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A NORM CONVERGENCE RESULT ON RANDOM PRODUCTS OF RELAXED PROJECTIONS IN HILBERT SPACE

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ABSTRACT. Suppose X is a Hilbert space and $C_1, \ldots, C_N$ are closed convex intersecting subsets with projections $P_1, \ldots, P_N$. Suppose further r is a mapping from $\mathbb{N}$ onto $\{1, \ldots, N\}$ that assumes every value infinitely often. We prove (a more general version of) the following result:

If the N-tuple $(C_1, \ldots, C_N)$ is "innately boundedly regular", then the sequence (x_n), defined by

$$x_0 \in X \text{ arbitrary}, \quad x_{n+1} := P_{r(n)}x_n, \text{ for all } n \geq 0,$$

converges in norm to some point in $\bigcap_{i=1}^N C_i$.

Examples without the usual assumptions on compactness are given. Methods of this type have been used in areas like computerized tomography and signal processing.

1. INTRODUCTION, FACTS, AND NOTATION

Numerous problems in mathematics [10] and physical sciences [9, 8, 26] can be described as follows. Let X be a real Hilbert space and suppose $T_1, \ldots, T_N$ are pairwise distinct nonexpansive self-mappings of some closed convex nonempty subset D of X; recall that a self-mapping T of D is called *nonexpansive*, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in D$. Suppose further that the *set of fixed points*, $\operatorname{Fix} T_i := \{x \in D: T_ix = x\}$, of each mapping T_i is nonempty and that $C := \bigcap_{i=1}^N \operatorname{Fix} T_i \neq \varnothing$. The aim is to find such a common fixed point. One frequently employed approach is the following:

Let r be a *random mapping* for $\{1, \ldots, N\}$, i.e., a surjective mapping from $\mathbb{N}$ onto $\{1, \ldots, N\}$ that takes each value in $\{1, \ldots, N\}$ infinitely often. Then generate a *random sequence* (x_n) by

$$x_0 \in D \text{ arbitrary}, \quad x_{n+1} := T_{r(n)}x_n, \quad \text{for all } n \geq 0,$$

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and hope that this sequence converges to some point in C. We also speak of a *random* or *unrestricted product* (resp. *iteration*). (For products generated by some form of control, there are many results: for instance, *cyclic control* arises when $r(n) = n + 1 \mod N$; see [6].)

This is, in general, a hopeless undertaking, as the example $X := \mathbb{R}$, $N := 1$, and $T_1 := -I$ shows (as usual, I denotes the identity).

So let us temporarily consider the important special case when $D = X$ and each mapping T_i is the projection onto some closed convex nonempty subset C_i of X; hence $\operatorname{Fix} T_i = C_i$. The problem of finding a common fixed point is then the famous *Convex Feasibility Problem.* This situation allows us to compare the following known results (in fact, all authors listed below have established (much) more general but less comparable results):

Amemiya and Ando [3]: If each set C_i is a closed subspace, then the random product converges weakly to the projection onto C.

Bruck [7]: If some set C_i is compact, then the random product converges in norm to some point in C. If $N = 3$ and each set C_i is symmetric, then the random product converges weakly to some point in C.

Dye [11]: If the sets C_i are finite-dimensional subspaces, then the random product converges in norm to some point in C.

Dye and Reich [14]: If the sets C_i have a common "weak internal point" or if $N = 3$, then the random product converges weakly to some point in C.

Youla [29]: If the sets C_i have a common "inner point", then the random product converges weakly to some point in C.

Aharoni and Censor [2], Flåm and Zowe [16], Tseng [27], Elsner et al. [15]: If X is finite dimensional, then the random product converges in norm to some point in C.

The objective of this paper is to provide a new applicable condition which guarantees norm convergent random products.

The paper is organized as follows: In Section 2, we discuss four important concepts: (*innate*) *bounded regularity* is a crucial geometric property of tuples of closed convex sets. *Fejér monotonicity* and *Baillon and Bruck's quasi-projection* capture essential properties of random sequences. Relaxed projections and Banach contractions are subsumed in the class of *projective mappings.* The third sections contains our main result and some examples.

