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THE DE BRANGES-ROVNYAK MODEL WITH FINITE-DIMENSIONAL COEFFICIENTS

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ABSTRACT. A characterization in terms of the canonical model spaces of L. de Branges and J. Rovnyak is obtained for Hilbert spaces of formal power series with vector coefficients which satisfy a difference-quotient inequality, thereby extending the closed ideal theorems of A. Beurling and P. D. Lax.

1. INTRODUCTION

This paper extends the well-known invariant subspace characterization of A. Beurling [3] and P. D. Lax [11] for the shift on the Hardy space of square summable power series with vector coefficients (cf. [10, 13–15]). The focus is instead on certain (not necessarily orthogonal) complements of contractively contained invariant manifolds of the shift. These are the spaces $\mathscr{H}(B)$ of L. de Branges and J. Rovnyak [6–8]. In the Beurling-Lax theory, the key point is a dimension inequality. The inequality is trivial when the coefficient space has infinite dimension, so the essential content is in the finite-dimensional case. Previously only special cases of the more abstract problem have been treated [6, 9], but our methods generalize an argument from [7, Theorem 6]. The main difficulty again comes down to a dimension inequality in the finite-dimensional case. The purpose here is to derive new results on the structure of $\mathscr{H}(B)$ spaces which reveal what is needed for the inequality to hold. As a consequence, we obtain a complete characterization of the spaces $\mathscr{H}(B)$.

2. $\mathscr{H}(B)$ SPACES

A basic concept in the de Branges-Rovnyak theory is complementation: A Hilbert space $\mathscr{F}$ is contained contractively in a Hilbert space $\mathscr{K}$ if $\mathscr{F}$ is a submanifold of $\mathscr{K}$ and if the inclusion map of $\mathscr{F}$ into $\mathscr{K}$ is a contraction. If $\mathscr{F}$ is contained contractively in $\mathscr{K}$, then the space complementary to $\mathscr{F}$ in $\mathscr{K}$ is the Hilbert space $\mathscr{G}$ of elements g of $\mathscr{K}$ with the property that

$$\|g\|_{\mathscr{G}}^2 = \sup\{\|g+f\|_{\mathscr{K}}^2 - \|f\|_{\mathscr{F}}^2 : f \in \mathscr{F}\}$$

is finite. The space $\mathscr{G}$ is contained contractively in $\mathscr{K}$. Moreover, $\mathscr{G}$ is the unique Hilbert space such that the inequality $\|k\|_{\mathscr{K}}^2 \le \|f\|_{\mathscr{F}}^2 + \|g\|_{\mathscr{G}}^2$ holds whenever $k = f + g$ is a decomposition of k in $\mathscr{K}$ into f in $\mathscr{F}$ and g in

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$\mathscr{G}$ and such that every element k in $\mathscr{K}$ admits a decomposition for which equality holds.

Let $\mathscr{C}$ be a finite-dimensional Hilbert space, and let $\mathscr{H}$ be a Hilbert space of formal power series $f(z)$ whose coefficients are in $\mathscr{C}$ such that

$$(1) \qquad \|[f(z)-f(0)]/z\|_{\mathscr{H}}^2 \le \|f(z)\|_{\mathscr{H}}^2 - |f(0)|_{\mathscr{C}}^2.$$

Then $\mathscr{H}$ is contained contractively in $\mathscr{C}(z)$, the Hilbert space of square summable power series $\sum a_n z^n$ with a_n in $\mathscr{C}$ and norm given by $\|\sum a_n z^n\|_{\mathscr{C}(z)}^2 = \sum |a_n|_{\mathscr{C}}^2$.

Let $B(z)$ be a power series whose coefficients are operators on $\mathscr{C}$ such that $\|B(z)f(z)\|_{\mathscr{C}(z)} \le \|f(z)\|_{\mathscr{C}(z)}$ whenever $f(z)$ is in $\mathscr{C}(z)$. Cauchy multiplication by $B(z)$ thus defines a contraction operator on $\mathscr{C}(z)$ which will be denoted by T_B. The range $\mathscr{M}(B)$ of T_B becomes a Hilbert space in the unique norm with the property that $\|T_B f\|_{\mathscr{M}(B)} = \|f\|_{\mathscr{C}(z)}$ whenever f is orthogonal to the kernel of T_B. Furthermore, $\mathscr{M}(B)$ is contained contractively in $\mathscr{C}(z)$, and multiplication by z is a contraction on $\mathscr{M}(B)$.

