THE NILPOTENCY CLASS OF FINITE GROUPS OF EXPONENT p

MICHAEL VAUGHAN-LEE

ABSTRACT. We investigate the properties of Lie algebras of characteristic p which satisfy the Engel identity $xy^n = 0$ for some $n < p$. We establish a criterion which (when satisfied) implies that if a and b are elements of an Engel- n Lie algebra L then ab^{n-2} generates a nilpotent ideal of L . We show that this criterion is satisfied for $n = 6$, $p = 7$, and we deduce that if G is a finite m -generator group of exponent 7 then G is nilpotent of class at most $51m^8$.

1. INTRODUCTION

In 1989 E. I. Zelmanov [18, 19] solved the restricted Burnside problem by proving that for every positive integer m and every prime-power p^k there is a bound on the possible orders of finite m -generator groups of exponent p^k . Together with results of Hall and Higman [3], this proves that for all positive integers m and n there is a bound on the possible orders of finite m -generator groups of exponent n . Nevertheless, the problem of obtaining explicit bounds remains open in most cases. If G is a finite m -generator group of exponent p^k then G is nilpotent, and in some ways it is more natural to look for bounds on the nilpotency class of G , rather than to look directly for bounds on the order of G . If we can show that the class of G is bounded by c , say, then it is easy to see that the order of G is at most $p^{k.m^c}$. However, neither Zelmanov's solution of the restricted Burnside problem for prime-power exponent, nor Kostrikin's 1959 solution of the problem for prime exponent [8], give explicit bounds for either the order or the class of G . Adjan and Razborov [1] gave the first primitive recursive bound for the class of a finite m -generator group of exponent p , and similar bounds may be found in [9]. Vaughan-Lee and Zelmanov [17] showed that if G is a finite m -generator group of exponent p then the class of G is at most

$$\underbrace{m \overbrace{m \dots m}^m}_{3^p},$$

and they showed that if G is a finite m -generator group of prime-power expo-

Received by the editors February 10, 1994.

1991 *Mathematics Subject Classification*. Primary 20D15, 20F12, 20F18; Secondary 20F40, 20F45, 20F50.

©1994 American Mathematical Society
0002-9947/94 \$1.00 + \$.25 per page

nent q then the class of G is at most

$$\underbrace{m^{m^{\cdot^{\cdot^{\cdot^m}}}}}_{q^{q^q}}.$$

It should be possible to improve these bounds considerably, and in special cases we can do much better. Groups of exponent 2 are abelian, and groups of exponent 3 are nilpotent of class at most 3 (Levi and van der Waerden [10]). Groups of exponent 4 are locally finite (Sanov [12]), and it follows from results of Gupta and Newman [2] and Razmyslov [11] that m -generator groups of exponent 4 have class at most $3m-2$. Higman [7] solved the restricted Burnside problem for exponent 5 in 1956, and he showed that there is a positive integer N such that if G is a finite m -generator group of exponent 5 then G is nilpotent of class at most Nm . Higman did not compute an explicit value for N , but Havas, Newman, and Vaughan-Lee [4] showed that it is possible to take $N = 6$ in Higman's result. The largest finite 2-generator group of exponent 5 has class 12 ([5]), and the largest finite 3-generator group of exponent 5 has class 17 ([15]). So $N = 6$ is best possible, at least for small values of m . In this article we show that if G is a finite m -generator group of exponent 7 then G is nilpotent of class at most $51m^8$. Although this result is unlikely to be best possible, it is still considerably better than the bound

$$\underbrace{m^{m^{\cdot^{\cdot^{\cdot^m}}}}}_{3^7}$$

mentioned above.

The results of Higman, Kostrikin, and Zelmanov all make use of the connection between groups of prime-power exponent and their associated Lie rings. If G is a finite m -generator group of exponent p , and if L is the associated Lie ring of G , then L is an m -generator Lie ring of characteristic p , and L satisfies the Engel identity $xy^{p-1} = 0$. (See [16] for a proof of this fact.) We can think of L as a Lie algebra over the field of integers modulo p , and Kostrikin solved the restricted Burnside problem for exponent p by proving that if L is a finitely generated Lie algebra over a field of characteristic p , and if L satisfies the Engel- $(p-1)$ identity, then L is nilpotent (of bounded class). Since the nilpotency class of a finite group of exponent p is the same as the nilpotency class of its associated Lie ring, his theorem proves that the class of a finite m -generator group of exponent p is bounded.

The connection with the restricted Burnside problem has motivated a number of researchers to investigate Engel Lie algebras. It is well known that if L is an Engel-2 Lie algebra over a field K , then L is nilpotent of class at most 3. (For a proof see [16].) If L is an Engel-3 Lie algebra over a field K , then L is nilpotent of class at most 4 provided $\text{char}(K) \neq 2, 5$ (see [6] and [13]). In the case when $\text{char}(K) = 2$, then an m -generator Engel-3 Lie algebra over K is nilpotent of class at most $2(m+1)^6$ (see [14]), and in the case when $\text{char}(K) = 5$, then an m -generator Engel-3 Lie algebra over K is nilpotent of

class at most $2m$. Engel-4 Lie algebras are nilpotent of class at most 8 provided the characteristic of the ground field is not 2, 3, or 5 (see [13]). Traustason [14] has shown that an m -generator Engel-4 Lie algebra over a field of characteristic 2 is nilpotent of class at most $2(m+1)^6$. Over fields of characteristic 3 and 5, the class of an m -generator Engel-4 Lie algebra is bounded by $3m$ and $6m$ respectively. It is easy to see that if L is an Engel- p Lie algebra over a field of characteristic p , then ab^{p-1} is central in L for all $a, b \in L$. So if we let $\zeta(L)$ be the centre of L , then $L/\zeta(L)$ is Engel- $(p-1)$. It follows from this and from the results on Engel-4 Lie algebras, that the class of an m -generator Engel-5 Lie algebra over a field of characteristic 5 is at most $6m+1$. Traustason [14] has shown that if $\text{char}(K) > 5$, then an m -generator Engel-5 Lie algebra over K is nilpotent of class at most $79m$.

A number of results mentioned above make use of a reduction due to Higman [7]. He introduced a Lie algebra L defined as follows. We let F be the free Lie algebra with free generators $x, a_1, a_2, \dots$ over a field K . If we let I be the ideal of F generated by $\{ab^n | a, b \in F\}$, then F/I is a (relatively) free Lie algebra in the variety of Lie algebras determined by the Engel- n identity $xy^n = 0$. We let $L = F/(I + J)$, where J is the ideal of F generated by $\{a_i a_j | i, j \geq 1\}$. So if we set $\bar{x} = x + I + J$, and $\bar{a}_i = a_i + I + J$ for $i = 1, 2, \dots$, then L is an Engel- n Lie algebra over K , L is generated by $\bar{x}, \bar{a}_1, \bar{a}_2, \dots$, and $\bar{a}_i \bar{a}_j = 0$ for all $i, j \geq 1$. It is easy to see that provided $\text{char}(K) > n$, then L is nilpotent if and only if the ideal of L generated by $\bar{x}$ is nilpotent, and Higman showed that if this ideal is nilpotent of class N then the ideal of F/I generated by $x + I$ is also nilpotent of class N . So, if $\text{Id}_L(\bar{x})$ is nilpotent of class N , it follows that if x is an arbitrary element of an arbitrary Engel- n Lie algebra M over K , then the ideal of M generated by x is nilpotent of class at most N . This implies that m -generator Engel- n Lie algebras over K are nilpotent of class at most Nm . Higman showed that L is nilpotent in the case when $n = 4$, provided that $\text{char}(K) \neq 2, 3$. Higman did not explicitly compute the class of L , but Havas, Newman, and Vaughan-Lee [4] used a computer program to compute L when $n = 4$ and $\text{char}(K) = 5$. They found that L has class 12 in this case, and that the ideal of L generated by $\bar{x}$ is nilpotent of class 6. It is from this result that we deduce that if G is a finite m -generator group of exponent 5 then G has class at most $6m$. Higman's algebra L is also known to be nilpotent for the case $n = 3$, $\text{char}(K) \neq 2$, and for the case $n = 5$, $\text{char}(K) \neq 2, 3$ (see [14]). On the other hand Traustason [13] has found Engel-3 Lie algebras of characteristic 2 in which the ideal generated by an element is not nilpotent, and I have similar examples of Engel-5 Lie algebras of characteristic 3.

In this article we consider a certain quotient algebra M of Higman's algebra L . We let X be the ideal of F generated by $\{xa_i a_j | i, j \geq 1\}$, and we let $M = F/(I + J + X)$. We show that if $\text{char}(K) > n$ then the nilpotency of M implies that $\text{Id}_A(ab^{n-2})$ is nilpotent whenever a, b are elements of an Engel- n Lie algebra A over K . I have used an implementation of the nilpotent quotient algorithm for graded Lie rings (see [4]) to compute M for $n = 6$ and $\text{char}(K) = 7$. The computations show that M is nilpotent of class 22 in this case. We are able to deduce from this, and from other properties of M , that m -generator Engel-6 Lie algebras of characteristic 7 are nilpotent of class at

most $51m^8$. It follows from this that finite m -generator groups of exponent 7 are nilpotent of class at most $51m^8$.