Suppose C is a closed convex nonempty subset of X. The *projection onto* C, denoted P_C, is the mapping which sends every point to its nearest point in C. The associated *distance function* is defined by $d(\cdot, C)\colon X \to [0, +\infty[\colon x \mapsto \|x - P_C x\| = \inf_{c \in C} \|x - c\|$. If $\alpha \in]0, 2[$, then the mapping $R := (1-\alpha)I + \alpha P_C$ is called a *relaxed projection.* For R, the following holds:

Facts 1.1. (i) [18] R is nonexpansive. (ii) [6, Lemma 2.4(iv)] For every $x \in X$ and every $c \in C$, $\|x - c\|^2 - \|Rx - c\|^2 \geq \alpha(2-\alpha)\|x - P_C x\|^2$.

A nonexpansive self-mapping T of some closed convex nonempty subset D of X is called a *Banach contraction* if there is some *contraction constant* $\kappa \in [0, 1[$, such that $\|Tx - Ty\| \leq \kappa\|x - y\|$, for all $x, y \in D$.

Fact 1.2. [30, Lemma A2]. Suppose T is a nonexpansive self-mapping of some closed convex nonempty subset D of X. If T has fixed points and C is a closed convex nonempty subset of $\operatorname{Fix} T$, then $\|x - Tx\| \leq 2d(x, C)$, for every $x \in D$.

Finally, "$\to$" abbreviates norm converge and "int" stands for the interior.

2. Four handy tools

Definition 2.1 (Tool 1: (innate) bounded regularity). An N-tuple $(C_1, \dots, C_N)$ of closed convex intersecting sets is called *boundedly regular* if for every bounded sequence (x_n) in X,

$$\max\{d(x_n, C_i) : i \in \{1, \dots, N\}\} \to 0 \text{ implies } d(x_n, C_1 \cap \cdots \cap C_N) \to 0.$$

We say that $(C_1, \dots, C_N)$ is *innately boundedly regular* if $(C_j)_{j \in J}$ is boundedly regular, for every nonempty subset J of $\{1, \dots, N\}$.

Facts 2.2. Suppose $C_1, \dots, C_N$ are closed convex intersecting sets in X. Then the N-tuple $(C_1, \dots, C_N)$ is innately boundedly regular, whenever at least one of the following conditions holds:

(i) All sets, except possibly one, are boundedly compact.
(ii) X is finite dimensional.
(iii) Each set is a closed subspace and the sum $\sum_{j \in J} C_j^{\perp}$ is closed, for every nonempty subset J of $\{1, \dots, N\}$.
(iv) Each set is a closed subspace and all sets, except possibly one, are finite dimensional.
(v) Each set is a closed subspace and all sets, except possibly one, have finite codimension.
(vi) Each set is a *polyhedron*, i.e. a finite intersection of half-spaces.
(vii) Each set is a hyperplane.
(viii) Each set is a half-space.
(ix) There is some $i \in \{1, \dots, N\}$ such that $C_i \cap \bigcap_{j \in \{1, \dots, N\} \setminus \{i\}} \operatorname{int} C_j \neq \varnothing$.

Proof. (i), (ii), … ,(ix), follow from [6, Proposition 5.4(i), Proposition 5.4(iii), Theorem 5.19, Corollary 5.21(i), Corollary 5.21(ii), Corollary 5.26, Corollary 5.22, Fact 5.23, Corollary 5.14], respectively. □

Definition 2.3 (Tool 2: Fejér monotone sequences). Suppose C is a closed convex nonempty subset of X and (x_n) is a sequence in X. We say that (x_n) *is Fejér monotone w.r.t.* C if

$$\|x_{n+1} - c\| \leq \|x_n - c\|, \quad \text{for every } c \in C \text{ and all } n.$$

Facts 2.4. Suppose the sequence (x_n) is Fejér monotone w.r.t. some closed convex nonempty subset C of X. Then (see [20] or [6]):

(i) The sequences $(d(x_n, C))$, $(\|x_n - c\|)$ are decreasing and convergent for every $c \in C$. In particular, (x_n) is bounded.
(ii) (x_n) converges in norm to some point in C if and only if there is some subsequence (x_{n_k}) of (x_n) with $d(x_{n_k}, C) \to 0$.

