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ASYMPTOTIC STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS BY LIAPUNOV FUNCTIONALS

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ABSTRACT. We consider the asymptotic stability in a system of functional differential equations $x'(t) = F(t, x_t)$ by Liapunov functionals V. The work generalizes some well-known results in the literature in that we only require the derivative of V to be negative definite on a sequence of intervals $I_n = [s_n, t_n]$. We also show that it is not necessary to require a uniform upper bound on V for nonuniform asymptotic stability.

1. INTRODUCTION

We consider a system of functional differential equations with finite delay

$$x'(t) = F(t, x_t), \qquad x \in R^n, \tag{1.1}$$

and obtain conditions on a Liapunov functional V to ensure that the zero solution of (1.1) is asymptotically stable or uniformly asymptotically stable. Our results generalize some well-known theorems in the literature in that we only require the following properties of V.

(P_1) The derivative of V along a solution of (1.1) is negative definite on a sequence of intervals $I_n = [s_n, t_n]$.

(P_2) V has a uniform upper bound on $J_n = [t_n - \sigma, t_n]$ with $\sigma > 0$.

For reference, the discussion here follows closely those of Burton [1] and Burton and Hatvani [3, 4]. Our work also has roots in the recent work of Burton and Makay [5] in which the asymptotic stability of (1.1) was obtained by a Liapunov functional V having property (P_1) and a uniform upper bound on a sequence $\{t_n\}$. A growth condition on $F(t, \varphi)$ is required in [5].

For $x \in R^n$, $|\cdot|$ denotes the Euclidean norm of x. The length of an interval $I_n = [a, b]$ is defined by $l(I_n)$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup\{|Ax| : |x| \le 1\}$. For a given $h > 0$, C will be the space of continuous functions $\varphi : [-h, 0] \to R^n$ with the supremum norm $\|\varphi\| = \sup\{|\varphi(s)| : -h \le s \le 0\}$. C_H denotes the set of $\varphi \in C$ with $\|\varphi\| < H$. If x is a continuous function of u defined on $-h \le u < A$, $A > 0$, and if t

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is a fixed number satisfying $0 \le t < A$, then x_t denotes the restriction of x to the interval $[t-h, t]$ so that x_t is an element of C defined by $x_t(s) = x(t+s)$ for $-h \le s \le 0$. For any $\varphi \in C$ we define

$$|\varphi|_2 = \left[\int_{-h}^{0} |\varphi(s)|^2 \, ds\right]^{1/2}.$$

In (1.1), $x'(t)$ denotes the right-hand derivative of x at t. It is assumed that $F : R^+ \times C_H \to R^n$, $R^+ = [0, +\infty)$, is continuous so that a solution will exist for each $(t_0, \varphi) \in R^+ \times C_H$. We denote by $x(t_0, \varphi)$ a solution of (1.1) with initial function $\varphi \in C_H$ where $x_{t_0}(t_0, \varphi) = \varphi$. The value of $x(t_0, \varphi)$ at t will be $x(t) = x(t, t_0, \varphi)$. For the continuation of solutions, we suppose that F takes bounded sets of $R^+ \times C_H$ into bounded sets of R^n. We also assume that $F(t, 0) = 0$ so that $x = 0$ is a solution of (1.1). Let $V : R^+ \times C_H \to R^+$ be a continuous functional and define the upper right-hand derivative of V along a solution of (1.1) by

$$V'_{(1.1)}(t, \varphi) = \limsup_{\delta \to 0^+} \{V(t+\delta, x_{t+\delta}(t, \varphi)) - V(t, \varphi)\}/\delta.$$

For reference on Liapunov's direct method and fundamental theorems of (1.1), we refer to the work of Burton [2], Hale [6], Kato [7], and Yoshizawa [8].

Definition 1.1. The zero solution of (1.1) is said to be stable if, for each $\varepsilon > 0$ and $t_0 \ge 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\varphi \in C_H, \|\varphi\| < \delta, t \ge t_0]$ imply that $|x(t, t_0, \varphi)| < \varepsilon$. If δ is independent of t_0, then the zero solution is uniformly stable.