2. REDUCTION THEOREMS

As above, we let K be an arbitrary field, and we let F be the free Lie algebra over K with free generators $x, a_1, a_2, \dots$. If u is any (Lie) product of these free generators then we define the multiweight of u to be $\underline{w} = (w_0, w_1, w_2, \dots)$ where w_0 is the number of times x appears in the product u , and where for each $i = 1, 2, \dots$, w_i is the number of times the free generator a_i occurs in the product. Note that $w_i = 0$ for all but finitely many values of i . If $\underline{w}$ is any given multiweight then we let $F_{\underline{w}}$ be the K -subspace of F spanned by products of multiweight $\underline{w}$. We call $F_{\underline{w}}$ the multihomogeneous component of F of multiweight $\underline{w}$. The free Lie algebra F is multigraded in the sense that it is the direct sum of its multihomogeneous components, and satisfies $F_{\underline{v}} F_{\underline{w}} \leq F_{\underline{v} + \underline{w}}$ for all multiweights $\underline{v}$ and $\underline{w}$. (Here addition of multiweights is defined componentwise.) We suppose that $\text{char}(K) > n$, which implies that the Engel- n identity $xy^n = 0$ is equivalent to the multilinear identity $S_n(x, y_1, y_2, \dots, y_n) = 0$, where

$$S_n(x, y_1, y_2, \dots, y_n) = \sum_{\sigma \in \text{Sym}(n)} x y_{\sigma(1)} y_{\sigma(2)} \cdots y_{\sigma(n)}.$$

To see this, substitute $y_1 + y_2 + \cdots + y_n$ for y in xy^n , expand, and pick out the terms which are multilinear in $y_1, y_2, \dots, y_n$. We obtain the multilinear word $S_n(x, y_1, y_2, \dots, y_n)$. On the other hand if we substitute y for $y_1, y_2, \dots, y_n$ in $S_n(x, y_1, y_2, \dots, y_n)$ then we obtain $n! xy^n$. So the ideal I of F generated by values of xy^n is generated by values of $S_n(x, y_1, y_2, \dots, y_n)$. It follows that I is a multigraded ideal of F , and hence that the relatively free Engel- n Lie algebra F/I is multigraded. The ideal J of F generated by $\{a_i a_j | i, j \geq 1\}$ is also multigraded. As above we let $L = F/(I + J)$ and we let $\bar{x} = x + I + J$, $\bar{a}_i = a_i + I + J$ for $i = 1, 2, \dots$. The following lemma was mentioned above, and is due to Higman [7].

Lemma 1. *Suppose that $\text{Id}_L(\bar{x})$ is nilpotent of class N . If u is a product of the free generators of F , and if u has multiweight $(w_0, w_1, w_2, \dots)$ where $w_0 > N$, then $u \in I$.*

Proof. The proof is by induction on $\sum_{i>0} w_i$. First note that the result is trivial if $\sum_{i>0} w_i = 0$. Now let u have multiweight $(w_0, w_1, w_2, \dots)$ where $w_0 > N$ and where $\sum_{i>0} w_i > 0$. Then

$$u + I + J \in (\text{Id}_L(\bar{x}))^{w_0} = \{0\},$$

and so $u \in I + J$. Since F, I , and J are all multigraded this means that we can express u in the form $u = a + b$, where $a \in I$ and $b \in J$, and where a and b have the same multiweight as u . Since $b \in J$ we can express b as a linear combination of products c of the form $c = a_i a_j v_1 v_2 \cdots v_m$ where $v_1, v_2, \dots, v_m$ are elements from the generating set $\{x, a_1, a_2, \dots\}$ of F , and where c has multiweight $(w_0, w_1, w_2, \dots)$. We let c be a product of this form, and we pick an index k such that $w_k = 0$. Then we let $d = a_k v_1 v_2 \cdots v_m$. Clearly d has multiweight $(w_0, w_1, \dots, w_i - 1, \dots, w_j - 1, \dots, w_k + 1, \dots)$ and so by induction $d \in I$. Now consider the endomorphism of the free Lie algebra F

which maps x to x , maps a_r to a_r for $r \neq k$, and maps a_k to $a_i a_j$. The product c is the image of d under this endomorphism, and so since $d \in I$ and since I is fully invariant, we see that $c \in I$. Since b is a linear combination of products c of this form, this implies that $b \in I$. Hence $u \in I$, as required. $\square$

In this article we make use of the following variation on Lemma 1. We let F , I , and J be as above, and we let X be the ideal of F generated by $\{x a_i a_j | i, j \geq 1\}$. Then we let $M = F/(I + J + X)$, and let $\tilde{x} = x + I + J + X$, $\tilde{a}_i = a_i + I + J + X$ for $i = 1, 2, \dots$.

Lemma 2. *Suppose that $\text{Id}_M(\tilde{x})$ is nilpotent of class N . If u is a product of the free generators of F , and if u has multiweight $(w_0, w_1, w_2, \dots)$ where $w_0 - \sum_{i>0} w_i > N$, then $u \in I$.*

Proof. The proof is by induction on $\sum_{i>0} w_i$. First note that the result is trivial if $\sum_{i>0} w_i = 0$. Now let u have multiweight $(w_0, w_1, w_2, \dots)$ where $w_0 - \sum_{i>0} w_i > N$ and where $\sum_{i>0} w_i > 0$. Then

$$u + I + J + X \in (\text{Id}_L(\tilde{x}))^{w_0} \leq (\text{Id}_L(\tilde{x}))^{N+1} = \{0\},$$

and so $u \in I + J + X$. Since F , I , J , and X are all multigraded this means that we can express u in the form $u = a + b + c$, where $a \in I$, $b \in J$, and $c \in X$, and where a , b , and c have the same multiweight as u . The proof that $b \in I$ is exactly the same as in the proof of Lemma 1, and the proof that $c \in I$ is very similar. Consider an element $c \in X$ with multiweight $(w_0, w_1, w_2, \dots)$. We can express c as a linear combination of products d of the form $d = (x a_i a_j) v_1 v_2 \cdots v_m$ where $v_1, v_2, \dots, v_m$ are elements from the generating set $\{x, a_1, a_2, \dots\}$ of F , and where d has multiweight $(w_0, w_1, w_2, \dots)$. We let d be a product of this form, and we pick an index k such that $w_k = 0$. Then we let $e = a_k v_1 v_2 \cdots v_m$. Clearly e has multiweight $(w_0 - 1, w_1, \dots, w_i - 1, \dots, w_j - 1, \dots, w_k + 1, \dots)$ and so by induction $e \in I$. Now consider the endomorphism of the free Lie algebra F which maps x to x , maps a_r to a_r for $r \neq k$, and maps a_k to $x a_i a_j$. The product d is the image of e under this endomorphism, and so since $e \in I$ and since I is fully invariant, we see that $d \in I$. Since c is a linear combination of products d of this form, we see that $c \in I$. We have already noted that $b \in I$, and so $u \in I$, as required. $\square$

We use Lemma 2 to establish the following reduction theorem.

Theorem 3. *Suppose that $\text{Id}_M(\tilde{x})$ is nilpotent of class N , and let A be an arbitrary Engel- n Lie algebra over the field K . If a and b are arbitrary elements of A then the ideal of A generated by ab^{n-2} is nilpotent of class at most $8(N - n) + 41$.*

Proof. The basic idea of the proof is very simple, although it relies on detailed calculations from §§3, 4, and 5. If x is any element of A and if $m > 2$, then it follows from Lemma 6 in §5 that $(\text{Id}_A(x))^m$ is contained in the ideal of A generated by products of the form

$$v = x a_{11} a_{12} \cdots a_{1k_1} x a_{21} a_{22} \cdots a_{2k_2} x \cdots x a_{r1} a_{r2} \cdots a_{rk_r},$$

where v has weight r in x with $\frac{3m+3}{8} \leq r \leq m$, and where v has total weight $r + k_1 + k_2 + \cdots + k_r < r + 2 + \frac{(r-1)(3n-10)}{3}$. If we let $x = ab^{n-2}$ then this

implies that $(Id_A(x))^m$ is contained in the ideal of A generated by products which have weight $r(n-2)$ in b for some r such that $\frac{3m+3}{8} \leq r \leq m$ and have total weight less than $r(n-1) + 2 + \frac{(r-1)(3n-10)}{3}$. Lemma 2 implies that these products are zero provided $r(n-2) - (r+2 + \frac{(r-1)(3n-10)}{3}) \geq N$, that is, if $r \geq 3(N-n) + 16$, or if $m \geq 8(N-n) + 42$. $\square$

Note that the conclusions of Lemma 2 and Theorem 3 are still valid (with the same proofs) under a slightly weaker hypothesis. We can replace the hypothesis that $Id_M(\tilde{x})$ is nilpotent of class N with the hypothesis that if v is a product in M with multiweight $(w_0, w_1, w_2, \dots)$ in the generators $\tilde{x}, \tilde{a}_1, \tilde{a}_2, \dots$, then $v = 0$ whenever $w_0 - \sum_{i>0} w_i > N$.

3. ENGEL- $(p-1)$ LIE ALGEBRAS

Let L be a Lie algebra over a field K of characteristic $p > 5$, and let L satisfy the Engel- $(p-1)$ identity. We suppose that L has an abelian subalgebra A , and an element x such that L is generated by A and x . The Engel- $(p-1)$ identity implies that $uv^{p-1} = 0$ for all $u, v \in L$. If we substitute $v_1 + v_2 + \dots + v_{p-1}$ for v , expand, and pick out the terms which are linear in $v_1, v_2, \dots, v_{p-1}$, then we obtain the identity

$$S_{p-1}(u, v_1, v_2, \dots, v_{p-1}) = 0,$$

where

$$S_{p-1}(u, v_1, v_2, \dots, v_{p-1}) = \sum_{\sigma \in \text{Sym}(p-1)} uv_{\sigma(1)}v_{\sigma(2)} \cdots v_{\sigma(p-1)}.$$

We will use this identity to establish a number of further identities that hold in L . In the identities that follow we let $a, b, c, a_1, a_2, \dots, b_1, b_2, \dots, c_1, c_2, \dots$ denote arbitrary elements of the abelian Lie subalgebra A of L . We state each identity in turn, and follow each statement with a proof.

$$(1) \quad ua_1a_2 \cdots a_{p-1} = 0 \quad \text{for all } u \in L.$$

This follows immediately from the identity $S_{p-1}(u, a_1, a_2, \dots, a_{p-1}) = 0$, using the fact that $a_i a_j = 0$ for $1 \leq i, j \leq p-1$. $\square$

$$(2) \quad xa^{p-3}xb^{p-2} = xa^{p-2}xb^{p-3} = 0.$$

Using the fact that $p-3$ is even, we write $p-3 = 2s$. Then

$$0 = (xa^s)(xa^s) = \sum_{r=0}^s (-1)^r \binom{s}{r} xa^{s+r}xa^{s-r},$$

and so

$$xa^{p-3}xb^{p-2} = \pm \sum_{r=0}^{s-1} (-1)^r \binom{s}{r} xa^{s+r}xa^{s-r}b^{p-2},$$

which is zero by (1). So $xa^{p-3}xb^{p-2} = 0$. Substituting $b_1 + b_2 + \dots + b_{p-2}$ for b , expanding, and picking out the terms which are linear in $b_1, b_2, \dots, b_{p-2}$, we obtain $xa^{p-3}xb_1b_2 \cdots b_{p-2} = 0$. (We refer to this as linearizing with respect

to b , and we will be making frequent use of this process.) Next consider $xa^{p-2}xb^{p-3}$. The identity $S_{p-1}(x, x, a, a, \dots, a) = 0$ gives

$$xa^{p-2}x = - \sum_{r=0}^{p-2} xa^r xa^{p-2-r},$$

and so

$$xa^{p-2}xb^{p-3} = - \sum_{r=0}^{p-3} xa^r xa^{p-2-r}b^{p-3} = 0,$$

by (1), and by the identity $xa^{p-3}xb_1b_2 \cdots b_{p-2} = 0$ just established. $\square$

