ANALYTIC FOURIER-FEYNMAN TRANSFORMS AND CONVOLUTION

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ABSTRACT. In this paper we develop an L_p Fourier-Feynman theory for a class of functionals on Wiener space of the form $F(x) = f(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx)$. We then define a convolution product for functionals on Wiener space and show that the Fourier-Feynman transform of the convolution product is a product of Fourier-Feynman transforms.

1. INTRODUCTION AND PRELIMINARIES

The concept of an L_1 analytic Fourier-Feynman transform was introduced by Brue in [1]. In [3] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [6] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ which extended the results in [1, 3] and gave various relationships between the L_1 and the L_2 theories.

In this paper we first develop an L_p Fourier-Feynman theory for a class of functionals not considered in [1, 3, 6]. We next define a convolution product for functionals on Wiener space and then show that the Fourier-Feynman transform of the convolution product is a product of Fourier-Feynman transforms.

In [3, 6] all of the functionals F on Wiener space and all the real-valued functions F on $\mathbb{R}^n$ were assumed to be Borel measurable. But, as was pointed out in [7, p. 170], the concept of scale-invariant measurability in Wiener space and Lebesgue measurability in $\mathbb{R}^n$ is precisely correct for the analytic Fourier-Feynman theory.

Let $C_0[0, T]$ denote Wiener space; that is, the space of real-valued continuous functions x on $[0, T]$ such that $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0, T]$, and let m denote Wiener measure. $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a functional F by

$$\int_{C_0[0, T]} F(x) m(dx).$$

A subset E of $C_0[0, T]$ is said to be scale-invariant measurable [4, 7] provided $\rho E \in \mathcal{M}$ for each $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property

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that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Let $\mathbb{C}$, $\mathbb{C}_+$, and $\mathbb{C}_+^\sim$ denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with non-negative real part. Let F be a $\mathbb{C}$ -valued scale-invariant measurable functional on $C_0[0, T]$ such that

$$J(\lambda) = \int_{C_0[0, T]} F(\lambda^{-1/2}x) m(dx)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0, T]$ with parameter λ and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0, T]}^{\text{anw}_\lambda} F(x) m(dx) = J^*(\lambda).$$

Let $q \neq 0$ be a real number, and let F be a functional such that

$$\int_{C_0[0, T]}^{\text{anw}_\lambda} F(x) m(dx)$$

exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0, T]}^{\text{anf}_q} F(x) m(dx) = \lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]}^{\text{anw}_\lambda} F(x) m(dx)$$

where $\lambda \rightarrow -iq$ through $\mathbb{C}_+$.

Notation. (i) For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0, T]$ let

$$(1.1) \quad (T_\lambda(F))(y) = \int_{C_0[0, T]}^{\text{anw}_\lambda} F(x + y) m(dx).$$

(ii) Given a number p with $1 \leq p \leq +\infty$, p and p' will always be related by $1/p + 1/p' = 1$.

(iii) Let $1 < p \leq 2$, and let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$(1.2) \quad \lim_{n \rightarrow \infty} \int_{C_0[0, T]} |H_n(\rho y) - H(\rho y)|^{p'} m(dy) = 0.$$

Then we write

$$(1.3) \quad \text{l. i. m.}_{n \rightarrow \infty} (w_s^{p'}) (H_n) \approx H$$

and we call H the scale invariant limit in the mean of order p' . A similar definition is understood when n is replaced by the continuously varying parameter λ . We are finally ready to state the definition of the L_p analytic Fourier-Feynman transform [6] and our definition of the convolution product.

Definition. Let $q \neq 0$ be a real number. For $1 < p \leq 2$ we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F by the formula ($\lambda \in \mathbb{C}_+$)

$$(1.4) \quad (T_q^{(p)}(F))(y) = \text{l.i.m.}_{\lambda \rightarrow -iq} (w_s^{p'}) (T_\lambda(F))(y)$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by the formula

$$(1.5) \quad T_q^{(1)}(F)(y) = \lim_{\lambda \rightarrow -iq} (T_\lambda(F))(y)$$

for s-a.e. y . We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_2) \approx T_q^{(p)}(F_1)$.

