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DIFFERENTIAL OPERATORS, n-BRANCH CURVE SINGULARITIES AND THE n-SUBSPACE PROBLEM

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ABSTRACT. Let R be the coordinate ring of a smooth affine curve over an algebraically closed field of characteristic zero k. For S a subalgebra of R with integral closure R denote by $\mathscr{D}(S)$ the ring of differential operators on S and by $H(S)$ the finite-dimensional factor of $\mathscr{D}(S)$ by its unique minimal ideal. The theory of diagonal n-subspace systems is introduced. This is used to show that if A is a finite-dimensional k-algebra and $t \geq 1$ is any integer there exists such an S with

$$H(S) \cong \begin{pmatrix} A & * \\ 0 & M_t(k) \end{pmatrix}.$$

Further, the Morita classes of $H(S)$ are classified for curves with few branches, and it is shown how to lift Morita equivalences from $H(S)$ to $\mathscr{D}(S)$.

0. INTRODUCTION

Suppose R is a finitely generated Dedekind domain (or the localization at a semimaximal ideal of such) over an algebraically closed field of characteristic zero k. We investigate two analytic invariants of a curve $\operatorname{Spec} S$ with normalization $\operatorname{Spec} R$. First, there is a $\max R$-diagonal (see §1 for a definition) which may alternatively be considered as a subspace system, that is, a vector space and a collection of n subspaces (where $\operatorname{Spec} S$ has n branches). Secondly, there is a finite-dimensional algebra $H(S)$ which is the endomorphism ring of this diagonal/subspace system when considered in the appropriate category. The connection with differential operators, mentioned in the title of the paper, arises from the following rephrasing of a result of K.A. Brown [Br]. Let $\mathscr{D}(S)$ be the ring of differential operators on S, and let $J(S)$ be the minimal ideal of $\mathscr{D}(S)$. Then $H(S) \cong \mathscr{D}(S)/J(S)$. Actually, this discussion reverses the historical development which began with the work of [SS] on the algebra $\mathscr{D}(S)/J(S)$. Brown's result allowed substantial progress on the structure of this algebra. In particular, he showed, in [Br], that $H(S)$ has a block upper triangular structure:

$$(*) \qquad H(S) \cong \begin{pmatrix} A & * \\ 0 & M_t(k) \end{pmatrix},$$

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for some finite-dimensional algebra A and some $t \geq 1$. In §2 we show that, rather surprisingly, any A can occur.

0.1. **Theorem.** *Suppose that R is a finitely generated Dedekind domain over an algebraically closed field of characteristic zero k. Let A be any finite-dimensional algebra and t any positive integer. Then there exists an algebra S with integral closure R such that $H(S)$ is as in* (*).

This shows that $H(S)$ is a rather rich invariant.

Section 1 explores the machinery of categories of finite $\max R$-diagonals. We will gloss over this here. The important point to note is that the category of finite diagonals embeds as a full subcategory of the category of subspace systems. In particular, it follows that the category of finite $\max R$-diagonals has a Krull-Schmidt theorem. Also, we can plug into the well-known theory of subspace systems (see [Be] and [GP]) to obtain a classification when S is the local ring of a curve singularity with ≤ 4 branches.

Of course, Theorem 0.1 follows from knowing precisely which $\max R$-diagonals arise from subalgebras S. To simplify the explanation of this, let us suppose from now on that R is semilocal. We denote by $k_{\max R}$ the indecomposable $\max R$-diagonal which considered as a subspace system is the vector space k and $|\max R|$ copies of the zero subspace.

0.2. **Theorem.** *Let R be a semilocal Dedekind domain of finite type over an algebraically closed field of characteristic zero k. Let $\mathscr{V}$ be a $\max R$-diagonal. Then $\mathscr{V}$ is isomorphic to the diagonal of a local ring S with integral closure R if and only if $\mathscr{V}$ has $k_{\max R}$ as a direct summand.*

In the final section we consider the question of Morita equivalences $\mathscr{D}(S) \sim \mathscr{D}(S')$ when S, S' are the local rings of curves with integral closure R. We examine circumstances in which these equivalences can be lifted from equivalences $H(S) = \mathscr{D}(S)/J(S) \sim \mathscr{D}(S')/J(S') = H(S')$. We prove that if the associated $\max R$-diagonals of S and S' contain the same indecomposables (albeit with different multiplicities), then $\mathscr{D}(S)$ and $\mathscr{D}(S')$ are Morita equivalent. This enables one to write down a large number of Morita equivalences of rings of differential operators. For example, we have the following consequence.

0.3. **Theorem.** *Let S and S' be the local rings of algebraic curve singularities with integral closure R. If $H(S)$ and $H(S')$ are both simple (and nonzero) then $\mathscr{D}(S) \sim \mathscr{D}(S')$.*

As another corollary we can obtain the following classification.