Definition 1.2. The zero solution of (1.1) is asymptotically stable (AS) if it is stable and if, for each $t_0 \in R^+$, there exists $\delta = \delta(t_0) > 0$ such that $\|\varphi\| < \delta$ implies that $x(t, t_0, \varphi) \to 0$ as $t \to +\infty$. The zero solution of (1.1) is uniformly asymptotically stable (UAS) if it is uniformly stable and if there is a $\sigma > 0$ and if for each $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that $[t_0 \in R^+, \|\varphi\| < \sigma, t \ge t_0 + T]$ imply that $|x(t, t_0, \varphi)| < \varepsilon$.

Definition 1.3. $W : R^+ \to R^+$ is called a wedge if W is continuous and strictly increasing with $W(0) = 0$. Throughout this paper W, W_j $(j = 0, 1, 2, \ldots)$ will denote the wedges.

Definition 1.4. A continuous function $G : R^+ \to R^+$ is convex downward if $G([t+s]/2) \le [G(t) + G(s)]/2$ for all $t, s \in R^+$.

Jensen's inequality. Let W be convex downward and let $f, p : [a, b] \to R^+$ be continuous with $\int_a^b p(s)\,ds > 0$. Then

$$\int_a^b p(s)\,ds\, W\left[\int_a^b p(s) f(s)\,ds \Big/ \int_a^b p(s)\,ds\right] \le \int_a^b p(s) W(f(s))\,ds.$$

2. Main theorems

The following lemmas will be used in the proof of our results.

Lemma 2.1. *If W_1 is a wedge, then for any $L > 0$ there is a convex downward wedge W_0 such that $W_0(r) \leq W_1(r)$ for all $r \in [0, L]$. In fact, $W_0(r) = \int_0^r W_1(s)\,ds/L$ will suffice.*

Lemma 2.2 [4, p. 286]. *Let $x : [t_0 - h, +\infty) \to R^n$ be a bounded continuous function and $\{t_n\}$ be an increasing sequence of real numbers with $t_n \to +\infty$ as $n \to +\infty$ (short notation $\{t_n\} \uparrow \infty$) such that $|x_{t_n}|_2 \to 0$ as $n \to +\infty$. Then there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a sequence $\{s_k\}$ with $s_k \in [t_{n_k} - h, t_{n_k}]$ and $|t_{n_k} - s_k| < \frac{1}{k}$ such that $|x(s_k)| + |x_{s_k}|_2 \to 0$ as $k \to +\infty$.*

Theorem 2.1. *Suppose that there exist a continuous functional $V : R^+ \times C_H \to R^+$, wedges W, W_i $(i = 1, 2, 3)$, a constant $\sigma > 0$ and a sequence $\{t_n\} \uparrow \infty$ such that*

(i) *$W_1(|\varphi(0)|) \leq V(t, \varphi)$, $V(t, 0) = 0$ for all $t \in R^+$ and $V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3(|\varphi|_2)$ for $t \in [t_n - \sigma, t_n]$,*

(ii) *$V'_{(1.1)}(t, x_t) \leq 0$ for all $t \geq t_0$ and $V'_{(1.1)}(t, x_t) \leq -W(|x(t)|)$ for $t \in [t_n - h, t_n]$, where $x(t) = x(t, t_0, \varphi)$ is any solution of (1.1) with $x_t \in C_H$ and $t_n - h \geq t_0$.*

Then the zero solution of (1.1) is AS.

Proof. Let $t_0 \in R^+$ and $\varepsilon > 0$. Then there exists $\delta > 0$ $(\delta < H)$ such that $V(t_0, \varphi) < W_1(\varepsilon)$ whenever $\|\varphi\| < \delta$. Let $x(t) = x(t, t_0, \varphi)$ be a solution of (1.1) with $\|\varphi\| < \delta$. It then follows that $W_1(|x(t)|) \leq V(t, x_t) \leq V(t_0, \varphi) < W_1(\varepsilon)$ and, therefore, $|x(t)| < \varepsilon$ for $t \geq t_0$. Thus the zero solution of (1.1) is stable.