The de Branges-Rovnyak space $\mathscr{H}(B)$ is defined to be the complementary space to $\mathscr{M}(B)$ in $\mathscr{C}(z)$. The space $\mathscr{H}(B)$ satisfies (1) and is an underlying space for canonical models of contractions on Hilbert space [1, 2, 12, 16, 17].

Multiplication by z is a contraction on the space $\mathscr{M}$ complementary to $\mathscr{H}$ in $\mathscr{C}(z)$. In [6] (cf. [5, Theorem 6]), de Branges extended the Beurling-Lax theorem by showing that if multiplication by z is isometric on $\mathscr{M}$, then $\mathscr{H}$ is isometrically equal to a space $\mathscr{H}(B)$. It should be further noted that when $\mathscr{C}$ is infinite dimensional, any space $\mathscr{H}$ which satisfies (1) is isometrically equal to a space $\mathscr{H}(B)$ [4, Theorem 11].

Let $\mathscr{H}(B)$ be a given space. Then $\mathscr{H}(B)$ is also contained contractively in $\mathscr{H}(zB)$. The space $\mathscr{H}(zB)$ may be obtained as those elements $h(z)$ of $\mathscr{C}(z)$ such that $[h(z)-h(0)]/z$ is in $\mathscr{H}(B)$ and $\|h(z)\|_{\mathscr{H}(zB)}^2 = \|[h(z)-h(0)]/z\|_{\mathscr{H}(B)}^2 + |h(0)|_{\mathscr{C}}^2$. The complementary space to $\mathscr{H}(B)$ in $\mathscr{H}(zB)$ is the space $B(z)\mathscr{C}$ with $\|B(z)c\|_{B(z)\mathscr{C}} = |c|_{\mathscr{C}}$ for every c orthogonal to $\mathscr{C} \cap \ker T_B$. Let us define linear transformations $J_\pm$ from $\mathscr{H}(B)$ into $\mathscr{C}$, with ranges denoted $\mathscr{C}_\pm$, as follows: $J_+ f = f(0)$ and J_- is the operator whose adjoint is given by $J_-^* c = [B(z) - B(0)]c/z$. Let $B(z) = \sum B_n z^n$, and let $\overline{B}_n$ be the adjoint of B_n on $\mathscr{C}$. Then $J_+^* c = [1 - B(z)\overline{B}(0)]c$; and since $\mathscr{C}$ is finite dimensional, $\mathscr{C}_+ = (1 - B_0\overline{B}_0)\mathscr{C}$ and $\mathscr{C}_- = (\bigvee_{n\ge1} \overline{B}_n\mathscr{C}) \subseteq (1 - \overline{B}_0 B_0)\mathscr{C}$.

Let $R(0)$ denote the difference-quotient transformation on $\mathscr{H}(B)$, which maps $f(z)$ into $[f(z) - f(0)]/z$. Then $R(0)^* f(z) = zf(z) - B(z)J_- f$ so that $[1 - R(0)R(0)^*]f(z) = [B(z) - B(0)](J_- f)/z$ and $[1 - R(0)^*R(0)]f(z) = (J_+ f) + B(z)J_- R(0)f$. Note that if $[1 - R(0)^*R(0)]f(z) = c + B(z)c_-$ with c in $\mathscr{C}$ and c_- in $\mathscr{C}_-$, then necessarily $c = J_+ f$ and $c_- = J_- R(0)f$. Therefore, since $\dim\mathscr{C}$ is finite,

$$\begin{aligned}
\operatorname{rank}[1 - R(0)^*R(0)] &= \dim\{(J_+ f,\ J_- R(0)f) : f \in \mathscr{H}(B)\} \\
&\ge \dim\mathscr{C}_+ = \operatorname{rank}(1 - \overline{B}_0 B_0) \qquad (2)\\
&\ge \dim\mathscr{C}_- = \operatorname{rank}[1 - R(0)R(0)^*]. \qquad (3)
\end{aligned}$$

More precisely, the following will turn out to be a defining property of the spaces $\mathscr{H}(B)$.