0.4. **Corollary.** *Let S be the local ring of an n-branch curve singularity with integral closure R. Then*

(a) *If $n = 2$ there is exactly one Morita class amongst the $\mathscr{D}(S)$.*
(b) *If $n = 3$ there are at most sixteen Morita classes amongst the $\mathscr{D}(S)$.*
(c) *If $n \geq 4$ there are infinitely many Morita classes amongst the $\mathscr{D}(S)$.*

We complete the introduction with a summary of the contents of the paper. In §1 we briefly develop the theory of diagonals independently of subspace systems. One reason for doing so is that the category of diagonals admits a duality which cannot be obtained from a duality of subspace systems. This duality turns out to be closely related to the Gorenstein property; see [CHM].

In §2 we consider the richness of the two invariants. In particular we prove Theorems 0.1 and 0.2.

The final section considers Morita equivalences of rings of differential operators and proves Theorems 0.3 and 0.4.

1. Categories of diagonals

This section provides a rapid exposition of the necessary generalities on diagonals. There is, for the most, part a parallel theory of 'subspace problems', so some of the proofs are omitted or else are sketchy.

We cover the definitions of the categories I-Diag in §1.1; each of which is contravariantly self-equivalent (§1.2) and, possesses direct sums and hence indecomposables (§1.3) and tensor products (§1.4). Finally we compare the categories of diagonals with those of the 'n-subspace problem' in §1.5.

1.1. Fix once and for all an indexing set I and a base-field k.

Suppose $\mathscr{W} = (W; W_i : i \in I)$ is a $1 + I$-tuple of vector spaces. We say that $\mathscr{W}$ is an *I-diagonal* if

(1) $W \subseteq \bigoplus_{i \in I} W_i$;

(2) $W \cap W_i = 0$ for each $i \in I$;

(3) $\pi_i(W) = W_i$ for each $i \in I$, where $\pi_i : \bigoplus_{j \in I} W_j \to \bigoplus_{j \in I} W_j$ is the projection onto the ith summand.

W is termed the *slant* of $\mathscr{W}$, and we will write $\operatorname{Slant}\mathscr{W}$ if useful. W_i is termed the ith *summand* of $\mathscr{W}$, and $\bigoplus_{i \in I} W_i$, its *sum*, is written $\operatorname{Sum}\mathscr{W}$. π_i will always mean the ith projection map and $\pi_i\mathscr{W}$ the ith summand. The inclusion map $\operatorname{Slant}\mathscr{W} \to \operatorname{Sum}\mathscr{W}$ is denoted by i, and the projection map $\operatorname{Sum}\mathscr{W} \to \operatorname{Sum}\mathscr{W}/\operatorname{Slant}\mathscr{W}$ is denoted by χ. We define $(\pi_i\mathscr{W})^\perp := \ker \pi_i = \bigoplus_{j \neq i} \pi_j \mathscr{W}$ and call it the ith *perp.* We say a diagonal is *finite dimensional* if all its summands are finite dimensional and *finite* if its sum is finite dimensional.

Given I-diagonals $\mathscr{W} = (W; W_i : i \in I)$ and $\mathscr{V} = (V; V_i : i \in I)$ we define

$$\begin{aligned}
\mathscr{E}(\mathscr{W}, \mathscr{V}) &:= \{\theta \in \operatorname{Hom}(W, V) : \theta(W \cap W_i^\perp) \subseteq V \cap V_i^\perp \text{ for } i \in I\} \\
&= \{\theta \in \operatorname{Hom}(\operatorname{Slant}\mathscr{W}, \operatorname{Slant}\mathscr{V}) : \\
&\qquad\qquad \theta(\operatorname{Slant}\mathscr{W} \cap \ker \pi_i) \subseteq \operatorname{Slant}\mathscr{V} \cap \ker \pi_i \text{ for all } i \in I\}, \\
\mathscr{F}(\mathscr{W}, \mathscr{V}) &:= \{\theta \in \operatorname{Hom}(\bigoplus_{i \in I} W_i, \bigoplus_{i \in I} V_i) : \theta(W_i) \subseteq V_i \text{ for } i \in I \text{ and } \theta(W) \subseteq V\} \\
&= \{\theta \in \operatorname{Hom}(\operatorname{Sum}\mathscr{W}, \operatorname{Sum}\mathscr{V}) : \theta\pi_i = \pi_i\theta \\
&\qquad\qquad \text{for all } i \in I \text{ and } \theta(\operatorname{Slant}\mathscr{W}) \subseteq \operatorname{Slant}\mathscr{V}\}, \\
\mathscr{G}(\mathscr{W}, \mathscr{V}) &:= \{\theta \in \operatorname{Hom}((\bigoplus_{i \in I} W_i)/W, (\bigoplus_{i \in I} V_i)/V) : \theta\chi(W_i) \subseteq \chi(V_i) \text{ for } i \in I\}.
\end{aligned}$$

We distinguish $\mathscr{E}(\mathscr{W}) := \mathscr{E}(\mathscr{W}, \mathscr{W})$, $\mathscr{F}(\mathscr{W}) := \mathscr{F}(\mathscr{W}, \mathscr{W})$, and $\mathscr{G}(\mathscr{W}) := \mathscr{G}(\mathscr{W}, \mathscr{W})$. Note that $\mathscr{E}(\mathscr{W})$ is an algebra while $\mathscr{E}(\mathscr{W}, \mathscr{V})$ is an $\mathscr{E}(\mathscr{V})$-$\mathscr{E}(\mathscr{W})$ bimodule and similarly for $\mathscr{F}$ and $\mathscr{G}$.

