , one can consider the composite functor $H^Q(-; M)$

$$(\Pi\text{-al}/X) \xrightarrow{\text{sab}} (\Pi\text{-al}/X)_{\text{sab}} \xrightarrow{\Phi} \mathbf{M}_{EX^{\text{op}}} \xrightarrow{M \otimes_{EX} -} (\text{abelian groups})$$

which (8.3) send an object $(Y \rightarrow X) \in (\Pi\text{-al}/X)$ to the abelian group $M \otimes_{EX} EX \otimes_{EY} IY = M \otimes_{EY} IY$. The *Quillen homology groups of X with coefficients in M* then will be defined as the left derived functors of the functor $H^Q(-; M)$, applied to the identity map of X , i.e. (2.4, 2.5, and [8]),

$$H_i^Q(X; M) = \pi_i(M \otimes_{EF, X} IF.X)$$

or equivalently (3.6 and 4.6)

$$H_i^Q(X; M) = \text{Tor}_i^{EF, X}(M, IF.X).$$

Moreover, combining this with 4.7, 5.1, and the short exact sequence $* \rightarrow IF.X \rightarrow EF.X \rightarrow Z \rightarrow *$ one gets

8.5. **Proposition.** *There are natural isomorphisms*

$$H_i^Q(X; M) \approx H_{i+1}(X; M), \quad i > 0,$$

and a natural exact sequence

$$0 \rightarrow H_1(X; M) \rightarrow H_0^Q(X; M) \rightarrow M \rightarrow H_0(X; M) \rightarrow 0.$$

“Dualizing” all this one obtains

8.6. **The Quillen cohomology of a Π -algebra.** Given a Π -algebra X and a left EX -module N , one can consider the composite functor $H_Q(-; N)$

$$(\Pi\text{-al}/X) \xrightarrow{\text{sab}} (\Pi\text{-al}/X)_{\text{sab}} \xrightarrow{\Phi} \mathbf{M}_{EX} \xrightarrow{\text{hom}_{EX}(-, N)} (\text{abelian groups})$$

which (8.3) sends an object $(Y \rightarrow X) \in (\Pi\text{-al}/X)$ to the abelian group

$$\text{hom}_{EX}(EX \otimes_{EY} IY, N) = \text{hom}_{EY}(IY, N).$$

The *Quillen cohomology groups* of X with coefficients in N then will be defined as the left derived functors of the functor $H_Q(-; N)$, applied to the identity map of X , i.e. (2.4, 2.5, and [8]),

$$H_Q^i(X; N) = \pi^i \text{hom}_{EF, X}(IF.X, N)$$

or equivalently (3.6 and 6.4)

$$H_Q^i(X; N) = \text{Ext}_{EF, X}^i(IF.X, N).$$

Moreover combining this with 6.5, 7.1, and the short exact sequence $* \rightarrow IF.X \rightarrow EF.X \rightarrow Z \rightarrow *$ one gets

8.7. **Proposition.** *There are natural isomorphisms*

$$H_Q^i(X; N) \approx H^{i+1}(X; N), \quad i > 0,$$

and a natural exact sequence

$$0 \rightarrow H^0(X; N) \rightarrow N \rightarrow H_Q^0(X; N) \rightarrow H^1(X; N) \rightarrow 0.$$

It thus remains to give

9. PROOFS OF PROPOSITIONS 8.2 AND 8.3

We start with a

9.1. **Proof of 8.2(i).** As EX is generated by the ex ($x \in X$), there is at most one such left EX -module structure on $B^\#$ and one thus has to show only that the formulas of 8.2 indeed determine such a structure. Most of this is straightforward. The only nontrivial part is to show that, for $b \in B_q$, $x \in X_p$, and $\alpha \in \pi_r S^p$ ($r > p > 1$)

$$\begin{aligned} e(x \circ \alpha)b &= H(\alpha)(ex)^2b, & p \text{ even and } r = 2p - 1, \\ &= 0, & \text{otherwise.} \end{aligned}$$

If $q > 1$, one notes [3, 1.3] that, unless p is even and $r = 2p - 1$, the element (2.3). $[\alpha, i_q] \in \pi_{q+r-1}(S^p \vee S^q)$ has finite order and hence [3, 3.4] is a finite sum of (i_p, i_q) -decomposable (i.e., elements which are Whitehead products in i_p and i_q , followed by a composition). Naturality then yields that $e(x \circ \alpha)b = 0$.

If p is even and $r = 2p - 1$, then similarly $[\alpha, i_q] = H(\alpha)[i_p, [i_p, i_q]]$ modulo (i_p, i_q) -decomposable, which implies that $e(x \circ \alpha)b = H(\alpha)(ex)^2b$.

