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THE CAUCHY PROBLEM IN $\mathbb{C}^N$ FOR LINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH DATA ON A QUADRIC SURFACE

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ABSTRACT. By means of a method developed essentially by Leray some global existence results are obtained for the problem referred to in the title. The partial differential equations are required to have constant principal part and the initial surface to be irreducible and not everywhere characteristic. The Cauchy data are assumed to be given by entire functions. Under these conditions the location of all possible singularities of solutions are determined. The sets of singularities can be divided into two types, K- and L-singularities. K, the set of K-singularities, is the global version of the characteristic tangent defined by Leray. The L-sets are here quadric surfaces which, in contrast to the K-sets, allow unbounded singularities. The L-sets are in turn divided into three types: initial, asymptotic, and latent singularities. The initial singularities appear when the characteristic points of the initial surface are exceptional according to Leray's local theory. These sets of singularity intersect the initial surface at characteristic points. The asymptotic case, where the set of singularities does not cut the initial surface, can be viewed as projectively equivalent to the initial case, the intersection taking place at infinite characteristic points. Finally the latent singularities are sets which intersect the initial surface, but where the solutions do not develop singularities initially. In the case of the Laplace equation with data on a real quadric surface it is shown that the K-singularities and the asymptotic singularities occur on the classical focal sets defined by Poncelet, Plücker, Darboux et al. There are also latent singularities appearing in coordinate subspaces of $\mathbb{R}^N$. As a corollary a new proof is given of the fact that ellipsoids have the Pompeiu property.

1. INTRODUCTION

For holomorphic partial differential equations the local theory of Cauchy problems is well developed. In the noncharacteristic case the classical Cauchy-Kovalevskaya theorem states existence and uniqueness of analytic solutions. In the neighborhood of a characteristic point of the initial surface Γ Leray's theorem [L] asserts in general existence and uniqueness outside the characteristic tangent K. (See §2 for precise definitions and statements.) Globally, if Γ is a hyperplane (in $\mathbb{C}^n$ or $\mathbb{R}^n$), the Cauchy-Kovalevskaya theorem can in certain cases be extended to yield entire solutions if the Cauchy data are entire (cf. results by Persson [P] and Miyake [Mi]).

When Γ is an arbitrary analytic hypersurface, the global problem is more complicated. Recently, however, Sternin and Shatalov have given explicit

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solutions of this Cauchy problem in the constant coefficient case in terms of an integral transformation, R_Γ, suitable for multivalued analytic functions with singularities on prescribed varieties and ramified around the initial surface Γ. (See [S-S 1, 2, 3] for details and further references.) The dependence of a transformed function $R_\Gamma f(p) = \int_{\gamma(p)} f(x)\,dx$ on its variables $p = (p_0, p_1, \dots, p_n) \in \mathbb{CP}^n$ ($\mathbb{CP}^n$ is the projective compactification of the dual complex space $\mathbb{C}^n$) is determined by the path (surface) of integration $\gamma(p)$ (see [S-S 1] for the definition of $\gamma(p)$) and the necessary globalization of the solutions is motivated by a topological lemma ensuring the existence of $\gamma(p)$, where the dependence of γ on p may be multivalued, for almost all $p \in \mathbb{CP}^n$. Since R_Γ commutes with differentiation, the equation $P(D)u = f$ is transformed by R_Γ to an ordinary differential equation in the variable p_0. The explicit solution u of the Cauchy problem $P(D)u = f$, $D_\alpha u = 0$ on Γ for $|\alpha| < m$ (m is the order of $P(D)$), is given by

$$u = R_\Gamma^{-1} \circ Q \circ R_\Gamma(f), \tag{1.1}$$

where the operation Q means solving the above ordinary differential equation.

The singularities of this solution u have been determined in some cases involving second order PDEs and quadric initial surfaces [S-S 2, 3]. (In fact all given examples fall within the scope of the present study.) It is not clear to what extent it is possible to get explicit sets of singularities outside the class of problems given in the examples.

The present study is an attempt to combine the method of Leray (cf. [L, G-K-L, G]) with the global existence theorems of Persson and Miyake in order to get explicit results on the location of the singularities. This approach turns out to be successful in the cases referred to in the title, but there seems to be little hope of extending the method to other cases because of lack of appropriate global existence theorems for these cases. However, given the restrictions to second order equations and quadric surfaces, the present approach is slightly more general than that of Sternin and Shatalov since it allows variable (i.e. entire) coefficients in the nonprincipal part of the equation.

The results given in $\mathbb{C}^N$ can be reinterpreted in $\mathbb{R}^N$. This is done in §5, mainly for the case of the Laplace equation,

$$P(D) = \Delta_N = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2},$$

or for Laplace-like parabolic equations,

$$P(D) = \Delta_n, \qquad n < N.$$

This study in $\mathbb{R}^N$ has connections with classical works on the gravitation potential of ellipsoids, since it can be shown that these potentials, when continued inside the ellipsoid, are identical modulo a holomorphic function to a solution of the Cauchy problem with data (describing the mass density) given on the boundary Γ of the ellipsoid. These classical studies (cf. articles by Laplace, Ivory, Gauss, Dirichlet et al. in [La]) establish that the singularities appear on the focal ellipse related to the given ellipsoid. Recently, H. Shahgholian [Sha] has generalized these results to n dimensions and polynomial Cauchy data. The present study slightly extends this result by allowing entire data.

There is still another classical connection to be pointed out. In [Her] Herglotz studied the gravitation potential in the two-dimensional case by means of analytic functions of one variable. He observed that the singularities appear at the foci, if the boundaries of the mass densities are algebraic curves, and furthermore that the singularities, which in general are bounded, may be unbounded if the foci are extraordinary according to Plücker, Darboux et al. (cf. [P1, Dar]). This observation can be reinterpreted in higher dimensions, as shown in §4, by means of the corresponding distinction between K- and L-singularities.

Finally we would like to mention another source of inspiration for this study. In [Sh 1] and [Kh-Sh 1] the concept of Schwarz potential is introduced. The Schwarz potential of the analytic surface Γ in $\mathbb{C}^n$ is the solution u of the Cauchy problem $\Delta u = 0$ in a neighborhood of Γ with data $u = (\sum_{j=1}^n x_j^2)/2$ and $\operatorname{grad} u = (x_1, x_2, \dots, x_n)$ on Γ. Shapiro and Khavinson conjecture that any solution of the Laplace equation with Cauchy data given on an analytic surface Γ can be analytically continued (in $\mathbb{R}^n$ or $\mathbb{C}^n$) as far as the Schwarz potential of Γ can be.

The investigation of this conjecture with regard to quadric surfaces has been an important impetus for this work, which has not contradicted the conjecture.

This report is organized as follows: In §2 notation is presented together with definitions and information from other sources. A summary of the main notions is included at the end of this section. Section 3 deals with the linear algebraic aspects which are needed to develop the theory. Section 4 contains the main results in $\mathbb{C}^N$ including the main theorem, Theorem 4.3, and the definitions of the various types of the singularities and their properties. Section 5 contains the results in $\mathbb{R}^N$ concerning mostly the Laplace equation. These results include the identification in $\mathbb{R}^N$ between the focal sets and the K-singularities and also a proof that ellipsoids have the Pompeiu property. Section 6 finally contains a discussion of some open problems.

2. Notation and preliminaries

We will work within $\mathbb{C}^N$, $N \geq 2$, and let x and y denote variable vectors in $\mathbb{C}^N$. (However, in §3 we change convention and let $x \in \mathbb{C}^n$, $1 \leq n \leq N$, $(x, x') \in \mathbb{C}^N$.) A general linear PDE is denoted

$$P(x, D)u = f, \tag{2.1}$$

or

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f(x), \tag{2.2}$$

where standard multi-index notation is used, e.g., $D^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}$, $|\alpha| = \sum_{j=1}^N \alpha_j$. $\partial u / \partial x$ denotes the gradient $(\partial u/\partial x_1, \dots, \partial u/\partial x_N)$. The principal part of the operator $P(x, D)$ in (2.2) is

$$P_m(x, D) = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha. \tag{2.3}$$

We often use g instead of P_m to denote the corresponding principal symbol:

$$g(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha, \qquad \xi \in \mathbb{C}^N. \tag{2.4}$$

We will be most concerned with second order equations $(m = 2)$ with constant principal parts:

$$\sum_{j,k=1}^{N} a_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u = f(x), \tag{2.5}$$

where the matrix a_{jk} is constant and symmetric. If a_{jk} has rank n, $1 \leq n \leq N$, equation (2.5) can be transformed by a regular complex linear transformation to the form

$$\Delta_n u + \sum_{j=1}^{N} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u = f(x), \tag{2.6}$$

where we have used

$$\Delta_n u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

We will use the notation

$$\varphi(x) = x^T A x + B^T x + C = 0 \tag{2.7}$$

to define a general quadratic surface, Γ.

We introduce some more notation regarding the quadrics in §3 needed to define a certain normal form.

A point $x^\circ \in \Gamma$ is called *characteristic* with respect to $P(x, D)$ if

$$g(x^\circ, \partial\varphi(x^\circ)/\partial x) = 0. \tag{2.8}$$

Here, $g = P_m$ and $\Gamma = \{x : \varphi(x) = 0\}$.

Note that condition (2.8) means that all nonregular points of Γ (i.e., x° such that $\partial\varphi(x^\circ)/\partial x = 0$) are formally characteristic. The set of characteristic points is denoted Γ_{CH}.

We can now formulate the general Cauchy problem $(*)$ involving the linear equation (2.1) and the initial surface Γ. Suppose that each function occurring in (2.1) is holomorphic in a neighborhood of the analytic surface $\Gamma = \{x : \varphi(x) = 0\}$. We seek the unique analytic solution $u(x)$ satisfying

(2.9*)
(i) u is a solution of (2.1) $P(x, D)u = f$ in a neighborhood of $\Gamma \backslash \Gamma_{\mathrm{CH}}$.
(ii) $D^\alpha(u - w) = 0$ on $\Gamma \backslash \Gamma_{\mathrm{CH}}$ for $|\alpha| \leq m - 1$, where w is a given entire function.

If we in addition assume that (2.1) be of form (2.6) we will refer to the corresponding Cauchy problem as (Q^*).

Note that $(*)$ always has a unique solution in a neighborhood of $\Gamma \backslash \Gamma_{\mathrm{CH}}$ in view of the Cauchy-Kovalevskaya theorem. This study is concerned with the possible global analytic continuation of this local solution of the (Q^*) problem.

We next describe a variant of the Leray transformation method adapted to the present situation. (We follow most closely the presentation in [G].) The first step transforms the problem $(*)$ to a related problem in $\mathbb{C}^{N+1}$ by introducing a new independent variable λ.

Set

$$\Gamma^\lambda = \{x : \varphi(x) = \lambda\} \tag{2.9}$$

and let D_x, $D_{x,\lambda}$, etc., denote differential operators with respect to the indicated variables. Consider the new Cauchy problem $(**)$:

$$(**)\quad \begin{aligned} &\text{(i)}\ \ P(x, D_x)u^*(x, \lambda) = f(x) \text{ in a neighborhood of } \Gamma^\lambda \backslash \Gamma^\lambda_{\mathrm{CH}}. \\ &\text{(ii)}\ \ D^\beta_{x,\lambda}(u^* - w) = 0 \text{ on } \Gamma^\lambda \backslash \Gamma^\lambda_{\mathrm{CH}} \text{ for } |\beta| \le m-1. \end{aligned}$$

The operator P and the functions f and w are identical with those of problem $(*)$. Note that the solution u of $(*)$ can be recovered from u^* by

$$u(x) = u^*(x, 0). \tag{2.10}$$

The second step consists of the introduction of a new variable t, in place of λ, by the transformation $\lambda = \lambda(x, t)$, where $\lambda(x, t)$ is defined to be the solution of the following Hamilton-Jacobi problem:

$$\lambda_t = -g(x, \partial\lambda/\partial x) \tag{2.11}$$

with initial condition

$$\lambda(x, 0) = \varphi(x). \tag{2.12}$$

Hence, $\lambda(x, t)$ is related both to the operator P and the surface Γ. It can be shown by classical Hamilton-Jacobi theory (cf. [G-K-L] or [G]) that the solution $\lambda(x, t)$ is

$$\lambda(x, t) = \lambda[y(x, t), t], \tag{2.13}$$

where

$$\lambda[y, t] = \varphi(y) + t(m-1)g(y, \partial\varphi/\partial y) \tag{2.14}$$

and where

$$y = y(x, t), \qquad y \in \mathbb{C}^N, \tag{2.15}$$

is defined implicitly by the relation

$$x = x(y, t), \tag{2.16}$$

which represents the solution of the Hamilton equations

$$\begin{aligned} &\text{(i)}\ \ dx/dt = \partial g/\partial\xi, \\ &\text{(ii)}\ \ d\xi/dt = -\partial g/\partial x \end{aligned} \tag{2.17}$$

with initial conditions

$$x(0) = y, \qquad \xi(0) = \partial\varphi/\partial y. \tag{2.18}$$

Note that in case (Q^*), $g(x, \xi) = \sum_{j=1}^n \xi_j^2$, and hence $\partial g/\partial x_j = 0$, $\partial g/\partial\xi_j = 2\xi_j$.

Curves of type (2.16) satisfying (2.17) are called *bicharacteristics.* Curves of type (2.16) satisfying both (2.17) and (2.18) are called *principal bicharacteristics* with respect to Γ and are denoted β_{y°, if they are issued from y°, i.e. if $x(y^\circ, 0) = y^\circ$. Note that the observation made above concerning $\partial g/\partial x$ and $\partial g/\partial\xi$ in the (Q^*) case implies that the bicharacteristics are straight lines.

We also observe that in view of (2.12) the surface Γ^λ is transformed to the hyperplane

$$X_t = \{(x, t) : t = 0\} \tag{2.19}$$

by the transformation $\lambda = \lambda(x, t)$.

Now set $U(x, t) = u^*(x, \lambda(x, t))$ and $V(x, t) = \partial u^*(x, \lambda(x, t))/\partial\lambda$.

It turns out that the so defined function V satisfies a certain equation $P_1(x, t, D_{x,t})V = 0$, which is noncharacteristic with respect to the hyperplane X_t (i.e., there is no characteristic point on X_t). The function U will also satisfy an equation $P_2(x, t, D_{x,t})U = f$, which however in general is characteristic unless P is a first order operator.

Hence, we consider the two alternative transformed Cauchy problems $(***)$ and $(***)'$, the second of which is considered only in the case of first order operators.

$$(***)\qquad \begin{aligned} &\text{(i)}\ \ P_1(x, t, D_{x,t})V(x, t) = 0 \text{ in a neighborhood of } X_t. \\ &\text{(ii)}\ \ D_t^p(V - w_1) = 0 \text{ on } X_t \text{ for } p = 0, 1, \dots, m-1. \end{aligned}$$

$$(***)'\qquad \begin{aligned} &\text{(i)}\ \ P_2(x, t, D_{x,t})U(x, t) = f(x) \text{ in a neighborhood of } X_t. \\ &\text{(ii)}\ \ U(x, 0) = w(x). \end{aligned}$$

The new operators P_1 and P_2 and the function w_1 will be determined in the next lemma.

Lemma 2.1. *If the equation $Pu = f$ is of type* (2.5), *the Cauchy problem* $(***)$ *takes the form*: *(set $V_j = \partial V/\partial x_j$ etc.)*

$$\text{(i)}\qquad \begin{aligned} V_{tt} &- \sum_{j,k=1}^{N} a_{jk}(\lambda_t V_{jk} - 2\lambda_j V_{kt} - \lambda_{jk} V_t) \\ &- \sum_{j=1}^{N} b_j(\lambda_t V_j - \lambda_j V_t) - c\lambda_t V = 0 \end{aligned}$$

in a neighborhood of X_t.

(ii) $V(x, 0) = 0$, $V_t(x, 0) = P(x, D_x)w(x) - f(x)$.

Moreover, if $P_2 u = \sum_{j=1}^{N} a_j u_j + cu$, *equation* $(***)'$ (i) *is transformed to*

(iii) $U_t + P_2 U = f$.

Proof. (i) Differentiating the relations $U(x, t) = u^*(x, \lambda(x, t))$ and $V(x, t) = u^*_\lambda(x, \lambda(x, t))$ gives

$$U_j = u^*_j + u^*_\lambda \lambda_j, \tag{2.20}$$

$$U_{jk} = u^*_{jk} + u^*_{j\lambda}\lambda_k + (u^*_{\lambda\lambda}\lambda_k + u^*_{\lambda k})\lambda_j + u^*_\lambda \lambda_{jk}, \tag{2.21}$$

$$V_j = u^*_{\lambda j} + u^*_{\lambda\lambda}\lambda_j. \tag{2.22}$$

Equation (2.20) yields

$$u^*_j = U_j - V\lambda_j. \tag{2.23}$$

Setting $W(x, t) = u^*_{\lambda\lambda}(x, \lambda(x, t))$ and introducing $V = u^*_\lambda$ in (2.21) and (2.22) gives

$$u^*_{\lambda j} = V_j - W\lambda_j \tag{2.24}$$

and

$$u^*_{jk} = U_{jk} - V_j\lambda_k - V_k\lambda_j - V\lambda_{jk} + W\lambda_j\lambda_k. \tag{2.25}$$

Hence,

$$\begin{aligned}\sum_{j,k=1}^{N} a_{jk}u^*_{jk} &+ \sum_{j=1}^{N} b_j u^*_j + cu^* - f \\ &= \sum_{j,k=1}^{N} a_{jk}(U_{jk} - V_j\lambda_k - V_k\lambda_j - V\lambda_{jk} + W\lambda_j\lambda_k) \\ &\quad + \sum_{j=1}^{N} b_j(U_j - V\lambda_j) + cU - f.\end{aligned}$$

Now, using (2.11) in this case, $\sum a_{jk}\lambda_j\lambda_k = -\lambda_t$ and $W\lambda_t = V_t$, we get

$$V_t = \sum_{j,k=1}^{N} a_{jk}(U_{jk} - 2V_j\lambda_k - V\lambda_{jk}) + \sum_{j=1}^{N} b_j(U_j - V\lambda_j) + cU - f. \tag{2.26}$$

Differentiating (2.26) with respect to t and using $U_t = V\lambda_t$ gives

$$V_{tt} = \sum_{j,k=1}^{N} a_{jk}(V_{jk}\lambda_t - 2V_{jt}\lambda_k - V_t\lambda_{jk}) + \sum_{j=1}^{N} b_j(V_j\lambda_t - V_t\lambda_j) + cV\lambda_t \tag{2.27}$$

where we have used

$$\frac{\partial^2(V\lambda_t)}{\partial x_j \partial x_k} = V_{jk}\lambda_t + V_k\lambda_{jt} + V_j\lambda_{kt} + V\lambda_{jkt}.$$

This proves formula (i).

Since $\lambda(x, 0) = \varphi(x)$, $V(x, 0)$ takes the values of $u^*_\lambda(x, \lambda)$ as $(x, \lambda) \in \Gamma_\lambda = \{(x, \lambda) : \lambda = \varphi(x)\}$. But since $u^*(x, \lambda)$ is the solution of Cauchy problem $(**)$ with Cauchy data given by $w(x)$ independent of λ on Γ_λ, we must have $u^*_\lambda(x, \lambda) = 0$ on Γ_λ, and hence $V(x, 0) \equiv 0$. The expression for $V_t(x, 0)$ is obtained from (2.26) by setting $t = 0$, using $V(x, 0) = V_j(x, 0) = 0$ and inserting $w(x)$ for $U(x, 0)$.