Definition. Let F_1 and F_2 be functionals on $C_0[0, T]$. For $\lambda \in \mathbb{C}_+^\sim$ we define their convolution product (if it exists) by

$$(1.6) \quad (F_1 * F_2)_\lambda(y) = \begin{cases} \int_{C_0[0, T]}^{\text{anw}_\lambda} F_1\left(\frac{y+x}{\sqrt{2}}\right) F_2\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda \in \mathbb{C}_+, \\ \int_{C_0[0, T]}^{\text{anf}_q} F_1\left(\frac{y+x}{\sqrt{2}}\right) F_2\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases}$$

Remark. Our definition of convolution is different than the definition given by Yeh in [9]. For one thing, our convolution product is commutative; that is to say $(F_1 * F_2)_\lambda = (F_2 * F_1)_\lambda$. Next we briefly describe a class of functionals for which we establish the existence of $T_q^{(p)}(F)$. Let n be a positive integer, and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be an orthonormal set of functions in $L_2[0, T]$. For $1 \leq p < \infty$ let $\mathcal{A}_n^{(p)}$ be the space of all functionals F on $C_0[0, T]$ of the form

$$(1.7) \quad F(x) = f\left(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx\right)$$

s-a.e. where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is in $L_p(\mathbb{R}^n)$ and the integrals $\int_0^T \alpha_j(t) dx(t)$ are Paley-Wiener-Zygmund stochastic integrals. Let $\mathcal{A}_n^{(\infty)}$ be the space of all functionals of the form (1.7) with $f \in C_0(\mathbb{R}^n)$, the space of bounded continuous functions on $\mathbb{R}^n$ that vanish at infinity. It is quite easy to see that if F is in $\mathcal{A}_n^{(p)}$, then F is scale-invariant measurable. If $p > 1$ the Feynman integral above should be interpreted as the scale-invariant limit in the mean of the analytic Wiener integral.

2. THE TRANSFORM OF FUNCTIONALS IN $\mathcal{A}_n^{(P)}$

In this section we show that the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ exists for all F in $\mathcal{A}_n^{(p)}$ and belongs to $\mathcal{A}_n^{(p')}$. We start with some preliminary lemmas.

Lemma 2.1. Let $1 \leq p \leq \infty$, and let $F \in \mathcal{A}_n^{(p)}$ be given by (1.7). Then for all $\lambda \in \mathbb{C}_+$,

$$(2.1) \quad (T_\lambda(F))(y) = g\left(\lambda; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy\right)$$

where

$$(2.2) \quad g(\lambda; w_1, \dots, w_n) \equiv g(\lambda; \vec{w}) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n (u_j - w_j)^2\right\} d\vec{u}.$$

Proof. For $\lambda > 0$, using a well-known Wiener integration theorem we obtain

$$\begin{aligned} (T_\lambda(F))(y) &= \int_{C_0[0, T]} F(\lambda^{-1/2}x + y) m(dx) \\ &= \int_{C_0[0, T]} f\left(\lambda^{-1/2} \int_0^T \alpha_1 dx + \int_0^T \alpha_1 dy, \dots, \lambda^{-1/2} \int_0^T \alpha_n dx + \int_0^T \alpha_n dy\right) m(dx) \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f\left(v_1 + \int_0^T dy, \dots, v_n + \int_0^T \alpha_n dy\right) \\ &\quad \times \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n v_j^2\right\} d\vec{v} \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n \left(u_j - \int_0^T \alpha_j dy\right)^2\right\} d\vec{u} \\ &= g\left(\lambda; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy\right) \end{aligned}$$

where g is given by (2.2). Now by analytic continuation in λ , (2.1) holds throughout $\mathbb{C}_+$. $\square$

Lemma 2.2. Let $F \in \mathcal{A}_n^{(1)}$ be given by (1.7), and let $g(\lambda; \vec{w})$ be given by (2.2). Then

- (i) $g(\lambda; \cdot) \in C_0(\mathbb{R}^n)$ for all $\lambda \in \mathbb{C}_+^\sim$;
- (ii) $g(\lambda; \vec{w})$ converges pointwise to $g(-iq; \vec{w})$ as $\lambda \rightarrow -iq$ through $\mathbb{C}_+$; and
- (iii) as elements of $C_0(\mathbb{R}^n)$, $g(\lambda; \vec{w})$ converges weakly to $g(-iq; \vec{w})$ as $\lambda \rightarrow -iq$ through values in $\mathbb{C}_+$.