We define I-$\operatorname{diag}_{\mathscr{E}}$, respectively I-$\operatorname{diag}_{\mathscr{F}}$, respectively I-$\operatorname{diag}_{\mathscr{G}}$ to be the category whose objects are the I-diagonals and whose morphisms are given by $\operatorname{Mor}(\mathscr{V}, \mathscr{W}) := \mathscr{E}(\mathscr{V}, \mathscr{W})$, respectively $\operatorname{Mor}(\mathscr{V}, \mathscr{W}) := \mathscr{F}(\mathscr{V}, \mathscr{W})$, respectively $\operatorname{Mor}(\mathscr{V}, \mathscr{W}) := \mathscr{G}(\mathscr{V}, \mathscr{W})$.

Define maps $\hat{} : \mathscr{F}(\mathscr{W}, \mathscr{V}) \longrightarrow \mathscr{E}(\mathscr{W}, \mathscr{V}) : \theta \longmapsto \hat{\theta}$, where $\hat{\theta}(w) = \theta(w)$ and $\check{} : \mathscr{F}(\mathscr{W}, \mathscr{V}) \longrightarrow \mathscr{G}(\mathscr{W}, \mathscr{V}) : \theta \longmapsto \check{\theta}$, where $\check{\theta}(w + W) = \theta(w) + V$.

Note that these maps are k-algebra homomorphisms when $\mathscr{W} = \mathscr{V}$. The next proposition shows that the three categories defined above are isomorphic, so henceforth where convenient we shall just use the generic I-diag.

Proposition. *Let $\mathscr{W}$, $\mathscr{V}$, and $\mathscr{U}$ be I-diagonals.*

(1) *The linear map $\hat{}$ is a bijection. In particular $\hat{}: \mathscr{F}(\mathscr{W}) \to \mathscr{E}(\mathscr{W})$ is an isomorphism of k-algebras.*

(2) *The linear map $\check{}$ is a bijection. In particular $\check{}: \mathscr{F}(\mathscr{W}) \to \mathscr{G}(\mathscr{W})$ is an isomorphism of k-algebras.*

(3) *We have a commuting diagram:*

$$\begin{tikzcd}
\mathscr{E}(\mathscr{V}, \mathscr{U}) \times \mathscr{E}(\mathscr{W}, \mathscr{V}) \arrow[r, "\hat{}^{-1} \times \hat{}^{-1}"] \arrow[d] & \mathscr{F}(\mathscr{V}, \mathscr{U}) \times \mathscr{F}(\mathscr{W}, \mathscr{V}) \arrow[r, "\check{} \times \check{}"] \arrow[d] & \mathscr{G}(\mathscr{V}, \mathscr{U}) \times \mathscr{G}(\mathscr{W}, \mathscr{V}) \arrow[d] \\
\mathscr{E}(\mathscr{W}, \mathscr{U}) \arrow[r, "\hat{}^{-1}"] & \mathscr{F}(\mathscr{W}, \mathscr{U}) \arrow[r, "\check{}"] & \mathscr{G}(\mathscr{W}, \mathscr{U})
\end{tikzcd}$$

where the vertical maps are given by composition of linear maps.

(4) *We have a commuting diagram:*

$$\begin{tikzcd}
\mathscr{E}(\mathscr{W}, \mathscr{V}) \times W \arrow[r, "(\hat{}^{-1}, i)"] \arrow[d, "\mu_{\mathscr{E}}"'] & \mathscr{F}(\mathscr{W}, \mathscr{V}) \times \bigoplus_i W_i \arrow[r, "(\check{}, \chi)"] \arrow[d, "\mu_{\mathscr{F}}"'] & \mathscr{G}(\mathscr{W}, \mathscr{V}) \times (\bigoplus_i W_i)/W \arrow[d, "\mu_{\mathscr{G}}"'] \\
V \arrow[r, "i"] & \bigoplus_i V_i \arrow[r, "\chi"] & (\bigoplus_i V_i)/V
\end{tikzcd}$$

where the $\mu_{\mathscr{E}}$, $\mu_{\mathscr{F}}$, and $\mu_{\mathscr{G}}$ are the evaluation maps.

Proof. (1), (2), and (4) are routine extensions of [CH2, Proposition 5.3]. (3) is immediate from the definitions.

Example. Given a vector space V we write V_I for any diagonal isomorphic to the diagonal which has all its summands equal to V and slant the diagonal copy of V.

