

ARITHMETIC CALCULUS OF FOURIER TRANSFORMS BY IGUSA LOCAL ZETA FUNCTIONS

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ABSTRACT. We show the possibility of explicit calculation of the Fourier transforms of complex powers of relative invariants of some prehomogeneous vector spaces over \mathbb{R} by using the explicit form of p -adic Igusa local zeta functions.

Let (G, ρ, V) be a regular prehomogeneous vector space defined over \mathbb{R} , and f_1, \dots, f_r the basic \mathbb{R} -relative invariants of (G, ρ, V) (cf. [F. Sato 2, p. 444]). For explicit calculation of Fourier transforms of the complex power

$$|f(x)|^s = \prod_{i=1}^r |f_i(x)|^{s_i}$$

with $s = (s_1, \dots, s_r) \in \mathbb{C}^r$, two general methods are known. When (G, ρ, V) has finitely many G -orbits, one can use a microlocal calculus established by M. Kashiwara according to M. Sato's idea (see p. 404 in [M], [K-K-M]). When $r = 1$ and the relative invariant is linear for each variable, one can use Igusa's method (see p. 8 in [Igusa 2]).

In this paper, we shall show the possibility of calculation for some cases by using the explicit form of the Igusa's local zeta function, based on the idea of Iwasawa-Tate theory [Iwasawa], [Tate].

As an example, we shall calculate the Fourier transform of $|f(x)|^s$ for $(GL_1^4 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1^* \oplus \Lambda_1^*, \text{Alt}_{2m+1} \oplus \text{Aff}^{2m+1} \oplus \text{Aff}^{2m+1} \oplus \text{Aff}^{2m+1})$ which has infinitely many orbits and $r = 4$ so that it has not been calculated by other methods.

The Igusa local zeta function of this prehomogeneous vector space has been explicitly calculated by [Hosokawa]. Our method is applicable for a prehomogeneous vector space with $Z_a = \tau Z_m$ for some constant $\tau > 0$. For example, irreducible (resp. simple, 2-simple of type I) regular prehomogeneous vector spaces with finitely many adelic open orbits satisfy $Z_a = \tau Z_m$ (see [Igusa 4] and [K-K]).

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1. PRELIMINARIES

Let \tilde{G} be a connected linear algebraic group and $\rho: \tilde{G} \rightarrow GL(V)$ a rational representation of \tilde{G} on a finite-dimensional vector space V . Put $G = \rho(\tilde{G}) \subset GL(V)$. When V has a Zariski-dense G -orbit Y , we call a triplet (\tilde{G}, ρ, V) (or a pair (G, V)) a prehomogeneous vector space (abbrev. P.V.). A point of Y is called a generic point and the isotropy subgroup at a generic point is called a generic isotropy subgroup which is unique up to isomorphisms. The complement S of Y is a Zariski-closed set which is called the singular set of (G, V) . An irreducible component S_i of codimension one is the zeros of some irreducible polynomial $f_i(x)$ ($i = 1, \dots, r$). Then these polynomials are algebraically independent relative invariants, i.e., $f_i(\rho(g)x) = \chi_i(g)f_i(x)$ for $g \in \tilde{G}$ and $x \in V$ with some rational characters χ_i of \tilde{G} . Moreover any relatively invariant rational function $f(x)$ is of the form $f(x) = cf_1(x)^{m_1} \cdots f_r(x)^{m_r}$ with $(m_1, \dots, m_r) \in \mathbb{Z}^r$ and some constant c (see p. 60 in [S-K]). We call f_1, \dots, f_r the basic relative invariants of (\tilde{G}, ρ, V) . Now a P.V. is called regular if the Hessian $H_f(x) = \det(\frac{\partial^2 f}{\partial x_i \partial x_j}(x))$ is not identically zero for some relative invariant $f(x)$. In this case, we have $\det \rho(g)^2 = \chi_1(g)^{2\kappa_1} \cdots \chi_r(g)^{2\kappa_r}$ for some $\kappa = (\kappa_1, \dots, \kappa_r) \in (\frac{1}{2}\mathbb{Z})^r$ (see p. 61 in [S-K]). When $r = 1$, we have $\kappa = \frac{n}{d}$ with $d = \deg f$ and $n = \dim V$.

