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A CONVERGENCE THEOREM FOR RIEMANNIAN SUBMANIFOLDS

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ABSTRACT. In this paper we study the convergence of Riemannian submanifolds. In particular, we prove that any sequence of closed submanifolds with bounded normal curvature and volume in a closed Riemannian manifold subconverge to a closed submanifold in the $C^{1,\alpha}$ topology. We also obtain some applications to irreducible homogeneous manifolds and pseudo-holomorphic curves in symplectic manifolds.

1. INTRODUCTION

Let $\mathscr{M}(n, K, v, D)$ be the class of manifolds (M, g) satisfying the bounds

$$|\sec(M)| \leq K, \quad \operatorname{vol}(M) \geq v, \quad \operatorname{diam}(M) \leq D. \tag{1}$$

The well-known convergence theorem says that given a sequence of manifolds $(M_i, g_i) \in \mathscr{M}(n, K, v, D)$, there is a subsequence, $(M_{i'}, g_{i'})$, a $C^{1,\alpha}$ Riemannian n-manifold (M_0, g_0), and $C^{2,\alpha}$ diffeomorphisms $\phi_{i'} : M_0 \to M_{i'}$, such that $\phi_{i'}^* g_{i'}$ converges to g_0 in the $C^{1,\alpha'}$ topology $(\alpha' < \alpha)$. We refer to, for example, [Ch, N1, N2, G1, Ps, GW, K] for details. M. Anderson has extended this theorem to a larger class of Riemannian manifolds (see [AM]).

In this paper we are going to establish a convergence theorem for Riemannian submanifolds. Fix a complete Riemannian m-manifold $(\widetilde{M}, \tilde{g})$. We will denote by (M, f) an n-dimensional submanifold in $(\widetilde{M}, \tilde{g})$, where M is a smooth n-manifold and f is an immersion from M into $\widetilde{M}$. One has the natural orthogonal decomposition $T_{f(x)}\widetilde{M} = f_*(T_xM) \oplus T_xM^\perp$. The normal curvature $II_f : T_xM \otimes T_xM \to T_xM^\perp$ is defined by the formula

$$II_f(u, v) = -(\widetilde{\nabla}_U V)^\perp,$$

where U and V are local extensions of $f_*(u)$ and $f_*(v)$ on $\widetilde{M}$ and $\widetilde{\nabla}$ denotes the Levi-Civita connection of $\tilde{g}$. Throughout this paper we will always denote by $\sec(f)$, $\operatorname{scal}(f)$, $\operatorname{inj}(f)$, and $\operatorname{vol}(f)$ the sectional curvature, the scalar curvature, the injectivity radius, and the volume of $(M, f^*\tilde{g})$, respectively.

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Let $\mathscr{S}_n(\Lambda, V)$ denote the class of n-dimensional closed submanifolds (M, f) in $\widetilde{M}$, which satisfy the bounds

$$|II_f| \le \Lambda, \quad \mathrm{vol}(f) \le V. \tag{2}$$

We have the following

Theorem 1. *Given any uniformly bounded sequence $(M_i, f_i) \in \mathscr{S}_n(\Lambda, V)$, namely, $\bigcup_{i=1}^{\infty} f_i(M_i)$ is contained in a compact subset of $\widetilde{M}$. There is a subsequence $(M_{i'}, g_{i'})$, a $C^{1,\alpha}$ Riemannian manifold (M_0, g_0), and $C^{2,\alpha}$ diffeomorphisms $\phi_{i'} : M_0 \to M_{i'}$, such that $f_{i'} \circ \phi_{i'}$ converges to a $C^{1,\alpha}$ immersion f_0 in the $C^{1,\alpha'}$ topology, for any $\alpha' < \alpha$, with $f_0^* \tilde{g} = g_0$.*