Now we let Y be the ideal of L generated by all elements of the form $xa_1a_2 \cdots a_{p-2}$, with $a_1, a_2, \dots, a_{p-2} \in A$. In the remainder of this section we will establish a number of identities which hold in the quotient algebra L/Y .

$$(3) \quad xa^{p-4}xb^{p-2} \in Y.$$

The identity $S_{p-1}(xa^{p-4}x, b, b, \dots, b) = 0$ gives

$$\sum_{r=0}^{p-2} xa^{p-4}b^rxb^{p-2-r} = 0,$$

and using the fact that $xa^{p-4}(xb^{p-2}) \in Y$ we obtain

$$\sum_{r=0}^{p-2} (-1)^r \binom{p-2}{r} xa^{p-4}b^rxb^{p-2-r} \in Y.$$

Now $xa^{p-4}b^r \in Y$ if $r > 1$, and so these two identities give

$$\begin{aligned} xa^{p-4}xb^{p-2} + xa^{p-4}bxb^{p-3} &\in Y, \\ xa^{p-4}xb^{p-2} - (p-2)xa^{p-4}bxb^{p-3} &\in Y. \end{aligned}$$

Identity (3) follows immediately. $\square$

$$(4) \quad xa^{p-3}xb^{p-3} \in Y.$$

Using the same argument as in the proof of (2) we see that if we let $s = (p-3)/2$ then

$$xa^{p-3}xb^{p-3} = \pm \sum_{r=0}^{s-1} (-1)^r \binom{s}{r} xa^{s+r}xa^{s-r}b^{p-3}.$$

Using (1) this gives

$$xa^{p-3}xb^{p-3} = \pm (-1)^{s-1} s xa^{p-4}xab^{p-3},$$

and so (4) follows from the linearization of (3) with respect to b . $\square$

In the remainder of this section we will establish a number of identities involving elements of the form

$$xa_1a_2 \cdots a_rxb_1b_2 \cdots b_sxc_1c_2 \cdots c_t,$$

where $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s, c_1, c_2, \dots, c_t \in A$, and where $r+s+t = 3p-11$. We let M be the subspace of L spanned by elements of this form. Our aim is to prove that all these products lie in Y , so that $M \leq Y$. Clearly such a product lies in Y if $r \geq p-2$. Identity (1) implies that the product is zero if $s \geq p-1$, or $t \geq p-1$. Identity (2) implies that the product is zero if $r+s \geq 2p-5$, and identities (3) and (4) imply that the product lies in Y if $r+s = 2p-6$. So we need only consider products with $r \leq p-3$, and with $2p-9 \leq r+s \leq 2p-7$.

$$(5) \quad xa^rxb^sxc^t \in Y \quad \text{if } r+s+t = 3p-11, r \leq p-5.$$

Note that we may assume that $r < p-2$, and that $s, t < p-1$, and so the linearization of (5) gives

$$xa_1a_2 \cdots a_rxb_1b_2 \cdots b_sxc_1c_2 \cdots c_t \in Y$$

if $r+s+t = 3p-11, r \leq p-5$.

First we deal with the case when $r \leq p-6$. In this case either $s = p-2$ or $t = p-2$. Suppose that $t = p-2$. Then $S_{p-1}(xa^rxb^s, x, c, c, \dots, c) = 0$ gives

$$\sum_{m=0}^{p-2} xa^rxb^sc^mxc^{p-2-m} = 0.$$

Also, using the fact that $xa^rxb^s(xc^{p-2}) \in Y$, we obtain

$$\sum_{m=0}^{p-2} (-1)^m \binom{p-2}{m} xa^rxb^sc^mxc^{p-2-m} \in Y.$$

Now if $r \leq p-6, r+s+t = 3p-11, t = p-2$, then $s \geq p-3$. So if $m > 1$ then $xa^rxb^sc^m = 0$ by (1). So the two identities above give

$$\begin{aligned} xa^rxb^sxc^{p-2} + xa^rxb^sxcxc^{p-3} &= 0, \\ xa^rxb^sxc^{p-2} - (p-2)xa^rxb^sxcxc^{p-3} &\in Y. \end{aligned}$$

The fact that $xa^rxb^sxc^{p-2} \in Y$ (when $r \leq p-6, r+s = 2p-9$) follows immediately from this. The case when $s = p-2$ follows similarly from the identity $S_{p-1}(xa^rx, x, b, b, \dots, b)c^t = 0$, and from the fact that $xa^rx(xb^{p-2})c^t \in Y$.

Now consider the case when $r = p-5$. Just as in the proof of (2), we use the fact that $p-5$ is even, and we let $s = (p-5)/2$. Then

$$0 = (xa^s)(xa^s) = \sum_{r=0}^s (-1)^r \binom{s}{r} xa^{s+r}xa^{s-r},$$

and so

$$xa^{p-5}x = \pm \sum_{r=0}^{s-1} (-1)^r \binom{s}{r} xa^{s+r}xa^{s-r}.$$

So the case when $r = p-5$ reduces to the case when $r \leq p-6$, and the proof of (5) is complete. $\square$

$$(6) \quad xa^rxa^{p-2-r}b^sxc^t \in Y \quad \text{if } 0 \leq r \leq p-2, s+t = 2p-9.$$

Identity (5) implies (6) if $r \leq p - 5$, and clearly (6) holds if $r = p - 2$. So we need only consider the cases when $r = p - 4$ or $p - 3$. The identity $S_{p-1}(x, x, a, a, \dots, a)b^sxc^t = 0$ gives

$$xa^{p-4}xa^2b^sxc^t + xa^{p-3}xab^sxc^t \in Y,$$

and the fact that $x(xa^{p-2})b^sxc^t \in Y$ gives

$$-\binom{p-2}{2}xa^{p-4}xa^2b^sxc^t + (p-2)xa^{p-3}xab^sxc^t \in Y.$$

This completes the proof of (6). $\square$

$$(7) \quad xa^{p-3}xb^{p-6}c^rxc^{p-2-r} \in Y \quad \text{for } 0 \leq r \leq p-2.$$

Identities (1)–(4) imply (7) for $r \geq 3$. The identity

$$S_{p-1}(xa^{p-3}, xb^{p-6}x, c, c, \dots, c) = 0$$

gives $xa^{p-3}(xb^{p-6}x)c^{p-2} \in Y$. If we expand this and use (1) then we obtain

$$2xa^{p-3}xb^{p-6}xc^{p-2} \in Y,$$

which gives (7) for the case $r = 0$. Using these results we see that the identity $S_{p-1}(xa^{p-3}xb^{p-6}, x, c, c, \dots, c) = 0$ gives

$$xa^{p-3}xb^{p-6}cxc^{p-3} + xa^{p-3}xb^{p-6}c^2xc^{p-4} \in Y,$$

and the fact that $xa^{p-3}xb^{p-6}(xc^{p-2}) \in Y$ implies that

$$-(p-2)xa^{p-3}xb^{p-6}cxc^{p-3} + \binom{p-2}{2}xa^{p-3}xb^{p-6}c^2xc^{p-4} \in Y.$$

This completes the proof of (7). $\square$

$$(8) \quad xa_1a_2 \dots a_rb^sxb^{p-2-s}c_1c_2 \dots c_txd_1d_2 \dots d_u \in Y$$

if $0 \leq s \leq p-2$, $r+t+u = 2p-9$.

We prove (8) by induction on r . The case $r = 0$ follows from (6), so suppose that $r > 0$ and suppose that (8) holds for all smaller values of r . Clearly (8) holds if $r+s \geq p-2$, so we suppose that $r+s < p-2$. We let $u = a_1 + a_2 + \dots + a_r + b$. Then (6) implies that

$$xu^{r+s}xu^{p-2-r-s}b^rc_1c_2 \dots c_txd_1d_2 \dots d_u \in Y.$$

If we expand this identity and pick out the terms which are linear in $a_1, a_2, \dots, a_r$ and have degree $p-2$ in b , then using the inductive hypothesis we obtain

$$r! \binom{r+s}{s} xa_1a_2 \dots a_rb^sxb^{p-2-s}c_1c_2 \dots c_txd_1d_2 \dots d_u \in Y.$$

This establishes (8). $\square$

$$(9) \quad xa^{p-3}xb_1b_2 \dots b_rc^sxc^{p-2-s}d_1d_2 \dots d_t \in Y$$

if $0 \leq s \leq p-2$, $r+t = p-6$.

We establish (9) by induction on t , using (7), in the same way as we used (6) to establish (8). $\square$

$$(10) \quad xa^{p-3}xb^{p-5}xc^{p-3} = \alpha xa^{p-3}xb^{p-5}cxc^{p-4} \text{ modulo } Y, \quad \text{for some } \alpha \neq 0.$$

If we use (4) and (7) then the identity $S_{p-1}(xa^{p-3}, x, x, c, c, \dots, c)b^{p-5} = 0$ gives

$$xa^{p-3}xc^{p-5}xc^2b^{p-5} + xa^{p-3}xc^{p-4}xc^2b^{p-5} \in Y.$$

We let $u = b + c$, and use (9), which gives

$$xa^{p-3}xu^{p-5}xu^3b^{p-6} \in Y.$$

We expand this identity, and pick out the terms which are of degree $p-5$ in b and degree $p-3$ in c . This gives

$$3xa^{p-3}xc^{p-5}xc^2b^{p-5} + (p-5)xa^{p-3}xb^{p-6}xc^3b^{p-6} \in Y.$$

Using (9) again we have $xa^{p-3}xbu^{p-6}xu^4b^{p-7} \in Y$. Expanding this we obtain

$$4xa^{p-3}xb^{p-6}xc^3b^{p-6} + (p-6)xa^{p-3}xb^2c^{p-7}xc^4b^{p-7} \in Y.$$

So we can see that $xa^{p-3}xc^{p-5}xc^2b^{p-5}$ is a nonzero linear multiple of $xa^{p-3}xb^2c^{p-7}xc^4b^{p-7}$. Continuing in this way we see that $xa^{p-3}xc^{p-5}xc^2b^{p-5}$ is a nonzero linear multiple of $xa^{p-3}xb^{p-5}xc^{p-3}$. Similarly we see that $xa^{p-3}xc^{p-4}xc^2b^{p-5}$ is a nonzero linear multiple of $xa^{p-3}xb^{p-5}cxc^{p-4}$. So (10) follows from the identity $xa^{p-3}xc^{p-5}xc^2b^{p-5} + xa^{p-3}xc^{p-4}xc^2b^{p-5} \in Y$ proved above. $\square$

$$(11) \quad xa^{p-4}xb^{p-6}c^rxc^{p-1-r} \in Y \quad \text{for } 0 \leq r \leq p-1.$$