1.2. **Self-duality.** Write V^* for $\operatorname{Hom}(V, k)$ when V is a vector space. We may define a (contravariant) equivalence of categories $*: I\text{-Diag} \longrightarrow I\text{-Diag}$ by

$$\mathscr{V}^* := (\{\theta \in \bigoplus(\pi_i\mathscr{V})^* : \theta(\operatorname{Slant}\mathscr{V}) = 0\}; (\pi_i\mathscr{V})^* : i \in I)$$

whenever $\mathscr{V}$ is an I-Diagonal and $\theta^* : \bigoplus(\pi_i\mathscr{W})^* \longrightarrow \bigoplus(\pi_i\mathscr{V})^* : \sum_i \psi_i \longmapsto \sum \psi_i\theta\pi_i$ whenever $\theta \in \mathscr{F}(\mathscr{V}, \mathscr{W})$. The main point to check is that $\mathscr{V}^*$ is a diagonal. Indeed if $\theta \in \operatorname{Slant}(\mathscr{V}^*) \cap (\pi_i\mathscr{V}^*)$, then

$$\theta(\bigoplus_{i \in I} \pi_i\mathscr{V}) \subseteq \theta(\operatorname{Slant}\mathscr{V} + (\pi_i\mathscr{V})^{\perp}) = 0$$

and so $\theta = 0$. Further if $x \in \operatorname{Sum}\mathscr{V}$ is such that $(\operatorname{Slant}\mathscr{V}^* + \bigoplus_{j\neq i}(\pi_j\mathscr{V})^*)(x) = 0$, then $x \in \pi_i\mathscr{V} \cap \operatorname{Slant}\mathscr{V} = 0$. Thus $\operatorname{Slant}(\mathscr{V}^*) + \bigoplus_{j\neq i}(\pi_j\mathscr{V})^* = \bigoplus(\pi_j\mathscr{V})^*$ and so $\pi_i(\operatorname{Slant}(\mathscr{V}^*)) = (\pi_i\mathscr{V})^* = \pi_i(\mathscr{V}^*)$.

We should point out that ** is naturally equivalent to the identity. In particular it follows that $\operatorname{Mor}(\mathscr{V}, \mathscr{V}) \simeq \operatorname{Mor}(\mathscr{V}^*, \mathscr{V}^*)^{op}$.

1.3. **Direct sums.** Suppose $\mathscr{V}_l$ is an I-diagonal for each $l \in L$. Define the direct sum of the $\mathscr{V}_l$ by

$$\bigoplus_{l\in L}\mathscr{V}_l := \left(\sum_{l\in L}\operatorname{Slant}\mathscr{V}_l;\ \bigoplus_{l\in L}\pi_i(\mathscr{V}_l) : i \in I\right).$$

Let $\sigma_l \in \operatorname{Mor}(\mathscr{V}_l, \bigoplus_{l\in L}\mathscr{V}_l)$ be the natural 'embedding' and let $\pi_l \in \operatorname{Mor}(\bigoplus_{l\in L}\mathscr{V}_l, \mathscr{V}_l)$ be the natural 'projection'. Let $e_l = \pi_l\sigma_l$. If $L = \{1, \dots, n\}$, $M = \{1, \dots, p\}$, and $\mathscr{W}_m$ is an I-diagonal for each $m \in M$, then we have

$$\begin{aligned}
&\operatorname{Mor}\left(\bigoplus_{l\in L}\mathscr{V}_l, \bigoplus_{m\in M}\mathscr{W}_m\right)\\
&= \bigoplus_{(l,m)\in L\times M} e_m\operatorname{Mor}\left(\bigoplus_{l\in L}\mathscr{V}_l, \bigoplus_{m\in M}\mathscr{W}_m\right)e_l\\
&\cong \bigoplus_{(l,m)\in L\times M} \sigma_m\operatorname{Mor}(\mathscr{V}_l, \mathscr{W}_m)\pi_l\\
&= \begin{pmatrix} \operatorname{Mor}(\mathscr{V}_1, \mathscr{W}_1) & \operatorname{Mor}(\mathscr{V}_2, \mathscr{W}_1) & \cdots & \operatorname{Mor}(\mathscr{V}_n, \mathscr{W}_1)\\ \operatorname{Mor}(\mathscr{V}_1, \mathscr{W}_2) & \operatorname{Mor}(\mathscr{V}_2, \mathscr{W}_2) & \cdots & \operatorname{Mor}(\mathscr{V}_n, \mathscr{W}_2)\\ \vdots & \vdots & \ddots & \vdots\\ \operatorname{Mor}(\mathscr{V}_1, \mathscr{W}_p) & \operatorname{Mor}(\mathscr{V}_2, \mathscr{W}_p) & \cdots & \operatorname{Mor}(\mathscr{V}_n, \mathscr{W}_p)\end{pmatrix}.
\end{aligned}$$

We say an I-diagonal $\mathscr{V}$ is *indecomposable* if it is nonzero and $\mathscr{V} \cong \mathscr{W}_1 \oplus \mathscr{W}_2$ implies $\mathscr{W}_1 \cong 0$ or $\mathscr{W}_2 \cong 0$.

The following result is easily derived.