Theorem 1. *Let* $R(0)$ *be the difference-quotient transformation on a given space* $\mathscr{H}(B)$. *Then*

$$\operatorname{rank}[1 - R(0)^*R(0)] = \dim\{c \in \mathscr{C} : B(z)c \in \mathscr{H}(B)\} + \operatorname{rank}[1 - R(0)R(0)^*].$$

Proof. Suppose that $B(z)c$ is in $\mathscr{H}(B)$. Then $c = (J_-f) + d$ where f is in $\mathscr{H}(B)$ and $[B(z) - B(0)]d/z = 0$. Moreover,

$$[1 - R(0)^*R(0)]\{[R(0)^*f] + B(z)c\} = (B_0 d) + B(z)J_-f. \tag{4}$$

Let $J_-f_1, \ldots, J_-f_{s_0}$ be a basis for the subspace $\mathscr{C}_-' = \{c \in \mathscr{C}_- : B(z)c \in \mathscr{H}(B)\}$, and let $J_+g_1, \ldots, J_+g_t$ be a basis for $\mathscr{C}_+$ where f_i and g_j are in $\mathscr{H}(B)$ for all i and j. Suppose that there are constants $\lambda_1, \ldots, \lambda_{s_0+t}$ such that

$$\begin{aligned} 0 = &\sum_1^{s_0} \lambda_i [1 - R(0)^*R(0)]\{[R(0)^*f_i] + B(z)J_-f_i\} \\ &+ \sum_1^t \lambda_{s_0+j}[1 - R(0)^*R(0)]g_j. \end{aligned}$$

Equivalently by (4) we have

$$0 = \left(\sum_1^t \lambda_{s_0+j}J_+g_j\right) + B(z)J_-\left[\sum_1^t \lambda_{s_0+j}R(0)g_j + \sum_1^{s_0}\lambda_i f_i\right]$$

so that $\sum \lambda_{s_0+j}J_+g_j = 0$ and hence $\lambda_{s_0+j} = 0$ $(j = 1, \ldots, t)$. It follows that $\sum \lambda_i J_- f_i = 0$ and thus $\lambda_i = 0$ for all i. Therefore,

$$\operatorname{rank}[1 - R(0)^*R(0)] \geq s_0 + t. \tag{5}$$

Let $c_i = J_-f_i$ $(i = 1, \ldots, s_0)$ and expand $\{c_i\}$ to a basis $c_1, \ldots, c_s$ of $\{c \in \mathscr{C} : B(z)c \in \mathscr{H}(B)\}$. For every $j > s_0$ let us write $c_j = (J_-f_j) + d_j$ as above where f_j is in $\mathscr{H}(B)$ and d_j is orthogonal to $\mathscr{C}_-$. By (4), $B_0 d_j$ is in $\mathscr{C}_+$, so it is in $(B_0\mathscr{C}) \cap (1 - B_0\overline{B}_0)\mathscr{C}$. But since $\mathscr{C}$ is finite dimensional, it follows that this intersection coincides with $B_0(1 - \overline{B}_0 B_0)\mathscr{C}$, and hence $B_0 d_j = B_0 e_j$ where e_j is in $(1 - \overline{B}_0 B_0)\mathscr{C}$. Thus $d_j - e_j$ is in $\ker B_0$, which is also contained in $(1 - \overline{B}_0 B_0)\mathscr{C}$, and consequently d_j is in $[(1 - \overline{B}_0 B_0)\mathscr{C}] \ominus \mathscr{C}_-$.