Finally, formula (iii) of the lemma follows from (2.20) using (2.11): $\sum_{j=1}^{N} a_j\lambda_j = -\lambda_t$ and $U_t = V\lambda_t$. The proof is complete.

It is clear that a solution $V(x, t)$ of $(***)$ also yields a solution $u(x)$ of $(*)$. First $U(x, t)$ can be determined by

$$U(x, t) = U(x, 0) + \int_0^t \frac{\partial u}{\partial s}(x, s)\, ds$$

and hence,

$$U(x, t) = w(x) + \int_0^t V(x, s)\lambda_t(x, s)\, ds \tag{2.28}$$

as long as the integral is taken along a path inside the domains of analyticity for V and λ. Note that $U_t(x, 0) = 0$ follows from $V(x, 0) = 0$.

Secondly, since $u(x) = u^*(x, 0)$ and $u^*(x, \lambda(x, t)) = U(x, t)$ we have

(2.29) $$u(x) = U(x, t(x)),$$

where $t(x)$ is defined by

(2.30) $$\lambda(x, t) = 0.$$

The following example shows how the transformation from $(*)$ to $(***)$ works in concrete example. We assume that $(*)$ is the Cauchy problem (i) $\Delta_n u = 0$, (ii) $D^\alpha(u - |x|^2/2) = 0$ $(|\alpha| \leq 1)$ on $B_1 = \{x \colon \varphi(x) = |x|^2 - 1 = 0\}$. The solution

$$u(x) = n/2(n-2) - |x|^{2-n}/(n-2)$$

is the Schwarz potential of B_1 [Kh-Sh 1].

$$u^*(x, \lambda) = \frac{n(1+\lambda)}{2(n-2)} - \frac{(1+\lambda)^{n/2}|x|^{2-n}}{n-2}$$

is the solution of the corresponding problem $(**)$ on $B_1^\lambda = \{x \colon |x|^2 - 1 = \lambda\}$.

Hence,

$$u_\lambda^* = \frac{n}{2(n-2)}(1 - |x|^{2-n}(1+\lambda)^{n/2-1}).$$

The λ-function for the present problem is $\lambda = |x|^2/(1+4t) - 1$. Therefore

$$V(x, t) = u_\lambda^*(x, \lambda(x, t)) = \frac{n}{2(n-2)}\left(1 - \frac{1}{(1+4t)^{(n-2)/2}}\right).$$

This can be seen to satisfy equation $(***)$(i), with Cauchy data

$$V(x, 0) = 0, \quad V_t(x, 0) = \Delta|x|^2/2 = n.$$

Finally

$$\begin{aligned} U(x, t) &= \frac{|x|^2}{2} + \int_0^t V(x, s)\lambda_t(x, s)\, ds \\ &= \frac{|x|^2}{2(n-2)}\left(\frac{n}{1+4t} - \frac{2}{(1+4t)^{n/2}}\right) \end{aligned}$$

and, as required

$$u(x) = U(x, t(x)) = n/2(n-2) - |x|^{2-n}/(n-2),$$

since $1 + 4t(x) = |x|^2$, which follows from $\lambda(x, t) = 0$.

We next collect some facts from Hamilton-Jacobi theory. First we make two definitions.

(2.31) $$Z_\lambda = \{x \colon \lambda(x, t) = \lambda_t(x, t) = 0 \text{ for some } t\},$$

(2.32) $$K = \bigcup_{y \in \Gamma_{\mathrm{CH}}} \beta_y.$$

K is the characteristic tangent defined in [L]. The set Z_λ is obviously the set of algebraic singularities of the function $x \mapsto t(x)$ defined by $\lambda(x, t) = 0$, as long as $\lambda(x, t(x))$ is defined.

The next lemma relates Z_λ to the characteristic tangent K.

Lemma 2.2. $Z_\lambda \subset K$.

Proof. We use the following basic facts:

(a) $g(x, \partial\lambda(x, t)/\partial x)$ is constant along bicharacteristics.

(b) $\partial\lambda(x^\circ, 0)/\partial x = \partial\varphi(x^\circ)/\partial y$.

Both (a) and (b) can be deduced from the fact that λ can be constructed as a function with derivatives $\lambda_t = -g(x, \xi(x, t))$ and $\partial\lambda/\partial x = \xi(x, t)$, where $\xi(x, t) = \xi[y(x, t), t], \xi[(y, t)]$ is the solution of (2.17)–(2.18) and $y(x, t)$ defined by (2.15)–(2.16). Hence,

$$\frac{d}{dt} g\left(x, \frac{\partial\lambda}{\partial x}(x, t)\right) = \frac{d}{dt} g(x, \xi(x, t)) = g_x x_t + g_\xi \xi_t = g_x g_\xi - g_\xi g_x = 0$$

on bicharacteristics, since (2.17) holds there. This proves (a).

As for (b) we have $\partial\lambda(x^\circ, 0)/\partial x = \xi(x^\circ, 0) = \partial\varphi(x^\circ)/\partial y$ from (2.18).

Now suppose $x^\circ \in Z_\lambda$. Then $\lambda(x^\circ, t^\circ) = 0$ for some t°. Then $y(x^\circ, t^\circ)$ is defined, since $\lambda(x^\circ, t^\circ) = \lambda[y(x^\circ, t^\circ), t^\circ]$. Therefore $x^\circ = x(y^\circ, t^\circ)$, which means that $x^\circ \in \beta_{y_0}$. We must show that $y^\circ \in \Gamma_{\mathrm{CH}}$ which implies $\beta_{y_0} \subset K$.

From (2.11), property (a), and property (b) we have

$$\begin{aligned} 0 = \lambda_t(x^\circ, t^\circ) &= -g\left(x^\circ, \frac{\partial\lambda}{\partial x}(x^\circ, t^\circ)\right) \\ &= -g\left(y^\circ, \frac{\partial\lambda}{\partial x}(y^\circ, 0)\right) = -g\left(y^\circ, \frac{\partial\varphi}{\partial y}(y^\circ)\right). \end{aligned}$$

Moreover, by (2.13)–(2.14),

$$\begin{aligned} \lambda(x^\circ, t^\circ) &= \varphi(y(x^\circ, t^\circ)) + t(m-1) g\left(y(x^\circ, t^\circ), \frac{\partial\varphi}{\partial y}(y(x^\circ, t^\circ))\right) \\ &= \varphi(y^\circ) + t(m-1) g\left(y^\circ, \frac{\partial\varphi}{\partial y}(y^\circ)\right) = 0, \end{aligned}$$

which implies $\varphi(y^\circ) = 0$. Hence,

$$\varphi(y^\circ) = g\left(y^\circ, \frac{\partial\varphi}{\partial y}(y^0)\right) = 0,$$

which means that $y^\circ \in \Gamma_{\mathrm{CH}}$ as required.

We also need

Lemma 2.3. (i) *Z_λ is everywhere characteristic and tangent to Γ along Γ_{CH}.*

(ii) *If x° is an exceptional point of Γ_{CH}, i.e., $\lambda(x^\circ, t) = 0$ in a neighborhood of $t = 0$, then $\beta_{x^\circ} \subset \Gamma_{\mathrm{CH}}$ in a neighborhood of x°.*

This lemma is proved in [G-K-L] using Hamilton-Jacobi theory and we omit the proof here.

Leray's result regarding existence of the solution is stated in the following theorem.

Theorem 2.4 [L, G-K-L]. *Consider the Cauchy problem $(*)$, where Γ is not everywhere characteristic. Assume also that x° is a nonexceptional regular point of Γ. Then $(*)$ has a unique local solution in $\Omega\backslash K$ for some neighborhood Ω of x°.*

The proof follows directly from the transformation method presented above and from the Cauchy-Kovalevskaya theorem. Note that the condition that x°

be nonexceptional implies that the equation $\lambda(x, t) = 0$ defines an algebraic function $x \mapsto t(x)$ in view of Weierstrass' Preparation Theorem.

These results have been extended by Hamada who shows that the same conclusion is valued if $g(x, \xi)$ contains a multiple factor and if x° is nonexceptional in a wider sense than in Leray's theory. (See [Ha] for details.) In this case however the singularities on K may be essential.

The detailed study of the problem (Q^*) is made possible by the fact that the λ-function in this case can be computed explicitly (this is done in §3) and that the transformed problem (Q^{***}) is covered by the following theorem, which claims existence of a global solution.

Theorem 2.5 (Miyake [Mi], Persson [P]). *Consider the following noncharacteristic Cauchy problem* $(t \in \mathbb{C},\ x \in \mathbb{C}^{n-1})$:

(i)

$$\frac{\partial^m u}{\partial t^m} + \sum_{p=0}^{m-1}\left(\sum_{|\alpha|=m-p} a_\alpha(x, t)\frac{\partial^p}{\partial t^p}(D_x^\alpha u)\right) + \sum_{|\beta|\le m-1} b_\beta(x, t) D_{x,t}^\beta u$$
$$= f(x, t).$$

(ii) $\partial^q u(x, 0)/\partial t^q = w_q(x)$, $0 \le q \le m-1$, *where* $w_q(x)$ *are entire and the coefficients* $a_\alpha(x, t)$ *are polynomials in* x *of type*: $a_\alpha(x, t) = \sum_{|\gamma|\le|\alpha|} c_\gamma(t) x^\gamma$, *where* $c_\gamma(t)$ *as well as* $b_\beta(x, t)$ *and* $f(x, t)$ *are analytic in* $\mathbb{C}_x^{n-1} \times \Omega$, $\Omega \subset \mathbb{C}_t$.

This problem has a unique analytic solution $u(x, t)$ *which may be analytically continued along any arc in* $\mathbb{C}_x^{n-1} \times \Omega$.

The equations (i) which satisfy condition (iii) are called Persson equations. Persson [P] proved this theorem for the case where all coefficients are entire. Miyake [Mi] proved a general theorem comprising even Goursat problems. In that theorem the coefficients are allowed to depend only continuously on t. An alternative proof of Theorem 2.5 appears in [J].

Example. The equation $u_{tt} + x^2 u_{xx} + (x+t)u_{xt} + u = 0$ is a Persson equation, whereas $u_{tt} + x^3 u_{xx} + u = 0$ is not.

We also state some classical results concerning singularities, which will be needed later.

Theorem 2.6 (Delassus, le Roux [De]). *Let* u *be a solution of problem* $(*)$ *where all functions appearing in equation* (2.1) $P(x, D)u = f$ *are analytic in some domain* $\Omega \subset \mathbb{C}^N$. *Suppose* u *is analytic in* $\Omega \backslash \Gamma$, *where* Γ *is a regular analytic set. If* u *is not analytically continuable across* $x^\circ \in \Gamma$, *then* Γ *is characteristic with respect to* P *at* x°.

The proof of this theorem follows directly from a theorem of Zerner.

We say that a surface $S = \{x : \Phi(x, \overline{x}) = 0\}$, where Φ is real-valued and hence S of real codimension 1, is *Zerner characteristic* at $x^\circ \in S$ with respect to P, if $P_m(x^\circ, \partial\Phi(x^\circ, \overline{x}^\circ)/\partial x) = 0$. $\overline{x}$ is the complex conjugate of x, i.e., $\overline{x} = (\overline{x}_1, \dots, \overline{x}_N) = (x_1' - ix_1'', \dots, x_N' - ix_N'')$, where x' and $x'' \in \mathbb{R}^N$ are the real and imaginary part respectively of x.

Theorem 2.7 (Zerner [Z], [Hö 2, Theorem 9.4.7]). *Let u be an analytic solution of $P(x, D)u = f$ in the domain $Z = \{x : \Phi(x, \overline{x}) < 0\}$, where all functions appearing in the equation are analytic in some domain Ω. Suppose $x^\circ \in \partial Z$, $x^\circ \in \Omega$ and ∂Z of class C^1 at x°. Then, if ∂Z is not Zerner characteristic at x°, $u(x)$ can be analytically continued across x°.*

Proof of Theorem 2.6. Let Γ be defined by $\varphi(x) = \alpha(x', x'') + i\beta(x', x'') = 0$. Set $S = \{x' + ix'' : \alpha(x', x'') = 0\}$. Hence $\Gamma \subset S$. Then, u is analytic in $\Omega\backslash\Gamma$ implies that u is analytic in $\Omega\backslash S$. Suppose that $x^\circ \in \Gamma$. The Cauchy-Riemann equations yield

$$\begin{aligned}
\frac{\partial \alpha}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial \alpha}{\partial x_j'} - i\frac{\partial \alpha}{\partial x_j''}\right) = \frac{1}{4}\left(2\frac{\partial \alpha}{\partial x_j'} - 2i\frac{\partial \alpha}{\partial x_j''}\right) \\
&= \frac{1}{4}\left(\frac{\partial \alpha}{\partial x_j'} + i\frac{\partial \beta}{\partial x_j'}\right) - \frac{i}{4}\left(\frac{\partial \alpha}{\partial x_j''} + i\frac{\partial \beta}{\partial x_j''}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\frac{\partial \varphi}{\partial x_j'} - \frac{i}{2}\frac{\partial \varphi}{\partial x_j''}\right) = \frac{1}{2}\frac{\partial \varphi}{\partial x_j}.
\end{aligned}$$

Hence, Γ is characteristic if and only if S is Zerner characteristic. The conclusion follows from Theorem 2.7.

The Delassus-le Roux theorem (Theorem 2.6) can be slightly strengthened by means of a classical result by Hartogs [Ha] concerning the sets of singularity of analytic functions, not necessarily single-valued.

We make the same assumptions on the analyticity of the coefficients in the equation $Pu = f$ as before.

Theorem 2.8. *Let u satisfy $P(x, D)u = f$ in $\Omega\backslash\Gamma$, where Γ is a regular C^1 surface of complex codimension 1 in the domain $\Omega \subset \mathbb{C}^N$. If u is analytic in $\Omega\backslash\Gamma$ and not analytically continuable across $x^\circ \in \Gamma$, then*

(i) *Γ is an analytic set in Ω.*
(ii) *u has singularities on all $\Gamma \cap \Omega$.*
(iii) *Γ is characteristic with respect to P in Ω.*

Proof. (i) and (ii) constitute the theorem of Hartogs in the version given in [B-T, Satz 19]. (iii) follows from Theorem 2.6.

We get directly the following corollary.

Corollary 2.9.

(i) *No proper subset $\Gamma' \neq \varnothing$ of an irreducible algebraic variety Γ can be the set of singularities of a solution of* (2.1).
(ii) *If u is analytic in $\Omega\backslash\Gamma$, where Γ is a C^1 set of real codimension > 2, then Γ is a removable set of singularities.*

This study concerns in general possible singularities. The actual development of a singularity depends in general on the Cauchy data. One can, however, in certain cases prove the existence of singularities at characteristic points as the following theorem due to Harold S. Shapiro shows:

Theorem 2.10 [Sh 1]. *Let* $u(x)$ *be a local solution of the equation* $P(x, D)u = f$ *with Cauchy data* $w \equiv 0$ *on the initial surface* $\Gamma = \{x : \varphi(x) = 0\}$. *Suppose that*

(i) φ *is irreducible and regular at* $x^\circ \in \Gamma_{\mathrm{CH}}$.
(ii) $f(x^\circ) \neq 0$.

Then $u(x)$ *must develop a singularity at* x°.

(Note that the condition $w \equiv 0$ is no restriction. With $w \not\equiv 0$ one can replace u by $v = u - w$).

Proof. Suppose that $u(x)$ is not singular at $x^\circ \in \Gamma$. Since the Cauchy data vanish up to the order $m - 1$ on Γ, u can be written $u = \varphi^m v$ for some holomorphic v in a neighborhood of x° (cf. [B-T, p. 92]). Hence, for $|\alpha| = m$, $D^\alpha u = \varphi A + m! D^\alpha \varphi \cdot v$ and for $|\alpha| < m$, $D^\alpha u = \varphi B$, where A and B are holomorphic near x°. Therefore,

$$P(x, D)u = \varphi \cdot C + m! v P_m(x, \partial\varphi/\partial x),$$

where C is another holomorphic function near x°. But since x is characteristic, $P_m(x^\circ, \partial\varphi(x^\circ)/\partial x) = 0$, and hence $P(x^\circ, D)u(x^\circ) = 0$, contradicting $f(x^\circ) \neq 0$.

We conclude this section with some examples taken mostly outside the class (Q^*) of problems illustrating the concept of λ-globalization and giving a preliminary view of the different K- and L-types of singularities which will be studied in §4.

In the case where the λ-function can be determined and where the transformed equation $(***)$(i) turns out to be a Persson equation with entire coefficients, the solution formula (2.29) $u(x) = U(x, t(x))$ can be given a global interpretation. Also, the singularities of u can be seen to depend only on λ since the equation $\lambda(x, t) = 0$ defines the function $x \mapsto t(x)$. In fact, if x° is a point not on Γ such that $\lambda(x^\circ, t^\circ) = 0$ for some $t^\circ \neq 0$ and if x° does not lie on the envelope of the family of surfaces in $\mathbb{C}^N$ defined by $\lambda(x, t) = 0$ (with t interpreted as the parameter of the family), then it is clear that $u(x)$ can be continued analytically from a point of Γ to x° along a curve lying in the set

$$\Lambda = \{x : \lambda(x, t) = 0 \text{ for some } t\}$$

and avoiding the envelope. This analytic continuation corresponds to a similar continuation of $U(x, t)$ from $(x^\circ, 0)$ to (x°, t°). We call this process *λ-globalization.* It is clear that the envelope of the surface family is identical with the characteristic tangent K and constitutes a possible set of singularities (cf. Examples 1 and 4).

Another possible set of singularities is the complement Λ^c of Λ. These sets will be called L-singularities in §4, (more precisely, initial or asymptotic L-singularities) (cf. Examples 2 and 3). It will also be shown (Example 5) that due to the multivaluedness of the function $x \mapsto t(x)$, there may appear yet another type of L-singularities, called in §4 latent singularities.

Note that the surface family $\{x : \lambda(x, t) = 0\}$ itself may have independent interest. Since $t = 0$ corresponds to the initial surface $\Gamma = \{x : \varphi(x) = \lambda(x, 0) = 0\}$ the family can be viewed as a deformation on the surface Γ induced by and characteristic of the principal part of operator P.

In the list of examples below we will give for each problem, i.e., couple of operator P and surface Γ, the corresponding λ-function, bicharacteristics β_y, Λ^c-set and characteristic tangent K.

Note first that for first order operators $(m = 1)$, formula (2.14) shows that

$$\lambda(x, t) = \varphi(y(x, t)).$$

Also, first order operators with coefficients linear in x give rise to a transformed equation $(***)'$ of Persson type.

Example 1. $P = a_1(\partial/\partial x_1) + a_2(\partial/\partial x_2)$, $\Gamma\colon x_1^2 + x_2^2 - 1 = 0$; $\beta_y\colon x_j = y_j + a_j t$, $j = 1, 2$; $\lambda = (x_1 - a_1 t)^2 + (x_2 - a_2 t)^2 - 1$ (i.e., the deformation consists of a translation of Γ in direction (a_1, a_2)). $\Lambda^c = \varnothing$, $K =$ translation envelop of Γ, i.e., two straight lines with direction (a_1, a_2).

Example 2. $P = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2)$, $\Gamma\colon x_1^2 + x_2^2 - 1 = 0$; $\beta_y\colon x_j = y_j e^t$, $j = 1, 2$ (i.e., straight lines through y in direction towards the origin but avoiding this point unless $y = 0$); $\lambda = (x_1^2 + x_2^2)e^{-2t} - 1$; $\Lambda^c = \{(x_1, x_2) : x_1^2 + x_2^2 = 0\}$, $K = \varnothing$ (no envelope due to the blow-up character of the deformation). Here, Λ^c is an L-singularity which does not meet Γ. This type will be called an asymptotic singularity in §4.