Proposition. (1) *An I-diagonal $\mathscr{V}$ is indecomposable if and only if* $\operatorname{Mor}(\mathscr{V}, \mathscr{V})$ *is a local ring.*

(2) *Every finite I-diagonal decomposes in an essentially unique way as a sum of indecomposable I-diagonals.*

(3) *Suppose $\mathscr{V} = \bigoplus_i \mathscr{V}_i^{n_i}$ where each n_i is a positive integer and each $\mathscr{V}_i$ is an indecomposable I-diagonal with $\mathscr{V}_i \not\cong \mathscr{V}_j$ when $i \neq j$. Then* $\operatorname{Mor}(\mathscr{V}, \mathscr{V}) \cong \bigoplus_i \operatorname{Mor}(\mathscr{V}, \mathscr{V}_i)^{n_i}$ *is a decomposition of* $\operatorname{Mor}(\mathscr{V}, \mathscr{V})$ *into indecomposable projective right ideals. In particular, the basic algebra corresponding to* $\operatorname{Mor}(\mathscr{V}, \mathscr{V})$ *is* $\operatorname{Mor}(\bigoplus_i \mathscr{V}_i, \bigoplus_i \mathscr{V}_i)$.

Example. k_I is an indecomposable I-diagonal. We say a diagonal $\mathscr{V}$ is *spanned by perps* if $\operatorname{Slant}\mathscr{V} = \sum_{i\in I}\operatorname{Slant}\mathscr{V} \cap (\pi_i\mathscr{V})^{\perp}$. It is easy to check that the only indecomposable I-diagonal which is not spanned by perps is k_I and further that an I-diagonal is spanned by perps if and only if it has no indecomposable direct summand which is isomorphic to k_I.

1.4. **Tensor products.** Suppose V, W are vector spaces, I, J are sets, $\mathscr{V}$ is an I-diagonal, and $\mathscr{W}$ is a J-diagonal. We have various tensor products.

First we define an I-diagonal $\mathscr{V} \otimes W := (\operatorname{Slant}\mathscr{V} \otimes W\,;\ \pi_i(\mathscr{V}) \otimes W : i \in I)$ and by similar means a J-diagonal $V \otimes \mathscr{W}$. At the same time we have an $I \times J$-diagonal $\mathscr{V} \otimes \mathscr{W} := (\operatorname{Slant}\mathscr{V} \otimes \operatorname{Slant}\mathscr{W}\,;\ \pi_i\mathscr{V} \otimes \pi_j\mathscr{W} : (i, j) \in I \times J)$.

If $\mathscr{V}$, $\mathscr{W}$ are finite I-diagonals and $\mathscr{V}'$, $\mathscr{W}'$ are J-diagonals and we write V, W, V', and W' for the slants of $\mathscr{V}$, $\mathscr{W}$, $\mathscr{V}'$, and $\mathscr{W}'$, respectively, then, after making the appropriate identifications, we have

$$\begin{aligned}
&\mathscr{E}(\mathscr{V}, \mathscr{V}') \otimes \mathscr{E}(\mathscr{W}, \mathscr{W}') \\
&\quad \subseteq \mathscr{E}(\mathscr{V} \otimes \mathscr{W}, \mathscr{V}' \otimes \mathscr{W}') \\
&\quad \subseteq \mathscr{E}(\mathscr{V} \otimes W, \mathscr{V}' \otimes W') \cap \mathscr{E}(V \otimes \mathscr{W}, V' \otimes \mathscr{W}') \\
&\quad = (\mathscr{E}(\mathscr{V}, \mathscr{V}') \otimes \operatorname{Hom}(W, W')) \cap (\operatorname{Hom}(V, V') \otimes \mathscr{E}(\mathscr{W}, \mathscr{W}')) \\
&\quad = \mathscr{E}(\mathscr{V}, \mathscr{V}') \otimes \mathscr{E}(\mathscr{W}, \mathscr{W}').
\end{aligned}$$

We thus have the following proposition.

Proposition. *If $\mathscr{V}$, $\mathscr{W}$ are finite I-diagonals and $\mathscr{V}'$, $\mathscr{W}'$ are J-diagonals, then there is a canonical isomorphism* $\operatorname{Mor}(\mathscr{V} \otimes \mathscr{W}, \mathscr{V}' \otimes \mathscr{W}') \simeq \operatorname{Mor}(\mathscr{V}, \mathscr{V}') \otimes \operatorname{Mor}(\mathscr{W}, \mathscr{W}')$.

Remark. If $\mathscr{V}$ or $\mathscr{W}$ is spanned by perps, then so is $\mathscr{V} \otimes \mathscr{W}$.

1.5. **The n-subspace problem.** The study of diagonals is closely related to the following much studied problem "Given a vector space V in what ways (up to appropriate isomorphism) can one choose n subspaces of V". We can use the extensive theory developed in the setting of this problem to give information on diagonals. In particular, we may obtain lists of indecomposable diagonals for $|I| \leq 4$.