Next let $t_0 \geq 0$ and find $\delta > 0$ of stability for $\varepsilon_0 = \min\{H, 1\}$. If $\|\varphi\| < \delta$, then $|x(t, t_0, \varphi)| < \varepsilon_0$ for $t \geq t_0$. We will show that $x(t, t_0, \varphi) \to 0$ as $t \to +\infty$. Without loss of generality, we may assume that $t_{n-1} + h \leq t_n$ for $n = 1, 2, \ldots$. By Lemma 2.1 there exists a convex downward wedge W_4 such that $W_4(r^2) \leq W(r)$ for $0 \leq r \leq 1$. For $t \geq t_n$, we have

$$\begin{aligned} V(t, x_t) &\leq V(t_0, \varphi) - \sum_{j=1}^{n} \int_{t_j - h}^{t_j} W(|x(s)|)\,ds \\ &\leq V(t_0, \varphi) - \sum_{j=1}^{n} \int_{t_j - h}^{t_j} W_4(|x(s)|^2)\,ds. \end{aligned}$$

Apply Jensen's inequality to obtain

$$V(t, x_t) \leq V(t_0, \varphi) - \sum_{j=1}^{n} h W_4\left(\frac{1}{h}|x_{t_j}|_2^2\right). \tag{2.1}$$

This implies that

$$\sum_{j=1}^{+\infty} W_4\left(\frac{1}{h}|x_{t_j}|_2^2\right) < +\infty \quad \text{and} \quad |x_{t_n}|_2 \to 0 \text{ as } n \to +\infty.$$

By Lemma 2.2, it follows that there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a sequence $\{s_k\}$ with $s_k \in [t_{n_k} - h, t_{n_k}]$ and $t_{n_k} - s_k < \frac{1}{k}$ such that

$$|x(s_k)| + |x_{s_k}|_2 \to 0 \quad \text{as} \quad k \to +\infty. \tag{2.2}$$

Without loss of generality, we may assume that $s_k \in [t_{n_k} - \sigma, t_{n_k}]$ for $k = 1, 2, \ldots$. For the given $\varepsilon > 0$ there exists $K > 0$ such that $W_2(|x(s_K)|) + W_3(|x_{s_K}|_2) < W_1(\varepsilon)$. We then have

$$W_1(|x(t)|) \le V(t, x_t) \le V(s_K, x_{s_K}) < W_1(\varepsilon) \text{ and } |x(t)| < \varepsilon$$

for $t \ge t_{n_K} \ge s_K$. Thus the proof is complete.

Theorem 2.2. *Suppose that there exist a continuous functional* $V : R^+ \times C_H \to R^+$, *wedges* W, W_i $(i = 1, 2, 3)$, *a constant* $\sigma > 0$ *and a sequence* $\{t_n\} \uparrow \infty$ *such that*

(i) $W_1(|\varphi(0)|) \le V(t, \varphi)$, $V(t, 0) = 0$ *for all* $t \in R^+$ *and* $V(t, \varphi) \le W_2(|\varphi(0)|) + W_3(|\varphi|_2)$ *for* $t \in [t_n - \sigma, t_n]$,

(ii) $V'_{(1.1)}(t, x_t) \le 0$ *for all* $t \ge t_0$ *and* $V'_{(1.1)}(t, x_t) \le -W(|x_t|_2)$ *for* $t \in I_n$ *and* $x_t \in C_H$, *where* $x(t) = x(t, t_0, \varphi)$ *is any solution of* (1.1) *with* $t_n - h \ge t_0$ *and* I_n *is a sequence of intervals* $I_n = [s_n, t_n]$ *with* $\sum_{n=1}^{\infty} l(I_n) = +\infty$.