Example 3. $P = x_1(\partial/pax_1) + x_2(\partial/\partial x_2)$, $\Gamma\colon x_2 - x_1^2 = 0$; $\beta_y\colon x_j = y_j e^t$, $j = 1, 2$, $\lambda = x_2 e^{-t} - x_1 e^{-2t}$; $\Lambda^c = \{(x_1, x_2)\colon x_1 = 0 \text{ or } x_2 = 0\} \setminus \{(0, 0)\}$, $K = \varnothing$.

Here $\Lambda^C \cup \{(0, 0)\}$ is an L-singularity which meets Γ at the characteristic (and exceptional) point $(0, 0)$. In §4 this will be called an initial L-singularity. (This example appears in [L] as an illustration of the exceptional case).

Example 4. $P = x_2(\partial/\partial x_1) - x_1(\partial/\partial x_2)$, $\Gamma\colon x_1^2/a_1^2 + x_2^2/a_2^2 - 1 = 0$; $\beta_y\colon x_1 = y_1 \cos t + y_2 \sin t$, $x_2 = -y_1 \sin t + y_2 \cos t$ (i.e., in $\mathbb{R}^2$: circles with radius $\sqrt{y_1^2 + y_2^2}$); $\lambda = (x_1 \cos t - x_2 \sin t)^2/a_1^2 + (x_1 \sin t + x_2 \cos t)^2/a_2^2 - 1$ (i.e., the ellipse Γ is rotated around the origin); $\Lambda^c = \varnothing$, $K = \{(x_1, x_2) : x_1^2 + x_2^2 = a_1^2\} \cup \{(x_1, x_2) : x_1^2 + x_2^2 = a_2^2\}$. In $\mathbb{R}^2$, K is obviously two circles as indicated.

Example 5. $P = \partial/\partial x_1$, $\Gamma\colon x_2 x_1^2 - x_1 - 1 = 0$; $\beta_y\colon x_1 = y_1 + t$; $x_2 = y_2$; $\lambda = x_2 t^2 + (1 - 2x_1 x_2)t + x_1^2 x_2 - x_1 - 1$; $\Lambda^C = \varnothing$, $K = \{(x_1, x_2) : x_2 = -\frac{1}{4}\}$. The function $x \mapsto t(x)$ defined by $\lambda = 0$ obviously has an unbounded singularity at $x_2 = 0$ on one of the two branches of its Riemann surface. This singularity, which clearly also appears in $u(x)$, is an example of what is called a latent L-singularity in §4.

In the case of first order operators it is well known that the analytic continuation of the solutions can be achieved along the bicharacteristics (cf. Kreiss [Kr]). Obviously the λ-globalization takes this form as shown by Examples 1–5. In the case of higher order operators, the picture is not that simple. This is shown by the fact that the λ-function given by (2.13)–(2.14) does not take the simple form $\varphi(y(x, t))$ as in the first order case.

The last two examples deal with second order operators. In Example 6 the transformed equation $(***)(\mathrm{i})$ is a Persson equation with singularities in the coefficients which however do not affect the solution $u(x)$. This will be shown in §4.

Example 6. $P = \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2$, $\Gamma\colon x_1^2/a_1^2 + x_2^2/a_2^2 - 1 = 0$; $\beta_y\colon x_j = y_j(1+4t/a_j)$, $\lambda = x_1^2/(a_1^2+4t)+x_2^2/(a_2^2-4t)-1$; $\Lambda^c = \varnothing$, $K\colon x_2 = \pm(x_1 = \pm 1)$ (uncoupled signs), where $1 = \sqrt{a_1^2 + a_2^2}$.

In $\mathbb{R}^2$ the deformation takes place within the square bounded by the four lines of K.

In §4 it is proved that in the corresponding problem for $P = \Delta_2$, K consists of the four lines $x_2 = \pm i(x_1 \pm l)$ $(l = \sqrt{a_1^2 - a_2^2}$, $a_1 > a_2)$, which cut $\mathbb{R}^2$ at the foci $x_1 = \pm 1$, $x_2 = 0$ of the ellipse Γ.

Example 7. $P = x_1(\partial^2/\partial x_n^2) - \sum_{j=1}^{n-1} \partial^2/\partial x_j^2$ (generalized Tricomi operator), $\Gamma\colon x_n = 0$.

Here, $g(x, \xi) = x_1\xi_n^2 - \sum_{j=1}^{n-1} \xi_j^2$ and $\lambda = \varphi(y) + tg(y, \partial\varphi/\partial y) = y_n + ty_1$, where y_1 and y_n depend on x and t in a way given by the solution $x = x(y, t)$ of the Hamilton equations (2.17)–(2.18). We get $x_1 = y_1 + t^2$, $x_j = y_j$ $(2 \le j \le n-1)$ and $x_n = y_n + 2y_1 t + \frac{2}{3}t^3$. Hence, $\lambda(x, t) = x_n - tx_1 + \frac{1}{3}t^3$.

The transformed equation $(***)$(i) here takes the form

$$V_{tt} = x_1(t^2 - x_1)V_{nn} - 2x_1 V_{nt} - 2tV_{1t} + (x_1 - t^2)\sum_{j=1}^{n-1} V_{jj},$$

which is a Persson equation with entire coefficients and hence yields entire solutions $V(x, t)$ and $U(x, t)$. The function $\lambda(x, t)$ indicates that the initial plane $x_n = 0$ is deformed into new planes $x_n - tx_1 + \frac{1}{3}t^3 = 0$, in such a way that the envelope, K, is defined by $9x_n^2 - 4x_1^3 = 0$, a cylindrical surface with a cusp at $x_1 = x_n = 0$.

Summary of main notions. $(*)$ is the main Cauchy problem defined in §2.

(Q^*) is the same Cauchy problem when $Pu = f$ is given by (2.6), a linear second order normalized equation.

$(**)$ and $(***)$ are transformations of $(*)$ defined in §2.

$(Q**)$ and $(Q***)$ are the corresponding problems specialized by (2.6).

$g(x, \xi) = P_m(x, \xi)$ is the principal symbol of P ((2.4)).

The quadratic surface Γ is defined by $\varphi(x) = x^T Ax + B^T x + C = 0$ ((2.7)).

A point of $x^\circ \in \Gamma$ is *characteristic* if $g(x^\circ, \partial\varphi(x^\circ)/\partial x) = 0$.

Γ_{CH} is the set of characteristic points of Γ.

The function λ, defined as the solution of (2.11)–(2.12), is determined by (2.13)–(2.18).

Its form in the (Q^*)-case is given by (3.2) and more precisely by (3.19)–(3.21) when $\varphi(x)$ is in normal form.

In (3.23) $T(x, t)$ and $N(t)$ are defined as the numerator and denominator respectively of $\lambda(x, t)$.

Also, in (3.23)–(3.24) the polynomials $\psi_j(x)$ and $\sigma_j(x, t)$ are defined.

β_y is the *principal bicharacteristic* defined by (2.16)–(2.17) issued from $y \in \Gamma_{\mathrm{CH}}$.

The *characteristic tangent* K is defined as $\bigcup_{y\in\Gamma_{\mathrm{CH}}} \beta_y$ ((2.32)).

The notion of *Persson equation* is defined in connection with Theorem 2.4.

The *normal form* of φ is defined in Lemma 3.2 and in connection with that the matrices I_p, S_p and the forms $I_p(x)$, $S_p(x)$, and $S_p^{(m)}(x)$.

The eigenvalues of A (the leading matrix in the normal form of φ) are denoted a_j, $0 \leq j \leq M$. If 0 is an eigenvalue it is denoted a_0.

p_j is the multiplicity of a_j and $\alpha_j = -1/4a_j$ ((3.22)) are the poles of $\lambda(x, t)$.

The variable notation ($x^{(j)}$, $x^{(jk)}$, $x^{(jkv)}$, etc.) is explained in (3.25)–(3.27).

$\varphi^{(2)}$ denotes $\sum_{j=1}^{n}(\partial\varphi/\partial x_j)^2$ ((3.34)).

$W(x, x')$ is defined by (3.48).

The sets of singularities of u and t are denoted S_u and S_t respectively.

The following is a list of the most important sets of singularities and their defining relations:

Ψ (3.37)
Disc T (3.41)
Ψ^{**} (3.43)
Σ_j (3.44), Σ (3.46), Σ^{**} (3.47), Σ^* (4.5)
$L = \Psi \cup \Sigma^*$ ((4.6))
L_I, L_A, and L_L are defined in the text before Lemma 4.5.
F_k (5.8) (focal sets)
X_k (5.15), Y_k (5.16), $\widetilde{Y}_k$ (5.17).

3. Algebraic preliminaries

In this section we turn to the study of the multivalued function $x \mapsto t(x)$ defined by the equation $\lambda(x, t) = 0$ and appearing in the solution formula (2.29) $u(x) = U(x, t(x))$. To this end we determine the λ-function corresponding to the problem (Q^*) in terms of the given quadric defined by

$$\varphi(x) = x^T Ax + B^T x + C = 0. \tag{3.1}$$

From now on we let $x \in \mathbb{C}^n$ be n-dimensional, where n is the number of variables that appear in the principal part Δ_n of the operator. We let $x' \in \mathbb{C}^{N-n}$ denote the remaining variables, which in the sequel will play the role of parameters. We assume that $1 \leq n \leq N$. Following this convention we let the vector B in (3.1) depend linearly on x', $B = B(x')$, and in like manner the constant C depends quadratically on x', $C = C(x')$. This dependence will in general not be indicated explicitly.

Lemma 3.1. *The λ-function corresponding to the principal operator Δ_n and the quadric Γ defined by* (3.1) *is*

$$\lambda(x, t) = \frac{1}{4t}x^T(I - G^{-1})x + (x - tB)^T G^{-1}B + C, \tag{3.2}$$

where

$$G = I + 4tA. \tag{3.3}$$

(Note that (3.2) is valid even when $t = 0$, since t factors out of the expression $x^T(I - G^{-1})x$. It is easily checked that $\lambda(x, 0) = \varphi(x)$ as required.)

Proof. We use formulas (2.13)–(2.14) to determine $\lambda(x, t)$, i.e., first (2.14)

$$\lambda[y, t] = \varphi(y) + tg(y, \partial\varphi/\partial y),$$

where we have set $m = 2$.

$$\varphi(y) = y^T Ay + B^T y + C \quad \text{implies} \quad \frac{\partial\varphi}{\partial y} = 2Ay + B.$$

Since $g(y, \xi) = \sum_{j=1}^{n} \xi_j^2$,

$$g\left(y, \frac{\partial \varphi}{\partial y}\right) = \left(\frac{\partial \varphi}{\partial y}\right)^T \left(\frac{\partial \varphi}{\partial y}\right) = (2y^T A^T + B^T)(2Ay + B)$$
$$= 4y^T A^2 y + 4B^T Ay + B^T B.$$

The Hamilton equations (2.17) here take the form

$$(3.4) \qquad \begin{aligned} &\text{(ia)} \quad \frac{dx_j}{dt} = 2\xi_j, \quad 1 \le j \le n, \qquad \text{(ib)} \quad \frac{dx'}{dt} = 0, \\ &\text{(ii)} \quad \frac{d\xi}{dt} = 0, \qquad \xi \in \mathbb{C}^N, \end{aligned}$$

with initial conditions

$$(3.5) \qquad \begin{aligned} &\text{(i)} \quad x(0) = y, \quad x'(0) = y', \qquad y \in \mathbb{C}^n, \ y' \in \mathbb{C}^{N-n}. \\ &\text{(ii)} \quad \xi(0) = \frac{\partial \varphi}{\partial y}(y). \end{aligned}$$

Equations (3.4)–(3.5)(ii) imply $\xi = \partial \varphi / \partial y$. Hence, from (3.4)(i),

$$(3.6) \qquad \begin{aligned} &\text{(a)} \quad dx/dt = 4Ay + 2B, \\ &\text{(b)} \quad dx'/dt = 0 \end{aligned}$$

which have the solution

$$(3.7) \qquad \begin{aligned} &\text{(a)} \quad x = y + t(4Ay + 2B), \\ &\text{(b)} \quad x' = y'. \end{aligned}$$

Note that (3.7) defines the principal bicharacteristics, when $y \in \Gamma_{\mathrm{CH}}$. Note also that (3.7)(b) implies $B(y') = B(x')$, $C(y') = C(x')$.

Inversion of (3.7)(a) gives

$$(3.8) \qquad y = (I + 4tA)^{-1}(x - 2Bt) = G^{-1}(x - 2Bt).$$

Moreover, from (2.14),

$$\lambda[y, t] = y^T Ay + B^T y + C + t(4y^T A^2 y + 4B^T Ay + B^T B),$$

or equivalently

$$\lambda[y, t] = y^T A(I + 4tA)y + B^T(I + 4tA)y + tB^T B + C,$$

i.e.,

$$(3.9) \qquad \lambda[y, t] = y^T AGy + B^T Gy + tB^T B + C.$$

Assuming $t \neq 0$ and using $A = (G - I)/4t$ (from (3.3)) we get

$$(3.10) \qquad \lambda[y, t] = (1/4t)y^T(G - I)Gy + B^T Gy + tB^T B + C.$$

Now, inserting (3.8) in (3.10) gives

$$\begin{aligned} \lambda(x, t) &= \frac{1}{4t}(x - 2Bt)^T G^{-1}(G - I)GG^{-1}(x - 2Bt) \\ &\quad + B^T GG^{-1}(x - 2Bt) + tB^T B + C \\ &= \frac{1}{4t}(x - 2Bt)^T(I - G^{-1})(x - 2Bt) + B^T(x - 2Bt) + tB^T B + C \\ &= \frac{1}{4t}x^T(I - G^{-1})x - B^T(I - G^{-1})x + tB^T(I - G^{-1})B \\ &\quad + B^T x - 2tB^T B + tB^T B + C \\ &= \frac{1}{4t}x^T(I - G^{-1})x + (x - tB)^T G^{-1}B + C, \end{aligned}$$

which is (3.2).

If $t = 0$, we have $x = y$ in (3.8), and (3.9) gives $\lambda = \varphi(x)$ as required. This completes the proof.

Note that the linearity of equations (3.7) is a consequence of the fact that Γ is a quadric. For surfaces defined by higher order polynomials, equations (3.7) would be at least quadratic in y causing $y(x, t)$ and $\lambda(x, t)$ to be multivalued with algebraic singularities. In that case no global existence result for the transformed equation $(***)(\mathrm{i})$ of §2 corresponding to Theorem 2.5 is known to the author.

We now proceed to the problem of transforming an arbitrary complex quadric to one given in a normal form using transformations which leave the operator Δ_n invariant, i.e., complex orthogonal transformations in the x-variables and translations. Obviously it is not possible to achieve a complete diagonalization in the complex case.

Some notation: I_p is the $p \times p$ unity matrix

$$(I_p)_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \qquad p \geq 1.$$

$S_p = \frac{1}{2}(U_p + iV_p)$, where U_p and V_p are $(p \times p)$-matrices such that

$$(U_p)_{jk} = \begin{cases} 1, & |j-k| = 1, \\ 0, & |j-k| \neq 1, \end{cases} \qquad p \geq 2,$$

$$(V_p)_{jk} = \begin{cases} 1, & j+k = p, \\ -1, & j+k = p+2, \\ 0, & \text{otherwise}, \end{cases} \qquad p \geq 2.$$

We define $S_1 = 0$. Hence S_p is of form $(p \geq 2)$

$$S_p = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & & & \\ & & & & & & 0 \\ 1 & 0 & 1 & & & & \\ 0 & 1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & & 0 & 1 \\ & 0 & & & & & \\ & & & & & 1 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} & & & & 0 & 1 & 0 \\ 0 & & & & & & \\ & & & & 1 & 0 & -1 \\ & & & & 0 & -1 & 0 \\ & & & \iddots & & & \\ 1 & 0 & & & & & \\ & & & & & 0 & \\ 0 & -1 & & & & & \end{pmatrix}.$$

We also occasionally write $I_p(x)$ for $x^T I_p x$, $S_p(x)$ for $x^T S_p x$, and $S_p^{(j)}(x)$ for $x^T S_p^j x$.

Lemma 3.2. *Any complex quadratic polynomial $\varphi(x, x')$ can be transformed under translations and complex orthogonal transformations in the x-variables to a normal form*

$$x^T A x + B^T x + C, \tag{3.11}$$

where A and B take the following forms: A is a constant symmetric $n \times n$ matrix consisting of diagonal blocks $A^{(j)}$:

(3.12) $\qquad A = [A^{(1)}, A^{(2)}, \dots, A^{(q)}]$ *where each $A^{(j)}$ is of form*

$$A^{(j)} = a_j I_p + \varepsilon_j S_p \text{ for some } p \geq 1. \tag{3.13}$$

In (3.13) a_j is an eigenvalue of A. ε_j can take the values 0 or 1. B is an n-dimensional vector with components depending linearly on x'. Moreover, $B_j \equiv 0$, unless A_{jj} is the first diagonal element of a block $A^{(k)}$ corresponding to the eigenvalue $a_v = 0$.

Before the proof of Lemma 3.2 we collect some facts on the S_p matrices. We use the following notation for Jordan boxes: J_p is a $(p \times p)$-matrix such that

$$(J_p)_{jk} = \begin{cases} 1, & k - j = 1, \\ 0, & k - j \neq 1. \end{cases}$$

Lemma 3.3.

(i) *S_p is linearly similar to J_p.*

(ii) $\mathrm{Det}(aI_p + S_p) = a^p$.

(iii) $\mathrm{Rank}(S_p) = p - 1$.

(iv) *The operation $S_p^j \mapsto S_p^{j+1}$ consists of letting the four superdiagonals of S_p^j take one step towards the nearest corner.*

In particular, $x^T S_p^p x = 0$, $x^T S_p^{p-1} x = \frac{1}{2}(x_1 - ix_p)^2$, $p \geq 2$, and $x^T S_p^{p-2} x = (x_1 - ix_p)(x_{p-1} + ix_2)$, $p \geq 4$.

(v) *The forms $x^T S_p^j x$, $0 \leq j \leq p - 1$, are linearly independent.*

(vi) *$x^T S_p^j x$ is reducible if and only if $j \geq p - 2$, $p \geq 2$.*

(vii) *For $a \neq 0$, $x^T(aI_p + S_p)x$ is reducible if and only if $p \leq 2$.*

(viii)

$$(I_p + 4t(aI_p + S_p))^{-1} = \sum_{j=0}^{p-1} \frac{(-4t)^j}{(1 + 4at)^{j+1}} S_p^j, \qquad p \geq 2.$$

(ix) *If $B^T = (B_1, 0, \ldots, 0)$ and $G_p = I_p + 4tS_p$, then*

$$(x - tB)^T G_p^{-1} B = \frac{B_1}{2} \sum_{j=1}^{p-1} (-4t)^j (x_{j+1} + ix_{p-j}) + B_1 x_1 - tB_1^2 + \frac{i}{8}(-4t)^p B_1^2.$$

Proof. Property (i) is given in [Ga]. Properties (ii) and (iii) are consequences of (i) and corresponding properties of J_p. Property (iv) also mimics the behaviour of J_p^j and is easily checked by explicit computation. The linear independence of (v) (in the vector space spanned by the monomials $x_j x_k$) follows from (iv) since this property implies that each form $x^T S_p^j x$ contains some generator $x_j x_k$ which does not appear in any other. (Incidentally, we assume that $S_p^0 = I_p$.) (vi) and (vii) follow from corresponding properties of J_p. As for (viii), we make the following ansatz

$$G_p^{-1} = (I_p + 4atI_p + 4tS_p)^{-1} = \sum_{j=0}^{p-1} c_j S_p^j.$$

Hence

$$G_p G_p^{-1} = c_0(1 + 4at)I_p + \sum_{j=1}^{p-1} ((1 + 4at)c_j + 4tc_{j-1}) S_p^j,$$

where we have used $S_p^p = 0$ from (iv). Identifying with I_p yields $c_0 = (1+4at)^{-1}$ and $c_j = -4(1+4at)^{-1}c_{j-1}$, which implies that

$$c_j = (-4t)^j(1+4at)^{-j-1}, \qquad j = 0, 1, \dots, p-1,$$

as required. (ix) follows from direct computation and from the information on the leftmost columns in S_p that can be extracted from (iv).