We say a $1 + I$-tuple $\mathscr{S} = (S\,;\ S_i : i \in I)$ of vector spaces is an I-subspace system provided $S_i \subseteq S$ for each $i \in I$. Given two I-subspace systems $\mathscr{S} = (S\,;\ S_i : i \in I)$ and $\mathscr{T} = (T\,;\ T_i : i \in I)$ we can define $\operatorname{Hom}(\mathscr{S}, \mathscr{T}) = \{\theta \in \operatorname{Hom}(S, T) : \theta S_i \subseteq T_i \text{ for all } i \in I\}$. In this way we create a category, 'the category of I-subspace systems'.

There is a functor, $\imath$, from $I\text{-diag}_{\mathscr{E}}$ to the category of I-subspace systems defined by $\imath(\mathscr{W}) := (\operatorname{Slant}\mathscr{W}\,;\ \operatorname{Slant}\mathscr{W} \cap \ker \pi_i : i \in I)$ and $\imath(\phi) = \phi$ whenever $\phi \in \mathscr{E}(\mathscr{W}, \mathscr{V})$. Given an I-subspace system $\mathscr{T} = (T\,;\ T_i : i \in I)$ the $1 + I$-tuple $\mathscr{W} := (T\,;\ T/T_i : i \in I)$ is a diagonal (with $T \hookrightarrow \bigoplus_{i \in I} T/T_i$ the canonical map) if and only if

$$(*) \qquad \bigcap_{j \neq i} T_j = 0 \quad \text{for all } i \in I.$$

Further when $\mathscr{W}$ is a diagonal we have $\imath(\mathscr{W}) \cong \mathscr{T}$. Thus we may regard I-diag as a subcategory of the I-subspace systems via $\imath$.

Proposition. *The functor $\imath$ makes I-diag isomorphic to a full subcategory of the category of I-subspace systems. A subspace system $\mathscr{T} = (T\,;\ T_i : i \in I)$ is in the image of $\imath$ if and only if $(*)$ is satisfied.*

The category of I-subspace systems is endowed with both direct sums and tensor products which extend the corresponding notions developed for I-diag. Further, because I-diag is a full subcategory of the I-subspace systems, its indecomposable objects are precisely those diagonals which are indecomposable when regarded as I-subspace systems. There are comparatively few indecomposable I-subspace systems which do not correspond to I-diagonals. Indeed by [Be, §6, Corollary to Lemma 2], if I is finite, then there are precisely $|I|+1$ such subspace systems, namely $(k;0,k,\dots,k),\dots,(k;k,\dots,k,0)$ and $(k;k,\dots,k)$ (after ordering I).

Using this observation we can classify I-diagonals for $|I|\leq 4$. Indeed if $I=\{1,2\}$, then the only indecomposable is k_I. For $I=\{1,2,3\}$ the indecomposables are k_I, k_I^*, $(k;k,k,0)$, $(k;k,0,k)$, and $(k;0,k,k)$. If $\delta,\delta^*,\epsilon_1,\epsilon_2,\epsilon_3\in\{0,1\}$ and $\mathscr{V}=k_I^{*\delta^*}\oplus(k;0,k,k)^{\epsilon_1}\oplus(k;k,0,k)^{\epsilon_2}\oplus(k;k,k,0)^{\epsilon_3}\oplus k_I^{\delta}$, then define $\mathscr{E}_{\delta,\delta^*,\epsilon_1,\epsilon_2,\epsilon_3}:=\mathscr{E}(\mathscr{V})$. For example,

$$\mathscr{E}_{1,1,1,1,1}\cong\left\{\begin{pmatrix}\alpha^* & 0 & \delta_1 & 0 & \delta_3 & \gamma_1\\ 0 & \alpha^* & -\delta_1 & \delta_2 & 0 & \gamma_2\\ 0 & 0 & \alpha_1 & 0 & 0 & \beta_1\\ 0 & 0 & 0 & \alpha_2 & 0 & \beta_2\\ 0 & 0 & 0 & 0 & \alpha_3 & \beta_3\\ 0 & 0 & 0 & 0 & 0 & \alpha\end{pmatrix} : \alpha,\alpha^*,\alpha_p,\beta_q,\gamma_r\in k\right\}.$$

The case $I=\{1,2,3,4\}$ is somewhat more complicated and in particular there are infinitely many indecomposables. These indecomposables may be read off from Gelfand and Ponomarev's classification of I-subspace systems in [GP]. When $|I|\geq 5$ the problem is wild. However, Kac's work in [KR] does describe the dimension vectors of the indecomposables.

2. The diversity of finite-dimensional factors

In this section we associate diagonals to curves and determine precisely which diagonals arise in this way.

2.1. Suppose R is a Dedekind domain of finite type over an algebraically closed field of characteristic zero k. Further, suppose that if $m\in\max R$, then $R/m\cong k$.

We recall some notation and results from [CH] and [CH2]. Suppose V is a subspace of R. We describe V as *dense* if it contains an ideal of R with finite codimension in R. Given m a maximal ideal of R let $V(m)=\bigcap_{n=0}^{\infty}V+m^n$. We say $V\subseteq R$ is *m-primary* if it contains a power of m or equivalently if $V=V(m)$. Define $V^+=\bigcap_{m\in\max R}V(m)$. We say V is *primary decomposable* if $V=V^+$ or equivalently [CH, 2.4] if it is the intersection of finitely many primary subspaces of R. Define V^- to be the sum of all the primary decomposable subspaces of V. We will have $V=V^-$ if and only if V is primary decomposable [CH, 2.14]. If S is a dense subalgebra of R, then so is S^+; further, Spec $R\to$ Spec S^+ is injective, Spec $S^+\to$ Spec S is unramified, and $S^-=\operatorname{ann} S^+/S$ (see [CH2, 2.4; CH, 2.16]).