When G is reductive, it is regular if and only if a generic isotropy subgroup is reductive, and it is so if and only if the singular set S is a hypersurface (see p. 73 in [S-K]). In this case, without essential loss of generality, we may assume that a generic isotropy subgroup is semisimple (cf. §3 and §4 in [K-K]). Let k be an algebraic number field. From now on, we assume that (G, V) is a reductive P.V. defined over k with a connected semisimple generic isotropy subgroup, and all coefficients of $f_i(x)$ are in k . We denote by G_A, V_A , etc., the adelization of G, V , etc., with respect to k . Let $\Omega(k_A^\times/k^\times)$ be the space of quasicharacters of the idele class group k_A^\times/k^\times of k and $\mathfrak{S}(V_A)$ the Schwarz-Bruhat space of V_A . For $\omega = (\omega_1, \dots, \omega_r) \in \Omega(k_A^\times/k^\times)^r$, we write $\omega(\chi(g)) = \omega_1(\chi_1(g)) \cdots \omega_r(\chi_r(g))$ and $\omega(f(x)) = \omega_1(f_1(x)) \cdots \omega_r(f_r(x))$ ($g \in G_A, x \in Y_A$) for simplicity. Now we define the two adelic zeta-functions $Z_a(\omega, \Phi)$ and $Z_m(\omega, \Phi)$ of (G, V) .

$$(1.1) \quad Z_a(\omega, \Phi) = \int_{G_A/G_k} \omega(\chi(g)) \sum_{\xi \in Y_k} \Phi(g\xi) d_{G_A}(g),$$

$$(1.2) \quad Z_m(\omega, \Phi) = \int_{Y_A} \omega(f(x)) \Phi(x) d_{Y_A}(x) \quad (\Phi \in \mathfrak{S}(V_A)).$$

Here d_{G_A} is a right-invariant measure on G_A while d_{Y_A} is a G_A -invariant measure on Y_A . Since a generic isotropy subgroup is connected semisimple, we may take the same convergence factor for d_{G_A} and d_{Y_A} . Note that, for the simplest P.V. (GL_1 , Aff^1), we have

$$Z_m = \int_{k_A^\times} \omega(x) \Phi(x) d^\times x = \int_{k_A^\times/k^\times} \omega(x) \sum_{\xi \in k^\times} \Phi(x\xi) d^\times x = Z_a$$

which appears in the original Iwasawa-Tate theory. For $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ and $x = (x_1, \dots, x_r) \in k_A^{\times r}$, we put $\omega_s(x) = |x_1|_A^{s_1} \cdots |x_r|_A^{s_r}$. For any $\omega \in \Omega(k_A^\times/k^\times)^r$, we have $|\omega(x)| = \omega_\sigma(x)$ for some $\sigma = (\sigma_1, \dots, \sigma_r) \in \mathbb{R}^r$. In this case, we write $\sigma(\omega) = (\sigma_1, \dots, \sigma_r) \in \mathbb{R}^r$. It is known that $Z_m(\omega, \Phi)$ is absolutely convergent when $\sigma(\omega) = (\sigma_1, \dots, \sigma_r) > \kappa = (\kappa_1, \dots, \kappa_r)$, i.e., $\sigma_i > \kappa_i$ for $i = 1, \dots, r$ (see [Ono], p. 90 in [F. Sato 1]). One can see that $Z_m(\omega, \Phi)$ has an Euler product $Z_m(\omega, \Phi) = \prod_{v \in \Sigma} Z_v(\omega_v, \Phi_v)$ for $\Phi = \bigotimes_{v \in \Sigma} \Phi_v$ where Σ denotes the set of places of k . We can express the local factor $Z_v(\omega_v, \Phi_v)$ by the Igusa local zeta function for almost all finite places v . We define $Z_v(s)$ to be the Igusa local zeta function $\int_{O_v^n} |f_1(x)|^{s_1} \cdots |f_r(x)|^{s_r} dx$ with $\int_{O_v^n} dx = 1$. It is a rational function of $t_i = q_v^{-s_i}$ ($i = 1, \dots, r$).

Theorem 1.1 ([Igusa 4], [K-K]). *Let (G, ρ, V) be an irreducible (resp. simple, 2-simple of type I) P.V. defined and split over an algebraic number field k satisfying $|G_A \backslash Y_A| < +\infty$. Then we have $Z_a(\omega, \Phi) = \tau Z_m(\omega, \Phi)$ for some constant $\tau > 0$.*

From now on, we assume that $Z_a = \tau Z_m$ for some $\tau > 0$ so that $Z_a(\omega, \Phi)$ converges absolutely when $\sigma(\omega) > \kappa$.

2. FUNCTIONAL EQUATION

We have assumed that (\tilde{G}, ρ, V) or (G, V) is a reductive regular P.V. defined and split over an algebraic number field with a connected semisimple generic isotropy subgroup and $Z_a(\omega, \Phi) = \tau Z_m(\omega, \Phi)$ for some $\tau > 0$. We may assume that $f_i(x) \in O_k[x]$ ($i = 1, \dots, r$) where O_k is the maximal order of k . Let ρ^* be the contragredient representation of ρ on the dual vector space V^* of V . Then (\tilde{G}, ρ^*, V^*) is also a reductive regular P.V. defined over k with the singular set S^* , and $Y^* = V^* - S^*$. Since \tilde{G} is reductive, we have the basic relative invariants f_1^*, \dots, f_r^* satisfying