Now $\{d_j : j > s_0\}$ is linearly independent: For suppose $\sum \alpha_j d_j = 0$. Then $\sum_{j>s_0} \alpha_j c_j = \sum_{j>s_0} \alpha_j J_- f_j$ is in $\mathscr{C}_-'$, so there exist β_i such that $\sum_{j>s_0}\alpha_j c_j = \sum_{i \leq s_0} \beta_i c_i$. Since $\{c_i\}$ is linearly independent, $\alpha_j = 0$ for all j, and hence

$$\begin{aligned} t = \dim \mathscr{C}_+ &= \operatorname{rank}(1 - B_0\overline{B}_0) = \operatorname{rank}(1 - \overline{B}_0 B_0) \\ &= \dim\{[(1 - \overline{B}_0 B_0)\mathscr{C}] \ominus \mathscr{C}_-\} + \dim \mathscr{C}_- \\ &\geq (s - s_0) + \operatorname{rank}[1 - R(0)R(0)^*]. \end{aligned}$$

In conjunction with (5) we have

$$\operatorname{rank}[1 - R(0)^*R(0)] \geq s + \operatorname{rank}[1 - R(0)R(0)^*].$$

To verify the reverse inequality, it suffices to show that there exist $r = \operatorname{rank}[1 - R(0)^*R(0)] - \operatorname{rank}[1 - R(0)R(0)^*]$ linearly independent vectors a_i in $\mathscr{C}$ such that $B(z)a_i$ is in $\mathscr{H}(B)$. By inequalities (2) and (3), it follows that $r = r_0 + r_1$ where $r_0 = \operatorname{rank}[1 - R(0)^*R(0)] - \dim \mathscr{C}_+$ and $r_1 = \dim\{[\operatorname{ran}(1 - \overline{B}_0 B_0)] \ominus \mathscr{C}_-\}$.

Suppose that $r_0 > 0$ and recall the basis $\{J_+ g_j\}$ of $\mathscr{C}_+$. As above, $\{[1 - R(0)^* R(0)]g_j\}$ is linearly independent, so if $\mathscr{G}$ is its span, then there are r_0 vectors $[1 - R(0)^* R(0)]\hat{g}_i$ $(i = 1, \dots, r_0)$, with $\hat{g}_i$ in $\mathscr{H}(B)$, which form a basis of $\operatorname{ran}[1 - R(0)^* R(0)] \ominus \mathscr{G}$. Now there exist constants λ_{ij} such that $J_+ \hat{g}_i = \sum_{j=1}^t \lambda_{ij} J_+ g_j$ for each i. Let us define $a_i = J_- R(0)(\hat{g}_i - \sum_j \lambda_{ij} g_j)$ for $i = 1, \dots, r_0$. Then $B(z)a_i = [1 - R(0)^* R(0)](\hat{g}_i - \sum_j \lambda_{ij} g_j)$ is in $\mathscr{H}(B)$, and $\{a_1, \dots, a_{r_0}\}$ is linearly independent: Suppose that $\sum \mu_i a_i = 0$. Then

$$\sum \mu_i [1 - R(0)^* R(0)]\hat{g}_i = \sum_i \mu_i [1 - R(0)^* R(0)] \left(\sum_j \lambda_{ij} g_j \right)$$

which must be zero since it is in both $\mathscr{G}$ and $\mathscr{G}^\perp$. Therefore $\mu_i = 0$ for every i.

Next, suppose that $r_1 > 0$ and let $\hat{d}_1, \dots, \hat{d}_{r_1}$ be a basis of $[\operatorname{ran}(1 - \overline{B}_0 B_0)] \ominus \mathscr{C}_-$. Then $B(z)\hat{d}_j = B_0 \hat{d}_j$ and $\hat{d}_j = (1 - \overline{B}_0 B_0) b_j$ for some b_j in $\mathscr{C}$. Let $\hat{f}_j(z) = [1 - B(z)\overline{B}(0)]B_0 b_j$ and define $a_{r_0+j} = \hat{d}_j + J_- R(0)\hat{f}_j$ for $j = 1, \dots, r_1$. Then $B(z)a_{r_0+j} = [1 - R(0)^* R(0)]\hat{f}_j$ is in $\mathscr{H}(B)$.