It is clear that $xa^{p-4}xb^{p-6}c^rxc^{p-1-r}$ must lie in the ideal of L generated by elements of the form $xa_1a_2 \cdots a_{p-3}$. So, modulo Y , $xa^{p-4}xb^{p-6}c^rxc^{p-1-r}$ can be expressed as a linear combination of elements of the form

$$xa_1a_2 \cdots a_{p-3}xb_1b_2 \cdots b_rxc_1c_2 \cdots c_s,$$

where $r+s = 2p-8$. Using (2), (4), (7), and (10) we may assume that $r = p-4$. So, modulo Y , $xa^{p-4}xb^{p-6}c^rxc^{p-1-r}$ can be expressed as a linear combination of terms of the form

$$xc^ra_1a_2 \cdots a_{p-3-r}xc^sb_1b_2 \cdots b_{p-4-s}xc^td_1d_2 \cdots d_{p-4-t}$$

where $r+s+t = p-1$. We prove that these elements all lie in Y by induction on t . If $t < 2$ then elements of this form lie in Y by (8). So suppose that $t \geq 2$. Let $u = b_1 + b_2 + \cdots + b_{p-4-s} + c$. Then (9) gives

$$xc^ra_1a_2 \cdots a_{p-3-r}xu^{p-4}xu^2c^{t-2}d_1d_2 \cdots d_{p-4-t} \in Y.$$

Expanding this and picking out the terms which are linear in $b_1, b_2, \dots, b_{p-4-s}$ and of degree $p-1$ in c we see that

$$xc^ra_1a_2 \cdots a_{p-3-r}xc^sb_1b_2 \cdots b_{p-4-s}xc^td_1d_2 \cdots d_{p-4-t}$$

is a linear combination of terms which have the same form, but have smaller values of t . So all these elements lie in Y by induction on t . This establishes (11). $\square$

$$(12) \quad xa^{p-4}xb^{p-5}c^rxc^{p-2-r} \in Y \quad \text{for } 0 \leq r \leq p-2.$$

Using (1) and (3) we see that $S_{p-1}(xa^{p-4}xb^{p-5}, x, c, c, \dots, c) = 0$ gives

$$(13) \quad xa^{p-4}xb^{p-5}xc^{p-2} + xa^{p-4}xb^{p-5}cxc^{p-3} + xa^{p-4}xb^{p-5}c^2xc^{p-4} \in Y.$$

Also the fact that $xa^{p-4}xb^{p-5}(xc^{p-2}) \in Y$ gives

$$(14) \quad xa^{p-4}xb^{p-5}xc^{p-2} + 2xa^{p-4}xb^{p-5}cxc^{p-3} + 3xa^{p-4}xb^{p-5}c^2xc^{p-4} \in Y.$$

Finally, $S_{p-1}(xa^{p-4}xb^{p-6}, x, c, c, \dots, c)b = 0$ together with (1) gives

$$(15) \quad xa^{p-4}xb^{p-6}cxc^{p-3}b + xa^{p-4}xb^{p-6}c^2xc^{p-4}b + xa^{p-4}xb^{p-6}c^3xc^{p-5}b \in Y.$$

If we set $u = b + c$, then (11) gives $xa^{p-4}xb^{p-6}uxu^{p-2} \in Y$. Expanding this, and picking out the terms of degree $p-5$ in b and of degree $p-2$ in c then we obtain

$$(p-2)xa^{p-4}xb^{p-6}cxc^{p-3}b + xa^{p-4}xb^{p-5}xc^{p-2} \in Y.$$

Similarly,

$$(p-3)xa^{p-4}xb^{p-6}c^2xc^{p-4}b + 2xa^{p-4}xb^{p-5}cxc^{p-3} \in Y,$$

and

$$(p-4)xa^{p-4}xb^{p-6}c^3xc^{p-5}b + 3xa^{p-4}xb^{p-5}c^2xc^{p-4} \in Y.$$

So (15) gives

$$\begin{aligned} & \frac{1}{p-2}xa^{p-4}xb^{p-5}xc^{p-2} + \frac{2}{p-3}xa^{p-4}xb^{p-5}cxc^{p-3} \\ & + \frac{3}{p-4}xa^{p-4}xb^{p-5}c^2xc^{p-4} \in Y. \end{aligned}$$

This, together with (13) and (14) gives (12). $\square$

$$(16) \quad xa^{p-3}xb^{p-5}xc^{p-3} \in Y.$$

Using (5), (7), and (12) we see that $S_{p-1}(xb^{p-5}, x, x, a, a, \dots, a)c^{p-3} = 0$ gives

$$(17) \quad xb^{p-5}axa^{p-4}xc^{p-3} + xb^{p-5}a^2xa^{p-5}xc^{p-3} \in Y.$$

If we set $u = a + b$ then (8) gives $xb^{p-6}u^2xu^{p-4}xc^{p-3} \in Y$. Expanding, and picking out the terms of degree $p-3$ in a and degree $p-5$ in b we obtain

$$2xb^{p-5}axa^{p-4}xc^{p-3} + (p-4)xb^{p-6}a^2xa^{p-5}bxc^{p-3} \in Y.$$

Similarly $xb^{p-4-r}u^r xu^{p-2-r}b^{r-2}xc^{p-3} \in Y$ gives

$$rxb^{p-3-r}a^{r-1}xa^{p-2-r}b^{r-2}xc^{p-3} + (p-2-r)xb^{p-4-r}a^rxa^{p-3-r}b^{r-1}xc^{p-3} \in Y$$

for $r = 3, 4, \dots, p-4$. Hence

$$xb^{p-5}axa^{p-4}xc^{p-3} - xa^{p-4}xab^{p-5}xc^{p-3} \in Y.$$

Similarly

$$xb^{p-5}a^2xa^{p-5}xc^{p-3} - \frac{2}{(p-4)(p-3)}xa^{p-3}xb^{p-5}xc^{p-3} \in Y,$$

and so (17) gives

$$xa^{p-4}xab^{p-5}xc^{p-3} + \frac{2}{(p-4)(p-3)}xa^{p-3}xb^{p-5}xc^{p-3} \in Y.$$

Now, as we saw in the proof of (2), if we let $s = (p - 3)/2$ then

$$0 = (xa^s)(xa^s) = \sum_{r=0}^s (-1)^r \binom{s}{r} xa^{s+r} xa^{s-r}.$$

It follows that

$$(-1)^{s-1} s xa^{p-4} xab^{p-5} xc^{p-3} + (-1)^s xa^{p-3} xb^{p-5} xc^{p-3} \in Y,$$

and this, together with the identity above, gives (16). $\square$

$$(18) \quad xa^{p-3} xb^r c^{p-4-r} xc^{r+1} d^{p-5-r} \in Y \quad \text{for } 0 \leq r \leq p-5.$$

We prove (18) by induction on r . The case $r = 0$ follows immediately from the identity $S_{p-1}(xa^{p-3}, x, x, c, c, \dots, c)d^{p-5} = 0$, using (4), (7), and (16). The remaining cases follow from the case $v = 0$ and (9) in the same way as (8) follows from (6). $\square$

$$(19) \quad xa^{p-3} xb^{p-4} xc^{p-4} + xa^{p-3} xc^{p-4} xb^{p-4} \in Y.$$

If we set $u = b + c$ then (18) gives

$$xa^{p-3} xb^r u^{p-4-r} xu^{r+1} b^{p-5-r} \in Y \quad \text{for } 0 \leq r \leq p-5.$$

If we expand this and pick out the terms which are of degree $p-4$ in both b and c , then we obtain

$$(p-4-r)xa^{p-3}xb^{r+1}c^{p-5-r}xc^{r+1}b^{p-5-r} + (r+1)xa^{p-3}xb^r c^{p-4-r}xc^r b^{p-4-r} \in Y$$

for $0 \leq r \leq p-5$. This gives (19). $\square$

$$(20) \quad xb^{p-4}xc^{p-4}xa^{p-3} \in Y.$$

Working modulo Y , and using (3), (4), (7), (16), and (19), we obtain

$$\begin{aligned} xb^{p-4}xc^{p-4}xa^{p-3} &= (xb^{p-4})(xc^{p-4})xa^{p-3} \\ &= (xb^{p-4})(xc^{p-4})(xa^{p-3}) + (p-3)(xb^{p-4})(xc^{p-4})axa^{p-4} \\ &= -(xa^{p-3})((xb^{p-4})(xc^{p-4})) + (p-3)(xb^{p-4}a)(xc^{p-4})xa^{p-4} \\ &\quad + (p-3)(xb^{p-4})(xc^{p-4}a)xa^{p-4} \\ &= -2xa^{p-3}xb^{p-4}xc^{p-4} + (p-3)xb^{p-4}axc^{p-4}xa^{p-4} \\ &\quad - (p-3)xc^{p-4}axb^{p-4}xa^{p-4} \\ &= -2xa^{p-3}xb^{p-4}xc^{p-4} - (p-3)xb^{p-4}axa^{p-4}xc^{p-4} \\ &\quad + (p-3)xc^{p-4}axa^{p-4}xb^{p-4}. \end{aligned}$$

Now we can use (8) and (19) to show that

$$xb^{p-4}axa^{p-4}xc^{p-4} = -\frac{1}{p-3}xa^{p-3}xb^{p-4}xc^{p-4}$$

modulo Y , and to show that $xc^{p-4}axa^{p-4}xb^{p-4} = -\frac{1}{p-3}xa^{p-3}xc^{p-4}xb^{p-4} = \frac{1}{p-3}xa^{p-3}xb^{p-4}xc^{p-4}$ modulo Y ; so $xb^{p-4}xc^{p-4}xa^{p-3} \in Y$, as required. $\square$

$$(21) \quad xa^{p-3}xb^{p-4}xc^{p-4} \in Y.$$

Using (7), (12), (16), and (20), the identity

$$S_{p-1}(xa^{p-4}, x, x, a, b, b, \dots, b)c^{p-4} = 0$$

gives

$$(22) \quad (p-3)xa^{p-4}xab^{p-4}xc^{p-4} + xa^{p-3}xb^{p-4}xc^{p-4} \\ + (p-4)xa^{p-4}bxab^{p-5}xc^{p-4} \in Y.$$