Then the zero solution of (1.1) *is AS.*

Proof. The fact that the zero solution of (1.1) is stable follows from the proof of Theorem 2.1. We now show that the zero solution of (1.1) is AS. Let $t_0 \ge 0$ and find $\delta > 0$ of stability for $\varepsilon_0 = \min\{H, 1\}$. Thus if $\|\varphi\| < \delta$, then $|x(t, t_0, \varphi)| < \varepsilon_0$ for $t \ge t_0$. We claim that

$$\liminf_{n \to +\infty} |x_{t_n}|_2 = 0. \tag{2.3}$$

Suppose that there exists $K > 0$ and a constant $\alpha > 0$ such that $|x_{t_n}|_2 \ge \alpha$ for all $n \ge K$. Define $P(t) = \int_{t-h}^{t} |x(s)|^2 \, ds$. Then $P'(t) = |x(t)|^2 - |x(t-h)|^2$ and there exists a constant $L > 0$ such that $|P'(t)| \le L$ for all $t \ge t_0$. For $|t_n - t| \le \alpha^2/2L$ we have $|P(t_n) - P(t)| \le L|t_n - t| \le \alpha^2/2$ and $P(t) \ge P(t_n) - \alpha^2/2 \ge \alpha^2/2$. This implies that $|x_t|_2 \ge \alpha/2$ for $t \in J_n = [t_n - \alpha^2/2L, t_n]$. We consider the following cases.

Case 1. There exist a constant $\gamma > 0$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $l(I_{n_k}) \ge \gamma$ with $n_1 \ge K$. We then choose $L > 0$ sufficiently large such that $J_{n_k} \subset I_{n_k}$. Let $t \ge t_{n_m}$ and integrate (ii) from t_0 to t to obtain

$$\begin{aligned} V(t, x_t) &\le V(t_0, \varphi) - \sum_{k=1}^{m} \int_{J_{n_k}} W(|x_s|_2)\, ds \\ &\le V(t_0, \varphi) - mW(\alpha/2)\alpha^2/2L \to -\infty \quad \text{as } m \to +\infty, \end{aligned} \tag{2.4}$$

a contradiction.

Case 2. Suppose that $l(I_n) \to 0$ as $n \to +\infty$. We may assume that $I_n \subset J_n$ for all $n \ge K$. Thus for $t \ge t_n$ we have

$$\begin{aligned} V(t, x_t) &\le V(t_0, \varphi) - \sum_{j=K}^{n} \int_{I_j} W(|x_s|_2)\, ds \\ &\le V(t_0, \varphi) - W(\alpha/2) \sum_{j=K}^{n} l(I_j) \to -\infty \quad \text{as } n \to +\infty, \end{aligned}$$

a contradiction. This proves (2.3). By Lemma 2.2, we again conclude that (2.2) holds for a sequence $\{s_k\}$ with $s_k \in [t_{n_k} - \sigma, t_{n_k}]$, where $\{t_{n_k}\}$ is a subsequence of $\{t_n\}$. Therefore, the zero solution of (1.1) is asymptotically stable by the proof following (2.2) in Theorem 2.1.

We now discuss the uniform asymptotic stability. Our result generalizes a well-known theorem of Burton [1].

Theorem 2.3. *Suppose that there exist a continuous functional $V : R^+ \times C_H \to R^+$, wedges W, W_i $(i = 1, 2, 3)$, a positive constant J and a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \le J$ such that*

(i) $W_1(|\varphi(0)|) \le V(t, \varphi) \le W_2(|\varphi(0)|) + W_3(|\varphi|_2)$ *for all* $(t, \varphi) \in R^+ \times C_H$,

(ii) $V'_{(1.1)}(t, x_t) \le 0$ *for all* $t \ge t_0$ *and* $V'_{(1.1)}(t, x_t) \le -W(|x(t)|)$ *for* $t \in [t_n - h, t_n]$, *where* $x(t) = x(t, t_0, \varphi)$ *is any solution of* (1.1) *with* $x_t \in C_H$ *and* $t_n - h \ge t_0$.

Then the solution of (1.1) *is UAS.*

Proof. There exists a wedge W^* such that $W_2(r) + W_3(r\sqrt{h}) \le W^*(r)$ for all $0 \le r \le 1$. The uniform stability is clear from (i). Next choose $\delta > 0$ of uniform stability for $\varepsilon_0 = \min\{H, 1\}$. Thus, if $x(t) = x(t, t_0, \varphi)$ is a solution of (1.1), then $|x(t, t_0, \varphi)| < \varepsilon_0$ whenever $\|\varphi\| < \delta$ and $t \ge t_0$. Moreover, $V(t, x_t) \le W^*(\delta)$ for all $t \ge t_0$. We will follow the proof of Theorem 2.1. For any $\varepsilon > 0$ find $\eta > 0$ such that $W_2(\eta) + W_3(\eta) < W_1(\varepsilon)$. From (2.1) we have

$$V(t, x_t) \le W^*(\delta) - \sum_{j=1}^{n} h W_4\left(\frac{1}{h}|x_{t_j}|_2^2\right). \tag{2.5}$$