Definition 2.5 (Tool 3: Baillon and Bruck's [4] quasi-projection). Suppose C is a closed convex nonempty subset of X and x_0 is a point in X. The *quasi-projection of* x_0 *onto* C, denoted $\mathcal{Q}_C x_0$, is defined by

$$\mathcal{Q}_C x_0 := \{x \in C : \|x - c\| \leq \|x_0 - c\|, \text{ for every } c \in C\}.$$

Proposition 2.6. *Suppose C is a closed convex nonempty subset of X and x_0 is a point in X. Then:*

(i) *$\mathscr{Q}_C x_0$ is a bounded closed convex nonempty subset of C.*
(ii) *$P_C x_0 \in \mathscr{Q}_C x_0 \subseteq \{x \in C\colon \|x - P_C x_0\| \le d(x_0, C)\}$.*
(iii) *If $x_0 \in C$, then $\mathscr{Q}_C x_0 = \{P_C x_0\} = \{x_0\}$.*
(iv) *$\mathscr{Q}_{C+z} x_0 = z + \mathscr{Q}_C(x_0 - z)$, for every $z \in X$.*
(v) *If C is a closed affine subspace, then $\mathscr{Q}_C \equiv P_C$.*
(vi) *If $(x_n)_{n \ge 0}$ is a Fejér monotone sequence w.r.t. C converging weakly to some point $x \in C$, then $x \in \mathscr{Q}_C x_0$.*

Proof. It is a straightforward to check (i)–(iv).

(v): In view of (iv), we need only consider the case when C is a closed subspace. Pick $\bar{c} \in \mathscr{Q}_C x_0$, fix an arbitrary real number t, and let $c := P_C x_0 + t(\bar{c} - P_C x_0)$. Then $c \in C$ and $\|\bar{c} - c\| \le \|x_0 - c\|$. Squaring yields $(1-t)^2\|P_C x_0 - \bar{c}\|^2 \le \|P_{C^\perp} x_0\|^2 + t^2\|P_C x_0 - \bar{c}\|^2$ or $\|P_C x_0 - \bar{c}\|^2 - 2t\|P_C x_0 - \bar{c}\|^2 \le \|P_{C^\perp} x_0\|^2$. Then letting $t \to -\infty$, we obtain a contradiction—unless $\bar{c} = P_C x_0$.

(vi) follows readily from the weak lower semicontinuity of the norm. □

Definition 2.7 (Tool 4: projective mappings). Suppose T is a nonexpansive self-mapping of some closed convex nonempty subset D of X. We say that T is *projective w.r.t.* c if $c \in \operatorname{Fix} T$ and if for every bounded sequence (x_n) in D,

$$\|x_n - c\| - \|Tx_n - c\| \to 0 \quad \text{implies} \quad d(x_n, \operatorname{Fix} T) \to 0.$$

If T has fixed points and is projective w.r.t. every one of them, then we simply speak of a *projective mapping.*

Lemma 2.8. *Suppose T is projective w.r.t. c. Then:*

(i) *If T is projective, then T is attracting* [6] (*see also* [25, *Corollary* 1.1]), *i.e., $\|Tx - c\| < \|x - c\|$, for every $x \in D \backslash \operatorname{Fix} T$ and every $c \in \operatorname{Fix} T$.*
(ii) *T has condition* (S) *w.r.t. c* [12]; *i.e., if (x_n) is a bounded sequence in D and $\|x_n - c\| - \|Tx_n - c\| \to 0$, then $x_n - Tx_n \to 0$.*
(iii) *For every $x_0 \in D$, the sequence of iterates $(T^n x_0)_{n \ge 0}$ converges in norm to some fixed point of T.*