Proof. We first note that for all $(\lambda, \vec{w}) \in \mathbb{C}_+^\sim \times \mathbb{R}^n$, $|g(\lambda; \vec{w})| \leq |\frac{\lambda}{2\pi}|^{n/2} \|f\|_1$. Then (i) follows from a standard argument and the dominated convergence theorem establishes (ii). To establish (iii) let $\mu \in M(\mathbb{R}^n)$, the dual of $C_0(\mathbb{R}^n)$.

By the dominated convergence theorem,

$$\begin{aligned}
 & \lim_{\lambda \rightarrow -iq} \int_{\mathbb{R}^n} g(\lambda; \vec{w}) d\mu(\vec{w}) \\
 &= \lim_{\lambda \rightarrow -iq} \int_{\mathbb{R}^n} \left(\frac{\lambda}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n (u_j - w_j)^2 \right\} d\vec{u} d\mu(\vec{w}) \\
 &= \int_{\mathbb{R}^n} \left(\frac{-iq}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ \frac{iq}{2} \sum_{j=1}^n (u_j - w_j)^2 \right\} d\vec{u} d\mu(\vec{w}) \\
 &= \int_{\mathbb{R}^n} g(-iq; \vec{w}) d\mu(\vec{w}). \quad \square
 \end{aligned}$$

Our first theorem, which is a direct consequence of Lemma 2.2, shows that the analytic L_1 Fourier-Feynman transform exists for all F in $\mathcal{A}_n^{(1)}$.

Theorem 2.1. Let $F \in \mathcal{A}_n^{(1)}$ be given by (1.7). Then $T_q^{(1)}(F)$ exists for all real $q \neq 0$ and

$$(2.3) \quad (T_q^{(1)}(F))(y) \approx g \left(-iq; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy \right) \in \mathcal{A}_n^{(\infty)}$$

where g is given by (2.2).

Remark. When $1 < p \leq 2$ and $\operatorname{Re} \lambda = 0$, the integral in (2.2) should be interpreted in the mean just as in the theory of the L_p Fourier transform [8].

Theorem 2.2. Let $1 < p \leq 2$, and let $F \in \mathcal{A}_n^{(p)}$ be given by (1.7). Then the L_p analytic Fourier-Feynman transform of F , $T_q^{(p)}(F)$ exists for all real $q \neq 0$, belongs to $\mathcal{A}_n^{(p')}$ and is given by the formula

$$(2.4) \quad (T_q^{(p)}(F))(y) \approx g \left(-iq; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy \right)$$

where g is given by (2.2).

Proof. We first note that for each $\lambda \in \mathbb{C}_+^\sim$, $g(\lambda; \vec{w})$ is in $L_{p'}(\mathbb{R}^n)$ [5, Lemma 1.1, p. 98]. Furthermore by [5, Lemma 1.2, p. 100]

$$(2.5) \quad \lim_{\lambda \rightarrow -iq} \|g(\lambda; \cdot) - g(-iq; \cdot)\|_{p'} = 0.$$

Now to show that $T_q^{(p)}(F)$ exists and is given by (2.4) it suffices to show that for each $\rho > 0$

$$\lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]} \left| g \left(\lambda; \rho \int_0^T \alpha_1 dy, \dots, \rho \int_0^T \alpha_n dy \right) - g \left(-iq; \rho \int_0^T \alpha_1 dy, \dots, \rho \int_0^T \alpha_n dy \right) \right|^{p'} m(dy) = 0.$$