Now, as we showed above, $xa^{p-4}xab^{p-4}xc^{p-4} = \frac{2}{p-3}xa^{p-3}xb^{p-4}xc^{p-4}$. Also (6) gives $xa^{p-4}bxab^{p-5}xc^{p-4} = -\frac{1}{p-3}xa^{p-3}xb^{p-4}xc^{p-4}$. So (21) follows from (22). $\square$

We are finally in a position to prove that the subspace M defined above lies in Y .

$$(23) \quad xa_1a_2 \cdots a_rxb_1b_2 \cdots b_sxc_1c_2 \cdots c_t \in Y \quad \text{if } r+s+t = 3p-11.$$

Clearly an element of this form lies in the ideal generated by elements of the form $xd_1d_2 \cdots d_{p-3}$. So we only need to consider the case when $r = p-3$. But then (7) and (16) imply that we may assume that $t \leq p-4$, and (4) implies that we may assume that $s \leq p-4$. This leaves only the case $s = t = p-4$, which is covered by (21). $\square$

4. ENGEL- n LIE ALGEBRAS ($n < p-1$)

Let L be a Lie algebra over a field K of characteristic $p > 5$, and let L satisfy the Engel- n identity for some $n < p-1$. We suppose that L has an abelian subalgebra A , and an element x such that L is generated by A and x . Just as in §3, the Engel- n identity implies that $uv^n = 0$ for all $u, v \in L$, and this is equivalent to the identity

$$S_n(u, v_1, v_2, \dots, v_n) = 0,$$

where

$$S_n(u, v_1, v_2, \dots, v_n) = \sum_{\sigma \in \text{Sym}(n)} uv_{\sigma(1)}v_{\sigma(2)} \cdots v_{\sigma(n)}.$$

The proofs that follow are similar to the proofs in §3, but shorter and easier for the following reason. The identity $S_n(u, v, w, w, \dots, w) = 0$ implies

$$(24) \quad \sum_{i=0}^{n-1} uw^i vw^{n-1-i} = 0.$$

This gives

$$\sum_{i=0}^{n-1} v(uw^i)w^{n-1-i} = 0.$$

Expanding uw^i in this identity gives

$$\sum_{i=0}^{n-1} \sum_{r=0}^i (-1)^r \binom{i}{r} vw^r uw^{n-1-r} = 0.$$

Interchanging the order of the summation we obtain

$$\sum_{r=0}^{n-1} \sum_{i=r}^{n-1} (-1)^r \binom{i}{r} vw^r uw^{n-1-r} = 0,$$

and since $\sum_{i=r}^{n-1} \binom{i}{r} = \binom{n}{r+1}$ this finally gives us

$$(25) \quad \sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1} v w^r u w^{n-1-r} = 0.$$

Note that we are assuming that $n < p-1$, so that the coefficients in this identity are different from those in (24). In the case when $n = p-1$, $\binom{n}{r+1} = (-1)^{r+1}$ for $0 \leq r \leq n-1$, and so identity (25) can be obtained from (24) by interchanging u and v . We will use the identity $S_n(u, v_1, v_2, \dots, v_n) = 0$, together with (24) and (25) to establish a number of further identities that hold in L . In the identities that follow we let $a, b, c, a_1, a_2, \dots, b_1, b_2, \dots, c_1, c_2, \dots$ denote arbitrary elements of the abelian Lie subalgebra A of L . As in §3, we state each identity in turn, and follow each statement with a proof.

$$(26) \quad u a_1 a_2 \cdots a_n = 0 \quad \text{for all } u \in L.$$

This follows immediately from the identity $S_n(u, a_1, a_2, \dots, a_n) = 0$, using the fact that $a_i a_j = 0$ for $1 \leq i, j \leq n$. $\square$

$$(27) \quad x a^{n-2} x b^{n-1} = x a^{n-1} x b^{n-2} = 0.$$

Identity (24) gives

$$\sum_{r=0}^{n-1} x a^{n-2} b^r x b^{n-1-r} = 0,$$

and identity (25) gives

$$\sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1} x a^{n-2} b^r x b^{n-1-r} = 0.$$

Now (26) implies that $x a^{n-2} b^r = 0$ if $r \geq 2$, and so these two identities imply that

$$x a^{n-2} x b^{n-1} = x a^{n-2} b x b^{n-2} = 0.$$

Similarly, (24) and (25) give

$$\begin{aligned} \sum_{r=0}^{n-1} x a^r x a^{n-1-r} b^{n-2} &= 0, \\ \sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1} x a^r x a^{n-1-r} b^{n-2} &= 0, \end{aligned}$$

and these identities, together with (26), imply that

$$x a^{n-1} x b^{n-2} = x a^{n-2} x a b^{n-2} = 0.$$

This establishes (27). $\square$

Now let Y be the ideal of L generated by elements of the form $x a_1 a_2 \cdots a_{n-1}$.

$$(28) \quad x a^{n-4} b^r x b^{n-1-r} \in Y \quad \text{for } 0 \leq r \leq n-1.$$

Clearly $x a^{n-4} b^r x b^{n-1-r} \in Y$ for $r \geq 3$. So (24) and (25) give

$$\begin{aligned} x a^{n-4} x b^{n-1} + x a^{n-4} b x b^{n-2} + x a^{n-4} b^2 x b^{n-3} &\in Y, \\ n x a^{n-4} x b^{n-1} - \binom{n}{2} x a^{n-4} b x b^{n-2} + \binom{n}{3} x a^{n-4} b^2 x b^{n-3} &\in Y. \end{aligned}$$

Also the fact that $xa^{n-4}(xb^{n-1}) \in Y$ gives

$$xa^{n-4}xb^{n-1} - (n-1)xa^{n-4}bxb^{n-2} + \binom{n-1}{2}xa^{n-4}b^2xb^{n-3} \in Y.$$

These three identities establish (28) since

$$\begin{vmatrix} 1 & 1 & 1 \\ n & -\binom{n}{2} & \binom{n}{3} \\ 1 & -(n-1) & \binom{n-1}{2} \end{vmatrix} = -\frac{n^2(n^2-1)}{12} \neq 0. \quad \square$$

$$(29) \quad xa_1a_2 \cdots a_r b^s x b^{n-1-s} c_1c_2 \cdots c_t \in Y \quad \text{for } 0 \leq s \leq n-1 \text{ if } r+t = n-4.$$

This follows from (28) in the same way as we showed in §3 that (8) follows from (6). $\square$

$$(30) \quad xa^{n-2}xb^{n-3} \in Y.$$

Using (28) we see that the identity $S_n(b, x, x, a, a, \dots, a)b^{n-4} = 0$ implies that

$$(31) \quad bxa^{n-4}xa^2b^{n-4} + bxa^{n-3}xab^{n-4} \in Y.$$

Now $bxa^{n-4}xa^2b^{n-4} = -xba^{n-4}xa^2b^{n-4}$. If we set $u = a + b$, then (29) implies that $xu^{n-3}xu^2b^{n-4} \in Y$. Expanding, and picking out the terms which are of degree $n-2$ in a and of degree $n-3$ in b , we obtain

$$(n-3)xba^{n-4}xa^2b^{n-4} + 2xa^{n-3}xab^{n-3} \in Y.$$

Similarly $bxa^{n-3}xab^{n-4} = -xba^{n-3}xab^{n-4} = \frac{1}{n-2}xa^{n-2}xb^{n-3}$. So (31) gives

$$(32) \quad \frac{2}{n-3}xa^{n-3}xab^{n-3} + \frac{1}{n-2}xa^{n-2}xb^{n-3} \in Y.$$

Now if n is odd, then $n-3$ is even. So if we let $s = (n-3)/2$ then

$$0 = (xa^s)(xa^s) = \sum_{r=0}^s (-1)^r \binom{s}{r} xa^{s+r} xa^{s-r}.$$

So

$$xa^{n-3}xab^{n-3} = - \sum_{r=0}^{s-1} (-1)^r \binom{s}{r} xa^{s+r} xa^{s-r+1} b^{n-3} \in Y.$$

Hence (32) implies that $xa^{n-2}xb^{n-3} \in Y$. On the other hand if n is even, then $n-2$ is even, and so if we let $s = (n-2)/2$ then $(xa^s)(xa^s) = 0$, and expanding this we obtain

$$xa^{n-2}xb^{n-3} - \frac{n-2}{2}xa^{n-3}xab^{n-3} \in Y.$$

Together with (32) this implies that $xa^{n-2}xb^{n-3} \in Y$. So, whether n is even or odd, we have shown that $xa^{n-2}xb^{n-3} \in Y$. $\square$

5. THE IDEAL GENERATED BY AN ELEMENT

Let L be an Engel- n Lie algebra over a field K with $\text{char}(K) > n$, and let x be an element of L . In this section we study the ideal $Id_L(x)$ of L generated by x . Clearly, $Id_L(x)$ is spanned by products in L such that at

least one of the elements in the product is x . More generally, it is easy to see that $(Id_L(x))^m$ is spanned by products in L such that x occurs at least m times in the product. Let u be a product of k elements $v_1, v_2, \dots, v_k \in L$, in some order with some bracketing. If x occurs m times in the sequence $v_1, v_2, \dots, v_k$, then we say that u has type $(m; k)$. Note that the elements $v_1, v_2, \dots, v_k$ could also be products, and that some of these products might involve x , so that type $(m; k)$ depends on the particular expression for u as a product of $v_1, v_2, \dots, v_k$. So, strictly speaking, we should call $(m; k)$ the type of the expression for u , rather than the type of u . Our aim in this section is to establish the following three lemmas.

Lemma 4. *If $m > 2$, then $(Id_L(x))^m$ is generated as an ideal by products of type $(m; k)$ where $k \leq m + 1 + \frac{(m-1)(2n-5)}{2}$.*

Lemma 5. *Let $m > 2$, and let*

$$v = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m}$$

be a product of type $(m; k)$ in L . Let Y be the ideal of L generated by elements of the form $xa_1a_2 \cdots a_{n-1}$ where $a_i \in \{a_{rs} | 1 \leq r \leq m, 1 \leq s \leq k_r\}$ for $i = 1, 2, \dots, n-1$, and let Z be the ideal of L generated by all products in L of type $(m; r)$ with $r < k$. If $k \geq m + 2 + \frac{(m-1)(3m-10)}{3}$ then $v \in Y + Z$.