Proof of Lemma 3.2. The properties of the matrix A are given in [Ga]. It remains to show that the components B_j may be set to 0 as stated. We can assume that A consists of only one block $A = aI_p + \varepsilon S_p$ ($\varepsilon = 0$ or 1). Set $x = y+v$, where v is a constant vector to be determined. Then $x^TAx+B^Tx = y^TAy + y^T(2Av+B) + v^Tv + B^TB$. We want $Av = -\frac{1}{2}B$, which is solvable for v if $a \neq 0$, since $\operatorname{Det} A = a^p$ (Lemma 3.3(ii)) regardless of the value of ε. Assume now that $A = S_p$, i.e., $a = 0$, $\varepsilon = 1$. Let A' be the result of removing the first row and column from A and let B' and v' be B and v with first components removed. A has rank $p-1$ (Lemma 3.3(iii)). Also A' has rank $p-1$, since the first row of A equals the last row times i and since, similarly, the first column of A equals the last times i and since, finally $A_{11} = A_{1p} = A_{p1} = A_{pp} = 0$. Hence, the equation $A'v' = -\frac{1}{2}B'$ is solvable for v' which means that all components of B except the first can be annihilated by the translation.

According to Lemma 3.2 the quadratic form $\varphi(x)$ in normal form can be expressed as a sum of $C(x')$ and blocks of either α, β, or γ type to be defined:

$$\psi_\alpha := aI_p(x), \qquad p \geq 1 \qquad (\alpha\text{-block}), \tag{3.14}$$

$$\psi_\beta := aI_p(x) + S_p(x), \qquad p \geq 2, \ a \neq 0 \qquad (\beta\text{-block}), \tag{3.15}$$

$$\psi_\gamma := S_p(x) + B_1x_1, \qquad p \geq 1 \qquad (\gamma\text{-block}). \tag{3.16}$$

Since the operation

$$\varphi \mapsto \Lambda(\varphi) = \lambda \tag{3.17}$$

is linear blockwise, the problem of determining $\Lambda(\varphi)$ for arbitrary φ in normal form is solved if $\Lambda(\psi_\alpha)$, $\Lambda(\psi_\beta)$ and $\Lambda(\psi_\gamma)$ are determined.

Lemma 3.4. *Suppose that the operation* Λ *of* (3.17) *corresponds to the principal operator* Δ_n. *Then*

$$\Lambda(\psi_\alpha) = aI_p(x)/(1+4at), \qquad p \geq 1. \tag{3.18}$$

$$\Lambda(\psi_\beta) = \frac{aI_p(x)}{1+4at} + \sum_{j=1}^{p-1}\frac{(-4t)^{j-1}S_p^{(j)}(x)}{(1+4at)^{j+1}}, \qquad p \geq 2. \tag{3.19}$$

$$\begin{aligned} &\text{(i)} \quad \Lambda(\psi_\gamma) = \sum_{j=1}^{p-1}(-4t)^{j-1}S_p^{(j)}(x) + \frac{B_1}{2}\sum_{j=1}^{p-1}(-4t)^j(x_{j+1}+ix_{p-j}) \\ &\qquad\qquad + \frac{i}{8}B_1^2(-4t)^p + B_1x_1 - B_1^2t, \qquad p \geq 2, \\ &\text{(ii)} \quad \Lambda(\psi_\gamma) = B_1x_1 - B_1^2t, \qquad p = 1. \end{aligned} \tag{3.20}$$

$$\Lambda(C(x')) = C(x'). \tag{3.21}$$

Proof. Follows immediately from the general formula (3.2) and from properties (viii) and (ix) of Lemma 3.3.

We assume from now on that $a_1, \dots, a_M$ are the M different nonzero eigenvalues of A. If 0 is an eigenvalue of A, it is denoted a_0. The multiplicity of the eigenvalue a_j is denoted p_j. Also set

$$\alpha_j := -1/4a_j, \qquad j \geq 1. \tag{3.22}$$

Hence α_j are the poles of $\lambda = \Lambda(\varphi)$.

We obtain immediately, as a corollary of Lemma 3.4,

Lemma 3.5. $\lambda = \Lambda(\varphi)$ *can be expressed in either one of the following forms*:

$$\lambda = \frac{T(x, t)}{N(t)} = \frac{\sum_{j=0}^{r} \psi_j(x) t^j}{\prod_{j=0}^{M}(1 + 4a_j t)^{p_j}}. \tag{3.23}$$

$$\lambda = \sum_{j=0}^{M} \frac{\sigma_j(x, t)}{(1 + 4a_j t)^{p_j}}. \tag{3.24}$$

Here, $\psi_j(x)$ are at most quadratic polynomials. $\sigma_j(x, t)$ are polynomials in x and t, at most quadratic in x.

We assume that (3.23) defines $T(x, t)$ and $N(t)$ by identification of numerators and denominators. It is assumed that T and N have no common nonconstant factor.

We introduce some further notation in order to facilitate the study of the polynomial $T(x, t)$. First set

$$\varphi(x) = \sum_{j=0}^{M} \varphi_j(x^{(j)}) + C, \tag{3.25}$$

where each φ_j contains the part of φ which involves the eigenvalue a_j. $x^{(j)}$ is the corresponding segment of the variable vector x. Moreover,

$$\varphi_j(x^{(j)}) = \sum_{k=1}^{M_j} \varphi_{jk}(x^{(jk)}), \tag{3.26}$$

where each φ_{jk} is the sum of all (j, k)-blocks, i.e., the blocks of form $a_j I_k(x) + S_k(x)$, and $x^{(jk)}$ the corresponding variables. Finally,

$$\varphi_{jk}(x^{(jk)}) = \sum_{v=1}^{M_{jk}} \varphi_{jkv}(x^{(jkv)}), \tag{3.27}$$

where each φ_{jkv} is a (j, k)-block and $x^{(jkv)}$ a variable segment containing k variables, $x_1^{(jkv)}, \dots, x_k^{(jkv)}$. Let also $B^{(k)}$ correspond to the variables $x^{(0k)}$ and $B^{(kv)}$ to $x^{(0kv)}$. Hence $B^{(kv)}$ is a segment of B of length k: $B_1^{(kv)}, \dots, B_k^{(kv)}$. By Lemma 3.2 all components of B corresponding to the variables $x^{(j)}$, $j \geq 1$, are zero. Only $B_1^{(kv)}$ do not necessarily vanish.

In order to determine the degree, $\operatorname{Deg} T$, of $T(x, t)$ as a polynomial in t, we introduce the indices $\zeta_{\alpha\beta}$ and ζ_γ which are related to the α- and β-parts and to the γ-parts of φ respectively.

First, set

$$\mu := \sum_{j=1}^{M} r_j, \tag{3.28}$$

where

$$r_j := \max\{p : S_p \text{ appears in } \varphi_j\}. \tag{3.29}$$

Then define

$$\zeta_{\alpha\beta} := \begin{cases} \mu - 1, & \text{if } C \equiv \varphi_0 \equiv 0, \\ \mu, & \text{otherwise.} \end{cases} \tag{3.30}$$

Next set

$$b_k := (B^{(k)})^T B^{(k)} \tag{3.31}$$

and

$$p := \max\{q : S_q \text{ appears in } \varphi_0\}. \tag{3.32}$$

It follows from (3.20) that the degree of $\Lambda(\varphi_0)$ depends on whether b_p and b_{p-1} vanish and whether $B_1^{(pv)} \not\equiv 0$ for some v.

We therefore set

$$\zeta_\gamma = \begin{cases} 0, & \text{if } \varphi_0 \equiv 0 \text{ (i.e., if } p = 1), \\ p, & \text{if } b_p \not\equiv 0, \\ p-1, & \text{if } b_p \equiv 0 \text{ and } (b_{p-1} \not\equiv 0 \text{ or } B_1^{(pv)} \not\equiv 0 \text{ for some } v), \\ p-2, & \text{if } b_p \equiv b_{p-1} \equiv 0 \text{ and } B_1^{(pv)} = 0 \text{ for all } v. \end{cases} \tag{3.33}$$

Lemma 3.6. (i) $\operatorname{Deg} T = \zeta_{\alpha\beta} + \zeta_\gamma \le n$.

(ii) *If* φ *is not everywhere characteristic, then* $\operatorname{Deg} T \ge 1$.

Proof. (i) follows directly from Lemma 3.4 and the definitions of $\zeta_{\alpha\beta}$ and ζ_γ. It is clear that $\zeta_{\alpha\beta} \le \mu$, $\zeta_\gamma \le p$, and hence $\deg T \le \mu + p \le n$.

(ii) We prove that if $\deg T = 0$, then φ is everywhere characteristic, i.e., $g(x, \partial\varphi/\partial x) = 0$ on $\Gamma = \{x : \varphi(x) = 0\}$. Since $\lambda(x, 0) = \varphi(x)$ and $N(0) = 1$, we have $\lambda(x, t) = \varphi(x)/N(t)$. Hence $\partial\varphi/\partial x = N(t)(\partial\lambda/\partial x)$ and $\lambda_t = -\varphi(x)N'(t)/N^2(t)$. The Hamilton-Jacobi equation gives $g(x, \partial\lambda/\partial x) = -\lambda_t$. Hence,

$$g\left(x, \frac{1}{N(t)} \cdot \frac{\partial\varphi}{\partial x}\right) = -\frac{\varphi(x)N'(t)}{N^2(t)}.$$

But since $g(x, \xi)$ is homogeneous in ξ of degree m, we have

$$\frac{1}{N^m} g\left(x, \frac{\partial\varphi}{\partial x}\right) = -\frac{\varphi(x)N'}{N^2}$$

and

$$g\left(x, \frac{\partial\varphi}{\partial x}\right) = -\varphi(x)N'(t)N^{m-2}(t) = 0$$

on Γ, which proves the lemma.

We next set

$$\varphi^{(2)}(x) := \sum_{j=1}^{n} \left(\frac{\partial\varphi}{\partial x_j}\right)^2 \tag{3.34}$$

and observe that the coefficients ψ_0 and ψ_1 in the representation $T(x, t) = \sum_{j=1}^{r} \psi_j(x)t^j$ satisfy the following lemma.

Lemma 3.7. (i) $\psi_0(x) = \varphi(x)$.

(ii) $\psi_1(x) = -\varphi^{(2)}(x) + k\varphi(x)$ *for some constant* k.

Proof. (i) follows from the relation $\lambda(x, 0) = \varphi(x)$.

(ii) We use the Hamilton-Jacobi equation (2.11), which for $P = \Delta_n$ means that

$$\lambda_t = -\sum_{j=1}^{n}\left(\frac{\partial\lambda}{\partial x_j}\right)^2,$$

i.e., for $t = 0$

$$\lambda_t(x, 0) = -\sum_{j=1}^{n}\left(\frac{\partial\lambda}{\partial x_j}(x, 0)\right)^2 = -\sum_{j=1}^{n}\left(\frac{\partial\varphi}{\partial x_j}(x)\right)^2 = -\varphi^{(2)}(x).$$

But

$$\lambda_t = -\frac{N'(t)}{N^2(t)}T(x, t) + \frac{T_t(x, t)}{N(t)}$$

and hence

$$\begin{aligned}-\varphi^{(2)}(x) = \lambda_t(x, 0) &= -N'(0)T(x, 0) + T_t(x, 0)\\ &= -N'(0)\varphi(x) + \psi_1(x),\end{aligned}$$

which proves the lemma.

Lemma 3.8. *If* $\varphi(x)$ *is irreducible, then* $T(x, t)$ *is irreducible.*

Proof. Since $\varphi(x)$ is irreducible and $\psi_j(x)$ are polynomials of at most degree 2, it is impossible that $T(x, t) = Q_1(x, t)\cdot Q_2(x, t)$, where both Q_1 and Q_2 depend on x. Suppose $t - a$ is a factor of $T(x, t)$. Then $a \neq \alpha_j$, since $\lim_{t\to\alpha_j}\lambda(x, t) = \infty$ for almost all x. Hence $\lambda(x, a)$ exists and vanishes identically, i.e., $\Lambda(\varphi_{jkv})|_{t=a} \equiv 0$ for each (j, k)-block φ_{jkv}. But this is impossible in view of Lemma 3.3(v) which assures linear independence of $S_p^{(j)}(x)$, $j = 0, \ldots, p-1$. Hence, $T(x, t)$ is irreducible.

We observe that the equation

$$T(x, t) = 0 \tag{3.35}$$

can be used instead of $\lambda(x, t) = 0$ as defining equation for the function $x \mapsto t(x)$. This is also more convenient since $T(x, t)$ is a polynomial defined everywhere.

We note also that for any $x^\circ \in \Gamma = \{x : \varphi(x) = 0\}$ there is at least one function element of $x \mapsto t(x)$ which takes the value $t = 0$ on $\Gamma\backslash\Gamma_{\mathrm{CH}}$, since $\varphi(x)$ is always identical with $\psi_0(x)$ in the representation

$$T(x, t) = \sum_{j=0}^{r}\psi_j(x)t^j. \tag{3.36}$$

It is also clear that for points $x^\circ \in \Gamma\backslash\Gamma_{\mathrm{CH}}$ this function element is unique, since $\psi_1(x) \neq 0$ in $\Gamma\backslash\Gamma_{\mathrm{CH}}$ by Lemma 3.7.

Consider the function $x \mapsto u(x)$ as a set W^u of function elements $(u_j(x), w_j)$, where $u_j(x)$ is holomorphic in the domain w_j. We assume that this set is complete in the sense that it contains a function element on each domain to which the function is analytically continuable.

Definition 3.1. *The initial branch of* $u(x)$ (solution of (Q^*)) is represented by the subset W^u_{in} of W^u such that for each element $(u_j(x), w_j)$ of W^u_{in}: $w_j \cap \Gamma \neq \varnothing$ and $u_j(x)$ satisfies the Cauchy data of (Q^*) on $\Gamma \backslash \Gamma_{\text{CH}}$. Let $B_\varepsilon(x^0)$ be the open ball centered in x^0: $B_\varepsilon(x^0) = \{x : |x - x^0| < \varepsilon\}$.

Definition 3.2. Let Z be an analytic set. $u(x)$ *develops an initial singularity on* Z if for some $\varepsilon > 0$ and $x^0 \in \Gamma \cap Z$ no initial function element (i.e. element of W^u_{in}) can be analytically continued across $Z \cap B_\varepsilon(x^0)$ in $B_\varepsilon(x^0)$.

We need also the following definitions:

$$\Psi = \{x : \psi_r(x) = 0\}, \tag{3.37}$$

$$Z_T = \{x : T(x, t) = T_t(x, t) = 0 \text{ for some } t\}. \tag{3.38}$$

(Recall the corresponding set Z_λ related to λ defined by (2.31).)

$$S_t = \text{the set of singular points of } x \mapsto t(x). \tag{3.39}$$

Classically we have (cf. [Gr-Fr]) $S_t = \Psi \cup Z_T$.

Here, the singular set Ψ allows unbounded singularities whereas Z_T only contains algebraic singularities. We can also relate S_t to the discriminant D_T of T. Recall that D_T is defined as the resultant of T and T_t (cf. [W]), i.e.,

$$D_T = \pm\frac{1}{\psi_r}\left|\begin{array}{cccccccccc} r\psi_r & (r-1)\psi_{r-1} & & \cdots & & \psi_1 & \psi_0 & 0 & \cdots & 0 \\ 0 & r\psi_r & & \cdots & & 2\psi_2 & \psi_1 & \psi_0 & \cdots & 0 \\ \cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ 0 & \cdot & \cdot & 0 & r\psi_r & \cdots & \cdots & & 2\psi_2 & \psi_1 \\ \psi_r & \cdot & \cdot & \psi_1 & \psi_0 & 0 & \cdot & \cdot & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdot & \cdot & 0 & \psi_r & & \cdots & & \psi_1 & \psi_0 \end{array}\right| \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} r \text{ rows} \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \begin{array}{l} r-1 \\ \text{rows} \end{array} \end{array}. \tag{3.40}$$

The sign of (3.40) does not concern us here since we are only interested in the zero set $\operatorname{Disc} T$ of D_T:

$$\operatorname{Disc} T = \{x : D_T(x) = 0\}, \qquad r \geq 2. \tag{3.41}$$

D_T is only defined if $r \geq 2$. For the case $r = 1$ we set

$$\operatorname{Disc} T = \Gamma_{\text{CH}}, \qquad (r = 1). \tag{3.42}$$

D_T is clearly a polynomial since ψ_r always factors out of the determinant in (3.40).

It is well known that $x^\circ \in \operatorname{Disc} T$, if

(i) $T(x^\circ, t)$ has a multiple zero, i.e., $x^\circ \in Z_T$, or

(ii) $\psi_r(x^\circ) = \psi_{r-1}(x^\circ) = 0$, i.e., $x^\circ \in \Psi^{**}$, where

$$\Psi^{**} = \{x : \psi_r(x) = \psi_{r-1}(x) = 0\}. \tag{3.43}$$

Clearly $\Psi^{**} \subset \Psi$.

Case (ii) above corresponds to the case where formally $t = \infty$ is a double zero of $T(x, t)$.

The above definitions mean that $\operatorname{Disc} T = Z_T \cup \Psi^{**}$ and $S_t = \operatorname{Disc} T \cup \Psi$.

We finally relate S_t to the characteristic tangent $K = \bigcup_{y \in \Gamma_{\text{CH}}} \beta_y$. Define first

$$\Sigma_j = t^{-1}(\alpha_j), \tag{3.44}$$

$$\Sigma_j^{**} = \begin{cases} \Sigma_j, & \text{if } t = \alpha_j \text{ is a multiple zero of } T(x, t) \text{ for all } x \in \Sigma_j, \\ \varnothing & \text{otherwise.} \end{cases} \tag{3.45}$$

($\Sigma_j^{**} \neq \varnothing$ means that Σ_j^{**} is defined by $Q(x) = 0$, where $T(x, t) = (t-\alpha_j)^2 Q_1(x, t) + Q(x)R(x, t)$; Q, Q_1, and R are polynomials.)

$$\Sigma = \bigcup_{j=1}^{M} \Sigma_j, \tag{3.46}$$

$$\Sigma^{**} = \bigcup_{j=1}^{M} \Sigma_j^{**}. \tag{3.47}$$

Note that Z_T can be decomposed as

$$Z_T = Z_\lambda \cup (K \cap \Sigma) \cup \Sigma^{**}.$$

Here, $K \cap \Sigma = K \backslash Z_\lambda$, since it follows from the proof of Lemma 2.1 that the only points of K which do not belong to Z_λ are those of form $x = x(y, \alpha_j)$, $y \in \Gamma_{\mathrm{CH}}$. Since $\lambda[y, \alpha_j] = 0$ if $y \in \Gamma_{\mathrm{CH}}$, these x are elements of $\Sigma = \bigcup_{j=1}^{M} t^{-1}(\alpha_j)$. Hence, we have $Z_T = K \cup \Sigma^{**}$ and $S_t = \Psi \cup Z_T = \Psi \cup K \cup \Sigma^{**}$. Also, since $\operatorname{Disc} T = Z_T \cup \Psi^{**}$, $\operatorname{Disc} T = K \cup \Sigma^{**} \cup \Psi^{**}$.