But

$$\begin{aligned} & \int_{C_0[0, T]} \left| g \left(\lambda; \rho \int_0^T \alpha_1 dy, \dots, \rho \int_0^T \alpha_n dy \right) - g \left(-iq; \rho \int_0^T \alpha_1 dy, \dots, \rho \int_0^T \alpha_n dy \right) \right|^{p'} m(dy) \\ &= \rho^{-n} \int_{\mathbb{R}^n} |g(\lambda; \vec{u}) - g(-iq; \vec{u})|^{p'} \exp \left\{ -\frac{1}{2\rho^2} \sum_{j=1}^n u_j^2 \right\} d\vec{u} \\ &\leq \rho^{-n} \|g(\lambda; \cdot) - g(-iq; \cdot)\|_{p'}^{p'}, \end{aligned}$$

which goes to zero as $\lambda \rightarrow -iq$ by (2.5). Thus $T_q^{(p)}(F)$ exists, belongs to $\mathcal{A}_n^{(p')}$, and is given by (2.4). $\square$

The following example generates an interesting set of functionals belonging to $\mathcal{A}_n^{(p)}$.

Example. Let $1 \leq p \leq +\infty$ be given, and let $\alpha_1, \alpha_2, \dots$ be an orthonormal set of functions from $L_2[0, T]$. Let $F \in L_p(C_0[0, T])$, and for each n define f_n by

$$f_n \left(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx \right) \equiv E \left[F(x) \mid \int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx \right].$$

Then, by the definition of conditional expectation, $f_n(\xi_1, \dots, \xi_n)$ is a Borel measurable function, and $\|f_n\|_p \leq \|F\|_p$, where

$$\|f_n\|_p = E \left[\left| f_n \left(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx \right) \right|^p \right],$$

and

$$\|F\|_p^p = E[|F(x)|^p].$$

Thus $f_n \in \mathcal{A}_n^{(p)}$, and so the analytic Fourier-Feynman transform $T_q^{(p)}(f_n)$ exists for all real $q \neq 0$.

We finish this section by obtaining an inverse transform theorem for F in $\mathcal{A}_n^{(p)}$.

Theorem 2.3. Let $1 \leq p \leq 2$, and let $F \in \mathcal{A}_n^{(p)}$. Let $q \neq 0$ be given. Then (i) for each $\rho > 0$,

$$\lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]} |T_{\bar{\lambda}} T_{\lambda}(F)(\rho y) - F(\rho y)|^p m(dy) = 0,$$

and (ii) $T_{\bar{\lambda}} T_{\lambda} F \rightarrow F$ s-a.e. as $\lambda \rightarrow -iq$ through $\mathbb{C}_+$.

Proof. Proceeding as in the proof of Lemma 2.1, we obtain for all $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} (T_{\bar{\lambda}}(T_{\lambda}(F)))(y) &= \left(\frac{\bar{\lambda}}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} g(\lambda; \vec{w}) \exp \left\{ -\frac{\bar{\lambda}}{2} \sum_{j=1}^n \left(w_j - \int_0^T a_j dy \right)^2 \right\} d\vec{w} \\ &= \left(\frac{\bar{\lambda}}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n (u_j - w_j)^2 \right\} \\ &\quad \times \exp \left\{ -\frac{\bar{\lambda}}{2} \sum_{j=1}^n \left(w_j - \int_0^T \alpha_j dy \right)^2 \right\} d\vec{u} d\vec{w} \\ &= k \left(\lambda, \bar{\lambda}; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy \right) \end{aligned}$$

where $g(\lambda; \vec{w})$ is given by (2.2) and

$$\begin{aligned} k(\lambda, \bar{\lambda}; v_1, \dots, v_n) &\equiv k(\lambda, \bar{\lambda}; \vec{v}) \\ &= \left| \frac{\lambda}{2\pi} \right|^n \int_{\mathbb{R}^{2n}} f(\vec{u}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n (u_j - w_j)^2 - \frac{\bar{\lambda}}{2} \sum_{j=1}^n (w_j - v_j)^2 \right\} d\vec{u} d\vec{w}. \end{aligned}$$

But [2, p. 525]

$$\begin{aligned} &\int_{\mathbb{R}} \exp \left\{ -\frac{\lambda}{2} (u_j - w_j)^2 - \frac{\bar{\lambda}}{2} (w_j - v_j)^2 \right\} dw_j \\ &= \left(\frac{\pi}{\operatorname{Re} \lambda} \right)^{1/2} \exp \left\{ -\frac{|\lambda|^2}{4 \operatorname{Re} \lambda} (u_j - v_j)^2 \right\}. \end{aligned}$$