$$f_i^*(\rho^*(g)y) = \chi_i(g)^{-1} f_i^*(y) \quad (i = 1, \dots, r)$$

for $g \in \tilde{G}$ and $y \in V^*$. By taking a basis, compatible with the k -structure of (G, ρ, V) , we may identify V and V^* with Aff^n so that $\rho^*(\tilde{g}) = {}^t\rho(\tilde{g})^{-1}$ for $\tilde{g} \in \tilde{G}$. We write $g^* \cdot y = {}^t g^{-1} y$ for $g = \rho(\tilde{g}) \in G = \rho(\tilde{G})$, and $y \in V^*$. We define $Z_a^*(\omega, \Psi)$ and $Z_m^*(\omega, \Psi)$ as follows:

$$(2.1) \quad Z_a^*(\omega, \Psi) = \int_{G_A/G_k} \omega(\chi^{-1}(g)) \sum_{\eta \in Y_k^*} \Psi(g^* \eta) d_{G_A}(g),$$

$$(2.2) \quad Z_m^*(\omega, \Psi) = \int_{Y_A^*} \omega(f^*(y)) \Psi(y) d_{Y_A^*}(y) \quad (\Psi \in \mathfrak{S}(V_A^*)).$$

For any place $v \in \Sigma$ of k , let k_v be the local field corresponding to v . For $\Phi_v \in \mathfrak{S}(V_v)$ with $V_v = k_v^n$, let $\hat{\Phi}_v$ be its Fourier transform with respect to the self-dual measure so that $\hat{\Phi}_v(x) = \Phi_v(-x)$ holds. For a finite place v , the self-dual measure $d_1 x$ satisfies $\int_{O_v^n} d_1 x = N(d_v)^{-\frac{n}{2}}$ where d_v is the different of k_v , and hence note that it is not the measure dx satisfying $\int_{O_v^n} dx = 1$ which appears in the definition of the Igusa local zeta function. For $\omega_v = (\omega_v^1, \dots, \omega_v^r)$ where ω_v^i is a quasicharacter of k_v^\times , we define the local zeta function Z_v by

$$Z_v(\omega_v, \Phi_v) \stackrel{\text{def}}{=} \int_{Y_v} \omega_v(f(x)) \Phi_v(x) d_{Y_v}(x).$$

Similarly we define Z_v^* by

$$Z_v^*(\omega_v, \Psi_v) \stackrel{\text{def}}{=} \int_{Y_v^*} \omega_v(f^*(y)) \Psi_v(y) d_{Y_v^*}(y).$$

Lemma 2.1. *For any infinite place $v \in \Sigma_\infty$, there exists $\Phi_v \in \mathfrak{S}(V_v)$ satisfying $\Phi_v \neq 0$, $\Phi_v|_{S_v} = 0$, and $\hat{\Phi}_v|_{S_v^*} = 0$.*

Proof. Put $F^* = f_1^* \cdots f_r^*$ so that S^* is the zeros of F^* . Take a nonzero $\Phi_0 \in C_0^\infty(Y_v)$ and put $\Phi_v = F^*(\text{grad}_x) \Phi_0$. Since $\hat{\Phi}_v(y) = \pm F^*(y) \hat{\Phi}_0(y)$, this Φ_v satisfies our conditions. This is a well-known argument. \square

We denote by A_f the restricted direct product over the finite places.

Lemma 2.2. *There exists $\Phi_f \in \mathfrak{S}(V_{A_f})$ satisfying $\Phi_f \neq 0$, $\Phi_f|_{S_{A_f}} = 0$, and $\hat{\Phi}_f|_{S_{A_f}^*} = 0$.*

Proof. For a finite place v , take $a \in V(O_v) = O_v^n$ satisfying $|F(a)|_v = 1$ with $F = f_1 \cdots f_r$. Since $|F(a + \pi b)|_v = |F(a) + \pi c|_v = 1$ for any $b \in O_v^n$, we have $a + \pi O_v^n \subset Y_v$. Let Φ_v be the characteristic function of $a + \pi O_v^n$. Then we have $\Phi_v \in \mathfrak{S}(V_v)$, $\Phi_v \neq 0$, and $\Phi_v|_{S_v} = 0$. Take a finite place $v' \neq v$ and let $\Phi_{v'}$ be the Fourier transform of the characteristic function of $a' + \pi O_{v'}^n$ with $|F^*(a')|_{v'} = 1$. Since $\Phi_{v'} \neq 0$ and $\hat{\Phi}_{v'}|_{S_{v'}^*} = 0$, if we put $\Phi_f = \Phi_v \cdot \Phi_{v'} \cdot \prod_{w \neq v, v'} \text{ch}_{O_w^n}$ where $\text{ch}_{O_w^n}$ is the characteristic function of O_w^n , this satisfies our condition. \square

To prove the following proposition, we shall use the argument similar to the one on p. 468 in [F. Sato 2].