If $q = 1$, one considers the element $(\alpha - \tau\alpha) \in \pi_r(S^p \vee S^1)$, where $\tau = \tau_{i_1}$. Again, unless p is even and $r = 2p - 1$, this element has finite order and hence is a finite sum of $(i_p, \tau i_p)$ -decomposables, which implies that $e(x \circ \alpha)b = 0$. If p is even and $r = 2p - 1$, then [10, Chapter XI]

$$(\alpha - \tau\alpha) + H(\alpha)[i_p - \tau i_p, i_p - \tau i_p] = (i_p - \tau i_p) \circ \alpha + H(\alpha)[i_p, i_p - \tau i_p]$$

and from this one readily deduces that $e(x \circ \alpha)b = H(\alpha)(ex)^2b$.

9.2. Proof of 8.2(ii). The main problem here is to construct a potential inverse for the functor Φ . To do this we recall from [3, §2] that a Π -algebra can be considered as a contravariant functor from the category of finitely generated free Π -algebras to the category of pointed sets, which sends coproducts to products. Given a (≥ 0) -graded left EX -module A , we then consider the contravariant functor which sends a finitely generated free Π -algebra V to the (pointed) set of pairs (f, g) , where f is a map $f: V \rightarrow X \in \Pi\text{-al}$ and g is a map of left EV -modules $g: IV \rightarrow (Ef)^*A$. In view of 3.6 this functor sends coproducts to products and hence determines a Π -algebra, say Y' , and one readily verifies that there are obvious maps $p: Y' \rightarrow X$ and $j: X \rightarrow Y' \in \Pi\text{-al}$ such that $pj = \text{id}$ and such that the kernel of p is the strongly abelian Π -algebra, obtained from A by raising the degrees by 1. It remains to show that the resulting functor to $(\Pi\text{-al}/X)_{\text{sab}}$ is actually a two-sided inverse of Φ , but that is straightforward.

We end, as promised, with a

9.3. Proof of 8.3. Let

$$* \rightarrow B \xrightarrow{i} Y' \xrightleftharpoons[j]{p} X \rightarrow *$$

and $B^\#$ be as in 8.2. Given an object $f: Y \rightarrow X \in (\Pi\text{-al}/X)$ and a map $k: EX \otimes_{EY} IY \rightarrow B^\#$ of left EX -modules, let $tk: Y \rightarrow Y'$ be the function defined (in the notation of 8.2) by

$$\begin{aligned} (tk)y &= k(1 \otimes (ey - 1))fy, & y \in Y_1, \\ &= k(1 \otimes ey) + fy, & y \in \tilde{Y}. \end{aligned}$$

Then a straightforward calculation using [10, XI, 8.5] shows that tk is actually a map in $\Pi\text{-al}$ such that $p(tk) = f$. Moreover the function t so defined is, in view of 3.6, a natural 1-1 correspondence between the left EX -module maps $EX \otimes_{EY} IY \rightarrow B^\#$ and the maps $f \rightarrow p \in (\Pi\text{-al}/X)$, whenever Y is a free Π -algebra, and a direct limit argument readily implies that this restriction on Y is superfluous. Thus the functor which send an object $f: Y \rightarrow X \in (\Pi\text{-al}/X)$ to the left EX -module $EX \otimes_{EY} IY$ is left adjoint to the composition of Φ^{-1} and the forgetful functor $(\Pi\text{-al}/X)_{\text{sab}} \rightarrow (\Pi\text{-al}/X)$ and hence canonically naturally equivalent to the functor $\Phi(\text{sab})$.

REFERENCES

1. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localization*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin and New York, 1972.
2. W. G. Dwyer and D. M. Kan, *Normalizing the cyclic modules of Connes*, Comment. Math. Helv. **60** (1985), 582-600.

3. ———, *The enveloping ring of a Π -algebra*, Advances in Homotopy Theory, London Math. Soc. Lecture Note Ser., vol. 139, Cambridge Univ. Press, Cambridge, 1989, pp. 49–60.
4. D. M. Kan, *On the homotopy relation for c.s.s. maps*, Bul. Soc. Mat. Mexicana 2 (1957), 75–81.
5. S. Mac Lane, *Homology*, Springer, New York, 1963.
6. J. W. Milnor, *On the construction FK* , London Math. Soc. Lecture Note Ser., vol. 4, Cambridge Univ. Press, Cambridge, 1972, pp. 119–136.
7. D. G. Quillen, *Homotopical algebra*, Lecture Notes in Math., vol. 43, Springer-Verlag, Berlin and New York, 1967.
8. ———, *On the (co-) homology of commutative rings*, Proc. Sympos. Pure Math., vol. 17, Amer. Math. Soc., Providence, RI, 1970, pp. 65–87.
9. C. R. Stover, *A van Kampen spectral sequence for higher homotopy groups*, Topology (to appear).
10. G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math., vol. 61, Springer-Verlag, Berlin and New York, 1978.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556
E-mail address: dwyer.1@nd.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE,
MASSACHUSETTS 02139