Hence

$$\begin{aligned} k(\lambda, \bar{\lambda}; \vec{v}) &= \left| \frac{\lambda}{2\pi} \right|^n \int_{\mathbb{R}^n} f(\vec{u}) \left(\frac{\pi}{\operatorname{Re} \lambda} \right)^{n/2} \exp \left\{ -\frac{|\lambda|^2}{4 \operatorname{Re} \lambda} \sum_{j=1}^n (u_j - v_j)^2 \right\} d\vec{u} \\ &= (f * \phi_{\varepsilon})(v_1, \dots, v_n) \end{aligned}$$

where

$$\phi(v_1, \dots, v_n) \equiv (2\pi)^{-n/2} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n v_j^2 \right\}, \quad \varepsilon \equiv \frac{\sqrt{2 \operatorname{Re} \lambda}}{|\lambda|},$$

and

$$\phi_{\varepsilon}(v_1, \dots, v_n) = \frac{1}{\varepsilon^n} = \frac{1}{\varepsilon^n} \phi \left(\frac{v_1}{\varepsilon}, \dots, \frac{v_n}{\varepsilon} \right).$$

Now

$$\int_{\mathbb{R}^n} \phi(v_1, \dots, v_n) dv_1 \cdots dv_n = 1 \quad \text{and} \quad \phi(v_1, \dots, v_n) > 0,$$

so using [8, Theorem 1.18, p. 10] it follows that

$$\begin{aligned} &\lim_{\lambda \rightarrow -iq} \int_{\mathbb{R}^n} |k(\lambda, \bar{\lambda}; v_1, \dots, v_n) - f(v_1, \dots, v_n)|^p d\vec{v} \\ (2.6) \quad &= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^n} |(f * \phi_\varepsilon)(v_1, \dots, v_n) - f(v_1, \dots, v_n)|^p d\vec{v} \\ &= \lim_{\varepsilon \rightarrow 0^+} \|f * \phi_\varepsilon - f\|_p^p = 0 \end{aligned}$$

since $\varepsilon \rightarrow 0^+$ as $\lambda \rightarrow -iq$ through $\mathbb{C}_+$. But now (i) of the theorem follows easily since for each fixed $\rho > 0$,

$$\begin{aligned} &\int_{C_0[0, T]} |T_{\bar{\lambda}} T_{\lambda}(F)(\rho y) - F(\rho y)|^p m(dy) \\ &= \rho^{-n} \int_{\mathbb{R}^n} |k(\lambda, \bar{\lambda}; \vec{v}) - f(\vec{v})|^p \exp \left\{ -\frac{1}{2\rho^2} \sum_{j=1}^n v_j^2 \right\} d\vec{v} \\ &\leq \rho^{-n} \|f * \phi_\varepsilon - f\|_p^p. \end{aligned}$$

Finally, (ii) of the theorem follows since by [8, Theorem 1.25, p. 13] it follows that the function $k(\lambda, \bar{\lambda}; v_1, \dots, v_n) = (f * \phi_\varepsilon)(v_1, \dots, v_n)$ converges pointwise to the function $f(v_1, \dots, v_n)$ as $\lambda \rightarrow -iq$ through $\mathbb{C}_+$. $\square$

Note that in the case $p = 2$, $p' = 2$, and so for F in $\mathcal{A}_n^{(2)}$, $T_q^{(2)}(F)$ is in $\mathcal{A}_n^{(2)}$ by Theorem 2.2. Hence we have the following theorem.

Theorem 2.4. *Let $F \in \mathcal{A}_n^{(2)}$ be given by (1.7). Then for all real $q \neq 0$,*

$$T_{-q}(T_q(F)) \approx F.$$

3. CONVOLUTIONS AND TRANSFORMS OF CONVOLUTIONS

Our first lemma gives an expression for $(F_1 * F_2)_\lambda$ for $\lambda \in \mathbb{C}_+$.