This implies that there exists a positive integer $K = K(\delta)$ such that $|x_{t_k}|_2 < \eta/2$ for some k with $1 \le k \le K$. Consequently, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $|x_{t_{n_k}}|_2 < \eta/2$ and $t_{n_k} - t_{n_{k-1}} \le KJ$ with $t_{n_0} = t_0$ for $k = 1, 2, \ldots$. For brevity, we rename $\{t_{n_k}\}$ by $\tau_k = t_{n_k}$. Since $P(t) = \int_{t-h}^{t} |x(s)|^2\, ds$ is uniformly continuous on R^+, there exists $\gamma > 0$ $(\gamma < h)$ such that $|x_t|_2 < \eta$ on $[\tau_k - \gamma, \tau_k]$ for $k = 1, 2, \ldots$. Let M be a positive integer such that $W^*(\delta) - MW(\eta)\gamma < 0$. We claim that there exists an integer m with $1 \le m \le M$ and a $s_m \in [\tau_m - \gamma, \tau_m]$ with $|x(s_m)| < \eta$. In fact, if $|x(s)| \ge \eta$ on $[\tau_k - \gamma, \tau_k]$ for $k = 1, 2, \ldots, M$, then

$$\begin{aligned} V(\tau_M, x_{\tau_M}) &\le W^*(\delta) - \sum_{k=1}^{M} \int_{\tau_k - \gamma}^{\tau_k} W(|x(s)|)\, ds \\ &\le W^*(\delta) - W(\eta)M\gamma < 0, \end{aligned}$$

a contradiction. Thus, such s_m exists and

$$\begin{aligned} V(t, x_t) &\le V(s_m, x_{s_m}) \le W_2(|x(s_m)|) + W_3(|x_{s_m}|_2) \\ &\le W_2(\eta) + W_3(\eta) < W_1(\varepsilon) \end{aligned}$$

for $t \ge \tau_M \ge s_m$. This implies that $|x(t)| < \varepsilon$ for $t \ge t_0 + T \ge \tau_M$, $T = KJM$, and the zero solution of (1.1) is UAS.

Our next theorem is a refinement of a simple version of Theorem 3 in [3] on uniform asymptotic stability.

Theorem 2.4. *Suppose that there exist a continuous functional $V : R^+ \times C_H \to R^+$, wedges W, W_i $(i = 1, 2, 3)$, positive constants σ, J and a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \le J$ such that*

(i) $W_1(|\varphi(0)|) \le V(t, \varphi) \le W_2(|\varphi(0)|) + W_3(|\varphi|_2)$ *for all* $(t, \varphi) \in R^+ \times C_H$,

(ii) $V'_{(1.1)}(t, x_t) \leq 0$ *for all* $t \geq t_0$ *and* $V'_{(1.1)}(t, x_t) \leq -W(|x_t|_2)$ *for* $t \in [t_n - \sigma, t_n]$, *where* $x(t) = x(t, t_0, \varphi)$ *is any solution of* (1.1) *with* $x_t \in C_H$ *and* $t_n - \sigma \geq t_0$.

Then the zero solution of (1.1) *is UAS.*

Proof. There exists a wedge W^* such that $W_2(r) + W_3(r\sqrt{h}) \leq W^*(r)$ for all $0 \leq r \leq 1$. The zero solution of (1.1) is uniformly stable by (i). Next choose $\delta > 0$ of uniform stability for $\varepsilon_0 = \min\{H, 1\}$. Thus if $x(t) = x(t, t_0, \varphi)$ is a solution of (1.1), then $|x(t, t_0, \varphi)| < \varepsilon_0$ whenever $\|\varphi\| < \delta$ and $t \geq t_0$. Moreover, $V(t, x_t) \leq W^*(\delta)$ for all $t \geq t_0$. We will follow the proof of Theorem 2.2. For any $\varepsilon > 0$, choose $\eta > 0$ such that $W_2(\eta) + W_3(\eta) < W_1(\varepsilon)$. We first claim that there exist a constant $K = K(\delta)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ with $t_{n_k} - t_{n_{k-1}} \leq KJ$ and