Proposition 2.3. *For $\Phi \in \mathfrak{S}(V_A)$ satisfying $\Phi|_{S_A} = \hat{\Phi}|_{S_A^*} = 0$, the functions $Z_a(\omega, \Phi)$ and $Z_a^*(\hat{\omega}, \hat{\Phi})$ can be analytically continued to the whole $\Omega(k_A^\times/k^\times)^r$ and they satisfy a functional equation $Z_a(\omega, \Phi) = Z_a^*(\hat{\omega}, \hat{\Phi})$ where $\hat{\omega} = \omega_\kappa \omega^{-1}$ and $\hat{\Phi}$ is the Fourier transform of Φ .*

Proof. Note that $Z_a(\omega, \Phi)$ (resp. $Z_a^*(\hat{\omega}, \hat{\Phi})$) converges absolutely on $A = \{\omega \in \Omega(k_A^\times/k^\times)^r; \sigma(\omega) > \kappa\}$ (resp. $A^* = \{\omega \in \Omega(k_A^\times/k^\times)^r; \sigma(\omega) < 0\}$). Take $\omega_b \in A$ and $\omega_{b^*} \in A^*$ with $b, b^* \in \mathbb{Z}^r$. For $c = b - b^*$ and $\varepsilon = \pm 1$,

put

$$Z_a^\varepsilon(\omega, \Phi) = \int_{\substack{G_A/G_k \\ \omega_c(\chi(g))^\varepsilon \geq 1}} \omega(\chi(g)) \sum_{\xi \in Y_k} \Phi(g\xi) d_{G_A}(g)$$

and

$$Z_a^{*\varepsilon}(\hat{\omega}, \hat{\Phi}) = \int_{\substack{G_A/G_k \\ \omega_c(\chi(g))^\varepsilon \geq 1}} \omega(\chi(g)) \omega_\kappa(\chi^{-1}(g)) \sum_{\eta \in Y_k^*} \hat{\Phi}(g^*\eta) d_{G_A}(g).$$

We have $Z_a(\omega, \Phi) = Z_a^+(\omega, \Phi) + Z_a^-(\omega, \Phi)$ and $Z_a^*(\hat{\omega}, \hat{\Phi}) = Z_a^{*+}(\hat{\omega}, \hat{\Phi}) + Z_a^{*-}(\hat{\omega}, \hat{\Phi})$. Then $Z_a^\varepsilon(\omega, \Phi)$ converges absolutely on

$$B^\varepsilon = \{\omega \in \Omega(k_A^\times/k^\times)^r; \omega\omega_c^{\varepsilon t} \in A \text{ for some } t \geq 0\}$$

and $Z_a^{*\varepsilon}(\hat{\omega}, \hat{\Phi})$ converges absolutely on

$$B^{*\varepsilon} = \{\omega \in \Omega(k_A^\times/k^\times)^r; \omega\omega_c^{\varepsilon t} \in A^* \text{ for some } t \geq 0\}.$$

Note that we have $B^+ = B^{*-} = \Omega(k_A^\times/k^\times)^r$ since $\sigma(\omega_c) > 0$. For example, for any $\omega \in \Omega(k_A^\times/k^\times)^r$, take $t > 0$ satisfying $\sigma(\omega\omega_c^t) > \kappa$, i.e., $\omega\omega_c^t \in A$. Then we have

$$\begin{aligned} \infty &> Z_a^+(\omega\omega_c^t, \Phi) = \int_{\substack{G_A/G_k \\ \omega_c(\chi(g)) \geq 1}} \omega_c(\chi(g))^t \cdot \omega(\chi(g)) \sum_{\xi \in Y_k} \Phi(g\xi) d_{G_A}(g) \\ &\geq Z_a^+(\omega, \Phi). \end{aligned}$$

On $B^\varepsilon \cap B^{*\varepsilon}$, by the adelic Poisson summation formula

$$\sum_{\xi \in Y_k} \Phi(g\xi) = \omega_\kappa(\chi^{-1}(g)) \sum_{\eta \in Y_k^*} \hat{\Phi}(g^*\eta)$$

for our Φ , we have $Z_a^\varepsilon(\omega, \Phi) = Z_a^{*\varepsilon}(\hat{\omega}, \hat{\Phi})$ so that $Z_a^\varepsilon(\omega, \Phi)$ and $Z_a^{*\varepsilon}(\hat{\omega}, \hat{\Phi})$ are analytically continued to $B^\varepsilon \cup B^{*\varepsilon} = \Omega(k_A^\times/k^\times)^r$. Hence $Z_a(\omega, \Phi) = Z_a^+(\omega, \Phi) + Z_a^-(\omega, \Phi)$ and $Z_a^*(\hat{\omega}, \hat{\Phi}) = Z_a^{*+}(\hat{\omega}, \hat{\Phi}) + Z_a^{*-}(\hat{\omega}, \hat{\Phi})$ are analytically continued to the whole $\Omega(k_A^\times/k^\times)^r$ and they satisfy the functional equation $Z_a(\omega, \Phi) = Z_a^*(\hat{\omega}, \hat{\Phi})$. \square

Theorem 2.4. For our P.V. (G, V) , the Euler product $\prod_v Z_v(\omega_v, \Phi_v)$ has a functional equation:

$$(2.3) \quad \prod_{v \in \Sigma} Z_v(\omega_v, \Phi_v) = \prod_{v \in \Sigma} Z_v^*(\hat{\omega}_v, \hat{\Phi}_v)$$

where $\hat{\omega}_v = (\omega_v)_\kappa \omega_v^{-1}$.