If in addition $\mathrm{inj}(f_i) \ge i_0 > 0$ for all i, the assertion of Theorem 1 follows immediately, see §3 or [K]. It is well known that for a Riemannian manifold, the sectional curvature bound does not give a lower bound on the injectivity radius. A flat torus is a simple example. For a Riemannian submanifold (M, f), contrary to the sectional curvature case, the normal curvature bound $|II_f| \le \Lambda$ does imply $\mathrm{inj}(f) \ge i_0$ for some number i_0 depending on Λ and the ambient space. This fact is somewhat surprising. In [AL] L. Andersson has given such a lower bound on $\mathrm{inj}(f)$ when the ambient space is Euclidean. Then, using results by Cheeger [Ch] and Andersson [AL], R. Howard [H] asserts that there are only finitely many diffeomorphism types among n-dimensional closed Riemannian submanifolds (M, f) in $\mathbb{R}^m$ with $|II_f| \le \Lambda$ and $\mathrm{vol}(f) \le V$.

The main idea of Theorem 1 is to prove a priori estimates on the injectivity radius for Riemannian submanifolds in a general Riemannian manifold. Our treatment is quite different from Andersson's.

Theorem 1 has a number of applications. For a closed submanifold (M, f) in $(\widetilde{M}, \tilde{g})$, we denote by $\mathrm{tr}\, II_f$ the mean curvature of f, which is defined by

$$\mathrm{tr}\, II_f = \sum_i II_f(e_i, e_i),$$

where $\{e_i\}$ is an orthonormal basis on $(M, f^* \tilde{g})$. Suppose that (M, f) satisfies the bounds

$$|\mathrm{tr}\, II_f| \le H, \quad \mathrm{scal}(f) \ge -\lambda, \quad \mathrm{vol}(f) \le V. \tag{3}$$

Further, suppose that the sectional curvature of $\widetilde{M}$ is bounded from above by K along $f(M)$. Then it follows from the Gauss equation that $|II_f| \le \Lambda(n, K, H, \lambda)$.

Let $\mathscr{V}_n(H, \lambda, V)$ be the class of all n-dimensional closed submanifolds (M, f) in $(\widetilde{M}, \tilde{g})$, which satisfy (3). We have the following

Corollary 1. *Given positive numbers H, λ, and V, the class $\mathscr{V}_n(H, \lambda, V)$ is precompact in the $C^{1,\alpha}$ topology $(\alpha < 1)$ in the sense of Theorem 1.*

Remark. The lower bound on the scalar curvature in Corollary 1 is necessary even when $n = 2$. See [G2] for some examples.

Let (N, ω) be a closed symplectic manifold with an ω-tamed almost complex structure J, i.e.,

$$\omega(v, Jv) > 0, \qquad v \ne 0.$$

Then there is a naturally induced Hermitian metric g_J defined by

$$g_J(u, v) = \frac{1}{2}(\omega(u, Jv) + \omega(v, Ju)).$$

Let (Σ, j) be a closed Riemannian surface, where j is a complex structure j, and $f : \Sigma \to N$ be a pseudo-holomorphic map, i.e., $df \circ j = J \circ df$. It is well known that f has isolated branch points (p is called a branch point of f if df_p is not injective, otherwise p is called a regular point). It was proved in [G2] that at regular points of f,

$$|\operatorname{tr} II_f| \le C_1, \tag{4}$$

where C_1 is a constant independent of f. Further if f represents a class $\alpha \in H_2(N, \mathbb{Z})$, then

$$\operatorname{vol}(f) \le V \tag{5}$$

where V is a constant independent of representatives f in α.

By the Gauss equation, $\operatorname{scal}(f) \le C_2$, where C_2 is also independent of f. However, there is no lower bound on the $\operatorname{scal}(f)$. Gromov's compactness theorem [G2] says that any sequence of (regular) pseudo-holomorphic curves (Σ, f_i) satisfying (5), subconverges to a cusp-curve (see also [PW, Y]). In particular, the limit may have singularities. Our next corollary asserts that there is no singular point in the limit if in addition $\operatorname{scal}(f_i) \ge -\lambda$ for all i. This property follows from Corollary 1 and (4). More precisely, we have the following

Corollary 2. *Let (Σ, j) and (N, ω, J) be as above. Let $f_i : \Sigma \to N$ be a sequence of regular J-holomorphic maps satisfying*

$$\operatorname{scal}(f_i) \ge -\lambda, \quad \operatorname{vol}(f_i) \le V.$$

Then there exists a sequence of diffeomorphisms $\phi_{i'}$ of Σ such that $f_{i'} \circ \phi_{i'}$ converges to a $C^{1,\alpha}$ regular pseudo-holomorphic map f_0.