Lemma 6. *If $m > 2$, then $(Id_L(x))^m$ is contained in the ideal generated by products of type $(r; k)$ where $\frac{3m+3}{8} \leq r \leq m$ and $k < r + 2 + \frac{(r-1)(3n-10)}{3}$.*

Let L be an Engel- n Lie algebra over a field K with $\text{char}(K) > n$, and let x be an element of L . Then $Id_L(x)$ is spanned by elements of the form $xa_1a_2 \cdots a_k$ with $a_1, a_2, \dots, a_k \in L$, and with $k \geq 0$. So $(Id_L(x))^m$ is spanned by elements of the form

$$(xa_{11}a_{12} \cdots a_{1k_1})(xa_{21}a_{22} \cdots a_{2k_2}) \cdots (xa_{m1}a_{m2} \cdots a_{mk_m}).$$

Note that this element has type $(m; k)$ where $k = m + k_1 + k_2 + \cdots + k_m$. We want to show that $(Id_L(x))^m$ is generated as an ideal by elements of type $(m; k)$ where $k \leq m + 1 + \frac{(m-1)(2n-5)}{2}$. So we let Z be the ideal of L generated by these elements, and we prove by induction on k that elements of type $(m; k)$ lie in Z for all k . We let u be a product of type $(m; k)$ for some $k > m + 1 + \frac{(m-1)(2n-5)}{2}$ and we suppose that all products of type $(m; r)$ lie in Z for all $r < k$. The product u can be written as a linear combination of products of type $(m; k)$ of the form

$$xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m}$$

(where $m + k_1 + k_2 + \cdots + k_m = k$), and so it is sufficient to prove that

$$xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} \in Z$$

for all $a_{ij} \in L$. Clearly it is sufficient to prove this in the case when L is the relatively free Engel- n Lie algebra freely generated by $x, a_{11}, a_{12}, \dots, a_{mk_m}$. Now we let J be the ideal of L generated by $\{a_{ij}a_{rs} | 1 \leq i, r \leq m, 1 \leq j \leq k_i, 1 \leq s \leq k_r\}$. If we can show that

$$v = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} \in Z + J$$

then, by an argument similar to the argument used in the proof of Lemma 1, it follows that $v \in Z$. Since L , Z , and J are all multigraded, if $v \in Z + J$ then v can be expressed in the form $v = b + c$, where $b \in Z$ and $c \in J$, and where b and c have the same multiweight as that of v in the free generators of L . But then c can be written as a linear combination of products of the form $(a_{ij}a_{rs})v_3v_4 \cdots v_k$ where $v_3, v_4, \dots, v_k$ are free generators of L , and where x occurs m times in the sequence $v_3, v_4, \dots, v_k$. If we treat $a_{ij}a_{rs}$ as a single element of L then $(a_{ij}a_{rs})v_3v_4 \cdots v_k$ has type $(m; k-1)$ and so lies in Z by the inductive hypothesis. Hence $v \in Z$ as required.

So from now on we work modulo the ideal $Z + J$. In other words, we assume that all products of type $(m; r)$ are zero for all $r < k$, and we assume that $a_{ij}a_{rs} = 0$ for all i, j, r, s such that $1 \leq i, r \leq m$, $1 \leq j \leq k_i$, $1 \leq s \leq k_r$. We make one further inductive hypothesis. If

$$w = xb_{11}b_{12} \cdots b_{1r_1}xb_{21}b_{22} \cdots b_{2r_2}x \cdots xb_{m1}b_{m2} \cdots b_{mr_m}$$

is a product of type $(m; k)$ with $b_{ij} \in L$ for all i, j such that $1 \leq i \leq m$, $1 \leq j \leq r_i$, and with $m + r_1 + r_2 + \cdots + r_m = k$, then we define the index of w to be $(r_1, r_2, \dots, r_m)$. We order the set of possible indexes lexicographically, so that

$$(r_1, r_2, \dots, r_m) > (k_1, k_2, \dots, k_m)$$

if for some s with $1 \leq s < m$ we have $r_i = k_i$ for $1 \leq i < s$ and $r_s > k_s$. We make the additional inductive hypothesis that $w = 0$ for all such w with index $(r_1, r_2, \dots, r_m) > (k_1, k_2, \dots, k_m)$.

Now we show that these assumptions imply that $v = 0$. First note that if $k_m > 0$ then

$$u = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m-1}$$

has type $(m; k-1)$. The inductive hypothesis implies that $u = 0$, and so $v = ua_{mk_m} = 0$. Next, note that we can use the results of §§3 and 4 since $a_{ij}a_{rs} = 0$ for all i, j, r, s such that $1 \leq i, r \leq m$, $1 \leq j \leq k_i$, $1 \leq s \leq k_r$. Identities (1) and (26) imply that $v = 0$ if $k_r \geq n$ for any r with $1 \leq r < m$. We can also show that $v = 0$ if $k_r = n-1$ for some r with $1 < r < m$. To see this suppose that $1 < r < m$ and that $k_r = n-1$. Set

$$u = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{r-11}a_{r-12} \cdots a_{r-1k_{r-1}},$$

and use the identity

$$S_n(u, x, a_{r1}, a_{r2}, \dots, a_{r_{n-1}})xa_{r+11}a_{r+12} \cdots a_{r+1k_{r+1}}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0.$$

This identity enables us to express v as a sum of products of type $(m; k)$ with higher index than v . So $v = 0$. Identities (2) and (27) imply that $v = 0$ if $k_1 = n-1$ and $k_2 = n-2$. Identities (2) and (27) can also be used to show that $v = 0$ if $k_r = k_{r+1} = n-2$ for some r with $1 < r < m$. For they imply that if $x, u, a_1, a_2, \dots, a_{n-2}, b_1, b_2, \dots, b_{n-2}$ are elements of an Engel- n Lie algebra over K then $uxa_1a_2 \cdots a_{n-2}xb_1b_2 \cdots b_{n-2} = -xua_1a_2 \cdots a_{n-2}xb_1b_2 \cdots b_{n-2}$ lies in the ideal generated by the elements ua_i, ub_i, a_ib_j ($1 \leq i, j \leq n-2$). If $k_r = k_{r+1} = n-2$ for some r with $1 < r < m$ then we substitute

$$xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{r-11}a_{r-12} \cdots a_{r-1k_{r-1}}$$

for u , and substitute a_{rj} for a_j and a_{r+1j} for b_j ($1 \leq j \leq n-2$) in this identity. The identity then implies that v lies in the ideal of L generated by elements of the form

$$xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{r-11}a_{r-12} \cdots a_{r-1k_{r-1}}a_{ij}$$

($i = r, r+1, 1 \leq j \leq n-2$). Since L is multigraded this implies that v can be expressed as a linear combination of products of type $(m; k)$ all of which have higher index than v . By the inductive hypothesis, all products with higher index than v are zero, and so $v = 0$. Finally note that if $k_{m-1} = n-2$ then setting

$$u = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m-21}a_{m-22} \cdots a_{m-2k_{m-2}}$$

we see that the identity $S_n(u, x, x, a_{m-11}, a_{m-12}, \dots, a_{m-1n-2}) = 0$ enables us to express v as a linear combination of products some of which have type $(m; k)$ but have higher index than that of v , and some of which lie in the ideal generated by products of type $(m; k-1)$. So $v = 0$.

To summarize, $v = 0$ unless the following conditions are satisfied:

$$\begin{aligned} k_1 &< n, \\ k_r &< n-1 \text{ for } 1 < r < m-1, \\ k_{m-1} &< n-2, \\ k_m &= 0, \\ k_1 &< n-1 \text{ or } k_2 < n-2, \end{aligned}$$

If $1 < r < m$ then $k_r < n-2$ or $k_{r+1} < n-2$.

But if all these conditions are satisfied then $k = m + k_1 + k_2 + \cdots + k_m \leq m + 1 + \frac{(m-1)(2n-5)}{2}$, contrary to the initial assumption. So $v = 0$ as required. This completes the proof of Lemma 4. $\square$

Now we prove Lemma 5. Let $m > 2$, and let

$$v = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m}$$

be a product of type $(m; k)$ in L . Let Y be the ideal of L generated by elements of the form $xa_1a_2 \cdots a_{n-1}$ where $a_i \in \{a_{rs} | 1 \leq r \leq m, 1 \leq s \leq k_r\}$ for $i = 1, 2, \dots, n-1$, and let Z be the ideal of L generated by all products in L of type $(m; r)$ with $r < k$. We show that if $k \geq m + 2 + \frac{(m-1)(3m-10)}{3}$ then $v \in Y + Z$. As above, we let J be the ideal of L generated by $\{a_{ij}a_{rs} | 1 \leq i, r \leq m, 1 \leq j \leq k_i, 1 \leq s \leq k_r\}$. Just as in the proof of Lemma 4 we see that we can assume that L is multigraded, and that $Y = Z = J = \{0\}$. We suppose that $k \geq m + 2 + \frac{(m-1)(3m-10)}{3}$ and we show that this (together with the assumption $Y = Z = J = \{0\}$) implies that $v = 0$. We can assume by induction that $w = 0$ if

$$w = xb_{11}b_{12} \cdots b_{1r_1}xb_{21}b_{22} \cdots b_{2r_2}x \cdots xb_{m1}b_{m2} \cdots b_{mr_m}$$

is a product of type $(m; k)$ with $b_{ij} \in \{a_{rs} | 1 \leq r \leq m, 1 \leq s \leq k_r\}$ for $i = 1, 2, \dots, m, j = 1, 2, \dots, r_i$, and with $(r_1, r_2, \dots, r_m) > (k_1, k_2, \dots, k_m)$.

As in the proof of Lemma 4, we see that $v = 0$ if $k_r \geq n-1$ for some r with $1 < r < m$, or if $k_m > 0$. Since we are making the assumption that $Y = \{0\}$, it is clear that $v = 0$ if $k_1 \geq n-1$. The assumption that $Y = \{0\}$ also means

that we can make use of identities (3), (4), and (23) from §3 and identity (30) from §4. We show that these identities imply that $v = 0$ if $k_r = n - 2$ for any r with $1 < r < m$.