Proof. For any $\Phi_f \in \mathcal{S}(V_{A_f})$, put $\Phi = \Phi_\infty^0 \otimes \Phi_f$ where $\Phi_\infty^0|_{S_\infty} = \hat{\Phi}_\infty^0|_{S_\infty} = 0$ and $\Phi_\infty^0 \neq 0$ (cf. Lemma 2.1). Then we have $\Phi|_{S_A} = \hat{\Phi}|_{S_A^*} = 0$ and by

Proposition 2.3, we obtain

$$(2.4) \quad Z_\infty(\omega_\infty, \Phi_\infty^0) \cdot Z_f(\omega_f, \Phi_f) = Z_\infty^*(\hat{\omega}_\infty, \hat{\Phi}_\infty^0) \cdot Z_f^*(\hat{\omega}_f, \hat{\Phi}_f) \\ \text{for any } \Phi_f.$$

On the other hand, for any $\Phi_\infty \in \mathfrak{S}(V_\infty)$, put $\Phi = \Phi_\infty \otimes \Phi_f^0$ with Φ_f^0 as in Lemma 2.2. Similarly we have

$$(2.5) \quad Z_\infty(\omega_\infty, \Phi_\infty) \cdot Z_f(\omega_f, \Phi_f^0) = Z_\infty^*(\hat{\omega}_\infty, \hat{\Phi}_\infty) \cdot Z_f^*(\hat{\omega}_f, \hat{\Phi}_f^0) \\ \text{for any } \Phi_\infty.$$

Multiplying equations (2.4) and (2.5), we have $Z_m(\omega, \Phi)Z_m(\omega, \Phi^0) = Z_m^*(\hat{\omega}, \hat{\Phi})Z_m^*(\hat{\omega}, \hat{\Phi}^0)$ with $\Phi = \Phi_\infty \otimes \Phi_f$ and $\Phi^0 = \Phi_\infty^0 \otimes \Phi_f^0$. By (2.4) with $\Phi_f = \Phi_f^0$, we have $Z_m(\omega, \Phi^0) = Z_m^*(\hat{\omega}, \hat{\Phi}^0)$ and hence $Z_m(\omega, \Phi) = Z_m^*(\hat{\omega}, \hat{\Phi})$ holds for any Φ . \square

3. UNRAMIFIED Γ -FACTORS AND IGUSA LOCAL ZETA FUNCTIONS

By Theorem 2.4, we have

$$(3.1) \quad Z_v(\omega_v, \Phi_v) \cdot \prod_{w \neq v} Z_m(\omega_w, \Phi_w) = Z_v^*(\hat{\omega}_v, \hat{\Phi}_v) \cdot \prod_{w \neq v} Z_m^*(\hat{\omega}_w, \hat{\Phi}_w).$$

$$(3.2) \quad Z_v^*(\hat{\omega}_v, \hat{\Psi}_v) \cdot \prod_{w \neq v} Z_m^*(\hat{\omega}_w, \hat{\Phi}_w) = Z_v(\omega_v, \Psi_v) \cdot \prod_{w \neq v} Z_m(\omega_w, \Phi_w).$$

Multiplying (3.1) and (3.2), we have

$$(3.3) \quad Z_v(\omega_v, \Phi_v) \cdot Z_v^*(\hat{\omega}_v, \hat{\Psi}_v) = Z_v^*(\hat{\omega}_v, \hat{\Phi}_v) \cdot Z_v(\omega_v, \Psi_v).$$

Namely, there exists $\Gamma_v(\omega_v)$ satisfying

$$(3.4) \quad Z_v(\omega_v, \Phi_v) = \Gamma_v(\omega_v) Z_v^*(\hat{\omega}_v, \hat{\Phi}_v) \quad \text{for any } \Phi_v \in \mathfrak{S}(V_v).$$

Thus we can obtain the local functional equation. However, it has already been obtained in more general cases [Igusa 1], [Gyoja].

In this section, we express a Γ -factor $\Gamma_v(\omega_v)$ by the Igusa local zeta function $Z_v(s)$ when ω_v is an unramified quasicharacter $\omega_{v,s} = (| \cdot |_v^{s_1}, \dots, | \cdot |_v^{s_r})$ for a finite place v of k . For one variable case ($r = 1$), including ramified case, see [Igusa 2].