We shall apply Theorem 1 to a certain class of homogeneous Riemannian manifolds.

Proposition 1. *Given positive numbers λ, v, and D, let $\mathscr{H}_n(\lambda)$ denote the class of irreducible homogeneous Riemannian n-manifolds (M, g) which satisfy the bounds*

$$\operatorname{scal}(M) \ge -\lambda, \quad \operatorname{diam}(M) = 1. \tag{6}$$

Then $\mathscr{H}_n(\lambda)$ is compact in the C^∞ topology. In particular, $\mathscr{H}_n(\lambda)$ contains finitely diffeomorphism types.

The proof of Proposition 1 will be given in the last section. It is not known to the author whether there are only finitely many diffeomorphism types among irreducible homogeneous Riemannian manifolds in each dimension.

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2. Isoperimetric inequalities

Let (M, g) be a complete Riemannian n-manifold. For any positive number $\mu > 0$, define the μ-isoperimetric constant of (M, g) by the formula

$$I_\mu(M) = \inf\left\{\frac{\operatorname{vol}(\partial\Omega)}{\operatorname{vol}(\Omega)^{(n-1)/n}}, \ \operatorname{vol}(\Omega) \le \mu\right\}.$$

Put $I(M) = I_\mu(M)$ for $\mu = \frac{1}{2}\mathrm{vol}(M)$. This is the so-called isoperimetric constant of (M, g).

It is easy to see that for a closed Riemannian n-manifold M with $I_\mu(M) > 0$, $\mathrm{vol}(M) > \mu$. Furthermore one has the following

Lemma 1. *Let (M, g) be a complete Riemannian n-manifold. Suppose that $I_\mu(M) \geq I_0 > 0$ for some $\mu > 0$. Then for all $r \leq r_0$, where $r_0 = nI_0^{-1}\mu^{1/n}$,*

$$\mathrm{vol}(B(p, r)) \geq n^{-n} I_0^n r^n.$$

Proof. First note that $I_\mu(M) \geq I_0$ is equivalent to the inequality

$$\mathrm{vol}(\partial\Omega) \geq I_0 \,\mathrm{vol}(\Omega)^{\frac{n-1}{n}} \tag{7}$$

for all domains Ω with $\mathrm{vol}(\Omega) \leq \mu$.

Let $B(p, r)$ be a metric ball in M with radius $r \leq r_0$. First we suppose that $\mathrm{vol}(B(p, r)) > \mu$. Then

$$\mathrm{vol}(B(p, r)) > n^{-n} I_0 r_0^n \geq n^{-n} I_0^n r^n.$$

Suppose that $\mathrm{vol}(B(p, r)) \leq \mu$. Then (7) implies the function $f(t) := \mathrm{vol}(B(p, t))$ satisfies

$$f'(t) \geq I_0 f(t)^{(n-1)/n}, \qquad t \leq r.$$

It follows from the above inequality that $f(t) \geq n^{-n} I_0^n t^n$, $t \leq r$. In particular, one obtains

$$\mathrm{vol}(B(p, r)) = f(r) \geq n^{-n} I_0^n r^n. \quad \square$$

In order to estimate the injectivity radius we need the following

Lemma 2 [CGT, Theorem 4.3]. *Let (M, g) be a complete n-manifold with $H \leq \sec(M) \leq K$. Suppose that at a point $p \in M$,*

$$\mathrm{vol}(B(p, s)) \geq c_0$$

for some $s \leq \pi/4\sqrt{K}$. Then the injectivity radius at p satisfies

$$\mathrm{inj}(p) \geq \frac{1}{1 + V^H(3s)/c_0} \cdot s,$$

where $V^H(r)$ denotes the volume of r-ball in the space form of constant curvature H.