We collect these facts in a lemma:

Lemma 3.9. (i) $S_t = \Psi \cup Z_T = \Psi \cup \operatorname{Disc} T = \Psi \cup K \cup \Sigma^{**}$.
(ii) $\operatorname{Disc} T = K \cup \Psi^{**} \cup \Sigma^{**}$.

We finally present lists of polynomials in normal form representing three different classes of surfaces that are crucial to our theory.

Lemma 3.10. *Consider quadratic polynomials in normal form. The following are reducible*:

(i) $a_1 x_1^2 + C_1^2(x')$,
(ii) $a_1 x_1^2 + a_2 x_2^2$,
(iii) $a_1 x_1^2 + S_2(x^{(0)})$,
(iv) $a_1 I_2(x^{(1)}) + S_2(x^{(1)})$,
(v) $S_2(x) + C_2^2(x')$,
(vi) $S_2(x^{(021)}) + S_2(x^{(022)})$,
(vii) $S_3(x)$,
(viii) $B_1(x')x_1 + B_1(x')C_3(x')$,
(ix) $C_4(x')C_5(x')$.

The following define everywhere characteristic surfaces:

(x) $I_p(x)$, $p \geq 2$,
(xi) $B^T x^{(01)} + \sum_{v=1}^{q} S_2(x^{(02v)}) + C(x')$, where $B^T B \equiv 0$ and $q \geq 0$.

The following define everywhere noncharacteristic surfaces;

(xii) $I_p(x) + c_0$, $0 \neq c_0 \in \mathbb{C}$, $p \geq 1$.
(xiii) $S_3(x) + c_1$, $0 \neq c_1 \in \mathbb{C}$,
(xiv) $B^T x^{(01)} + \sum_{v=1}^{q} S_2(x^{(02v)}) + c(x')$, where $0 \neq B^T B \in \mathbb{C}$, $q \geq 0$,
(xv) $B^T x + c_2$, where $B = k(x')B^0$, k is linear in x', B^0 is a constant vector $(B^{0T} B^0 \neq 0)$, and $0 \neq c_2 \in \mathbb{C}$.

Remarks. We let $q = 0$ in the above sums signify that the sum is identically zero. From now on we let W denote the sum in (xi):

$$W(x, x') := B^T x^{(01)} + \sum_{v=1}^{q} S_2(x^{(02v)}) + C(x'), \tag{3.48}$$
$$\text{where } B^T B \equiv 0, \quad q \geq 0.$$

This polynomial is identically characteristic in the sense that $\sum_{v=1}^{n} \partial^2 W/\partial x_j^2 \equiv 0$, whereas $\sum_{j=1}^{n} \partial^2 I_p/\partial x_j^2 = 4I_p$, implying that I_p $(p \geq 2)$ is characteristic but not identically characteristic. Also, it follows from (3.20)–(3.21) that $\Lambda(W) = W$.

Proof of Lemma 3.10. (i)–(ix). We use the fact that the sum of two blocks, $\varphi_{(1)}(x) + \varphi_{(2)}(y)$, is reducible if and only if $\varphi_{(1)}$ and $\varphi_{(2)}$ are squares. This together with properties (vi) and (vii) of Lemma 3.3 reduces the proof to an inspection of a small number of cases.

(x)–(xi). A general quadric $\varphi(x) = x^T A x + B^T x + C$ is everywhere characteristic if and only if

$$\left(\frac{\partial \varphi}{\partial x}\right)^T \frac{\partial \varphi}{\partial x} = \varphi^{(2)} = k\varphi \quad \text{for some } k \in \mathbb{C}.$$

Since $\partial\varphi/\partial x = 2Ax + B$, $\varphi^{(2)} = 4x^T A x + 4B^T A x + B^T B$. This condition then leads to

$$\text{(i)}\quad A^2 = kA/4, \qquad \text{(ii)}\quad B^T A = kB^T/4, \qquad \text{(iii)}\quad B^T B = kC. \tag{3.49}$$

These conditions must hold for each (j, k)-block. The only such blocks which satisfy (3.49)(i) are, by Lemma 3.3, those with second order part given by $I_p(x)$, $S_2(x)$, or 0. Inspection of the possible cases shows that only (x) and (xi) are everywhere characteristic.

(xii)–(xv). The condition for a quadratic polynomial $\varphi(x)$ to be everywhere noncharacteristic is

$$1 \in \operatorname{Rad} \operatorname{Id}(\varphi, \varphi^{(2)}), \tag{3.50}$$

where $\operatorname{Id}(\varphi, \varphi^{(2)})$ is the ideal in $\mathbb{C}[x, x']$ generated by φ and $\varphi^{(2)}$ and $\operatorname{Rad}\operatorname{Id}$ its radical. We refer to Lemma 4.7 which shows that only polynomials $\varphi(x)$ for which $\deg T \leq 2$ need to be checked. We omit the details of the proof, which again is reduced to checking a small number of cases.

We conclude with explicit versions of some reducible quadratic forms appearing in Lemma 3.10.

$$S_2(x) = (i/2)(x_1 - ix_2)^2, \tag{3.51}$$

$$aI_2(x) + S_2(x) = (x_1 - ix_2)((a + \tfrac{1}{2})x_1 + (ia + \tfrac{1}{2})x_2), \tag{3.52}$$

$$S_3(x) = (1 + i)x_2(x_1 - ix_3). \tag{3.53}$$

4. Singularities in $\mathbb{C}^N$

Assume that equation (2.1), $P(x, D)u = f$, is of form(2.6), i.e. has principal part $P_m = \Delta_n$. Then the transformed equation of problem (Q^{***}) reduces to

$$V_{tt} - \sum_{j=1}^{n} (\lambda_t V_{jj} - 2\lambda_j V_{jt} - \lambda_{jj} V_t) - \sum_{j=1}^{n} b_j(\lambda_t V_j - \lambda_j V_t) - c\lambda_t V = 0. \tag{4.1}$$

Since λ_t is a quadratic polynomial and λ_j are linear polynomials in x, we can conclude that (4.1) is a Persson equation and Theorem 2.4 can be applied. This means that the solution $U(x, t)$ of (Q^{***}) exists in $\mathbb{C}_x^N \times \Omega$, where $\Omega = \mathbb{C}_t \backslash \{\alpha_1, \dots, \alpha_M\}$, since $\{\alpha_j\}_{j=1}^M$ is the set of poles of the coefficients $\lambda, \lambda_t, \lambda_j$, and λ_{jj} of equation (4.1). Hence, the solution $u(x) = U(x, t(x))$ of the original problem has possible singularities in $\Sigma = \bigcup_{j=1}^M t^{-1}(\alpha_j)$ which can be attributed to the outer function $(x, t) \mapsto U(x, t)$. Then, if S_u is the total set of singularities of $u(x)$, we must have

$$S_u \subset S_t \cup \Sigma = K \cup \Psi \cup \Sigma, \tag{4.2}$$

since $S_t = K \cup \Psi \cup \Sigma^{**}$ from Lemma 3.10 and $\Sigma^{**} \subset \Sigma$.

Before stating the main theorem we investigate the sets Ψ and Σ more closely.

Lemma 4.1. *The following list comprises the cases of nonempty Ψ-sets:*

(i) ($\alpha\beta$*-case*) *If*

$$\varphi = \varphi_{\alpha\beta} = \sum_{j=1}^M (a_j I_{p_j}(x^{(j)}) + R_j(x^{(j)})),$$

where R_j is a sum of S_p-terms in the $x^{(j)}$-variables, then

$$\Psi = \left\{ x : \sum_{j=1}^M I_{p_j}(x^{(j)}) = 0 \right\}.$$

(ii) ($\alpha\beta W$*-case*) *If $\varphi = \varphi_{\alpha\beta} + W(x, x')$, then*

$$\Psi = \{(x, x') : W(x, x') = 0\}.$$

(iii) (γ*-case*) *Let ζ_γ be the index defined in* (3.33), *M_q the number of S_q-blocks appearing in φ_0, and p the maximal order of blocks in φ_0. Moreover, $b_q = (B^{(q)})^T B^{(q)}$,*

$$\varphi_0 = \sum_{q=1}^p \sum_{v=1}^{M_q} (S_q(x^{(0qv)}) + B_1^{(qv)} x_1^{(0qv)}).$$

There are three cases according as $\zeta_\gamma = p$, $p-1$, or $p-2$:

(a) $\zeta_\gamma = p$ $(b_p \not\equiv 0)$:

$$\Psi = \{(x, x') : b_p(x') = 0\}.$$

(b) $\zeta_\gamma = p - 1$ ($b_p \equiv 0$ *and* ($b_{p-1} \not\equiv 0$ *or* $B_1^{(pv)} \not\equiv 0$ *for some* v)):

$$\Psi = \left\{ \{(x, x') : \frac{i}{8}(-4)^{p-1} b_{p-1}(x') \right.$$
$$\left. + \frac{(-4)^{p-1}}{2} \sum_{v=1}^{M_p} B_1^{(pv)}(x_p^{(0pv)} + i x_1^{(0pv)}) = 0 \right\}.$$

(c) $\zeta_\gamma = p-2$ ($b_p \equiv b_{p-1} \equiv 0$ *and* $B_1^{(pv)} \equiv 0$ *for all* v):

$$\Psi = \Bigg\{(x, x') : \frac{i}{8}(-4)^{p-2}b_{p-2}(x') + \frac{(-4)^{p-2}}{2}\sum_{v=1}^{M_p} i(x_1^{(0pv)} - ix_p^{(opv)})^2 + \frac{(-4)^{p-2}}{2}\sum_{v=1}^{M_{p-1}} B_1^{(p-1,v)}(x_{p-1}^{(0,p-1,v)} + ix_1^{(0,p-1,v)}) = 0\Bigg\}.$$

Moreover, the cases of nonempty Ψ^{**}*-sets are case* (ii)(c) *above in the special case where* $M_p = 1$, $M_{p-1} = 0$, $b_p \equiv b_{p-1} \equiv b_{p-2} \equiv b_{p-3} \equiv 0$. *In that case* $\Psi^{**} = \{x : x_1^{(0p)} - ix_p^{(0p)} = 0\}$.

Proof. Follows from the formulas (3.18)–(3.21) from which in each case the leading coefficient ψ_r of $T(x, t)$ can be read off. In the case of the Ψ^{**}-sets, recall that $\Psi^{**} = \{x : \psi_r(x) = \psi_{r-1}(x) = 0\}$ and the fact that of the blocks $S_p^{(j)}(x)$, $j = 0, 1, \dots, p-1$, only $S_p^{(p-2)}(x)$ and $S_p^{(p-1)}(x)$ have a common factor (Lemma 3.3).

Lemma 4.2. *The following list comprises the cases of nonempty* Σ*-sets. For each term* φ_j *of* φ *we define the corresponding set* $\Sigma_j = t^{-1}(\alpha_j)$. *Here* M_q *is the number of* S_q*-blocks appearing in* φ_j *and* p *the maximal order of blocks in* φ_j.

(i) (α*-case*). *If* $\varphi_j = a_j I_{p_j}(x^{(j)})$, *then*

$$\Sigma_j = \{x \colon I_{p_j}(x^{(j)}) = 0\}.$$

(ii) ($\alpha\beta$*-case*). *If*

$$\varphi_j = a_j I_{p_j}(x^{(j)}) + \sum_{q=2}^{p}\sum_{v=1}^{M_q} S_q(x^{(jqv)}),$$

then

$$\Sigma_j = \left\{x : \sum_{v=1}^{M_p}(x_1^{(jpv)} - ix_p^{(jpv)})^2 = 0\right\}.$$

Moreover, the cases of nonempty Σ^{**}*-sets are case* (ii) *above, when* $M_p = 1$, $M_{p-1} = 0$. *In that case* $\Sigma^{**} = \{x : x_1^{(jp)} - ix_p^{(jp)} = 0\}$.

Proof. (i) follows directly from the representation $\Sigma_j = \{x : T(x, \alpha_j) = 0\}$. As for (ii), assume that $\varphi_j = a_j I_p + S_p(x)$. From (3.19) it follows that the corresponding part of T

$$T_j(x, t) = (1 + 4a_j t)^{p-1}a_j I_p(x) + \sum_{j=0}^{p-2}(1 + 4a_j t)^j(-4t)^{p-2-j}S_p^{(p-1-j)}(x) \tag{4.3}$$

and hence $T_j(x, \alpha_j) = (-4\alpha_j)^{p-2}S_p^{(p-1)}(x)$. Obviously $T_k(x, \alpha_j) = 0$ for $k \neq j$. For a general φ_j as given in the statement of the lemma, Σ_j is defined by a sum of $S_p^{(p-1)}$-blocks as indicated. In the case of Σ^{**}, we have a nonempty set when $t = \alpha_j$ is at least a double root of $T(x, t) = 0$ on a set of complex codimension one. This happens if $\psi_0(x)$ and $\psi_1(x)$ have a common factor, where ψ_j are the coefficients of $(1+4a_j t)^j$ in (4.3): $T_j = \sum_{j=0}^{p-2} \psi_j (1+4a_j t)^j$. Hence, Σ^{**} is defined as the zero set of the common factor of $S_p^{(p-2)}(x)$ and $S_p^{(p-1)}(x)$ in the case $M_p = 1$, $M_{p-1} = 0$.

Inspection of the lists in Lemmas 4.1 and 4.2 shows that all Ψ-sets are characteristic. The Σ-sets are characteristic except in case (i), when $p_j = 1$. Since any possible set of singularities must be characteristic (Theorem 2.5), we can delete these sets defining

$$\Sigma_j^* = \begin{cases} \varnothing, & \text{if } p_j = 1 \text{ in case (i) of Lemma 4.2,} \\ \Sigma_j, & \text{otherwise.} \end{cases} \tag{4.4}$$

$$\Sigma^* = \bigcup_{j=1}^{M} \Sigma_j^*. \tag{4.5}$$

The set of possible singularities of $u(x)$ is therefore reduced to $K \cup \Psi \cup \Sigma^*$.

We define

$$L = \Psi \cup \Sigma^*, \tag{4.6}$$

$$L^* = \Psi \cup \Sigma. \tag{4.7}$$

The set $L^* \backslash L$ may for natural reasons be called a locus of a *ghost singularity.*

We can now state the main theorem:

Theorem 4.3. *Suppose that $u(x)$ is a solution of Cauchy problem (Q^*). Then,*

(i) *$u(x)$ can be analytically continued along any path in $\mathbb{C}^N \backslash (K \cup L)$.*

(ii) *The set of singularities of u, S_u, is a union of irreducible components of $K \cup L$.*

(iii) *$u(x)$ is defined at each point of $K \backslash L$ and has there at most a bounded singularity.*

Proof. Part (i) has already been proved. (ii) follows from (i) and Corollary 2.9. The statement in (iii) is clearly true for $x^\circ \in K \backslash (\Psi \cup \Sigma)$. If x° lies on a "ghost singularity" $\Sigma \backslash \Sigma^*$, i.e., $x^\circ \in K \cap (\Sigma \backslash \Sigma^*)$, then $U(x, t(x))$ is not a priori bounded at x°, since we do not know the character of the singularities of $U(x, t)$. However, $U(x, t)$ must annihilate a bounded, algebraic singularity of $t(x)$ and must therefore also be bounded at $(x^\circ, t(x^\circ))$. The proof is complete.

We next collect some information on the set K in a theorem. Let Γ_{CH}^j be an irreducible component of Γ_{CH}, the set of characteristic points of Γ. Γ_{CH}^j is said to be *exceptional* if each $x^\circ \in \Gamma_{\mathrm{CH}}^j$ is exceptional, i.e., $T(x^\circ, t) \equiv 0$ in some neighborhood of $t = 0$ for all $x^\circ \in \Gamma_{\mathrm{CH}}^j$. We say that K is *degenerated along* Γ_{CH}^j if $\bigcup_{y \in \Gamma_{\mathrm{CH}}^j} \beta_y \subset \Gamma_{\mathrm{CH}}^j$. K is *degenerated* if K is degenerated along each component of Γ_{CH}.

Theorem 4.4.

(i) *K is empty if and only if Γ is everywhere noncharacteristic.*

(ii) *K is degenerated along Γ_{CH}^j if and only if Γ_{CH}^j is exceptional. If $\deg(T) = 1$, then K is degenerated.*

In the remaining part of the theorem we assume that K *is neither empty nor degenerated.*

(iii) K *is a ruled, everywhere characteristic surface tangent to* Γ *along* Γ_{CH}.

(iv) $K = \overline{\operatorname{Disc} T \backslash L} = \overline{\operatorname{Disc} T \backslash (\Psi^{**} \cup \Sigma^{**})}$.

(v) *The defining polynomial,* Q_K, *of* K *is a factor of the discriminant* D_T *of* T *and has degree* $\deg(Q_K) \leq 4r - 4 \leq 4n - 4$, *where* $r = \deg(T) = \zeta_{\alpha\beta} + \zeta_\gamma$.

(vi) K *is reducible if and only if* Γ_{CH} *is reducible.*

Proof. (i) From the definition of K, $K = \bigcup_{y \in \Gamma_{\mathrm{CH}}} \beta_y$, it is clear that $K = \varnothing$ if and only if $\Gamma_{\mathrm{CH}} = \varnothing$, since β_y, $y \in \Gamma_{\mathrm{CH}}$, is always a line or a point, i.e., nonempty.

(ii) If Γ^j_{CH} is exceptional, then K is degenerated along K by Lemma 2.3(ii). If, conversely, K is degenerated along Γ^j_{CH}, then for fixed $x^\circ \in \Gamma^j_{\mathrm{CH}}$ $(y(x^\circ, t) \in \Gamma^j_{\mathrm{CH}}$ in a neighborhood of $t = 0$. Hence $\lambda[y, t] = \varphi(y) + t(m-1)g(y, \partial\varphi/\partial y) = 0$ in a neighborhood of $t = 0$ and $\lambda(x^\circ, t) \equiv 0$, i.e., Γ^j_{CH} is exceptional.

(iii) That K is ruled follows from the definition. The rest follows from Lemma 2.3(i).

(iv) is proved in §3 (Lemma 3.9).

(v) follows from the expression (3.40) of D_T observing that each component of the determinant has at most degree 2.

(vi) We use the fact that an analytic set, S, is irreducible if and only if its set of regular points, S_{REG}, is connected (cf. [He]). Since K is a ruled surface and its set of nonregular points has codimension > 1, it is always possible to connect a regular point of K with a regular point of Γ_{CH} by a path that avoids singular points of K. Hence the problem of connectivity for K_{REG} is reduced to that for $(\Gamma_{\mathrm{CH}})_{\mathrm{REG}}$.

It should be mentioned in connection with the characteristic tangent K that a kind of latency phenomenon occur when $\deg(T) \geq 3$. In that case the function $x \mapsto t(x)$ has at least three branches. A point $x^\circ \in K$ may have the property that two branches of t, t_1 and t_2, take the same value there and hence may develop a singularity, whereas the third branch t_3 does not become singular at x°. That this nonsingular branch may be the initial one when $x^\circ \in \Gamma$ is shown in the example of the three-dimensional ellipsoid, where K intersects $\mathbb{R}^3$ along the so called focal hyperbola which cuts the ellipsoid Γ at points not on Γ_{CH} and hence cannot be the locus of initial singularities. This phenomenon, which obviously does not occur when $\deg(T) = 2$ will be discussed in §5 in the ellipsoid case and presently also in connection with L-singularities, where we will give precise definitions of latent and initial singularities.