First suppose that $k_2 = n - 2$. If $k_1 = n - 2$ then $v = 0$ by (4) and (30). If $k_1 < n - 2$ then, using the fact that products of type $(m; k)$ are zero if they have index greater than $(k_1, k_2, \dots, k_m)$, we see that

$$\begin{aligned} v &= xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} \\ &= (xa_{11}a_{12} \cdots a_{1k_1})(xa_{21}a_{22} \cdots a_{2k_2})x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} \\ &= -(xa_{21}a_{22} \cdots a_{2k_2})(xa_{11}a_{12} \cdots a_{1k_1})x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} \\ &= 0. \end{aligned}$$

Next suppose that $k_r = n - 2$ for some r with $2 < r < m$, and let $u = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{r-21}a_{r-22} \cdots a_{r-2k_{r-2}}$. If $k_{r-1} = 0$ then the identity

$$S_n(u, x, x, a_{r1}, a_{r2}, \dots, a_{rn-2}) \cdot xa_{r+11}a_{r+12} \cdots a_{r+1k_{r+1}}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0$$

enables us to express v as a linear combination of products of type $(m; k)$ with index greater than $(k_1, k_2, \dots, k_m)$. Hence $v = 0$ if $k_{r-1} = 0$. Now suppose that $k_{r-1} = s > 0$, and let b, c be elements in the linear span of $\{a_{ij} | 1 \leq i \leq m, 1 \leq j \leq k_i\}$. Using the fact that products of type $(m; k)$ are zero if they have index greater than $(k_1, k_2, \dots, k_m)$, the identity

$$S_n(uxb^{s-1}, x, c, c, \dots, c)xa_{r+11}a_{r+12} \cdots a_{r+1k_{r+1}}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0$$

gives

$$\begin{aligned} &uxb^{s-1}xc^{n-1}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} \\ &+ uxb^{s-1}cxc^{n-2}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0. \end{aligned}$$

Similarly the identity

$$uxb^{s-1}(xc^{n-1})xa_{r+11}a_{r+12} \cdots a_{r+1k_{r+1}}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0$$

gives

$$\begin{aligned} &uxb^{s-1}xc^{n-1}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} \\ &- (n-1)uxb^{s-1}cxc^{n-2}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0. \end{aligned}$$

So

$$\begin{aligned} &uxb^{s-1}xc^{n-1}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} \\ &= uxb^{s-1}cxc^{n-2}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0. \end{aligned}$$

Linearizing this we see that

$$(33) \quad uxb_1b_2 \cdots b_{s-1}xc_1c_2 \cdots c_{n-1}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0$$

and

$$(34) \quad uxb_1b_2 \cdots b_{s-1}cxc^{n-2}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0$$

for all $b_1, b_2, \dots, b_{s-1}, c_1, c_2, \dots, c_{n-1}, c$ in the linear span of $\{a_{ij} | 1 \leq i \leq m, 1 \leq j \leq k_i\}$. If we let $0 \leq t \leq s-1$, then (34) gives

$$uxb^t c^{s-1-t}(b+c)x(b+c)^{n-2}xa_{r+11} \cdots xa_{m1}a_{m2} \cdots a_{mk_m} = 0.$$

Expanding, and picking out the terms of degree s in b and of degree $n-2$ in c , we obtain

$$\begin{aligned} & \binom{n-2}{s-t} u x b^t c^{s-t} x b^{s-t} c^{n-2-s+t} x a_{r+1} \cdots x a_{m1} a_{m2} \cdots a_{mk_m} \\ & + \binom{n-2}{s-t-1} u x b^{t+1} c^{s-1-t} x b^{s-1-t} c^{n-1-s+t} x a_{r+1} \cdots x a_{m1} a_{m2} \cdots a_{mk_m} = 0. \end{aligned}$$

These identities for $t = 0, 1, \dots, s-1$ show that

$$u x b^s x c^{n-2} x a_{r+1} \cdots x a_{m1} a_{m2} \cdots a_{mk_m}$$

is a nonzero multiple of

$$u x c^s x c^{n-2-s} b^s x a_{r+1} \cdots x a_{m1} a_{m2} \cdots a_{mk_m}.$$

Now consider the identity

$$S_n(u, x, x, c, c, \dots, c) b^s x a_{r+1} a_{r+1} \cdots a_{r+1} x \cdots x a_{m1} a_{m2} \cdots a_{mk_m} = 0.$$

Expanding this, using (33), and ignoring terms which have index greater than $(k_1, k_2, \dots, k_m)$, we obtain

$$u x c^s x c^{n-2-s} b^s x a_{r+1} a_{r+1} \cdots a_{r+1} x \cdots x a_{m1} a_{m2} \cdots a_{mk_m} = 0.$$

By the above remarks, this implies that

$$u x b^s x c^{n-2} x a_{r+1} a_{r+1} \cdots a_{r+1} x \cdots x a_{m1} a_{m2} \cdots a_{mk_m} = 0,$$

and linearizing we obtain $v = 0$ as required.

So far, we have shown that $v = 0$ unless

$$\begin{aligned} k_1 &\leq n-2, \\ k_r &\leq n-3 \text{ for } r = 2, 3, \dots, m-1, \\ k_m &= 0. \end{aligned}$$

If these conditions are satisfied then $k_1 + k_2 + k_3 \leq 3n-8$, and $k_r + k_{r+1} + k_{r+2} \leq 3n-9$ for $r = 2, 3, \dots, m-3$. Identities (23) and (30) imply that $v = 0$ if $k_1 + k_2 + k_3 = 3n-8$. They also imply that $v = 0$ if $k_r + k_{r+1} + k_{r+2} = 3n-9$ for some r with $2 \leq r \leq m-3$. To see this, let $2 \leq r \leq m-3$ and let $k_r + k_{r+1} + k_{r+2} = 3n-9$. Then let

$$u = x a_{11} a_{12} \cdots a_{1k_1} x a_{21} a_{22} \cdots a_{2k_2} x \cdots x a_{r-11} a_{r-12} \cdots a_{r-1k_{r-1}}.$$

Identities (23) and (30), together with the above calculations, show that

$$\begin{aligned} v &= u x a_{r1} a_{r2} \cdots a_{rk_r} x \cdots x a_{m1} a_{m2} \cdots a_{mk_m} \\ &= -x u a_{r1} a_{r2} \cdots a_{rk_r} x \cdots x a_{m1} a_{m2} \cdots a_{mk_m} \end{aligned}$$

lies in the ideal generated by elements of the form bc and $x b_1 b_2 \cdots b_{n-1}$, where $b, c, b_1, b_2, \dots, b_{n-1}$ are elements of the set $\{u, a_{r1}, a_{r2}, \dots, a_{mk_m}\}$. Now, by assumption, $bc = 0$ if $b, c \in \{a_{r1}, a_{r2}, \dots, a_{mk_m}\}$, and $x b_1 b_2 \cdots b_{n-1} = 0$ if $b_1, b_2, \dots, b_{n-1} \in \{a_{r1}, a_{r2}, \dots, a_{mk_m}\}$. So, using the fact that L is multigraded, we see that v lies in the ideal generated by elements of the form uc and $u x b_1 b_2 \cdots b_{n-2}$ where $c, b_1, b_2, \dots, b_{n-2} \in \{a_{r1}, a_{r2}, \dots, a_{mk_m}\}$. Since L is multigraded, this implies that v is a linear combination of products of type $(m; k)$ of the form $u c v_1 v_2 \cdots v_s$ and of the form $u x b_1 b_2 \cdots b_{n-2} w_1 w_2 \cdots w_t$

where $v_1, v_2, \dots, v_s, w_1, w_2, \dots, w_t \in \{x, a_{r1}, a_{r2}, \dots, a_{mk_m}\}$. Note that we have shown above that $v = 0$ if $k_r = n - 2$. So we may assume that $k_r < n - 2$, which implies that all these products have index greater than that of v . So $v = 0$ as required.

Putting all this together we have shown that $v = 0$ unless

$$\begin{aligned} k_1 &\leq n - 2, \\ k_r &\leq n - 3 \text{ for } r = 1, 2, \dots, m - 1, \\ k_m &= 0, \\ k_1 + k_2 + k_3 &\leq 3n - 9, \\ k_r + k_{r+1} + k_{r+2} &\leq 3n - 10 \text{ for } r = 2, 3, \dots, m - 3. \end{aligned}$$

But if all these conditions are satisfied then $k = m + k_1 + k_2 + \dots + k_m < m + 2 + \frac{(m-1)(3n-10)}{3}$, contrary to our initial assumptions. So $v = 0$ in every case. This completes the proof of Lemma 5. $\square$

Finally we prove Lemma 6. We let x be an element of an Engel- n Lie algebra L over K , we let $m > 2$, and we let T be the ideal of L generated by products of type $(r; k)$ where $\frac{3m+3}{8} \leq r \leq m$ and $k < r + 2 + \frac{(r-1)(3n-10)}{3}$. We need to show that $(Id_L(x))^m \leq T$.