Corollary 3. *Suppose that (M, g) is a complete Riemannian n-manifold satisfying the bounds*

$$I_\mu(M) \geq I_0, \quad |\sec(M)| \leq K.$$

Then

$$\mathrm{inj}(M) \geq c(n, \mu, I_0, K) > 0.$$

Proof. This follows from Lemmas 1 and 2. $\square$

Let (M, g) be a closed Riemannian n-manifold, and let $C_M(\varepsilon)$ denote the maximal number of disjoint metric $\varepsilon/2$-balls in M. Suppose that M satisfies the bounds

$$I_\mu(M) \geq I_0, \quad \mathrm{vol}(M) \leq V. \tag{8}$$

Then it follows from Lemma 1 that for all $\varepsilon \leq \varepsilon(n, I_0, \mu)$

$$C_M(\varepsilon) \leq C(n, I_0, V)\varepsilon^{-n}. \tag{9}$$

Let $\mathscr{C}(n, \mu, I_0, V)$ denote the class of all closed n-manifolds satisfying (8). The following proposition follows from (9) and [G1].

Proposition 2. *$\mathscr{E}(n, \mu, I_0, V)$ is pre-compact in the Gromov-Hausdorff topology.*

From now on we will only consider immersions in a complete Riemannian m-manifold $(\widetilde{M}, \tilde{g})$. Suppose that f is complete, i.e., the pull-back metric $f^*\tilde{g}$ is complete, and it has bounded mean curvature. Then $(M, f^*\tilde{g})$ has positive μ-isoperimetric constant for some $\mu > 0$, provided that $\widetilde{M}$ has bounded geometry along the image of f. Let $\widetilde{U}_f = \{y \in \widetilde{M}, \operatorname{dist}(y, f(M)) < 1\}$, and $I_\mu(f) = I_\mu(M, f^*\tilde{g})$.

Theorem 2 (Hoffman and Spruck [HS]). *Suppose that*

$$\sec(\widetilde{U}_f) \le K, \quad \operatorname{inj}(\widetilde{U}_f) \ge i_0, \quad |\operatorname{tr} II_f| \le \Lambda.$$

Then there are constants $c(n) > 0$ and $\mu = \mu(n, K, i_0, \Lambda) > 0$, such that

$$I_\mu(f) \ge c(n).$$

Remark. This type of theorem in the case of $\widetilde{M} = \mathbb{R}^m$ was first obtained by J. Michael and L. Simon [MS].

Proposition 3. *Suppose that*

$$|\sec(\widetilde{U}_f)| \le K, \quad \operatorname{inj}(\widetilde{U}_f) \ge i_0, \quad |II_f| \le \Lambda.$$

Then there is a constant $c(n, K, i_0, \Lambda) > 0$ such that

$$\operatorname{inj}(f) \ge c(n, K, i_0, \Lambda).$$

Proof. By Theorem 2 one has $I_\mu(f) \ge c(n)$ for some $\mu = \mu(n, K, i_0, \Lambda) > 0$. Then Proposition 3 follows from the Gauss equation and Corollary 3. □

Remark. Proposition 3 generalizes a result of L. Andersson who treated the case when the ambient space is Euclidean. Our treatment, however, is quite different from his.

3. Proof of Theorem 1

Before going into proof, we shall recall first some basic facts on harmonic coordinates. We refer the reader to [JK] and [GW] for details.