The canonical map $V^+/V^-\longrightarrow\bigoplus V^+(m)/V^-(m)$ is an isomorphism of vector spaces [CH, 2.7], which we shall treat as an identification. With this in mind, we may associate to V a finite $\max R$-diagonal $\mathscr{V}:=(V/V^-;V^+(m)/V^-(m):m\in\max R)$. *Henceforth where we denote a dense subspace of R by a latin letter the corresponding diagonal will be denoted by the corresponding script letter.*

Example. Let $R = k[x]_{(x-1)} \cap k[x]_{(x+1)}$, with maximal ideals $m_1 = (x-1)R$, $m_2 = (x+1)R$. Consider $S = k + (x^2-1)^2 R$. Then

$$\begin{aligned} S^+ &= S(m_1) \cap S(m_2) = (k + (x-1)^2 R) \cap (k + (x+1)^2 R) \\ &= k + k(x^3 - 3x) + (x^2-1)^2 R \end{aligned}$$

and $S^- = (x^2-1)^2 R$. Note that Spec S has a unique singularity, with two cuspidal branches. Pulling the branches apart yields Spec S^+, which has two cuspidal singular points.

Lemma. *Let S be a subalgebra of R with integral closure R, and let ψ : $\max R \to \max S$ be the normalisation map. Let $\operatorname{Sing} S$ denote the subset of $\max S$ consisting of singular points. Then*

$$\mathscr{S} = \bigoplus_{x \in \operatorname{Sing} S} \mathscr{S}_x,$$

where $\mathscr{S}_x$ is a $\max R$-diagonal with $\pi_m(\mathscr{S}_x) = 0$ for $\psi(m) \neq x$.

Set-theoretic knowledge of any branching is enough to determine some of the indecomposables occurring. Suppose T is a finite set of maximal ideals of R which is not a singleton. For a vector space Z define a diagonal $Z_{\max R, T} := (Z\,;\, Z_{\max R, T, m} : m \in \max R)$ where $Z_{\max R, T, m}$ is a copy of Z if $m \in T$ and is zero otherwise.

Proposition. *Suppose S is a dense subalgebra of R and we have normalization map $\psi : \max R \longrightarrow \max S$. Suppose $x \in \max S$ is a singular point over which ψ is not injective. Then $k_{\max R, \psi^{-1}(x)}$ is an indecomposable summand of $\mathscr{S}$.*

Proof. By virtue of the lemma we may suppose that S is local. Define $K = \sum_m (\operatorname{Slant} \mathscr{S} \cap (\pi_m \mathscr{S})^\perp)$, and let $\operatorname{Slant} \mathscr{S} = K \oplus L$. As ψ is not injective over x, if $m \in \max R$ there exists $m' \neq m$ with $\psi(m') = x$. Also, $S^- \neq S$. Thus, for each $m \in \max R$ we have that

$$S \cap \bigcap_{\substack{n \in \max R \\ n \neq m}} S^-(n) \subseteq S \cap (S^- + m') = x.$$

It follows that K lies inside x/S^- and so $L \neq 0$. It is not hard to see that

$$\mathscr{S} \cong (K\,;\, \pi_m K : m \in \max R) \oplus L_{\max R, \psi^{-1}(x)},$$

as required.

2.2. Amazingly 'almost all' (see Proposition of §2.1) finite $\max R$-diagonals occur as diagonals corresponding to curves.

Theorem. *Suppose $\mathscr{V}$ is a finite* $\max R$*-diagonal, and let* Presing $:= \{m \in \max R : \pi_m \mathscr{V} \neq 0\}$. *Then there is a dense subalgebra $S \subseteq R$ such that*

(1) *the normalization map $\max R \longrightarrow \max S$ is injective whenever $\pi_m \mathscr{V} = 0$ and identifies together all the points m for which $\pi_m \mathscr{V} \neq 0$.*
(2) $\mathscr{S} \cong \mathscr{V} \oplus k_{\max R, \text{Presing}}$.

Proof. Let

$$n := \max\{\dim \pi_m \mathscr{V} : m \in \max R\}.$$

Without loss of generality we may assume $\pi_m \mathscr{V} \subseteq m^n/m^{2n}$ for each $m \in \max R$ and $\operatorname{Slant} \mathscr{V} \subseteq \mu^n/\mu^{2n}$ where $\mu = \bigcap \text{Presing}$ and μ^n/μ^{2n} has been identified with $\bigoplus_{m\in\text{Presing}} m^n/m^{2n}$. Thus $\text{Slant}\,\mathscr{V} = \Omega/\mu^{2n}$ where $\mu^{2n} \subseteq \Omega \subseteq \mu^n$. Now take $S = k + \Omega$. It is easy to verify that S has the required properties.