By Lemma 4, $(Id_L(x))^m$ is generated as an ideal by products of type $(m; k)$ with $k \leq m + 1 + \frac{(m-1)(2n-5)}{2}$, and we show that all these products lie in T . So let

$$v = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}xa_{31} \cdots xa_{m1}a_{m2} \cdots a_{mk_m}$$

be a product of type $(m; k)$, where $k \leq m + 1 + \frac{(m-1)(2n-5)}{2}$, and suppose by induction that all products of type $(m; r)$ lie in T for all $r < k$. If $k < m + 2 + \frac{(m-1)(3n-10)}{3}$ then $v \in T$, and we are done. If $k \geq m + 2 + \frac{(m-1)(3n-10)}{3}$, then $v \in Y + Z$ by Lemma 5, where Z is the ideal generated by products of type $(m; r)$ with $r < k$. By our inductive hypothesis, $Z \leq T$. So $v = b + c$ for some $b \in Y$ and some $c \in T$. As usual, we may assume that b and c have the same multiweight as v . So b can be written as a linear combination of products of type $(m; k)$ of the form

$$xa_1a_2 \cdots a_{n-1}v_{n+1}v_{n+2} \cdots v_k$$

where $a_1, a_2, \dots, a_{n-1} \in \{a_{ij} | 1 \leq i \leq m, 1 \leq j \leq k_i\}$ and $v_{n+1}, v_{n+2}, \dots, v_k \in \{x\} \cup \{a_{ij} | 1 \leq i \leq m, 1 \leq j \leq k_i\}$. Treating $xa_1a_2 \cdots a_{n-1}$ as a single element of L , we see that $xa_1a_2 \cdots a_{n-1}v_{n+1}v_{n+2} \cdots v_k = (xa_1a_2 \cdots a_{n-1})v_{n+1}v_{n+2} \cdots v_k$ can be viewed as a product of type $(m-1; k-n+1)$. So v can be written modulo T as a linear combination of products of type $(m-1; k-n+1)$. We now want to repeat the argument and show that products of type $(m-1; k-n+1)$ can be written modulo T as linear combinations of products of type $(m-2; k-2n+2)$. But to do this we need to be able to assume that all products of type $(m-1; r)$ with $r < k-n+1$ lie in T . So we extend our inductive hypothesis to include the assumption that all products of type $(r; s)$ lie in T for all pairs r, s with $\frac{3m+3}{8} \leq r \leq m$ and $s < k - (m-r)(n-1)$. Then by repeated use of the argument above we see that if $\frac{3m+3}{8} \leq r \leq m$ then v can be written modulo T as a linear combination of products of type $(r; k - (m-r)(n-1))$. Now $k \leq m + 1 + \frac{(m-1)(2n-5)}{2}$, and so $k - (m-r)(n-1) \leq$

$m + 1 + \frac{(m-1)(2n-5)}{2} - (m-r)(n-1) = \frac{7}{2} - \frac{m}{2} - r + (r-1)n$. If we choose r minimal subject to the condition $r \geq \frac{3m+3}{8}$ then $r \leq \frac{3m+10}{8}$, and so $m \geq \frac{8r-10}{3}$. Hence $k - (m-r)(n-1) \leq \frac{7}{2} - \frac{8r-10}{6} - r + (r-1)n < r + 2 + \frac{(r-1)(3n-10)}{3}$. But this implies that products of type $(r; k - (m-r)(n-1))$ all lie in T . So $v \in T$ as required, and this completes the proof of Lemma 6. $\square$

6. ENGEL-6 LIE ALGEBRAS OF CHARACTERISTIC 7

In this section we prove the following theorem.

Theorem 7. *If L is an m -generator Engel-6 Lie algebra of characteristic 7 then L is nilpotent of class at most $51m^8$.*

As an immediate corollary we obtain a polynomial bound on the class of a finite m -generator group of exponent 7.

Theorem 8. *If G is a finite m -generator group of exponent 7 then G is nilpotent of class at most $51m^8$.*

I made use of the nilpotent quotient algorithm for graded Lie rings described in Havas, Newman, and Vaughan-Lee [4] to compute the Lie algebra M described in §2. The computations showed that if $n = 6$ and $\text{char}(K) = 7$ then M is nilpotent of class 22. In addition, the computations showed that if v is a product with multiweight $(w_0, w_1, w_2, \dots)$ in the generators $\tilde{x}, \tilde{a}_1, \tilde{a}_2, \dots$ of M , then $v = 0$ if $w_0 - \sum_{i>0} w_i > 10$. So, using Theorem 3 and the remark that follows immediately after its proof, we see that if a, b are elements of an Engel-6 Lie algebra of characteristic 7, then the ideal generated by ab^4 is nilpotent of class at most 73. We can actually do slightly better than this.

In §5 we let L be an Engel- n Lie algebra over a field K with $\text{char}(K) > n$, and we let x be an element of L . In Lemma 5 we considered a product

$$v = xa_{11}a_{12} \cdots a_{1k_1}xa_{21}a_{22} \cdots a_{2k_2}x \cdots xa_{m1}a_{m2} \cdots a_{mk_m}$$

of type $(m; k)$ in L . We let Y be the ideal of L generated by elements of the form $xa_1a_2 \cdots a_{n-1}$ where $a_i \in \{a_{rs} | 1 \leq r \leq m, 1 \leq s \leq k_r\}$ for $i = 1, 2, \dots, n-1$, and we let Z be the ideal of L generated by all products in L of type $(m; r)$ with $r < k$. We showed that if $k \geq m + 2 + \frac{(m-1)(3m-10)}{3}$ then $v \in Y + Z$. In the proof we made a number of inductive hypotheses, and we showed that these implied that $v = 0$ unless

$$\begin{aligned} k_1 &\leq n - 2, \\ k_r &\leq n - 3 \text{ for } r = 1, 2, \dots, m - 1, \\ k_m &= 0, \\ k_1 + k_2 + k_3 &\leq 3n - 9, \\ k_r + k_{r+1} + k_{r+2} &\leq 3n - 10 \text{ for } r = 2, 3, \dots, m - 3. \end{aligned}$$

I have used the nilpotent quotient algorithm for graded Lie rings to show that when $n = 6$ and $\text{char}(K) = 7$ then the inductive hypotheses used in the proof of Lemma 5 also imply that $v = 0$ if $k_r = 2$, $k_{r+1} = 3$, $k_{r+2} = 3$ for some r with $1 < r < m - 2$. This, together with the inequalities above, imply that $v \in Y + Z$ if $k > \frac{7m-1}{2}$. We can then use the same argument as in the proof of Lemma 6 to show that if x is an element of an Engel-6 Lie algebra of characteristic

7 then $(Id_L(x))^{55}$ is generated by products of type $(r; k)$ with $19 \leq r \leq 55$, where $k \leq 65$ when $r = 19$, and where $k \leq \frac{7r-1}{2}$ when $20 \leq r \leq 55$. Then using the same argument as in the proof of Theorem 3, and taking $N = 10$, we see that if $a, b \in L$ then $Id_L(ab^4)$ has class at most 54.

Now let L be the (relatively) free m -generator Engel-6 Lie algebra of characteristic 7. By Kostrikin's theorem [8], L is nilpotent. Let I be the ideal of L generated by $\{ab^4 | a, b \in L\}$. Traustason [13] has shown that L/I is nilpotent of class 7, which implies that $L^8 \leq I$. Now let S be a subset of the generators of L , let L_S be the subalgebra of L generated by S , and let I_S be the ideal of L generated by $\{ab^4 | a, b \in L_S\}$. Then any product of eight elements from S must lie in I_S . This implies that $L^8 \leq \sum I_S \leq I$, where the sum is taken over all subsets S of the free generators of L such that $|S| \leq 8$. In fact $\sum I_S = I$, but we do not need to use this fact. I used the nilpotent quotient algorithm for graded Lie rings to compute the number of elements ab^4 which are needed to generate I_S for $2 \leq |S| \leq 8$, and the computations showed that $\sum I_S$ can be generated by d_m elements, where

$$d_m = 322 \binom{m}{8} + 1197 \binom{m}{7} + 1725 \binom{m}{6} \\ + 1219 \binom{m}{5} + 441 \binom{m}{4} + 153 \binom{m}{3} + 6 \binom{m}{2}.$$

Since each element ab^4 generates an ideal which is nilpotent of class at most 54, it follows that $\sum I_S$ is nilpotent of class at most $54d_m$. Since $L^8 \leq \sum I_S$, we see that L^8 is also nilpotent of class at most $54d_m$. Now if J is any ideal of L , then

$$J \underbrace{LL \cdots L}_{36} \leq J(L^8).$$

(For a proof of this see, for example, Proposition 4.6 of Chapter 1 of [9].) It follows that $L^{8+36k} \leq (L^8)^{k+1}$ for $k = 1, 2, \dots$, and so the fact that L^8 is nilpotent of class at most $54d_m$ implies that L is nilpotent of class at most $7 + 1944d_m$. Now $7 + 1944d_m$ is a polynomial of degree 8 in m , with leading coefficient 15.525. It is a straightforward calculation to show that $7 + 1944d_m < 51m^8$ for all values of m . This proves Theorem 7. $\square$

REFERENCES

1. S. I. Adjan and A. A. Razborov, *Periodic groups and Lie algebras*, Uspekhi Mat. Nauk **42** (1987), 3–68.
2. N. D. Gupta and M. F. Newman, *The nilpotency class of finitely generated groups of exponent four*, Lecture Notes in Math., vol. 372, Springer-Verlag, Berlin, 1974, pp. 330–332.
3. P. Hall and G. Higman, *On the p -length of p -soluble groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. **6** (1956), 1–42.
4. G. Havas, M. F. Newman, and M. R. Vaughan-Lee, *A nilpotent quotient algorithm for graded Lie rings*, J. Symbolic Comput. **9** (1990), 653–664.
5. G. Havas, G. E. Wall, and J. W. Wamsley, *The two generator restricted Burnside group of exponent five*, Bull. Austral. Math. Soc. **10** (1974), 459–470.
6. P. J. Higgins, *Lie rings satisfying the Engel condition*, Proc. Cambridge Philos. Soc. **50** (1954), 8–15.
7. G. Higman, *On finite groups of exponent five*, Proc. Cambridge Philos. Soc. **52** (1956), 381–390.

8. A. I. Kostrikin, *The Burnside problem*, Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 3–34.
9. ———, *Around Burnside* (translated by J. Wiegold), Ergeb. Math. Grenzgeb., Springer-Verlag, Berlin, 1990.
10. F. Levi and B. L. van der Waerden, *Über eine besondere Klasse von Gruppen*, Abh. Math. Sem. Univ. Hamburg **9** (1933), 154–158.
11. Ju. P. Razmyslov, *On a problem of Hall-Higman*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 833–847.
12. I. N. Sanov, *Solution of Burnside's problem for exponent four*, Leningrad State Univ. Ann. Math. Ser. **10** (1940), 166–170.
13. G. Traustason, *Engel Lie algebras*, Quart. J. Math. (2) **44** (1993), 355–384.
14. ———, *Engel Lie algebras*, D. Phil. thesis, Oxford Univ., 1993.
15. M. R. Vaughan-Lee, *Lie rings of groups of prime exponent*, J. Austral. Math. Soc. **49** (1990), 386–398.
16. ———, *The restricted Burnside problem*, Oxford Univ. Press, 1990.
17. M. R. Vaughan-Lee and E. I. Zelmanov, *Upper bounds in the restricted Burnside problem*, J. Algebra **161** (1993).
18. E. I. Zelmanov, *The solution of the restricted Burnside problem for groups of odd exponent*, Izv. Math. USSR **36** (1991), 41–60.
19. ———, *The solution of the restricted Burnside problem for 2-groups*, Mat. Sb. **182** (1991), 568–592.

CHRIST CHURCH, OXFORD, OX1 1DP, ENGLAND
E-mail address: vlee@math.ox.ac.uk