Lemma 3.1. *For a finite place v of k , we have $Z_v(\omega_{v,s}, \text{ch}_{O_v^n}) = c_v Z_v(s - \kappa)$ where $c_v^{-1} = \lim_{s_1, \dots, s_r \rightarrow \infty} Z_v(s)$.*

Proof. We write $|f(x)|_v^s = \omega_{v,s}(f(x)) = |f_1(x)|_v^{s_1} \cdots |f_r(x)|_v^{s_r}$. Then

$$Z_v(\omega_{v,s}, \text{ch}_{O_v^n}) = \int_{Y_v \cap O_v^n} |f(x)|_v^s dY_v(x)$$

where dY_v is a G_v -invariant measure with $\int_{Y_v^0} dY_v = 1$. Here $Y_v^0 = \{x \in O_v^n; |F(x)|_v = 1\}$ with $F(x) = f_1(x) \cdots f_r(x)$. Therefore if we put $dY_v(x) =$

$c_v \frac{dx}{|f(x)|_v^\kappa}$ with $\int_{O_v^n} dx = 1$, we have

$$Z_v(\omega_{v,s}, \text{ch}_{O_v^n}) = c_v \int_{Y_v \cap O_v^n} |f(x)|_v^{s-\kappa} dx = c_v Z_v(s - \kappa).$$

The last equality holds when $\text{Re}(s - \kappa) > 0$, i.e., $|0|_v^{s-\kappa} = 0$, and then it holds for all s by analytic continuation. We have

$$\begin{aligned} 1 &= \int_{Y_v^0} dY_v = c_v \int_{Y_v^0} dx = c_v \int_{|f(x)|_v=1} dx \\ &= c_v \lim_{s \rightarrow \infty} \int_{O_v^n} |f(x)|_v^s dx = c_v \lim_{s \rightarrow \infty} Z_v(s) \end{aligned}$$

since $f_i(x) \in \pi O_v$ implies $\lim_{s_i \rightarrow \infty} |f_i(x)|_v^{s_i} = 0$. \square

Lemma 3.2. For a finite place v of k ,

$$Z_v^*(\hat{\omega}_{v,s}, \hat{\text{ch}}_{O_v^n}) = N(d_v)^{-(d,s)+\frac{n}{2}} \cdot c_v Z_v^*(-s)$$

where d_v = the different of k_v , and $(d, s) = s_1 \deg f_1 + \dots + s_r \deg f_r$ where $Z_v^*(s)$ is the Igusa local zeta function $Z_v(s)$ for f^* .

Proof. By using the self-dual measure, we have $\hat{\text{ch}}_{O_v^n} = N(d_v)^{-\frac{n}{2}} \text{ch}_{(d_v^{-1})^n}$ and hence

$$\begin{aligned} Z_v^*(\hat{\omega}_{v,s}, \hat{\text{ch}}_{O_v^n}) &= Z_v^*(\omega_{v,s-\kappa}, N(d_v)^{-\frac{n}{2}} \text{ch}_{(d_v^{-1})^n}) \\ &= N(d_v)^{-\frac{n}{2}} \cdot c_v \int_{Y_v^* \cap (d_v^{-1})^n} |f^*(x)|_v^{-s} dx. \end{aligned}$$

Now if $d_v^{-1} = \pi_v^{-t} O_v$, then $N(d_v) = q_v^t$. By the change of variables $x = \pi_v^{-t} y$, we have $dx = N(d_v)^n dy$ and $|f^*(x)|_v^{-s} = N(d_v)^{-(d,s)} \cdot |f^*(y)|_v^{-s}$. Hence we obtain our result. \square

By these lemmas, we obtain the following theorem.

Theorem 3.3. For any finite place v of k , we have

$$Z_v(\omega_{v,s}, \Phi_v) = \Gamma_v(\omega_{v,s}) Z_v^*(\omega_{v,s-\kappa}, \hat{\Phi}_v)$$

for any $\Phi_v \in \mathfrak{S}(V_v)$ where Γ_v is given by $\Gamma_v(\omega_{v,s}) = N(d_v)^{(d,s)-\frac{n}{2}} \cdot Z_v(s - \kappa) / Z_v^*(-s)$ where $Z_v(s) = \int_{O_v^n} |f(x)|_v^s dx$ and $Z_v^*(s) = \int_{O_v^n} |f^*(x)|_v^s dx$ with $\int_{O_v^n} dx = 1$ is the Igusa local zeta function and $(d, s) = s_1 \deg f_1 + \dots + s_r \deg f_r$.