Proof. (i) follows easily from the definitions. (ii) follows from Fact 1.2(iii): $(T^n x_0)$ is Fejér monotone w.r.t. $\operatorname{Fix} T$ and $\|T^n x_0 - c\| - \|T^{n+1} x_0 - c\| \to 0$, for every $c \in \operatorname{Fix} T$; thus $d(T^n x_0, \operatorname{Fix} T) \to 0$ and the result follows from Facts 2.4. □

Remarks 2.9. (1) Suppose T is a nonexpansive self-mapping of some closed convex nonempty subset D of X with $\operatorname{Fix} T \ne \varnothing$. The following condition appears in Petryshyn and Williamson's [25, Theorem 1.2] and [24, Proposition 1]: (PW) For every bounded sequence (x_n) in D, $x_n - Tx_n \to 0$ implies $d(x_n, \operatorname{Fix} T) \to 0$. Clearly, if T is projective w.r.t. some fixed point, then T satisfies condition (PW). In contrast, let $X = D = \mathbb{R}$ and $Tx = 1 - x$. This mapping satisfies condition (PW) and the sequence $(T^n x)$ does not converge, for every $x \notin \operatorname{Fix} T = \{\frac{1}{2}\}$; thus T is not projective w.r.t. $\frac{1}{2}$.

(2) If $T\colon l_2 \to l_2\colon x = (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$, then the sequence $(T^n x)$ converges in norm to the (only) fixed point 0, for every $x \in l_2$. However, T is not projective w.r.t. 0 (consider the sequence of unit vectors); hence the converse of (iii) does not hold in general.

Theorem 2.10. *The class of projective mappings includes* (i) *Banach contractions and* (ii) *relaxed projections.*

Proof. (i) If $\kappa < 1$ is a contraction constant for a Banach contraction T and $\{c\} = \operatorname{Fix} T$, then $\|Tx - c\| \le \kappa\|x - c\|$ and hence (i) follows from $d^2(x, \operatorname{Fix} T) = \|x - c\|^2 \le (\|x - c\|^2 - \|Tx - c\|^2)/(1 - \kappa^2)$, for every $x \in D$.

(ii) If $\alpha \in]0, 2[$ and $T = (1-\alpha)I + \alpha P$, where $P = P_C$ is the projection onto some closed convex nonempty set C in $D = X$, then (Facts 1.1) $\|x - Px\|^2 = d^2(x, \operatorname{Fix} T) \le (\|x - c\|^2 - \|Tx - c\|^2)/(\alpha(2 - \alpha))$, for every $x \in X$ and every $c \in C$. □

Example 2.11. Genel and Lindenstrauss [17] constructed a *firmly nonexpansive* (see [18] or [19, Section 11]) self-mapping T of $X := l_2$ and some point $x_0 \in X$ such that $0 \in \operatorname{Fix} T$, $(T^n x_0)_{n \ge 0}$ converges weakly to 0 but not in norm: $\inf_n \|T^n x_0\| > 0$. Therefore, by Lemma 2.8(iii), T is not projective.

Remark 2.12. Using Lemma 2.8(i) and [6, Proposition 2.10], one can show the following: Suppose D is a closed convex nonempty subset of X and T_1, T_2 are projective self-mappings of D. If the pair $(\operatorname{Fix} T_1, \operatorname{Fix} T_2)$ is boundedly regular, then $T_2 T_1$ is projective.

Example 2.13. On the real line, let $Tx = \frac{1}{2}|x|^2$, if $|x| \le 1$ and $Tx = |x| - \frac{1}{2}$ otherwise. Then T is projective but does not belong to any of the standard classes of "nice" nonexpansive mappings (cf. [6, Example 2.3]).

3. The main result

Hypothesis. From now on, we always assume that D is a closed convex nonempty subset of X, that $N \ge 1$, that $T_1, \dots, T_N \colon D \to D$ are pairwise distinct and projective w.r.t. some common fixed point $c \in C := \bigcap_{i=1}^N C_i$, where each C_i equals $\operatorname{Fix} T_i$, and that the N-tuple $(C_1, \dots, C_N)$ is innately boundedly regular.