Finally, $\{a_i \colon i = 1, \dots, r = r_0 + r_1\}$ is linearly independent: Suppose that there are constants $\nu_1, \dots, \nu_r$ such that

$$0 = \sum \nu_i a_i = \sum_1^{r_0} \nu_i a_i + \sum_1^{r_1} \nu_{r_0+j}[\hat{d}_j + J_- R(0)\hat{f}_j].$$

It follows that $\sum_1^{r_1} \nu_{r_0+j}\hat{d}_j = 0$ since a_i $(1 \le i \le r_0)$ and $J_- R(0)\hat{f}_j$ $(1 \le j \le r_1)$ are in $\mathscr{C}_-$, and $\hat{d}_j$ is orthogonal to $\mathscr{C}_-$ for every j. Therefore $\nu_{r_0+j} = 0$ $(j = 1, \dots, r_1)$, and consequently $\sum_1^{r_0} \nu_i a_i = 0$ so that $\nu_i = 0$ for all i. □

3. The characterization

Let $\mathscr{H}$ be a space which satisfies (1), and let $\mathscr{H}'$ be the Hilbert space of all power series $h(z)$ such that $[h(z) - h(0)]/z$ is in $\mathscr{H}$ with $\|h(z)\|_{\mathscr{H}'}^2 = \|[h(z) - h(0)]/z\|_{\mathscr{H}}^2 + |h(0)|_{\mathscr{C}}^2$. Then $\mathscr{H}'$ satisfies (1), and $\mathscr{H}$ is contained contractively in $\mathscr{H}'$. Let $\mathscr{R}$ be the complementary space to $\mathscr{H}$ in $\mathscr{H}'$, and let $i_{\mathscr{H}}$ and $i_{\mathscr{R}}$ denote the respective inclusion maps of $\mathscr{H}$ and $\mathscr{R}$ into $\mathscr{H}'$. Then every h in $\mathscr{H}'$ admits the unique decomposition $h = (i_{\mathscr{H}}^* h) + (i_{\mathscr{R}}^* h)$ where $\|h\|_{\mathscr{H}'}^2 = \|i_{\mathscr{H}}^* h\|_{\mathscr{H}}^2 + \|i_{\mathscr{R}}^* h\|_{\mathscr{R}}^2$.

A fundamental result from the theory of $\mathscr{H}(B)$ spaces is: *$\mathscr{H}$ is isometrically equal to a space $\mathscr{H}(B)$ if and only if the dimension of $\mathscr{R}$ does not exceed the dimension of $\mathscr{C}$* [6]. More generally, if $\mathscr{C} \subseteq \tilde{\mathscr{C}}$ and $\dim \mathscr{R} \le \dim \tilde{\mathscr{C}}$, then $\mathscr{H}$ is a space $\mathscr{H}(\tilde{B})$ where the coefficients of $\tilde{B}(z)$ act on $\tilde{\mathscr{C}}$.

Lemma. *Let $\mathscr{F}$ be the subspace of elements of $\mathscr{H}$ for which equality holds in* (1). *Then $\mathscr{R}$ and $\mathscr{H} \cap \mathscr{R}$ are contained in $\mathscr{H}' \ominus \mathscr{F}$ and $\mathscr{H} \ominus \mathscr{F}$ respectively. Moreover,* $\dim \mathscr{R} = \dim \mathscr{H}' \ominus \mathscr{F}$ *and* $\dim \mathscr{H} \cap \mathscr{R} = \dim \mathscr{H} \ominus \mathscr{F}$.

Proof. As in [9], $\mathscr{F}$ is a (closed) subspace of $\mathscr{H}$ and is contained isometrically in $\mathscr{H}'$. Therefore for any f in $\mathscr{F}$ and g in $\mathscr{R}$, we have

$$\langle f, g \rangle_{\mathscr{H}'} = \langle f, i_{\mathscr{R}} g \rangle_{\mathscr{H}'} = \langle i_{\mathscr{R}}^* f, g \rangle_{\mathscr{R}} = \langle 0, g \rangle_{\mathscr{R}} = 0.$$

Hence $\mathscr{R}$ is a subset of $\mathscr{H}' \ominus \mathscr{F}$.

The restriction of $i^*_{\mathscr{R}}$ to $\mathscr{H}' \ominus \mathscr{F}$ is linear and continuous and has trivial kernel: if $i^*_{\mathscr{R}}h = 0$ for some h in $\mathscr{H}' \ominus \mathscr{F}$, then $i^*_{\mathscr{H}}h = h$, so h is also in $\mathscr{F}$, and thus $h = 0$. It follows that $\dim \mathscr{H}' \ominus \mathscr{F} = \dim i^*_{\mathscr{R}}(\mathscr{H}' \ominus \mathscr{F}) \leq \dim \mathscr{R}$, and hence $\dim \mathscr{R} = \dim \mathscr{H}' \ominus \mathscr{F}$.