Lemma 3.1. *Let $1 \leq p \leq \infty$, and let $F_j \in \bigcup_{1 \leq p \leq \infty} \mathcal{A}_n^{(p)}$ for $j = 1, 2$ be given by (1.7). Then for all $\lambda \in \mathbb{C}_+$,*

$$(3.1) \quad (F_1 * F_2)_\lambda(y) = h \left(\lambda; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy \right)$$

where

(3.2)

$$h(\lambda; w_1, \dots, w_n) \equiv h(\lambda; \vec{w}) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1\left(\frac{\vec{w} + \vec{u}}{\sqrt{2}}\right) f_2\left(\frac{\vec{w} - \vec{u}}{\sqrt{2}}\right) \\ \times \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n u_j^2\right\} d\vec{u}.$$

Proof. For $\lambda > 0$, using a well-known Wiener integration formula we obtain

$$(F_1 * F_2)_\lambda(y) = \int_{C_0[0, T]} F_1\left(\frac{y + \lambda^{-1/2}x}{\sqrt{2}}\right) F_2\left(\frac{y - \lambda^{-1/2}x}{\sqrt{2}}\right) m(dx) \\ = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1\left(2^{-1/2} \left[\int_0^T \alpha_1 dy + u_1\right], \dots, \right. \\ \left. 2^{-1/2} \left[\int_0^T \alpha_n dy + u_n\right]\right) \\ \times f_2\left(2^{-1/2} \left[\int_0^T \alpha_1 dy - u_1\right], \dots, 2^{-1/2} \left[\int_0^T \alpha_n dy - u_n\right]\right) \\ \times \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n u_j^2\right\} d\vec{u} \\ = h\left(\lambda; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy\right)$$

where h is given by (3.2), so (3.1) holds for $\lambda > 0$. Now by analytic continuation in λ , we see that (3.1) holds for all λ in $\mathbb{C}_+$. $\square$

Our next theorem establishes an interesting relationship involving convolutions and analytic Wiener integrals.

Theorem 3.1. Let $1 \leq p \leq \infty$, and let $F_j \in \bigcup_{1 \leq p \leq \infty} \mathcal{A}_n^{(p)}$ for $j = 1, 2$ be given by (1.7). Then for all $\lambda \in \mathbb{C}_+$,

$$(3.3) \quad (T_\lambda(F_1 * F_2)_\lambda)(z) = (T_\lambda(F_1))(2^{-1/2}z)(T_\lambda(F_2))(2^{-1/2}z).$$

Proof. It will suffice to establish (3.3) for $\lambda > 0$ since $T_\lambda(F_1 * F_2)_\lambda$, $T_\lambda(F_1)$, and $T_\lambda(F_2)$ all have analytic extensions throughout $\mathbb{C}_+$. So let $\lambda > 0$ be given.

Then by (3.1) and (3.2),

$$\begin{aligned} (T_\lambda(F_1 * F_2)_\lambda)(z) &= \int_{C_0[0, T]} (F_1 * F_2)_\lambda(\lambda^{-1/2}x + z)m(dx) \\ &= \int_{C_0[0, T]} h\left(\lambda; \int_0^T \alpha_1 d[\lambda^{-1/2}x + z], \dots, \right. \\ &\qquad \qquad \qquad \left. \int_0^T \alpha_n d[\lambda^{-1/2}x + z]\right) m(dx) \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} h\left(\lambda; v_1 + \int_0^T \alpha_1 dz, \dots, v_n + \int_0^T \alpha_n dz\right) \\ &\qquad \qquad \qquad \times \left\{-\frac{\lambda}{2} \sum_{j=1}^n v_j^2\right\} d\vec{v} \\ &= \left(\frac{\lambda}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} f_1\left(2^{-1/2}\left[v_1 + u_1 + \int_0^T \alpha_1 dz\right], \dots, \right. \\ &\qquad \qquad \qquad \left. 2^{-1/2}\left[v_n + u_n + \int_0^T \alpha_n dz\right]\right) \\ &\qquad \times f_2\left(2^{-1/2}\left[v_1 - u_1 + \int_0^T \alpha_1 dz\right], \dots, \right. \\ &\qquad \qquad \qquad \left. 2^{-1/2}\left[v_n - u_n + \int_0^T \alpha_n dz\right]\right) \\ &\qquad \times \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^\infty [u_j^2 + v_j]\right\} d\vec{u} d\vec{v}. \end{aligned}$$