4. ARITHMETIC CALCULUS (EXAMPLE)

Theorem 4.1 (Principle of Calculus of Fourier transforms). Let (G, ρ, V) be a reductive regular P.V. defined over an algebraic number field k satisfying $Z_a =$

τZ_m . Then for any $\Phi_\infty \in \mathfrak{S}(V_\infty)$, we have

$$(4.1) \quad \begin{aligned} & Z_\infty(| \cdot |_\infty^s, \Phi_\infty) \cdot \prod_{v \in \Sigma_f} c_v Z_v(s - \kappa) \\ &= |D_k|^{-(d, s) + \frac{n}{2}} \cdot Z_\infty^*(| \cdot |_\infty^{\kappa-s}, \hat{\Phi}_\infty) \cdot \prod_{v \in \Sigma_f} c_v Z_v^*(-s) \end{aligned}$$

where D_k is the discriminant of k , $Z_v(s)$ = the Igusa local zeta function, and Σ_f = the set of finite places of k .

Proof. By Theorem 2.4, Lemmas 3.1 and 3.2 for $\Phi = \Phi_\infty \otimes (\bigotimes_{v \in \Sigma_f} \text{ch}_{O_v^n})$ and $\omega = | \cdot |_\infty^s \cdot \prod_{v \in \Sigma_f} \omega_{v, s}$, we obtain our result. \square

This theorem shows that we can obtain Fourier transforms over \mathbb{R} or \mathbb{C} if we have the explicit form of the Igusa local zeta function. In this section, we shall calculate the Fourier transform of the complex power of the relative invariants of the following (\tilde{G}, ρ, V) where $V = \{\tilde{x} = (X; y, z, w); {}^tX = -X \in M_{2m+1}, y, z, w \in M_{2m+1, 1}\}$, and $\rho(g)\tilde{x} = (\alpha A X^t A; \beta A y, \gamma^t A^{-1} z, \delta^t A^{-1} w)$ for $g = (\alpha, \beta, \gamma, \delta; A) \in \tilde{G} = GL_1^4 \times SL_{2m+1}$ with $m \geq 2$. Then $\ker \rho = \{1, (1, -1, -1, -1; -I_{2m+1})\}$. Put

$$\xi = \left(\left(\begin{array}{cc|c} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right); e_{2m+1}, e_1 + e_{2m+1}, e_{m+1} + e_{2m+1} \right)$$

where

$$e_i = {}^t(0, \dots, 0, \overset{i}{1}, 0, \dots, 0).$$

Then

$$f_1(\tilde{x}) = \text{Pfaffian of } \left(\begin{array}{c|c} X & y \\ \hline -{}^ty & 0 \end{array} \right)$$

is a relative invariant corresponding to $\chi_1(g) = \alpha^m \beta$. Similarly, $f_2(\tilde{x}) = \langle y, z \rangle$ (resp. $f_3(\tilde{x}) = \langle y, w \rangle$, $f_4(\tilde{x}) = {}^t z X w$) is a relative invariant corresponding to $\chi_2(g) = \beta \gamma$ (resp. $\chi_3(g) = \beta \delta$, $\chi_4(g) = \alpha \gamma \delta$). Then we have $S = \{f_1 = 0\} \cup \{f_2 = 0\} \cup \{f_3 = 0\} \cup \{f_4 = 0\}$ with $V - S = \rho(G) \cdot \xi$ and a generic isotropy subgroup $\rho(G_\xi) = Sp_{m-1}$ is connected semisimple. We have $\det \rho(g)^2 = \chi_1^{2(2m-1)} \chi_2^2 \chi_3^2 \chi_4^{4m}$, i.e., $\kappa = (2m-1, 1, 1, 2m)$. By [K-K], this P.V. satisfies $Z_a = \tau Z_m$ for some $\tau > 0$.

Proposition 4.2 (Hosokawa). *Let K be a p -adic field with the maximal compact subring O_K . Let dx be the Haar measure on V_K satisfying $\int_{V(O_K)} dx = 1$. Then the Igusa local zeta function $Z_K(s) = \int_{V(O_K)} |f_1(x)|_K^{s_1} \cdots |f_4(x)|_K^{s_4} dx$ is given by*

$$(4.2) \quad \begin{aligned} Z_K(s) = & \prod_{i=1}^m \frac{(2i-1)}{1 - q^{-(s_1+2i-1)}} \cdot \prod_{k=2}^4 \frac{(1)}{1 - q^{-(s_k+1)}} \\ & \cdot \frac{(2m)}{1 - q^{-(s_4+2m)}} \cdot \frac{(2m+1)}{1 - q^{-(s_5+2m+1)}} \end{aligned}$$

where $(i) = 1 - q^{-i}$, $s_5 = s_1 + \dots + s_4$ and q = the module of K .