L, the set of L-singularities was defined (4.6) as the union $L = \Psi \cup \Sigma^*$. This union suggests a possible classification of the irreducible components of L, but it seems more natural to make the following distinction among the L-components:

L_L, latent singularities
L_I, initial singularities
L_A, asymptotic singularities

We make the precise definitions below.

Let L^j be an irreducible component of L.

Definition. L^j is is *initial* if $L^j \cap \Gamma \neq \varnothing$ and if some solution $u(x)$ of (Q^*) develops an initial singularity on L^j. (See Definition 3.2.)

Definition. L^j is *latent,* if $L^j \cap \Gamma \neq \varnothing$ and L^j is not initial.

Definition. L^j is *asymptotic,* if $L^j \cap \Gamma = \varnothing$.

It follows immediately that each irreducible component of L is either latent, initial, or asymptotic. We say that the above types of components of L comprise the sets L_L, L_I, and L_A respectively. Hence, $L = L_L \cup L_I \cup L_A$.

Lemma 4.5. *Let L^j be an irreducible component of L. If $L^j \cap \Gamma \not\subset \Gamma_{\mathrm{CH}}$, then L^j is latent.*

Proof. Let $x^\circ \in L^j \cap \Gamma \backslash \Gamma_{\mathrm{CH}}$. Then by the Cauchy-Kovalevskaya theorem each solution exists initially in some neighborhood of x°. Hence L^j is latent.

The initial singularities are related to the exceptional points of Γ_{CH} according to the following lemma.

Lemma 4.6. *Suppose Γ^j_{CH} is an irreducible component of Γ_{CH}.*

(i) *If $x^\circ \in \Gamma^j_{\mathrm{CH}}$ is an exceptional point, then $x^\circ \in L^j$ for each component L^j of L.*
(ii) *If Γ^j_{CH} is exceptional then $\Gamma^j_{\mathrm{CH}} \subset L_I$.*

Proof. (i) If x° is exceptional, then $\psi_j(x^\circ) = 0$, $0 \leq j \leq r$. Hence, if $S^j(x)$ is the defining polynomial of L^j, $S^j(x^\circ) = 0$, since $S^j(x) = \sum_{j=0}^q c_j \psi_j(x)$ for some c_j.

(ii) If Γ^j_{CH} is exceptional, K is degenerated along Γ^j_{CH} (Theorem 4.4(ii)). But from Theorem 2.10 it follows that (Q^*) in general, (for almost all Cauchy data) has a solution with singularities initially on Γ_{CH}. These singularities must propagate along an analytic surface of complex codimension one (Corollary 2.9) and hence L_I must be nonempty. From part (i) it follows that $\Gamma^j_{\mathrm{CH}} \subset L_I$.

Note that Γ_{CH} is always exceptional, when $T(x, t) = \varphi(x) + t\psi_1(x)$, i.e., when $\deg(T) = 1$. When $\deg(T) = 2$, Γ_{CH} may be exceptional. This occurs, e.g., when $\psi_1(x)$ and $\psi_2(x)$ have a common factor as is the case when $\Psi^{**} \neq \varnothing$. It can be shown that if $\deg(T) \geq 3$, Γ_{CH} is never exceptional. The degree of T can in fact be used as a measure of the complexity of the quadric Γ. The general rule is that with higher complexity only latent singularities exist, as shown by the following lemma:

Lemma 4.7. *If* $\deg(T) \geq 3$, *all components L^j of L are latent.*

The lemma follows from the fact that, if $S^j(x)$ is the defining polynomial of L^j, $S^j(x)$ is not an element of the ideal $\operatorname{Rad}\operatorname{Id}(\varphi, \psi_1)$, which corresponds to the variety Γ_{CH}. This can in turn be shown from the geometric fact that if $\deg(T) \geq 3$ there is always an $x^\circ \in L^j \cap \Gamma_{\mathrm{CH}}$ such that the gradients $\partial S^j/\partial x$, $\partial\varphi/\partial x$, and $\partial\psi_1/\partial x$ do not vanish at x° and are not linearly dependent. We omit the tedious proof of this fact.

We also state without proof that $S_j(x) = 0$ defines L^j, asymptotic or initial, if and only if $\lambda(x, t)$ can be written

$$\lambda(x, t) = \frac{N_1(t)Q_1(x, t) + S_j(x)Q_2(x, t)}{N(t)},$$

where $N_1(t)$ is a factor of the polynomial $N(t)$. $Q_1(x, t)$ and $Q_2(x, t)$ are polynomials. This shows that $L_I \cup L_A$ is identical with $\Lambda^C \cup \Gamma_{\mathrm{CH}}$, where Λ^C is the lacunary set of the λ-globalization defined in §2.

Lemma 4.7 shows that in order to find initial and asymptotic components of L it is sufficient to check the cases where $\deg(T) \leq 2$. This makes it possible to list all nonempty occurrences of L_A- and L_I-sets. If $\deg(T) = 1$, clearly all L^j are nonlatent, since $\varphi + \psi_1 t = 0$ defines a function t with singularities at $\{x : \psi_1(x) = 0\}$ on the unique branch of t. If $\deg(T) = 2$, the Ψ-singularities are in general latent: $\psi_2 t^2 + \psi_1 t + \varphi = 0$ defines the function t with two branches, $t = -\psi_1/2\psi_2 \pm \sqrt{(\psi_1^2 - 4\varphi\psi_2)/4\psi_2^2}$. In general only one branch of t develops a singularity at $\{x : \psi_2(x) = 0\} \cap \Gamma$. The other branch takes the value $t(x) = -\varphi(x)/\psi_1(x)$ on $\{x : \psi_2(x) = 0\}$ in a neighborhood of Γ, which means that this nonsingular branch is the initial one. However this latency property fails if ψ_2 and ψ_1 have a common factor. This case can in fact occur. It is equivalent to the case when $\Psi = \Psi^{**}$. Hence, in case $\deg(T) = 2$ and $L^j \subset \Psi^{**}$, L^j is nonlatent.

In like manner it can be shown that the Σ-singularities also in general are latent. If we rewrite the equation $t^2\psi_2 + t\psi_1 + \varphi = 0$ by means of the substitution $s = 1/(t - \alpha_j)$ it reads $c_2 s^2 + c_1 s + c_0 = 0$ where $c_0 = \psi_2$, $c_1 = 2\alpha_j\psi_2 + \psi_1$, $c_2 = T(x, \alpha_j) = \psi_2\alpha_j^2 + \psi_1\alpha_j + \varphi$. Here, if c_2 and c_1 have a common factor, s tends to ∞, i.e., t tends to α_j, on both branches. Hence $\Sigma_j = \{x : c_0(x) = 0\}$ is a nonlatent singularity if c_2 and c_1 have a common factor. This happens in Examples L6 and L7 below and is equivalent to the case when $\Sigma_j = \Sigma_j^{**}$.

With these remarks we present the list of nonlatent L-singularities. For each quadric $\Gamma = \{x : \varphi(x) = 0\}$ we indicate the corresponding λ-function (with respect to $P_m = \Delta_n$), the L^j-components and in some cases the characteristic tangent K. In the first four examples $\deg(T) = 1$ which means that all L^j-components are nonlatent and K is degenerated. In the following five examples $\deg(T) = 2$. Here latent L^j-components may occur and K is not degenerated.

Recall the notation $b^{(k)} = (B^{(k)})^T B^{(k)}$. We also use the notation $W(x^*, x')$ to indicate that the x-variables appearing in W, x^*, do not appear anywhere else in the polynomials φ.

Example L1.

$$\varphi = aI_p(x) + W(x^*, x').$$

$$\lambda = \frac{aI_p(x)}{1 + 4at} + W.$$

$$L^1 = \begin{cases} \{x : I_p(x) = 0\}, & \text{if } p \geq 2, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$L^2 = \{(x, x') : W(x^*, x') = 0\}.$$

If W is a constant $\neq 0$, then $L^1 \subset L_A$ and $L^2 = \varnothing$. Otherwise $L^1 \subset L_I$. Also, $L^2 \subset L_I$.

Example L2.

$$\varphi = a_1 I_{p_1}(x^{(1)}) + a_2 I_{p_2}(x^{(2)}).$$

$$\lambda = \frac{a_1 I_{p_1}(x^{(1)})}{1+4a_1 t} + \frac{a_2 I_{p_2}(x^{(2)})}{1+4a_2 t} = \frac{4a_1 a_2 t(I_{p_1}+I_{p_2}) + a_1 I_{p_1} + a_2 I_{p_2}}{(1+4a_1 t)(1+4a_2 t)}.$$

$$L^1 = \begin{cases} x : I_{p_1}(x^{(1)}) = 0\}, & \text{if } p_1 \geq 2, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$L^2 = \begin{cases} x : I_{p_2}(x^{(2)}) = 0\}, & \text{if } p_2 \geq 2, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$L^3 = \{x : I_{p_1}(x^{(1)}) + I_{p_2}(x^{(2)}) = 0\}; \qquad L^j \subset L_I, \ j = 1, 2, 3.$$

Example L3.

$$\varphi = B^{(1)^T} x^{(0)} + \sum_{\nu=1}^{M_2} (S_2(x^{(02\nu)}) + B_1^{(2\nu)} x_1^{(02\nu)}) + \sum_{\nu=1}^{M_3} S_3(x^{(03\nu)}) + W(x^*, x'),$$

$$\lambda = \psi_1 t + \varphi,$$

where

$$\psi_1 = -b^{(1)} - 2i \sum_{\nu=1}^{M_2} B_1^{(2\nu)} (x_1^{(02\nu)} - i x_2^{(02\nu)}) - 2i \sum_{\nu=1}^{M_3} (x_1^{(03\nu)} - i x_3^{(03\nu)})^2,$$

$$L = \{x : \psi_1(x) = 0\}.$$

Here the noncharacteristic surfaces (xiii), (xiv), and (xv) of Lemma 3.10 are included. In case (xiv) $L = \varnothing$. In case (xiii) and (xv) $L = L_A$. Otherwise $L = L_I$.

Example L4.

$$\varphi = a_1 I_p(x^{(1)}) + \sum_{\nu=1}^{M_2} S_2(x^{(12\nu)}), \qquad p \geq 2.$$

$$\lambda = \frac{4a_1^2 t I_p(x^{(1)}) + \varphi(x)}{(1+4a_1 t)^2}.$$

$$L^1 = \{x : I_p(x^{(1)}) = 0\} = L_I.$$

Example L5.

$$\varphi = a_1 I_2(x^{(1)}) + S_2(x^{(1)}) + W(x^*, x').$$

$$\lambda = \frac{W(1+4a_1 t)^2 + a_1 I_2(1+4a_1 t) + S_2}{(1+4a_1 t)^2}.$$

$$L^1 = \{x : x_1^{(1)} - i x_2^{(1)} = 0\}, \qquad K = \{x : a_1^2(x_1 + i x_2)^2 - 2iW = 0\}.$$

$L^1 = L_A$, if $W = \text{constant} \neq 0$ (φ is reducible if $W \equiv 0$).

$L^1 = L_I$, if W is not constant.

$L^2 = \{(x, x') : W(x^*, x') = 0\}$ is latent, i.e., $L^2 = L_L$.

Example L6.

$$\varphi = a_1 I_p(x^{(1)}) + S_3(x^{(0)}), \qquad p \geq 1.$$

$$\begin{aligned}\lambda &= \frac{a_1 I_p}{1+4a_1 t} + S_3(x^{(0)}) - 4tS_3^{(2)}(x^{(0)}) \\ &= \frac{1}{1+4a_1 t}(\varphi + 4t(x_1^{(0)} - ix_3^{(0)})(a_1(1+i)x_2^{(0)} - 2i(x_1^{(0)} - ix_3^{(0)})) \\ &\qquad - 8ia_1 t^2 (x_1^{(0)} - ix_3^{(0)})^2).\end{aligned}$$

$$L^1 = \{x : x_1^{(0)} - ix_3^{(0)}\} = L_I.$$

$$L^2 = \begin{cases} x : I_p(x^{(1)}) = 0\}, & \text{if } p \geq 2, \\ \varnothing & \text{otherwise.}\end{cases}$$

$$L^2 = L_L.$$

Example L7.

$$\varphi = a_1 I_p(x^{(1)}) + S_3(x^{(13)}), \qquad p \geq 3.$$

$$\begin{aligned}\lambda &= \frac{a_1 I_p}{1+4a_1 t} + \frac{S_3}{(1+4a_1 t)^2} - \frac{4tS_3^{(2)}}{(1+4a_1 t)^3} \\ &= \frac{1}{(1+4a_1 t)^3}(a_1 I_p (1+4a_1 t)^2 + (1+4a_1 t)(x_1^{(13)} - ix_3^{(13)}) \\ &\qquad \cdot ((1+i)x_2^{(13)} - \frac{i}{2a_1}(x_1^{(13)} - ix_3^{(13)})) + \frac{i}{2a_1}(x_1^{(13)} - ix_3^{(13)})^2).\end{aligned}$$

$$L^1 = \{x : x_1^{(13)} - ix_3^{(13)}\} = 0 = L_I.$$

$$L^2 = \{x : I_p(x^{(1)}) = 0\} = L_L.$$

Example L8.

$$\varphi = S_4(x) + W(x^*, x') = (x_1 - ix_4)(x_2 + ix_3) + \frac{i}{2}(x_2 - ix_3)^2 + W.$$

$$\lambda = 8it^2(x_1 - ix_4)^2 - 4it(x_1 - ix_4)(x_2 - ix_3) + \varphi.$$

$$L = \{x : x_1 - ix_4 = 0\} = L_I.$$

$$K = \{x : (x_1 - ix_4)(x_2 + ix_3) + W\} \quad \text{if } W \not\equiv 0.$$

($W \equiv 0$ implies that $K = \{x : x_2 + ix_3 = 0\}$.)

Example L9.

$$\varphi = \sum_{\nu=1}^{M_2}(S_2(x^{(02\nu)} + B_1^{(2\nu)} x_1^{(02\nu)})) + W(x^*, x').$$

$$\lambda = 2ib^{(2)}t^2 + \left(b^{(2)} - 2i\sum_{\nu=1}^{M_2} B_1^{(2\nu)}(x_1^{(02\nu)} - ix_2^{(02\nu)})\right) t + \varphi.$$

If $B_1^{(2\nu)} = k_\nu q(x')$, where $k_\nu \in \mathbb{R}$ and $q(x')$ is linear in x', then $L = \{(x, x') : q(x') = 0\} = L_I$.

Remarks corresponding to the nine cases.

1. This case includes the sphere $\sum_{j=1}^N x_j^2 = 1$ and various cylinders, $\sum_{j=1}^p x_j^2 = 1$, $p < N$. If $n = N$, we have asymptotic singularities in these cases on $\{x : \sum_{j=1}^N x_j^2 = 0\}$ and $\{x : \sum_{j=1}^p x_j^2 = 0\}$ respectively. The classical Newton potential for the sphere and the Schwarz potential for the above surfaces determined in [Kh-Sh 1] show that unbounded singularities actually occur. in [S-S3] appears an example where the singularities exhibit a very strong growth, in fact $u(x) > ce^{1/|x|}$ along a certain path.

If $n < p$, another type of singularity appear:

Example. $P_m = \Delta_1$, i.e., $P(D)U = U_{x_1x_1} +$ lower order derivatives. $\varphi = x_1^2 + x_2^2 - 1$, $N = 2$. Since $n = 1$, φ must be written $\varphi = x_1^2 + W$, where $W = x_2^2 - 1$. Hence $L^1 = \varnothing$ and $L^2 = \{x : W = 0\} = \{x : x_2 = \pm 1\}$. Here K is degenerated and L equals two lines which are tangent to Γ at $\Gamma_{\text{CH}} = \{(0, \pm 1)\}$. Moreover the lines L are clearly the only characteristic curves tangent to Γ at Γ_{CH}. Now a result of Dunau [Du] states that under certain conditions which in this case imply that the surface L is the unique characteristic surface tangent to Γ at a point $x^\circ \in \Gamma_{\text{CH}}$, the same conclusion is valid as in Leray's theorem with L in place of K, even if x° is exceptional (thereby slightly extending Leray's Theorem 2.3). Dunau's conditions involve even the nonprincipal part of P but are satisfied in this second order case, e.g., if the first order part of P is identically zero.

This example, together with some other of type $\varphi = I_p(x) + W$, are the only ones among the quadrics to which Dunau's theorem is applicable.

2. In [Kh-Sh 1, p. 20] the Schwarz potential in $\mathbb{R}^3$ (continuable in $\mathbb{C}^3$) of the conical surface $x_1^2 + x_2^2 - x_3^2 = 0$ is determined. The result is

$$U_\Gamma(x) = C_1(3x_3^2 - |x|^2) + C_2(3x_3^2 - |x|^2)\left[\log\frac{|x| - x_3}{|x| + x_3} + 3x_3|x|\right],$$

where $|x|^2 = x_1^2 + x_2^2 + x_3^2$ and C_1, C_2 two given constants. In agreement with the result here, U_Γ has singularities in $L^1 = \{x : x_1^2 + x_2^2 = 0\}$ and in $L^3 = \{x : |x| = 0\}$. The singularities of U_Γ are logarithmic in L^1 and algebraic in L^3, which shows that L-singularities may be bounded or unbounded.

3. As noted in the example $L = \{x : x_1 - ix_3 = 0\}$ is an asymptotic singularity, if $\varphi = S_3(x) + c_0 = (1 + i)x_2(x_1 - ix_3) + c_0$, $c_0 \neq 0$. This can be observed in $\mathbb{R}^3$ after the substitution $x_3 \mapsto ix_3$ which transforms Δ_3 to $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 - \partial^2/\partial x_3^2$ (i.e., the wave operator) and φ to $\varphi' = c_1x_2(x_1 + x_3) + c_0$. The plane $x_1 + x_3 = 0$ is clearly characteristic with respect to the wave operator and asymptotic to the transformed surface $\Gamma' = \{x : \varphi'(x) = 0\}$.

5. A similar subexample as in case 3 can be given here. If $W = 1$ and if ix_2 is introduced in place of x_2, we get the two-dimensional wave equation and the hyperbola $\Gamma = \{x : (x_1 - x_2)(x_1 + ax_2) + 1 = 0\}$. Here $x_1 - x_2 = 0$ is a characteristic asymptote. It can be shown that K represent the two characteristic tangents of Γ of form $x_1 + x_2 = c$.

6, 7. These two similar examples have the same singular sets. But note the difference: In Example 6 the I_p and S_3 variables are not overlapping whereas they are in Example 7. In case $p = 3$ it can be shown that in both examples there are two different characteristic surfaces which are tangent to Γ along

Γ_{CH}: $L^1 = L_I$ and $L^2 = L_L$. These cases are together with those discussed in Example 1 the only cases of tangent L-sets among the quadrics. The exhibited lack of uniqueness means however that the result of Danau cannot be applied. Moreover, in these cases the latent set L_L satisfies $L_L \cap \Gamma \subset \Gamma_{\mathrm{CH}}$ and is therefore a counterexample to the converse of Lemma 4.5.

To make the example more explicit, suppose that $p = 3$ in Example L6 and $a_1 = a(1+i)$. Moreover we make the same substitution $ix_3 \to x_3$ as in case 3 above to make the example real. Γ_{CH} has two components,

$$\Gamma^1_{\mathrm{CH}}\colon x_2 = x_1 - ix_3 = 0.$$

$$\Gamma^2_{\mathrm{CH}}\colon x_2 + \frac{1}{4a}(x_1 - x_3) = x_1\left(1 - \frac{3}{16a^2}\right) + x_3\left(1 + \frac{3}{16a^2}\right) = 0.$$

L^1 (initial) defined by $x_1 - ix_3 = 0$ and L^2 (latent) defined by $x_1^2 + x_2^2 + x_3^2 = 0$ are both tangent to Γ along Γ^1_{CH}, whereas K, defined by $x_1(1 - 1/16a^2) + x_2/2a + x_3(1 + 1/16a^2) = 0$, is tangent along Γ^2_{CH}. Clearly Γ^1_{CH} is an exceptional component and Γ^2_{CH} is not.