A coordinate $\mathbf{x} = (x^A) : U \subset \widetilde{M} \to \mathbb{R}^m$ is said to be harmonic if $\Delta_{\tilde{g}} x^A = 0$, $1 \le A \le m$. We suppose that $(\widetilde{M}, \tilde{g})$ satisfies the bounds

$$|\sec(\widetilde{M})| \le K, \quad \operatorname{inj}(\widetilde{M}) \ge i_0. \tag{10}$$

Then there is a number $\delta(m, K, i_0) > 0$ such that given any metric ball $B_\delta(p)$, $\delta < \delta(m, K, i_0)$, there is a harmonic coordinate $\mathbf{x} = (x^A)$ on $B_\delta(p)$ which has the following properties:

(i) $(1 + \eta_0(m, K\delta))^{-1}\delta_{AB} \le g_{AB} \le (1 + \eta_0(m, K\delta))\delta_{AB}$, where $g_{AB} = \tilde{g}(\partial_A, \partial_B)$;

(ii) $\|g_{AB}\|_{C^{1,\alpha}(B_\delta(p))} \le \eta_1(m, K, i_0, \alpha)$; and

(iii) $\|x^A\|_{C^{2,\alpha}(B_{\delta/2}(p))} \le \eta_2(m, K, i_0, \alpha)\|x^A\|_{C^0(B_\delta(p))}$.

Remark. Using the idea of [AL] and the above properties for harmonic coordinates, one can also prove a priori estimates on the injectivity radius for Riemannian submanifolds.

Now we start to prove Theorem 1. Given any sequence of immersions $(M_i, f_i) \in \mathscr{S}_n(\Lambda, V)$. By assumption we may assume that $\bigcup_i f_i(M_i) \subset F$ for some compact subset F in $\widetilde{M}$. Let $\widetilde{U}_F = \{\tilde{x} \in \widetilde{M}, \operatorname{dist}(\tilde{x}, F) < 1\}$. Suppose that

$$|\sec(\widetilde{U}_F)| \leq K, \quad \operatorname{inj}(\widetilde{U}_F) \geq i_0.$$

By Proposition 3 and the Gauss equation, one has

$$|\sec(f_i)| \leq K + 2\Lambda^2, \quad \operatorname{inj}(f_i) \geq c(n, K, \Lambda, i_0).$$

Put $g_i = f_i^* \tilde{g}$. Consider the sequence of Riemannian manifolds (M_i, g_i). By Gromov's compactness theorem (see, e.g., [K, AM]), there exist a subsequence (M_k, g_k), a closed $C^{1,\alpha}$ Riemannian n-manifold (M_0, g_0), and $C^{2,\alpha}$ diffeomorphisms $\phi_k : M_0 \to M_k$ such that $\tilde{g}_k := \phi_k^* g_k$ converges to g_0 in the $C^{1,\alpha'}$ topology $(\alpha' < \alpha)$. Fix a harmonic coordinate system (x^A) on $(\widetilde{M}, \tilde{g})$ and a harmonic coordinate system (x^a) on (M_0, g_0). Put $\tilde{f}_k = f_k \circ \phi_k$. We have

$$(\tilde{g}_k)_{ab} := \partial_a \tilde{f}_k^A \partial_b \tilde{f}_k^B g_{AB} \to g_{ab},$$

in the $C^{1,\alpha'}$ topology $(\alpha' < \alpha)$, where $g_{AB} = \tilde{g}(\partial_A, \partial_B)$, $g_{ab} = g_0(\partial_a, \partial_b)$. Thus one obtains a uniform C^0 bound on $\partial_a \tilde{f}_k$. Let $II_k = II_{\tilde{f}_k}$. By the definition of II_k, one has

$$\partial_a \partial_b \tilde{f}_k^C + \partial_a \tilde{f}_k^A \partial_b \tilde{f}_k^B \Gamma_{AB}^C(\tilde{f}_k) = \partial_c \tilde{f}_k^C \widetilde{\Gamma}_{ab}^c - (II_k)_{ab}^C \tag{11}$$

where $\widetilde{\Gamma}_{ab}^c$ are Christoffel's symbols of $\tilde{g}_k$ and $II_k(\partial_a, \partial_b) = (II_k)_{ab}^C \partial_C$. By assumption,