$$|x_{t_{n_k}}|_2 < \eta/2. \tag{2.6}$$

Since $P(t) = \int_{t-h}^{t} |x(s)|^2\, ds$ is uniformly continuous on R^+, there exists $\gamma > 0$ ($\gamma < \min\{\sigma, h\}$) such that

$$|P(t) - P(s)| < \eta^2/8 \tag{2.7}$$

whenever $|t - s| < \gamma$. Let K be the first positive integer such that $W^*(\delta) - \gamma K W(\eta/4) < 0$. We will show that there exists an integer n_1, $1 \leq n_1 \leq K$, with $|x_{t_{n_1}}|_2 < \eta/2$. Suppose that $|x_{t_n}| \geq \eta/2$ for $n = 1, 2, \ldots, K$. By (2.7) with $t = t_n$, we have $|x_s|_2 \geq \eta/4$ for all $s \in [t_n - \gamma, t_n]$. Integrate (ii) from t_0 to t_K to obtain

$$\begin{aligned} V(t_K, x_{t_K}) &\leq W^*(\delta) - \sum_{j=1}^{K} \int_{t_j-\gamma}^{t_j} W(|x_s|_2)\, ds \\ &\leq W^*(\delta) - \gamma K W(\eta/4) < 0, \end{aligned}$$

a contradiction. Using the same argument, we obtain a subsequence $\{x_{t_{n_k}}\}$ satisfying (2.6). Moreover, $|x_s|_2 < \eta$ for $s \in [t_{n_k} - \gamma, t_{n_k}]$ by (2.7). We again rename $\{t_{n_k}\}$ by $\tau_k = t_{n_k}$. Next find a positive integer M such that $W^*(\delta) - MW(\eta\sqrt{\gamma/2})\gamma/2 < 0$. We show that there exists m, $1 \leq m \leq M$, with $s_m \in [\tau_m - \gamma, \tau_m]$ and $|x(s_m)| < \eta$. In fact, if $|x(s)| \geq \eta$ on $[\tau_k - \gamma, \tau_k]$ for $k = 1, 2, \ldots, M$, then for $t \in [\tau_k - \gamma/2, \tau_k]$ we have

$$\begin{aligned} \int_{t-h}^{t} |x(s)|^2\, ds &\geq \int_{\tau_k-\gamma}^{t} |x(s)|^2\, ds \\ &\geq \int_{\tau_k-\gamma}^{\tau_k-\gamma/2} |x(s)|^2\, ds \geq \gamma\eta^2/2 \end{aligned}$$

and $|x_t|_2 \geq \eta\sqrt{\gamma/2}$. This then yields

$$\begin{aligned} V(\tau_M, x_{\tau_M}) &\leq W^*(\delta) - \sum_{k=1}^{M} \int_{\tau_k-\gamma/2}^{\tau_k} W(|x_s|_2)\, ds \\ &\leq W^*(\delta) - MW(\eta\sqrt{\gamma/2})\gamma/2 < 0, \end{aligned}$$

a contradiction. Thus, such s_m exists and

$$\begin{aligned} V(t, x_t) &\leq V(s_m, x_{s_m}) \leq W_2(|x(s_m)|) + W_3(|x_{s_m}|_2) \\ &\leq W_2(\eta) + W_3(\eta) < W_1(\varepsilon) \end{aligned}$$

for $t \geq \tau_M \geq s_m$. We then have $|x(t)| < \varepsilon$ for $t \geq t_0 + T \geq \tau_M$, $T = KJM$, and the zero solution of (1.1) is UAS.