Definition 3.1. A mapping $T\colon D \to D$ is called a *full word*, denoted $T \in \mathscr{F} := \mathscr{F}(T_1, \dots, T_N)$, if T can be written as a finite product of the mappings in $\{T_1, \dots, T_N\}$, where each mapping T_i occurs at least once. We say that T is an *M-word*, denoted $T \in \mathscr{W}_M := \mathscr{W}_M(T_1, \dots, T_N)$, if T can be written as a finite product where at most M different factors $T_{i_1}, \dots, T_{i_M}$ occur, for some $M \in \{1, \dots, N\}$ and some subset $\{i_1, \dots, i_M\}$ of $\{1, \dots, N\}$.

Note that the identity (the product with 0 factors) is always in $\mathscr{W}_M$ and that $\mathscr{F} \subseteq \mathscr{W}_N$.

Proposition 3.2. *In addition to the hypothesis, suppose that (x_n) is a bounded sequence in D, that (W_n) is a sequence in $\mathscr{W}_N$, and that $\|x_n - c\| - \|W_n x_n - c\| \to 0$. Then* $(*)$ $x_n - W_n x_n \to 0$. *Moreover, if each $W_n \in \mathscr{F}$, then $d(x_n, C) \to 0$.*

Proof. We assume without loss of generality that $c = 0$ (otherwise, we translate). For $M \in \{1, \dots, N\}$ define the statement $(*, M)$ by

> For every bounded sequence (x_n) in D and every sequence of words (W_n) in $\mathscr{W}_M$: if $\|x_n\| - \|W_n x_n\| \to 0$, then $x_n - W_n x_n \to 0$.

Hence, the main statement, $(*)$, holds exactly when $(*, N)$ does.

Step 1. $(*, 1)$ holds. Otherwise, there is a bounded sequence (x_n) in D, a sequence of words (W_n) in $\mathscr{W}_1$, some $i \in \{1, \ldots, N\}$, and a sequence of (strictly) positive integers (l_n) such that $\|x_n\| - \|W_n x_n\| \to 0$, $\inf_n \|x_n - W_n x_n\| > 0$, and $W_n = T_i^{l_n}$, for all n. Now $\|x_n\| \geq \|T_i x_n\| \geq \|W_n x_n\|$, hence $\|x_n\| - \|T_i x_n\| \to 0$. Because T_i is projective w.r.t. $c = 0$, we conclude (Fact 1.2)

$$0 \leftarrow d(x_n, C_i) = d(x_n, \operatorname{Fix} T_i) \geq d(x_n, \operatorname{Fix} T_i^{l_n}) \geq \tfrac{1}{2}\|x_n - W_n x_n\|,$$

which contradicts $\inf_n \|x_n - W_n x_n\| > 0$. Hence Step 1 is verified.

Step 2. If $M \in \{2, \ldots, N\}$ and $(*, M-1)$ holds, then so does $(*, M)$. Otherwise, there is a bounded sequence (x_n) in D, a sequence of words (W_n) in $\mathscr{W}_M \backslash \mathscr{W}_{M-1}$, and some indices $\{i_1, \ldots, i_M\} \subseteq \{1, \ldots, N\}$ such that $W_n \in \mathscr{F}(T_{i_1}, \ldots, T_{i_M})$, for all n, and $\|x_n\| - \|W_n x_n\| \to 0$, but $\inf_n \|x_n - W_n x_n\| > 0$. Fix $m \in \{1, \ldots, M\}$ and write $W_n = L_n T_{i_m} R_n$, where $R_n \in \mathscr{W}_{M-1}$, for all n. Since $\|x_n\| \geq \|R_n x_n\| \geq \|T_{i_m} R_n x_n\| \geq \|W_n x_n\|$, we get (i) $\|x_n\| - \|R_n x_n\| \to 0$ and (ii) $\|R_n x_n\| - \|T_{i_m} R_n x_n\| \to 0$. The fact that $(*, M-1)$ holds and (i) imply $x_n - R_n x_n \to 0$; thus

$$T_{i_m} R_n x_n - T_{i_m} x_n \to 0,$$

by nonexpansivity of T_{i_m}. Since T_{i_m} is projective w.r.t. $c = 0$, (ii) and Lemma 2.8(ii) yield