$$|(II_k)_{ab}^C| \leq C_0 \Lambda \sum_A |\partial_a \tilde{f}_k^A \partial_b \tilde{f}_k^A|,$$

where C_0 is a constant independent of k. Thus one obtains a uniform C^0 bound on $\partial_a \partial_b \tilde{f}_k^C$. Since $\bigcup_k \tilde{f}_k(M_0)$ is contained in a compact subset in $\widetilde{M}$, one concludes that $\tilde{f}_k$ subconverges to a $C^{1,\alpha}$ map f_0 in the $C^{1,\alpha'}$ topology $(\alpha' < \alpha)$. Clearly, $f_0^* \tilde{g} = g_0$, which implies $f_0 : (M_0, g_0) \to (\widetilde{M}, \tilde{g})$ is an isometric immersion. □

Remark. Without much difficulty, one can also show that if (M, f) is almost totally geodesic in a closed Riemannian m-manifold $(\widetilde{M}, \tilde{g})$, that is $\operatorname{vol}(f) \leq V$ and $|II_f| \leq \varepsilon(n, V, \widetilde{M})$, then there is a $C^{2,\alpha}$ immersion $\hat{f} : M \to \widetilde{M}$ such that $\hat{f}$ is totally geodesic. One can prove this fact by a limit argument. See [JK] for the regularity of the limit immersion.

4. Irreducible homogeneous manifolds

A connected Riemannian manifold (M, g) is said to be (isotropy) irreducible if for each point $p \in M$ the isotropy group I_p acts irreducibly on T_pM via its isotropy representation. (M, g) must be a homogeneous space. Hence we call it an irreducible homogeneous Riemannian manifold. It is known that g must be an Einstein metric. Further, by a theorem of Takahashi [T], any compact irreducible homogeneous Riemannian manifold can be isometrically minimally immersed into some $S^K(r) \subset \mathbb{R}^{K+1}$ using an orthonormal basis for an arbitrary eigenspace. See [L1, L2] for further discussion.

Proof of Proposition 1. Let (M, g) be a closed irreducible homogeneous Riemannian n-manifold satisfying (6). Let λ_1 denote the first nonzero eigenvalue and E_{λ_1} the corresponding eigenspace. It is a finite-dimensional Hilbert space with L^2-norm. Take any orthonormal basis $\phi_1, \dots, \phi_{K+1}$ for E_{λ_1}. Then for some constant $\alpha > 0$, $f = \alpha \cdot (\phi_1, \dots, \phi_{K+1})$ is an isometric minimal immersion into $S^K(r) \subset \mathbb{R}^{K+1}$ with $r^2 = n/\lambda_1$.

Regard (M, f) as a submanifold in $\mathbb{R}^{K+1}$. Let II_f denote the normal curvature of f. Then $\operatorname{tr} II_f := \sum_{i=1}^n II_f(e_i, e_i) = \frac{n}{r}\eta$, where η denotes the unit normal vector field on $S^K(r) \subset \mathbb{R}^{K+1}$. By the Gauss equation,

$$\operatorname{scal}(M) = n\lambda_1 - \sum_{i=1}^n |II_f(e_i, e_i)|^2.$$

Thus

$$|II_f| \le \sqrt{n\lambda_1 + \lambda}.$$

Notice that (M, g) is Einstein and $\operatorname{diam} = 1$. It follows from [Che] that

$$\lambda_1 \le c_1(n)\lambda + c_2(n). \tag{12}$$

Thus

$$|II_f| \le C_1(n, \lambda). \tag{13}$$

It follows from (12), (13), and the Gauss equation that

$$|\sec(M)| \le C_2(n, \lambda).$$

It follows from Proposition 3 that

$$\operatorname{inj}(M) \ge C_3(n, \lambda) > 0.$$

Therefore, $\mathscr{H}(n, \lambda)$ is precompact in the C^∞ topology, since all manifolds in $\mathscr{H}(n, \lambda)$ are Einstein. □

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