3. Example

Consider the scalar equation

$$x'(t) = -a(t)x(t) + b(t)x(t-1) \tag{3.1}$$

where $a(t) = 3(|\sin(\pi t/2)| - \sin(\pi t/2))^2 g^2(t)$,

$$b(t) = 2(|\sin(\pi t/2)| - \sin(\pi t/2))(|\cos(\pi t/2)| + \cos(\pi t/2))g(t)g(t-1)$$

and $g : R \to R^+$ is any continuous function with $1 \leq g(t)$ for all $t \in [4k - \frac{3}{2}, 4k - \frac{1}{2}]$ and $g(t) \leq B$ on $[4k - \frac{3}{2} - \sigma, 4k - \frac{1}{2}]$ for some positive numbers σ and B, where $k = 1, 2, \ldots$. Then the zero solution of (3.1) is AS.

Proof. Define

$$V(t, \varphi) = \frac{1}{2}|\varphi(0)|^2 + \frac{1}{3}\int_{-1}^{0} a(t+s)\varphi^2(s)\,ds$$

for $(t, \varphi) \in R^+ \times C$ and $W_1(r) = r^2/2$. It follows that

$$W_1(|\varphi(0)|) \leq V(t, \varphi). \tag{3.2}$$

If $x(t) = x(t, t_0, \varphi)$ is a solution of (3.1), then

$$V(t, x_t) = \frac{1}{2}|x(t)|^2 + \frac{1}{3}\int_{t-1}^{t} a(s)|x(s)|^2\,ds$$

and

$$\begin{aligned} V'_{(3.1)}(t, x_t) &= x(t)[-a(t)x(t) + b(t)x(t-1)] \\ &\quad + \frac{1}{3}a(t)x^2(t) - \frac{1}{3}a(t-1)x^2(t-1) \\ &= -\frac{2}{3}a(t)x^2(t) + b(t)x(t)x(t-1) \\ &\quad - \frac{1}{3}a(t-1)x^2(t-1). \end{aligned}$$

Notice that $a(t-1) = 3(|\cos(\pi t/2)| + \cos(\pi t/2))^2 g^2(t-1)$ and $b(t)x(t)x(t-1) \leq \frac{1}{3}a(t)x^2(t) + \frac{1}{3}a(t-1)x^2(t-1)$. Thus,

$$V'_{(3.1)}(t, x_t) \leq -\frac{1}{3}a(t)x^2(t) \tag{3.3}$$

for $t \geq t_0$ and

$$V'_{(3.1)}(t, x_t) \leq -2x^2(t) \quad \text{on } [t_k - 1, t_k] \tag{3.4}$$

where $t_k = 4k - \frac{1}{2}$. Define $W_2(r) = r^2/2$ and $W_3(r) = 4B^2 r^2$. Since $g(t) \leq B$ on $[t_k - 1 - \sigma, t_k]$, it follows that

$$V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3(|\varphi|_2) \quad \text{for } t \in [t_k - \sigma, t_k]. \tag{3.5}$$

By (3.2) through (3.5), all conditions of Theorem 2.1 are satisfied. Therefore, the zero solution of (3.1) is AS.

Remark 3.1. For $V(t, \varphi)$ defined above, it is clear that $V'_{(3.1)}(t, x_t) \equiv 0$ on $[4(k-1), 4k-2]$. If we choose $g(4k - \frac{1}{4}) = k$, then $V(t, \varphi)$ is not bounded for fixed φ.

Remark 3.2. If g is bounded in the above example, then the zero solution of (3.1) is UAS.

Remark 3.3. The asymptotic stability for (3.1) discussed above will not follow from Hale [6, p. 108] because he requires that $a(t) \geq \delta > 0$ and $(2a(t)-\delta)\delta - b^2(t) \geq \gamma$. It will not follow from the work of Burton and Hatvani [4, Corollaries 3.1–3.3] because they require that $\int_{t-1}^{t} a^2(s)\,ds$ be bounded in Corollary 3.1, $\int_{t-1}^{t} b^4(s)\,ds$ be bounded in Corollary 3.2, and $\liminf_{t\to+\infty} a(t) > 0$ in Corollary 3.3. It will not follow from the result of Burton and Makay [5] because they require a growth condition on $a(t)$ and $b(t)$.

Remark 3.4. For brevity, we omit examples for Theorems 2.2 and 2.4. We refer to [3, p. 84] and [4, p. 289] for equations with integral terms such as

$$x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} \lambda(s)x(s)\,ds.$$

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