8. Example 8 can also be given a real interpretation. It shows that the Cauchy problem for $U_{x_1x_1} - U_{x_2x_2} + U_{x_3x_3} - U_{x_4x_4} = 0$ with entire Cauchy data on $(x_1 - x_4)(x_2 + x_3) - \frac{1}{2}(x_2 - x_3)^2 = c_0$ may have an L-singularity on the plane $x_1 - x_4 = 0$.

From the list of nonlatent L-singularities we can also extract the fact, not proven so far, that no irreducible component L^j of L is identical with some component of K. For $\deg(T) \le 2$ this follows from the list, we need only check L-sets that are tangent to Γ. For $\deg(T) \ge 3$ it follows from Lemma 4.7.

The list also proves most of the following theorem.

Theorem 4.8. *All solutions of Cauchy problem* (Q^*) *are*

(i) *entire, when* Γ *is identically noncharacteristic, i.e.,*

$$\varphi(x) = B^T x^{(0)} + \sum_{\nu=1}^{q} S_2(x^{(02\nu)}) + C(x'), \quad \text{where } 0 \ne B^T B \in \mathbb{C},\ q \ge 0.$$

(ii) *Defined everywhere, i.e.,* $L = \varnothing$, *when*

(a)

$$\varphi = \varphi_\alpha = \sum_{j=1}^{q} a_j x_j^2 + c_0, \qquad 0 \ne c_0 \in \mathbb{C},\ \text{or}$$

(b)

$$\varphi = \varphi_\gamma = \sum_{q=1}^{p} \sum_{\nu=1}^{M_q} (S_q(x^{(0q\nu)}) + B_1^{(q\nu)} x_1^{(0q\nu)}), \qquad 0 \ne b^{(p)} \in \mathbb{C}$$

$(B^{(p)} \equiv 0$ *and* $0 \ne b^{(p-1)} \in \mathbb{C})$, *or*

(c)

$$\varphi = \varphi_\alpha + \varphi_\gamma.$$

(iii) *Single valued, when* $\varphi = \varphi_3(x)$, *where* φ_3 *is the polynomial given in case* 3 *of the* L*-list.*

Proof. (i) All solutions are entire when $\Gamma_{\mathrm{CH}} = \varnothing$, i.e., Γ is everywhere noncharacteristic, and when $L_A = \varnothing$. Comparing Lemma 3.11(xii)–(xv) and the L-list yields the given polynomial which is identical with case (xiv) of Lemma 3.11.

(ii) It follows from the definition of L, $L = \Psi \cup \Sigma^*$, that $L = \varnothing$ only if ψ_r is a nonzero constant and all nonzero eigenvalues are simple. These conditions yield the three cases.

(iii) $u(x)$ is necessarily single value if (a) $U(x, t)$ is entire and (b) $t(x)$ is single valued. (a) implies that φ must only contain φ_γ-terms. (b) implies that $\deg(T) = 1$.

These conditions imply that the given polynomial is the only remaining possibility.

Remark. In [Kh] part (i) of Theorem 4.8 is proved for all cylindric surfaces, not only quadric.

5. Singularities in $\mathbb{R}^N$

In this section we mostly study equations with principal part $\Delta_N u$ and the singularities in $\mathbb{R}^N$ of the solutions of the corresponding Cauchy problems (Q^*).

To show that the results for solutions of Cauchy problems have relevance for the classical Newton potential of a bounded domain in $\mathbb{R}^N$, we state the following lemma, versions of which can be found in [Kh-Sh 1] and [Kh-Sh 2].

Let u be the Newton potential of the bounded domain Ω in $\mathbb{R}^N$, with mass density f, i.e.,

$$u(x) = \int_\Omega \frac{f(y)\,dy}{|x-y|^N} \qquad (N > 2) \tag{5.1}$$

where f and $\partial\Omega$ are assumed to be real analytic.

Let $u = u_i$ in Ω and $u = u_e$ in Ω^C. It is well known that $u \in C^1(\mathbb{R}^N)$. Let v be the solution of the following Cauchy problem in $\mathbb{R}^N$:

$$\begin{aligned} \Delta_N v &= f && \text{in a neighborhood of } \partial\Omega, \\ v &= \frac{\partial v}{\partial x_j} = 0 && \text{on } \partial\Omega,\ j = 1, \dots, N. \end{aligned}$$

Assume that v can be continued real analytically in a domain Ω_v.

Lemma 5.1. (i) $u_i = v + u_e$ *in* $\Omega^C \cap \Omega_v$.

(ii) $u_e = -v + u_i$ *in* $\Omega \cap \Omega_v$.

This means that since u_e, which satisfies $\Delta u_e = 0$ in Ω^C, is real analytic in Ω^C and u_i, which satisfies $\Delta u_i = f$ in Ω, is real analytic in Ω, v has precisely the same singularities as u_i in Ω^C and the same singularities as u_e in Ω. It turns out that in the quadric case that concerns us here, i.e., ellipsoids in $\mathbb{R}^N$, u_i lacks singularities in Ω^C, whereas u_e in general develops singularities at the focal sets.

Before entering into this study, however, we make a short review of some classical results in this direction for the two-dimensional case and the Laplace equation. Already Herglotz [Her] studied the two-dimensional Newton potential by means of the function which later was named the Schwarz function. For the corresponding Cauchy problem studies have been made by Davis [Da] and

Khavinson-Shapiro in [Kh-Sh 1–2]. We follow here most closely the presentation in Karp [Ka].

Let Γ be a real analytic curve in $\mathbb{R}^2$, defined by $\varphi(x, y) = 0$. (We use here x and y as variables in $\mathbb{R}$. Also, $z = x + iy$, $\overline{z} = x - iy$.). Setting $x = (z+\overline{z})/2$, $y = (z-\overline{z})/2i$, we obtain $\varphi(x, y) = \Phi(z, \overline{z}) = 0$. Solving this equation for $\overline{z}$ gives a function $z \mapsto \overline{z} = s(z)$, called the Schwarz function for Γ. By means of this function the following two-dimensional Cauchy problem for the Laplace equation can be solved.

$$(5.2) \qquad \begin{aligned} &\Delta_2 u(x, y) = 0 \text{ in a neighborhood of } \Gamma. \\ &u = w, \quad \frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} \qquad \text{on } \Gamma, \end{aligned}$$

where $w(x, y)$ is real analytic in a neighborhood of Γ.

Lemma 5.2 [Da, Kh-Sh 1, Ka]. *Problem* (5.2) *has a solution* $u(x, y) = \operatorname{Re} F(z)$, *where*

$$(5.3) \qquad \frac{dF}{dz} = \frac{\partial w}{\partial x}\left(\frac{z+s(z)}{2}, \frac{z-s(z)}{2i}\right) - i\frac{\partial w}{\partial y}\left(\frac{z+s(z)}{2}, \frac{z-s(z)}{2i}\right),$$

where $s(z)$ *is the Schwarz function for* Γ.

The proof can be found in the indicated references. We have immediately

Corollary 5.3. *If* $w(x, y)$ *is real entire* (*i.e., entire when extended to a function in* $\mathbb{C}^2$), *then* S_u, *the set of singularities of* u *is* $S_u = \{(x, y) : z = x + iy \text{ is a singularity of } s(z)\}$.

Herglotz showed [Her] that the two-dimensional Newton potential with mass density given by an entire function has the same property.

Using Corollary 5.3 we locate the singularities of solutions of (5.2) in three different cases.

Example 1. Γ: $a_1x^2 + a_2y^2 - 1 = 0$, $a_1 \neq a_2$.

$$\phi(z, s) = s^2(a_1 - a_2) + 2sz(a_1 + a_2) + z^2(a_1 - a_2) - 4$$

$s(z)$ has algebraic singularities in $\operatorname{Disc}\phi$, the zero set of the discriminant D_ϕ of ϕ as a polynomial in s. It follows that $\operatorname{Disc}\phi = \{z : z^2 = 1/a_1 - 1/a_2\}$ which implies that the singularities are located at the two foci of the ellipse Γ. These singularities are obviously K-singularities in the present terminology.

Example 2. Γ: $x^2+y^2-1 = 0$. $\phi(z, s) = zs-1$, here $s(z) = 1/z$, which gives an unbounded singularity of s at the origin coming from the leading coefficient in ϕ and corresponding to an asymptotic singularity set L_A when interpreted in $\mathbb{C}^2$.

Example 3. Γ: $(x^2 + y^2)^2 + 4x^2 + 2 = 0$. (This curve belongs to the class of bicircular curves also studied by Herglotz [Her].) Here $\phi(z, s) = s^2(z^2 + 1) + 2sz + z^2 + 2$. Here, apart from the algebraic singularities given by the discriminant (computation shows $\{(x, y)\colon z^2 = -1 \pm i\}$), there is also unbounded singularities of s at $z = \pm i$, i.e., at $(x, y) = (0, \pm 1)$, due to the leading coefficient. These singularities are however latent since they only occur on one of two branches of $s(z)$.

The three examples show singularities of type K, L_A, and L_L respectively. (The unbounded singularities of $s(z)$ may or may not induce unbounded singularities of $U(x, y)$. In any case they should be classified as L-singularities.) There are no L_I-singularities in the real two-dimensional quadric case, a fact that also can be read off from the L-list in §4. In order to find latent singularities one has to use at least third order curves as shown by the Schwarz function method.

Herglotz observed the corresponding distinction between bounded and unbounded singularities of $s(z)$ in the case of the Newton potential. He attributed the difference to the fact that for two-dimensional quadrics the bounded singularities appear at ordinary foci, whereas the unbounded appear at extraordinary foci (e.g., the origin). According to a classical definition an ordinary focus of a curve in $\mathbb{R}^2$ is a point (a_1, a_2) such that the isotropic cone $(x_1 - a_1)^2 + (x_2 - a_2)^2 = 0$ in $\mathbb{C}^2$ is tangent to the complexified extension of Γ to $\mathbb{C}^2$ in at least two finite points. The focus is extraordinary if the tangential points lie at infinity (cf. [Pl, Dar]).

Note that the tangential conditions can be interpreted in terms of the Schwarz function s. The isotropic cone is transformed to an expression of type $(z - c_1)(\overline{z} - c_2) = 0$ by the introduction of the z- and $\overline{z}$-variables. Hence ordinary foci correspond to points $z = c_1$, $\overline{z} = c_2$ such that the lines $z = c_1$, $s = c_2$ are tangent to the curve defined by $\Phi(x, s) = 0$. Extraordinary foci occur when lines of this type are tangent at infinity, as are $z = 0$, $s = 0$ in the case of the circle $x^2 + y^2 - 1 = 0$, i.e., $sz - 1 = 0$.

Since the classical definitions of foci can be extended (cf. [Pl]) to spaces with higher dimensions than 2, it is an interesting question to investigate whether the couplings ordinary focus—bounded singularity and extraordinary focus—possibly unbounded singularity hold even in higher dimensions. The results in §4 indicate that they do, as far as singularities appearing at the foci, or focal sets, are concerned. But the results also show that singularities also appear at nonfocal sets. Such is the case, in general, for the initial and latent L-singularities. Hence, the above focal singularity principle cannot be generalized to higher dimensions without restrictions.

We now turn to the study of (Q^*) in $\mathbb{R}^N$, when Γ is a real quadric. In this case it is actually possible to diagonalize the quadratic polynomial and we get essentially only three cases:

$$(5.4)\qquad \begin{aligned} &\text{(i)} && \sum_{j=1}^{M} \frac{I_{p_j}(x^{(j)})}{b_j} = 1, \\ &\text{(ii)} && \sum_{j=1}^{M} \frac{I_{p_j}(x^{(j)})}{b_j} = 0, \\ &\text{(iii)} && \sum_{j=1}^{M} \frac{I_{p_j}(x^{(j)})}{b_j} = x_1^{(0)}. \end{aligned}$$

Note that we have replaced a_j by $1/b_j$.

We assume here that the equation $P(x, D)u = f$ is nonparabolic, i.e., its principal part is $\Delta_N u$. This is no serious restriction. Results corresponding to Theorems 5.4 and 5.6 can easily be proved in the parabolic case. We can write the three versions of (5.4) in a single form

$$\varphi(x) = \sum_{j=1}^{M} \frac{I_{p_j}(x^{(j)})}{b_j} = c_0, \tag{5.5}$$

where $c_0 = 1, 0$ and $x_1^{(0)}$ in cases (i), (ii), and (iii) respectively. Accordingly the equation $\varphi^{(2)} = 0$ can be written (modulo a constant):

$$\sum_{j=1}^{M} \frac{I_{p_j}(x^{(j)})}{b_j^2} = c_1, \tag{5.6}$$

where $c_1 = 0, 0$ and $\frac{1}{4}$ respectively.

The principal bicharacteristics β_y, $y \in \Gamma_{CH}$, are defined by (from (3.6))

$$\text{(i)}\quad x_\nu^{(j)} = y_\nu^{(j)} + \frac{4t}{b_j} y_\nu^{(j)} = y_\nu^{(j)} \left(1 + \frac{4t}{b_j}\right), \qquad j = 1, 2, \dots, M. \tag{5.7}$$

In case (iii) we have also

$$\text{(ii)}\quad x_1^{(0)} = y_1^{(0)} = 2t.$$

We now define the focal sets, F_k, by

$$F_k = \left\{ x \in \mathbb{R}^N : I_{p_k}(x^{k)}) = 0 \quad\text{and}\quad \sum_{\substack{j=1\\ j\neq k}}^{M} \frac{I_{p_j}(x^{(j)})}{b_j - b_k} = c_2 \right\}, \tag{5.8}$$

where $c_2 = 1, 0$ or $x_1^{(0)} - b_k/4$ in cases (i), (ii), and (iii) respectively.

We suppose that $b_1 > b_2 > \cdots > b_M$. In all cases we assume $b_1 > 0$, in case (ii) also $b_M < 0$.

Theorem 5.4. $K \cap \mathbb{R}^N = \bigcup_{k=1}^M F_k$.

This theorem holds for cases (i)–(iii). Note that some F_k may be empty as, e.g., F_1 in case (i).

Proof (case (i)). We first prove that $F_k \subset K \cap \mathbb{R}^N$. Suppose $x \in F_k$. Then we claim that there is a $y \in \Gamma_{CH}$ and a $t \in \mathbb{C}$ such that $x_\nu^{(j)} = y_\nu^{(j)}(1 + 4t/b_j)$ $\nu = 1, 2, \dots, p_j$. Choose $t = -b_k/4$ and $y_\nu^{(j)} = x_\nu^{(j)}/(1 - b_k/b_j)$ for $j \neq k$. Choose finally $y^{(k)}$ such that

$$I_{p_k}(y^{(k)}) = -b_k^2 \sum_{j\neq k} \frac{I_{p_j}(x^{(j)})}{(b_j - b_k)^2},$$

i.e., such that y satisfies (5.6). (Recall that $x \in \Gamma_{CH}$ if x satisfies (5.5) and (5.6).) We check that y satisfies (5.5):

$$\begin{aligned}\sum_{j=1}^{M} \frac{I_{p_j}(y^{(j)})}{b_j} &= \sum_{j\neq k} \frac{I_{p_j}(x^{(j)})}{b_j(1 - b_k/b_j)^2} - \frac{b_k^2}{b_k} \sum_{j\neq k} \frac{I_{p_j}(x^{(j)})}{(b_j - b_k)^2} \\ &= \sum_{j\neq k} \frac{I_{p_j}(x^{(j)})(b_j - b_k)}{(b_j - b_k)^2} = \sum_{j\neq k} \frac{I_{p_j}(x^{(j)})}{b_j - b_k} = 1,\end{aligned}$$

as required. Hence the claim is proved and therefore $F_k \subset K \cap \mathbb{R}^N$.

We next need a lemma.

Lemma 5.5. *β_y $(y \in \Gamma_{\mathrm{CH}})$ intersects $\mathbb{R}^N$ only for some real t.*

Proof. Suppose not. Then (from (5.7)) for some $t = \alpha + i\beta$, $\beta \neq 0$, there is an $x \in \mathbb{R}^N$ such that

$$\sum_{j=1}^{M} \frac{b_j I_{p_j}(x^{(j)})}{(b_j + 4t)^2} = 1 \tag{5.9}$$

and

$$\sum_{j=1}^{M} \frac{I_{p_j}(x^{(j)})}{(b_j + 4t)^2} = 0. \tag{5.10}$$

Note that all denominators are nonzero, when $t \notin \mathbb{R}$. Set

$$\begin{aligned} v_j = \frac{I_{p_j}(x^{(j)})}{(b_j + 4t)^2} &= \frac{I_{p_j}(x^{(j)})}{(b_j + 4(\alpha + i\beta))^2} \\ &= \frac{I_{p_j}(x^{(j)})}{(b_j + 4\alpha)^2 - 16\beta^2 + 8i\beta(b_j + 4\alpha)}. \end{aligned}$$

Hence,

$$\operatorname{Im} v_j = \frac{-8 I_{p_j}(x^{(j)})\beta(b_j + 4\alpha)}{((b_j + 4\alpha)^2 - 16\beta^2)^2 + 64\beta^2(b_j + 4\alpha)^2}$$

or

$$\operatorname{Im} v_j = -k_j\beta(b_j + 4\alpha), \quad \text{where } k_j \geq 0.$$

(5.9) and (5.10) imply

$$\sum_{j=1}^{M} \operatorname{Im} v_j = 0, \tag{5.11}$$

$$\sum_{j=1}^{M} b_j \operatorname{Im} v_j = 0. \tag{5.12}$$

Also (5.9) implies that at least one $I_{p_j}(x^{(j)}) \neq 0$ and hence (5.10) implies that at least two different I_{p_j}-terms, I_{p_r} and I_{p_s} do not vanish, i.e., $k_r \neq 0$ and $k_s \neq 0$.

From (5.11) and (5.12) it follows that for all c:

$$0 = \sum_{j=1}^{M} (b_j + c) \operatorname{Im} v_j = -\sum_{j=1}^{M} k_j\beta(b_j + c)(b_j + 4\alpha).$$

Now, take $c = 4\alpha$. Then, $-\beta\sum_{j=1}^{M} k_j(b_j + 4\alpha)^2 = 0$, which is impossible since k_j, b_j, and α are real, all $k_j \geq 0$ and k_r, $k_s \neq 0$. The lemma is proved.

We can now prove that $K \cap \mathbb{R}^N \subset \bigcup_{k=2}^{M} F_k$. Suppose that $x \in K \cap \mathbb{R}^N$. By Lemma 5.5,

$$x_\nu^{(j)} = y_\nu^{(j)}(1 + 4t/b_j) \quad \text{for some } y \in \Gamma_{\mathrm{CH}} \text{ and for some real } t. \tag{5.13}$$

Since y satisfies (5.5)–(5.6), not all $y_\nu^{(j)} \in \mathbb{R}$. Suppose $y_r^{(k)}$ and $y_s^{(1)} \notin \mathbb{R}$, $k \neq 1$. Then by (5.13) at least one of $x_r^{(k)}$ and $x_s^{(1)}$ is not real which is impossible. Hence only y-components in a single $y^{(k)}$-block are not real. Then $x_\nu^{(k)} \in \mathbb{R}$ only if $t = -b_k/4$, which means that $x_\nu^{(k)} = 0$ for all ν. This determines t. Hence $x_\nu^{(j)} = y_\nu^{(j)}(1 - b_k/b_j)$ for $j = 1, \dots, M$. This implies that $I_{p_k}(x) = 0$ and also, since $y \in \Gamma_{\mathrm{CH}}$ and satisfies (5.5) and (5.6), that $\sum_{j\neq k} I_{p_j}(x^{(j)})/(b_j - b_k) = 1$. (Do the calculations in the beginning of the proof in reverse order.)