Next we make the transformation

$$w_j = 2^{-1/2}(v_j + u_j)$$

and

$$r_j = 2^{-1/2}(v_j - u_j)$$

for $j = 1, 2, \dots, n$. The Jacobian of this transformation is one and

$$\sum_{j=1}^n [w_j^2 + r_j^2] = \sum_{j=1}^n [u_j^2 + v_j^2].$$

Hence for $\lambda > 0$, using (2.1) and (2.2), we see that

$$\begin{aligned}
& (T_\lambda(F_1 * F_2)_\lambda)(z) \\
&= \left(\frac{\lambda}{2\pi}\right) \int_{\mathbb{R}^{2n}} f_1 \left(w_1 + 2^{-1/2} \int_0^T \alpha_1 dz, \dots, w_n + 2^{-1/2} \int_0^T \alpha_n dz \right) \\
&\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n w_j^2 \right\} \\
&\quad \times f_2 \left(r_1 + 2^{-1/2} \int_0^T \alpha_1 dz, \dots, r_n + 2^{-1/2} \int_0^T \alpha_n dz \right) \\
&\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n r_j^2 \right\} d\vec{w} d\vec{r} \\
&= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1 \left(w_1 + 2^{-1/2} \int_0^T \alpha_1 dz, \dots, w_n + 2^{-1/2} \int_0^T \alpha_n dz \right) \\
&\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n w_j^2 \right\} d\vec{w} \\
&\quad \times \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_2 \left(r_1 + 2^{-1/2} \int_0^T \alpha_1 dz, \dots, r_n + 2^{-1/2} \int_0^T \alpha_n dz \right) \\
&\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n r_j^2 \right\} d\vec{r} \\
&= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1(\vec{w}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \left(w_j - 2^{-1/2} \int_0^T \alpha_j dz \right)^2 \right\} d\vec{w} \\
&\quad \times \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_2(\vec{r}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \left(r_j - 2^{-1/2} \int_0^T \alpha_j dz \right)^2 \right\} d\vec{r} \\
&= (T_\lambda(F_1))(2^{-1/2}z)(T_\lambda(F_2))(2^{-1/2}z).
\end{aligned}$$

Theorem 3.2. *The following hold for all $\lambda \in \mathbb{C}_+^\sim$.*

- (i) *If $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(1)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(1)}$.*
- (ii) *If $F_1 \in \mathcal{A}_n^{(2)}$ and $F_2 \in \mathcal{A}_n^{(2)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(\infty)}$.*
- (iii) *If $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(2)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(2)}$.*
- (iv) *If $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(1)} \cap \mathcal{A}_n^{(2)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(1)} \cap \mathcal{A}_n^{(2)}$.*
- (v) *If $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(\infty)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(\infty)}$.*

Proof. (i) Assume F_1 and F_2 belong to $\mathcal{A}_n^{(1)}$ and are given by (1.7). It will suffice to show that $h(\lambda; \cdot)$ given by (3.2) is in $L_1(\mathbb{R}^n)$ for every $\lambda \in \mathbb{C}_+^\sim$. But this follows from the calculations

$$\begin{aligned} \int_{\mathbb{R}^n} |h(\lambda; \vec{w})| d\vec{w} &\leq \left| \frac{\lambda}{2\pi} \right|^{n/2} \int_{\mathbb{R}^{2n}} |f_1(2^{-1/2}(\vec{w} + \vec{u})) f_2(2^{-1/2}(\vec{w} - \vec{u}))| d\vec{w} d\vec{u} \\ &= \left| \frac{\lambda}{2\pi} \right|^{n/2} \int_{\mathbb{R}^n} |f_1(\vec{v})| 2^{n/2} \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w} - \vec{v})| d\vec{w} d\vec{v} \\ &= \left| \frac{\lambda}{2\pi} \right|^{n/2} \|f_1\|_1 \|f_2\|_1. \end{aligned}$$