Now we take $k = \mathbb{Q}$ in Theorem 4.1 and we have

$$(4.3) \quad \int_{V_{\mathbf{R}} - S_{\mathbf{R}}} |f(x)|_{\mathbb{R}}^s \Phi_{\mathbb{R}}(x) dx = \gamma(s) \int_{V_{\mathbf{R}} - S_{\mathbf{R}}} |f(x)|_{\mathbb{R}}^{-s-\kappa} \Phi_{\mathbb{R}}(x) dx$$

with $\gamma(s) = \prod_p c_p Z_p(-s - \kappa) / \prod_p c_p Z_p(s)$ since

$$Z_{\mathbb{R}}(|\cdot|_{\mathbb{R}}^s, \Phi_{\mathbb{R}}) = \int_{V_{\mathbf{R}} - S_{\mathbf{R}}} |f(x)|_{\mathbb{R}}^{s-\kappa} \Phi_{\mathbb{R}}(x) dx.$$

Since

$$c_p^{-1} = \lim_{s_1, \dots, s_4 \rightarrow \infty} Z_{\mathbb{Q}_p}(s) = \prod_{i=1}^m (2i-1) \cdot (1)^3 \cdot (2m) \cdot (2m+1)$$

by Lemma 3.1 and Proposition 4.2, we have

$$\prod_p c_p Z_p(s) = \prod_{i=1}^m \zeta(s_1 + 2i - 1) \cdot \prod_{k=2}^4 \zeta(s_k + 1) \cdot \zeta(s_4 + 2m) \cdot \zeta(s_5 + 2m + 1)$$

where $\zeta(s)$ is the Riemann zeta function. Since $-s - \kappa = (-s_1 - 2m + 1, -s_2 - 1, -s_3 - 1, -s_4 - 2m)$, we have

$$\begin{aligned} \prod_p c_p Z_p(-s - \kappa) &= \prod_{i=1}^m \zeta(-s_1 - 2i + 2) \cdot \prod_{k=2}^3 \zeta(-s_k) \cdot \zeta(-s_4 + 1 - 2m) \\ &\quad \cdot \zeta(-s_4) \cdot \zeta(-s_5 - 2m). \end{aligned}$$

Thus, by using the functional equation of the Riemann zeta function:

$$\zeta(-s)/\zeta(1+s) = (2\pi)^{-s-1} \cdot 2 \cdot \left(-\sin \frac{\pi s}{2}\right) \cdot \Gamma(1+s),$$

we have

$$\begin{aligned} \gamma(s) &= \prod_p c_p Z_p(-s - \kappa) / \prod_p c_p Z_p(s) \\ &= \prod_{i=1}^m \frac{\zeta(-(s_1 + 2i - 2))}{\zeta(1 + (s_1 + 2i - 2))} \cdot \prod_{k=2}^4 \frac{\zeta(-s_k)}{\zeta(1 + s_k)} \cdot \frac{\zeta(-(s_4 + 2m - 1))}{\zeta(1 + (s_4 + 2m - 1))} \cdot \frac{\zeta(-(s_5 + 2m))}{\zeta(1 + (s_5 + 2m))} \\ &= \prod_{i=1}^m (2\pi)^{-s_1 - 2i + 1} \cdot 2 \cdot \left(-\sin \frac{\pi(s_1 + 2i - 2)}{2}\right) \cdot \Gamma(s_1 + 2i - 1) \\ &\quad \cdot \prod_{k=2}^4 (2\pi)^{-s_k - 1} \cdot 2 \cdot \left(-\sin \frac{\pi s_k}{2}\right) \cdot \Gamma(1 + s_k) \\ &\quad \cdot (2\pi)^{-s_4 - 2m} \cdot 2 \cdot \left(-\sin \frac{\pi(s_4 + 2m - 1)}{2}\right) \cdot \Gamma(s_4 + 2m) \\ &\quad \cdot (2\pi)^{-s_5 - 2m - 1} \cdot 2 \cdot \left(-\sin \frac{\pi(s_5 + 2m)}{2}\right) \cdot \Gamma(s_5 + 2m + 1) \end{aligned}$$

with $s_5 = s_1 + s_2 + s_3 + s_4$

$$\begin{aligned}
&= (2\pi)^{-t} \cdot (-2)^{m+5} \cdot \prod_{i=1}^m \sin \frac{\pi(s_1 + 2i - 2)}{2} \cdot \prod_{k=2}^4 \sin \frac{\pi s_k}{2} \cdot \sin \frac{\pi(s_4 + 2m - 1)}{2} \\
&\quad \cdot \sin \frac{\pi}{2}(s_1 + s_2 + s_3 + s_4 + 2m) \cdot \prod_{i=1}^m \Gamma(s_1 + 2i - 1) \cdot \prod_{k=2}^4 \Gamma(s_k + 1) \\
&\quad \cdot \Gamma(s_4 + 2m) \cdot \Gamma(s_1 + s_2 + s_3 + s_4 + 2m + 1) \\
&\quad \text{with } t = (m + 1)s_1 + 2s_2 + 2s_3 + 3s_4 + (m + 2)^2.
\end{aligned}$$

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