Next, let g be in $\mathscr{H} \cap \mathscr{R}$. Then g is in $\mathscr{H}' \ominus \mathscr{F}$ but also in $\mathscr{H} \ominus \mathscr{F}$ since for any f in $\mathscr{F}$

$$\langle f, g\rangle_{\mathscr{H}} = \langle i^*_{\mathscr{H}} f, g\rangle_{\mathscr{H}} = \langle f, i_{\mathscr{H}} g\rangle_{\mathscr{H}'} = \langle f, g\rangle_{\mathscr{H}'} = 0.$$

Therefore $(\mathscr{H} \cap \mathscr{R}) \subseteq (\mathscr{H} \ominus \mathscr{F})$. Finally $\dim \mathscr{H} \cap \mathscr{R} = \dim \mathscr{H} \ominus \mathscr{F}$ as above since $i^*_{\mathscr{R}}(\mathscr{H} \ominus \mathscr{F})$ is contained in $\mathscr{H} \cap \mathscr{R}$. □

The following will distinguish the spaces $\mathscr{H}(B)$.

Corollary 1. *Let $\mathscr{F}(B)$ be the subspace of elements of a given space $\mathscr{H}(B)$ for which equality holds in* (1). *Then*

$$\dim J_+\mathscr{F}(B) = \dim(\mathscr{C} \cap \ker T_B) + \operatorname{rank}[1 - R(0)R(0)^*].$$

Proof. Since $B(z)\mathscr{C}$ is finite dimensional, the lemma implies that $\mathscr{H}(B) \ominus \mathscr{F}(B)$ coincides with $\mathscr{H}(B) \cap B(z)\mathscr{C}$. By (1), the kernel of $1 - R(0)^*R(0)$ is contained in $\mathscr{F}(B)$ and is exactly the kernel of the restriction of J_+ to $\mathscr{F}(B)$. Thus since $1 - R(0)^*R(0)$ has finite rank and

$$J_+\mathscr{F}(B) = J_+\{\operatorname{ran}[1 - R(0)^*R(0)] \cap \mathscr{F}(B)\},$$

it follows that

$$\begin{aligned}\operatorname{rank}[1 - R(0)^*R(0)] &= \dim\{\operatorname{ran}[1 - R(0)^*R(0)] \cap \mathscr{F}(B)\} \\ &\quad + \dim[\mathscr{H}(B) \ominus \mathscr{F}(B)] \\ &= \dim J_+\mathscr{F}(B) + \dim[\mathscr{H}(B) \cap B(z)\mathscr{C}].\end{aligned}$$

The corollary now follows from Theorem 1 since we also have

$$\begin{aligned}\operatorname{rank}[1 - R(0)^*R(0)] &= \dim(\mathscr{C} \cap \ker T_B) + \dim[\mathscr{H}(B) \cap B(z)\mathscr{C}] \\ &\quad + \operatorname{rank}[1 - R(0)R(0)^*]. \quad \square\end{aligned}$$

By [7, Lemma 4], equality holds in (1) for a given space $\mathscr{H}(B)$ if and only if $\mathscr{H}(B)$ contains no nonzero element of the form $B(z)c$ with c in $\mathscr{C}$. An immediate consequence of the above results is

Corollary 2. *Let $\mathscr{H}(B)$ be a given space. Then* $\operatorname{rank}[1 - R(0)^*R(0)] = \operatorname{rank}[1 - R(0)R(0)^*]$ *if and only if equality holds in* (1) *for every $f(z)$ and there is no nonzero vector c such that $B(z)c = 0$.*

We now have the proposed characterization.

Theorem 2. *Let $\mathscr{H}$ be a Hilbert space of formal power series which satisfies* (1), *and let $\mathscr{F}$ be the subspace of those series for which equality holds in* (1). *Then $\mathscr{H}$ is isometrically equal to a space $\mathscr{H}(B)$ if and only if the dimension of the space of constant coefficients of elements of $\mathscr{F}$ is at least the rank of $1 - TT^*$ where T is the difference-quotient transformation on $\mathscr{H}$.*

Proof. Any space $\mathscr{H}(B)$ has the stated property by Corollary 1.