(ii) In this case for f_1, f_2 in $L_2(\mathbb{R}^n)$ we first note that $h(\lambda; \cdot)$ is in $L_\infty(\mathbb{R}^n)$ since for all $\vec{w} \in \mathbb{R}^n$,

$$\begin{aligned} |h(\lambda; \vec{w})| &\leq \left| \frac{\lambda}{2\pi} \right|^{n/2} \int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w} + \vec{u}))| |f_2(2^{-1/2}(\vec{w} - \vec{u}))| d\vec{u} \\ &\leq \left| \frac{\lambda}{2\pi} \right|^{n/2} \left\{ \int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w} + \vec{u}))|^2 d\vec{u} \right\}^{1/2} \\ &\quad \times \left\{ \int_{\mathbb{R}^n} |f_2(2^{-1/2}(\vec{w} - \vec{u}))|^2 d\vec{u} \right\}^{1/2} \\ &= \left| \frac{\lambda}{2\pi} \right|^{n/2} (\sqrt{2})^n \|f_1\|_2 \|f_2\|_2 \\ &= \left| \frac{\lambda}{\pi} \right|^{n/2} \|f_1\|_2 \|f_2\|_2. \end{aligned}$$

A standard argument now shows that h belongs to $C_0(\mathbb{R}^n)$.

(iii) Let $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(2)}$ be given by (1.7). It will suffice to show that $h(\lambda; \cdot)$ given by (3.2) is in $L_2(\mathbb{R}^n)$. But this follows from the calculations

$$\begin{aligned} \int_{\mathbb{R}^n} |h(\lambda; \vec{w})|^2 d\vec{w} &\leq \int_{\mathbb{R}^n} \left| \frac{\lambda}{2\pi} \right|^n \left[\int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w} + \vec{u})) f_2(2^{-1/2}(\vec{w} - \vec{u}))| d\vec{u} \right. \\ &\quad \times \left. \int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w} + \vec{v})) f_2(2^{-1/2}(\vec{w} - \vec{v}))| d\vec{v} \right] d\vec{w} \\ &= \left| \frac{\lambda}{2\pi} \right|^n \int_{\mathbb{R}^n} |f_1(\vec{r})| \int_{\mathbb{R}^n} |f_1(\vec{s})| \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w} - \vec{r}) \\ &\quad \times f_2(\sqrt{2}\vec{w} - \vec{s})| d\vec{w} d\vec{s} d\vec{r} \\ &\leq \left| \frac{\lambda}{2\pi} \right|^n \|f_1\|_1^2 \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w} - \vec{r})|^2 d\vec{r} \\ &= \left| \frac{\lambda}{2\pi} \right|^n (2)^{n/2} \|f_1\|_1^2 \|f_2\|_2^2. \end{aligned}$$

Hence $\|h\|_2 \leq |\lambda/\pi|^{n/2} \sqrt{2} \|f_1\|_1 \|f_2\|_2$.

Finally we note that (iv) follows directly from (i) and (iii) while (v) is immediate. $\square$

In our next theorem we show that the Fourier-Feynman transform of the convolution product is the product of transforms.

Theorem 3.3. (i) Let $F_1, F_2 \in \mathcal{A}_n^{(1)}$. Then for all real $q \neq 0$,

$$(3.4) \quad (T_q^{(1)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(1)}(F_2))(2^{-1/2}z).$$

(ii) Let $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(2)}$. Then for all real $q \neq 0$,

$$(3.5) \quad (T_q^{(2)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(2)}(F_2))(2^{-1/2}z).$$

(iii) Let $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(1)} \cap \mathcal{A}_n^{(2)}$. Then for all real $q \neq 0$,

$$(3.6) \quad (T_q^{(1)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(1)}(F_2))(2^{-1/2}z)$$

and

$$(3.7) \quad (T_q^{(2)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(2)}(F_2))(2^{-1/2}z).$$

Proof. Theorem 3.2 together with Theorem 2.2 assures us that all of the transforms on both sides of (3.4) through (3.7) exist. Equations (3.4) through (3.7) now follow from equation (3.3). $\square$

Remark. Throughout this paper, for simplicity we assumed that $\{\alpha_1, \dots, \alpha_n\}$ was an orthonormal set of functions in $L_2[0, T]$. However, all of our results hold provided that $\{\alpha_1, \dots, \alpha_n\}$ is a linearly independent set of functions from $L_2[0, T]$.

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