$$R_n x_n - T_{i_m} R_n x_n \to 0.$$

Adding the three preceding sequences gives $x_n - T_{i_m} x_n \to 0$; hence $\|x_n\| - \|T_{i_m} x_n\| \to 0$ and thus $d(x_n, C_{i_m}) \to 0$. Because m has been chosen arbitrarily, we conclude $\max_{m \in \{1, \ldots, M\}} d(x_n, C_{i_m}) \to 0$, and further (the M-tuple $(C_{i_m})_{m \in \{1, \ldots, M\}}$ is boundedly regular) $d(x_n, \cap_{m=1}^M C_{i_m}) \to 0$. Hence, by Fact 1.2, $0 \leftarrow d(x_n, \cap_{m=1}^M C_{i_m}) \geq d(x_n, \operatorname{Fix} W_n) \geq \frac{1}{2}\|x_n - W_n x_n\|$ which is the desired contradiction. Therefore, Step 2 is also verified.

Conclusion. $(*)$ holds.

Step 3. The "Moreover" part. Assume to the contrary that the "Moreover" part is wrong. Then there is some bounded sequence (x_n) in D and a sequence of full words (W_n) in $\mathscr{F}$ such that $\|x_n\| - \|W_n x_n\| \to 0$, but $\inf_n d(x_n, C) > 0$. Analogously to Step 2, we deduce $d(x_n, C) \to 0$, which is absurd. $\square$

Condition $(*)$ also appears as Dye and Reich's semigroup condition (S) in [13]. We are now ready for the main result:

Theorem 3.3. *In addition to the hypothesis, suppose r is a random mapping for $\{1, \ldots, N\}$. Then the random sequence (x_n), defined by*

$$x_0 \in D \text{ arbitrary}, \quad x_{n+1} := T_{r(n)} x_n, \quad \text{for all } n \geq 0,$$

converges in norm to some point in $\mathscr{Q}_C x_0$.

Proof. Since r is a random mapping, we can find a subsequence $(n_k)_k$ of $(n)_n$ such that $W_k := T_{r(n_{k+1}-1)} \cdots T_{r(n_k)} \in \mathscr{F}$, for all k. The sequence (x_{n_k}) is Fejér monotone w.r.t. C and the sequence $(\|x_{n_k} - c\|)$ converges; thus, by the last proposition, $d(x_{n_k}, C) \to 0$. On the other hand, (x_{n_k}) is a subsequence of (x_n); therefore, the result follows from Facts 2.4(ii) and Proposition 2.6(vi). □

The reader may deduce a variety of examples by putting together Facts 2.2, Proposition 2.6, Theorem 2.10, and Theorem 3.3; here, we give a rather small selection.

Example 3.4 ("Random Kaczmarz"). Suppose each set C_i is a hyperplane. Then the random product of relaxed projections onto these hyperplanes converges in norm to the projection onto C.

Remark 3.5. The cyclic control version with unrelaxed projections in Euclidean space was already known to Kaczmarz [22] in 1937.

Example 3.6 ("Random Agmon/Motzkin & Schoenberg"). If each set C_i is a half-space, then the random product of relaxed projections converges in norm to some point in $\mathscr{Q}_C x_0$.

Remark 3.7. The cyclic control version is due to Gubin et al. [20], whereas the "remotest set control" version is due to Agmon [1] and to Motzkin and Schoenberg [23]. In the field of image reconstruction, these methods are known as "AMS relaxation methods" or "ART for inequalities" [9, 8].

Example 3.8 ("Random von Neumann/Halperin"). Suppose each set C_i is a closed subspace and $\sum_{j\in J} C_j^{\perp}$ is closed, for every nonempty subset J of $\{1, \dots, N\}$. Then the random product of relaxed projections onto the subspaces C_i converges in norm to the projection onto C.

Remark 3.9. The cyclic control version is due to von Neumann [28] and to Halperin [21] and *does not require the assumption on the closedness of the sum of the complements*; see also Deutsch's survey [10] for applications and Baillon et al.'s [5, Corollary 2.4] for a more general (nonlinear) result.

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