Conversely, suppose that $\mathscr{H}$ is a space which satisfies (1) and the dimension hypothesis. Let $\mathscr{H}'$, $\mathscr{R}$, $i_{\mathscr{H}}$ and $i_{\mathscr{R}}$ be defined as above, and let $f(z)$ and $g(z)$ be in $\mathscr{H}$. Since

$$\langle i_{\mathscr{H}}^* z f(z), g(z)\rangle_{\mathscr{H}} = \langle z f(z), i_{\mathscr{H}} g(z)\rangle_{\mathscr{H}'} = \langle f(z), T g(z)\rangle_{\mathscr{H}},$$

it follows that $T^* f(z) = i_{\mathscr{H}}^* z f(z)$.

Let S denote the difference-quotient transformation on $\mathscr{H}'$. Then

$$\begin{aligned}(1 - TT^*) f(z) &= f(z) - T i_{\mathscr{H}}^* z f(z) \\ &= f(z) - S[z f(z) - i_{\mathscr{R}}^* z f(z)] = S i_{\mathscr{R}}^* z f(z).\end{aligned}$$

More generally, $S\mathscr{R}$ is contained in the range of $1 - TT^*$: Let $g(z)$ be in $\mathscr{R}$ such that $g(z)$ is orthogonal to $i_{\mathscr{R}}^* z f(z)$ for every $f(z)$ in $\mathscr{H}$. Then

$$0 = \langle g(z), i_{\mathscr{R}}^* z f(z)\rangle_{\mathscr{R}} = \langle g(z), z f(z)\rangle_{\mathscr{H}'} = \langle S g(z), f(z)\rangle_{\mathscr{H}}$$

for every $f(z)$ in $\mathscr{H}$. Letting $f(z) = S g(z)$, we conclude that $g(z)$ is constant. Hence $S\mathscr{R} = S \vee \{i_{\mathscr{R}}^* z f(z) \colon f(z) \in \mathscr{H}\}$, which is contained in $(1 - TT^*)\mathscr{H}$ since the rank of $1 - TT^*$ is finite by the hypothesis.

It follows that $\mathscr{R}$ is finite dimensional since

$$\dim \mathscr{R} \le \dim S\mathscr{R} + \dim \ker S \le \operatorname{rank}(1 - TT^*) + \dim \mathscr{C}.$$

Thus by the lemma $\mathscr{R} = \mathscr{H}' \ominus \mathscr{F}$.

Furthermore, since $\mathscr{H}'$ contains $\mathscr{C}$, the kernel of the restriction of S to $\mathscr{H}' \ominus \mathscr{F}$ is $\mathscr{C} \ominus \{f(0) \colon f(z) \in \mathscr{F}\}$. Hence, we have that

$$\begin{aligned}\dim \mathscr{R} &= \dim[\mathscr{C} \ominus \{f(0) \colon f(z) \in \mathscr{F}\}] + \dim S\mathscr{R} \\ &\le \dim \mathscr{C} - \dim\{f(0) \colon f(z) \in \mathscr{F}\} + \operatorname{rank}(1 - TT^*) \\ &\le \dim \mathscr{C}\end{aligned}$$

by the hypothesis. Therefore, $\mathscr{H}$ is isometrically equal to a space $\mathscr{H}(B)$. □

Finally, any space which satisfies (1) is at least a reducing subspace of $R(0)$ on some space $\mathscr{H}(B)$.

Corollary 3. *Let $\mathscr{H}$, $\mathscr{F}$ and T be defined as in Theorem 2, but assume on the other hand that*

$$\delta = \operatorname{rank}(1 - TT^*) - \dim\{f(0) \colon f(z) \in \mathscr{F}\}$$

is finite and positive. If $\tilde{\mathscr{C}}$ is any Hilbert space with dimension at least δ, then $\mathscr{H} \oplus \tilde{\mathscr{C}}(z)$ is isometrically equal to a space $\mathscr{H}(B)$.

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