This proves the theorem in case (i). The proofs of cases (ii) and (iii) are similar.

Now suppose that $b_k > 0$ and $-b_{k+1} < 0$. Then the λ-globalization of the solution $u(x)$ of (Q^*) is achieved as t varies between $-b_k/4 < 0$ and $-b_{k+1}/4 > 0$ on the real axis. More precisely, $u(x)$ can be analytically continued in $\Lambda_k \backslash K \subset \mathbb{R}^N$, where

$$\Lambda_k = \{x \in \mathbb{R}^N : \lambda(x, t) = 0 \text{ for some } t \in I_k\}, \tag{5.14}$$

where

$$I_k = \{t \in \mathbb{R} : -b_k/4 < t < b_{k+1}/4\}, \quad \text{if } k < M,$$

and

$$I_M = \{t \in \mathbb{R} : -b_M/4 < t\}.$$

Note that the L-singularities are never encountered here since these always correspond to constant t-values of form $t = -b_j/4$ or infinity. This principle of real λ-globalization has clearly a complex counterpart achieved by letting t vary in a complex domain including 0 and avoiding the α_j-values.

In order to describe the real analytic singularities we make the following definitions:

$$X_k := \{x \in \mathbb{R}^N : I_{p_k}(x^{(k)}) = 0\}, \tag{5.15}$$

$$Y_k := \left\{ x \in X_k : \sum_{\substack{j=1 \\ j\neq k}}^{M} \frac{I_{p_j}(x^{(j)})}{b_j - b_k} < c_2 \right\} \tag{5.16}$$

(c_2 takes the values 1, 0 or $x_1^{(0)} - b_k/4$ as in (5.8)).

$$\widetilde{Y}_k := \left\{ x \in X_k : \sum_{\substack{j=1 \\ j\neq k}}^{M} \frac{I_{p_j}(x^{(j)})}{b_j - b_k} > c_2 \right\}. \tag{5.17}$$

Note that $X_k = Y_k \cup F_k \cup \widetilde{Y}_k$ and that F_k is the boundary of both Y_k and $\widetilde{Y}_k$ and X_k.

Now write the equation $\lambda(x, t) = 0$ in the form

$$\sum_{j=1}^{k} \frac{I_{p_j}(x^{(j)})}{b_j + 4t} = \sum_{j=k+1}^{M} \frac{I_{p_j}(x^{(j)})}{-b_j - 4t} + c_3, \tag{5.18}$$

where c_3 takes the values 1, 0 and $x_1^{(0)} + t$ corresponding to the cases (i), (ii), and (iii) in (5.4).

Note that the left member of (5.18) is decreasing in the interval I_k, whereas the right member is increasing for fixed x. Moreover, if $I_{p_k}(x^{(k)}) \neq 0$ then the left member tends to $+\infty$ as t tends to $-b_k/4$ within I_k. Likewise, the right member tends to $+\infty$ as t approaches $-b_{k+1}/4$ in I_k. These observations are sufficient to motivate the claim that in $\mathbb{R}^N \backslash (X_k \cup X_{k+1})$ equation (5.18) has a solution $t \in I_k$. Moreover, if $x \in X_k$, then (5.18) has a solution in I_k if $x \in \widetilde{Y}_k$. Similarly, if $x \in X_{k+1}$, then (5.18) has a solution if $x \in Y_{k+1}$. We can now state the theorem on real existence domains.

Theorem 5.6. *Suppose that* Γ *is a quadratic surface in* $\mathbb{R}^N$ *given by* (5.4) *where* $b_k > 0$ *and* $b_{k+1} < 0$. *Then the real analytic solution of* (Q^*) *exists in* $\mathbb{R}^N \backslash (F_k \cup Y_k \cup F_{k+1} \cup \widetilde{Y}_{k+1})$. *If* $p_k = 1$ *the solution can be analytically continued across* Y_k. *The same holds for* $\widetilde{Y}_{k+1}$ *if* $p_{k+1} = 1$.

Proof. The proof follows directly from the observation motivated above that $\Lambda_k \backslash K$ is identical with the existence domain of the theorem. The continuation across Y_k and $\widetilde{Y}_{k+1}$ when $p_k = 1$ and $p_{k+1} = 1$ respectively follows from the fact that Y_k and $\widetilde{Y}_{k+1}$, normally parts of L-singularities, in the indicated cases are only ghost singularities, i.e., subsets of $L^* \backslash L$. See Definitions 4.6, 4.7.

Corollary 5.7 (for ellipsoids). *In case* $b_M > 0$ *in* (5.4)(i), *i.e., in case the quadrics are ellipsoids, the set of possible singularities of the solution of* (Q^*) *is* $F_M \cup Y_M$ *or (in case* $p_M = 1$*)* F_M.

Remarks. The set F_M in Corollary 5.7 is the focal ellipse (ellipsoid) of the ellipsoid Γ and Y_M the corresponding domain inside F_M. The real focal sets $F_1, \dots, F_{M-1}$, do not appear as sets of singularities in the real analytic case. Thus, e.g., the focal hyperbola in case $N = 3$,

$$\frac{x_1^2}{b_1 - b_2} - \frac{x_3^2}{b_2 - b_3} = 1, \qquad x_2 = 0,$$

is not a set of singularity of $u(x)$ as long as the analytic continuation is performed within $\mathbb{R}^3$. This is also seen by the fact that it intersects the initial ellipsoid at points in $\mathbb{R}^3$ and hence not at characteristic points. The Cauchy-Kovalevskaya theorem excludes the possibility of the hyperbola being an initial set of singularities. However, if the analytic continuation is performed in $\mathbb{C}^3$, partly outside $\mathbb{R}^3$, the focal sets $F_1, \dots, F_{M-1}$ may carry singularities. This is the latency phenomenon for K-sets mentioned in §4 and caused by the fact that the defining polynomial $T(x, t)$ for the function $x \mapsto t(x)$ is of degree > 2 allowing one branch of t to be analytic at a point where two other branches develop an algebraic singularity.

To illustrate Theorem 5.6 we give several examples.

Example 5.1 (cone).

$$\Gamma\colon \frac{I_1(x^{(1)})}{b_1} + \frac{I_2(x^{(2)})}{b_2} = 0, \qquad b_1 > 0, \ b_2 < 0.$$

Here $F_1 = F_2 = \varnothing$ and the set of singularities is $S_u = Y_1 \cup \widetilde{Y}_2 = X_1 \cup X_2$.

Example 5.2 (ellipse in $\mathbb{R}^2$).

$$\Gamma: \frac{x_1^2}{b_1} + \frac{x_2^2}{b_2} = 1, \qquad b_1 > b_2 > 0.$$
$$S_u = F_2 = \{(\pm\sqrt{b_1 - b_2}, 0)\}.$$

Example 5.3 (rotation paraboloid).

$$\Gamma: x_1^2 + x_2^2 = x_3,$$
$$S_u = F_1 \cup Y_1, \quad \text{where } F_1 = \{(0, 0, \tfrac{1}{4})\},$$

and

$$Y_1 = \{x : x_1 = x_2 = 0, \ x_3 > \tfrac{1}{4}\}.$$

Example 5.4 (prolate rotation ellipsoid).

$$\Gamma: \frac{x_1^2}{b_1} + \frac{x_2^2 + x_3^2}{b_2} = 1, \qquad b_1 > b_2 > 0,$$
$$S_u = F_2 \cup Y_2, \quad \text{where } F_2 = \{x : x_1^2 = b_1 - b_2, x_2 = x_3 = 0\},$$
$$Y_2 = \{x : |x_1|^2 < b_1 - b_2, \ x_2 = x_3 = 0\}.$$

Example 5.5 (oblate rotation ellipsoid).

$$\Gamma: \frac{x_1^2 + x_2^2}{b_1} + \frac{x_3^2}{b_2} = 1, \qquad b_1 > b_2 > 0,$$
$$S_u = F_2 = \{x : x_1^2 + x_2^2 = b_1 - b_2, \ x_3 = 0\}.$$

Example 5.6 (one-sheeted hyperboloid).

$$\Gamma: \frac{x_1^2}{b_1} + \frac{x_2^2}{b_2} + \frac{x_3^2}{b_3} = 1, \qquad b_1 > b_2 > 0 > b_3.$$
$$S_u = F_2 \cup F_3,$$
$$F_2 = \left\{x : x_2 = 0, \frac{x_1^2}{b_1 - b_2} - \frac{x_3^2}{b_2 - b_3} = 1\right\} \qquad \text{(hyperbola)}.$$
$$F_3 = \left\{x : x_3 = 0, \frac{x_1^2}{b_1 - b_3} + \frac{x_2^2}{b_2 - b_3} = 1\right\} \quad \text{(ellipse)}.$$

Example 5.7 (two-sheeted hyperboloid).

$$\Gamma: \frac{x_1^2}{b_1} + \frac{x_2^2}{b_2} + \frac{x_3^2}{b_3} = 1, \qquad b_1 > 0 > b_2 > b_3,$$

$$S_u = F_2 \qquad (F_1 = \varnothing, \ Y_1 = X_1 = \{x : x_1 = 0\}, \text{ and hence a ghost singularity}),$$
$$F_2 = \left\{x : x_2 = 0, \frac{x_1^2}{b_1 - b_2} - \frac{x_3^2}{b_2 - b_3} = 1\right\} \text{ (hyperbola, same as in Example 5.6)}.$$

Examples 5.4 and 5.5 can be compared to the following explicit versions of the outer Newton potential u_e of the corresponding ellipsoids considered as boundaries of masses of constant density one, i.e.,

$$u_e(x) = \int_\Omega \frac{dy}{|x-y|}, \tag{5.19}$$

where Ω is domain bounded by the real quadric $(x_1^2+x_2^2)/b_1 + x_3^2/b_2 = 1$. Following Kellogg [Ke, Chapter VII.6] the explicit outer potential can be written (we set $1 = \sqrt{b_1 - b_2}$, which is real if the ellipsoid is oblate, and $l_1 = \sqrt{b_2 - b_1}$, which is real in the prolate case)

$$\begin{aligned} u_e(x) = {} & \frac{1}{l^3}(2l^2 + 2x_3^2 - x_1^2 - x_2^2)\left(\frac{\pi}{2} - \arctan\frac{\sqrt{b_2+\tau}}{l}\right) \\ & + \frac{1}{l^2}\left(\sqrt{b_2+\tau} - \frac{3x_3^2}{\sqrt{b_2+\tau}}\right) \end{aligned} \tag{5.20}$$

or

$$\begin{aligned} u_e(x) = {} & \frac{1}{2l_1^3}(-2l_1^2 + 2x_3^2 - x_1^2 - x_2^2)\ln\left(\frac{\sqrt{b_2+\tau} - l_1}{\sqrt{b_2+\tau} + l_1}\right) \\ & - \frac{1}{l_1^2}\left(\sqrt{b_2+\tau} - \frac{3x_3^2}{\sqrt{b_2+\tau}}\right). \end{aligned} \tag{5.21}$$

$\tau(x)$ is here defined by $(x_1^2+x_2^2)/(b_1+\tau) + x_3^2/(b_2+\tau) = 1$ with the condition $\tau(x) = 0$ on Γ. As this function takes the value $\tau = -b_1$ on the interval $\{x : |x_3| \le l_1,\ x_1 = x_2 = 0\}$ in the prolate case, $u_e(x)$ in (5.21) has a logarithmic singularity on this interval. This conforms with the result in Example 5.4. No similar singularities appear in (5.20), the oblate case (compare Example 5.5).

The unboundedness of the singularities in the prolate case is explained by the fact that when interpreted in $\mathbb{C}^3$, the singular set Y_2 is a subset of an L-singularity. Any other singularity of $u_e(x)$ (in (5.20) or (5.21)) can be attributed to the function τ which has algebraic singularities of the K type. These are the F_2 sets in Examples 5.4 and 5.5. Note that the expression $\sqrt{b_2+\tau}$ does not introduce any new singularity, since $b_2 + \tau = x_3^2 \cdot h(x)$, where $h(x) \neq 0$ in a neighborhood of $\tau^{-1}(-b_2)$. In fact, the set $x_3 = \{x : x_3 = 0\}$ is a ghost singularity contained in $L^*\backslash L$ as mentioned in §4.

The results obtained here are related to the Pompeiu problem (cf. [Be] or [Z]) for ellipsoids. A bounded domain $\Omega \subset \mathbb{R}^N$ is said to have the Pompeiu property if $f \equiv 0$ is the only function f such that $\int_{\sigma(\Omega)} f\,dx = 0$ for all rigid motions σ of Ω. It has been proved (cf. [Be]) that under some mild restrictions on Ω, Ω fails to have the Pompeiu property if and only if the Cauchy problem

(5.22) $\Delta_N u + \alpha u = 1$, in a neighborhood of $\partial\Omega$, u and $\partial u/\partial x$ vanish on $\partial\Omega$,

has a solution in Ω for some complex $\alpha \neq 0$.

This fact together with the present results and Theorem 2.10 make it possible to prove the following theorem.

Theorem 5.8. *All ellipsoids,*

$$E\colon \sum_{j=1}^{M} \frac{I_{p_j}(x)}{b_j} \le 1 \qquad (b_1 > b_2 > \cdots > b_M > 0)$$

have the Pompeiu property.

Proof. Assume that $u(x)$ is any solution of problem (5.22) and that $x^* \in F_M$, the focal ellipsoid. We will show that $u(x)$ must develop a singularity at x^* when continued analytically from ∂E. Since problem (5.22) is covered by Theorem 2.10 on the existence of singularities, $u(x)$ must be singular on ∂E_{CH} and hence at a point $y^* \in \partial E_{\mathrm{CH}}$ such that $x^* \in \beta_{y^*}$. From the proof of Theorem 5.4 it follows that $y_\nu^{*(j)}$ is real if $j < M$.

We now define the following triangular two-dimensional set depending on x^* and y^*:

$$Z = \left\{ x : x_\nu^{(j)} = y_\nu^{*(j)} \left(1 + \frac{4t}{b_j}\right),\ -\frac{b_M}{4} < t < 0,\ j < M \text{ and} \right. \tag{5.23}$$
$$\left. x_\nu^{(M)} = \theta y_\nu^{*(M)} \left(1 + \frac{4t}{b_M}\right),\ -\frac{b_M}{4} < t < 0,\ 0 < \theta < 1 \right\}.$$

It is clear that Z contains a segment of E between y^* and a point $\tilde{y} \in E \cap \mathbb{R}^N \cap \overline{Z}$.

We now claim that an open neighborhood of Z can be taken as a domain of monodromy for the solution $u(x)$. Since this domain borders to the singular set β_{y^*} it is clear that $u(x)$ must develop a singularity at x^* even when continued in $\mathbb{R}^N$ from $\tilde{y}$ to x^*.

To prove the claim it is sufficient to show that $Z \subset \Lambda_\gamma \backslash K_\gamma$ where $\Lambda_\gamma = \{x : \lambda(x, t) = 0, t \in \gamma\}$, $\gamma = \{t \in \mathbb{R} : t > -b_M/4\}$, and $K_\gamma = K \cap \Lambda_\gamma$. K_γ can also be defined as $K_\gamma = \bigcup_{y \in E_{\mathrm{CH}}} \beta_y^\gamma$, where $\beta_y^\gamma = \{x : x(y, t) \in \beta_y,\ t \in \gamma\}$ ($x(y, t)$ given by (2.16).) First, $Z \subset \Lambda_\gamma$ is clear from the definition of Z. It remains to show that Z avoids any β_y^γ.

This follows from the following easily verified facts.

(i) In order that β_y^γ intersect Z, $y^{(j)}$ $(j < M)$ must be real.

(ii) β_y^γ intersects Z only if y is of form

$$y = (ky^{*\prime}, z^{(M)}), \quad \text{where } y^{*\prime} \text{ is defined by } y^* = (y^{*\prime}, y^{*(M)}). \tag{5.24}$$

(iii) If $y^* \in E_{\mathrm{CH}}$, the only other $y \in E_{\mathrm{CH}}$ of form (5.24) are those of form $(\pm y^{*\prime}, z^{(M)})$, $I_{p_M}(z^{(M)}) = I_{p_M}(y^{*(M)})$.

(iv) No β_y^γ, where y is of form $y = (\pm y^{*\prime}, z^{(M)})$, $I_{p_M}(z^{(M)}) = I_{p_M}(y^{*(M)})$, intersects Z.

6. Open problems

A complete investigation of the singularity sets should contain a study (not performed here) of their topological properties, e.g., homotopy groups of K, $K \cup L$, $\mathbb{C}^N \backslash (K \cup L)$, etc., in $\mathbb{C}^N$. Also, the question of reducibility for K is not quite settled here although the problem is reduced to the corresponding problem for Γ_{CH}.

It is natural to ask for results concerning the sets of singularity in the case of higher order or indeed analytic initial surfaces. As mentioned in §3 it seems impossible at present to achieve complete global existence results in these cases using Leray's theory. However, since it is possible to compute the λ-function explicitly in some of these cases, albeit the function may be algebraic and multiple-valued, the discriminant of λ may yield interesting possible singularity sets. The approach of Sternin and Shatalov [S-S 1, 2, 3] seems, however, more promising in this respect even if the singular sets thereby may be difficult to determine explicitly.

Another open problem concerns the possibility of weakening the requirement on the Cauchy data to be entire. In the real two-dimensional case the Schwarz function method makes it possible to follow the propagation from $\mathbb{C}^2$ to $\mathbb{R}^2$ of singularities originating from the Cauchy data. It is not clear whether a similar analysis using Leray's theory would be successful. By a different method, in [J] called the method of globalizing families (due essentially to Hörmander), it is possible to prove in some special cases, including the complex sphere and the cone $\sum_{\nu=1}^{p_1}(x_\nu^{(1)})^2 = a\sum_{\nu=1}^{p_2}(x_\nu^{(2)})^2$, that the same conclusions as in Theorem 4.3 are valid if the Cauchy data only are required to be holomorphic in a neighborhood of $\Gamma\backslash\Gamma_{\mathrm{CH}}$.

The local theory of Leray et al. [L, G-K-L] has been further elaborated by Dunau [Du] and Hamada [Ha], who obtain results in neighborhoods of some exceptional points. It would be interesting to know whether the present classification of the singularities into K, L_I, and L_L in these neighborhoods may play a role even for more general initial surfaces.

Khavinson and Shapiro have recently initiated an investigation on the global behaviour of the solutions to Dirichlet problems [Kh-Sh 3, Sh 2]. A natural interesting question, is whether the Dirichlet solutions from entire data may have singularities outside the class of singularity sets originating from the Cauchy problems studied here. Ebenfelt [E] has recently shown that this is actually the case by exhibiting examples (including $x^4 + y^4 = 1$) where the Dirichlet solutions in $\mathbb{R}^2$ develop singularities at infinite sets of points.

This discrepancy between Dirichlet and Cauchy solutions with entire data should be further investigated.

Finally we return to the Khavinson-Shapiro conjecture that the Schwarz potentials cover all possible singularities of solutions to Cauchy problems for the Laplace equation. Comparing the results of the present study with the known facts on Schwarz potentials (e.g., [Kh-Sh 1]) does not give rise to any counterexamples to the conjecture. It is obviously an interesting problem to continue investigations on singularity sets with this conjecture in mind.

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