For the rest of this section we suppose that $\max R$ is infinite.

Corollary. *Suppose A is a finite-dimensional algebra and n is a positive integer. Then there is a commutative algebra S with integral closure R such that*

$$\mathscr{D}(S)/J(S) \cong \begin{pmatrix} B & * \\ 0 & M_n(k) \end{pmatrix}$$

where B is a subalgebra of a matrix algebra which is isomorphic to A.

Proof. Choose $\mathscr{V}$ an I-diagonal where I is finite such that $\text{Mor}(\mathscr{V}, \mathscr{V}) \cong A$ (see [CH2, Theorem 5.5]). We have

$$\begin{aligned}\text{Mor}(\mathscr{V} \otimes k^*_{\{0,1,2\}}, \mathscr{V} \otimes k^*_{\{0,1,2\}}) &\cong \text{Mor}(\mathscr{V}, \mathscr{V}) \otimes \text{Mor}(k^*_{\{0,1,2\}}, k^*_{\{0,1,2\}}) \\ &\cong A.\end{aligned}$$

$\mathscr{V} \otimes k^*_{\{0,1,2\}}$ does not have $k_{I\times\{0,1,2\}}$ as an indecomposable summand by the remark in §1.4. Thus without loss of generality k_I is not an indecomposable summand of $\mathscr{V}$. Without loss of generality $I \subseteq \max R$. Define a $\max R$-diagonal $\mathscr{W} := (\text{Slant}\,\mathscr{V}; \mathscr{W}_m : m \in \max R)$ where $\mathscr{W}_m$ is $\pi_m\mathscr{V}$ when $m \in I$ and zero otherwise. By the theorem we may choose S a dense subalgebra of R such that $\mathscr{S} \cong \mathscr{W} \oplus k^n_{\max R, I}$. This S has the required properties.

Remark. The converse of this result also holds, see [Br, Theorem 5.5]. It also follows from the proposition of §2.1.

3. Morita equivalences

3.1. **Differential operators.** Retain the hypotheses of §2.1.

Given dense subspaces V, W we may define the differential operators from V to W by $\mathscr{D}(V, W) := \{\theta \in \mathscr{D}(Q) : \theta * V \subseteq W\}$ where $\mathscr{D}(Q)$ is the ring of differential operators on Q, the field of fractions of R. $\mathscr{D}(V)$ is a ring, and $\mathscr{D}(V, W)$ is a $\mathscr{D}(W)$-$\mathscr{D}(V)$-bimodule.

There is a close connection between the subbimodule structure of $\mathscr{D}(V, W)$ and the diagonals associated to V and W. Indeed $\mathscr{D}(V, W)$ has a unique minimal nonzero subbimodule $J(V, W)$ and the canonical map $\mathscr{D}(V, W) \longrightarrow \text{Mor}(\mathscr{V}, \mathscr{W})$ is surjective with kernel $J(V, W) := \mathscr{D}(V^+, W) = \mathscr{D}(V^+, W^-) = \mathscr{D}(V, W^-)$. For the details, see [CH2]. Further, $\mathscr{D}(V, W)J(U, V)$, $J(V, W)\mathscr{D}(U, V) \subseteq J(U, W)$ and the following diagram commutes

$$\begin{array}{ccc} \mathscr{D}(V, W)/J(V, W) \times \mathscr{D}(U, V)/J(U, V) & \longrightarrow & \mathscr{D}(U, W)/J(U, W) \\ \downarrow & & \downarrow \\ \text{Mor}(\mathscr{V}, \mathscr{W}) \times \text{Mor}(\mathscr{U}, \mathscr{V}) & \longrightarrow & \text{Mor}(\mathscr{U}, \mathscr{W}) \end{array}$$

A portion of this information may be expressed by saying there is a functor from the category of dense subspaces of R (with morphisms being differential operators) to the category of $\max R$-diagonals.

3.2. **Morita theory.** We are interested in finding pairs V, W of dense subspaces of R whose rings of differential operators are Morita equivalent. The most natural class of Morita equivalences arise when $\mathscr{D}(V, W)$ is the relevent progenerator. We examine the question "when does the Morita context

$$\begin{pmatrix} \mathscr{D}(V) & \mathscr{D}(W, V) \\ \mathscr{D}(V, W) & \mathscr{D}(W) \end{pmatrix} \subseteq M_2(\mathscr{D}_Q(Q))$$

give rise to a Morita equivalence between $\mathscr{D}(V)$ and $\mathscr{D}(W)$?" This problem can be reduced to one phrased in terms of the associated diagonals of V and W as follows.

Theorem. *Suppose V and W are dense subspaces of R with corresponding diagonals $\mathscr{V}$ and $\mathscr{W}$. Consider the following statements:*

(1) *The canonical maps*

$$\frac{\mathscr{D}(V, W)}{J(V, W)} \times \frac{\mathscr{D}(W, V)}{J(W, V)} \longrightarrow \frac{\mathscr{D}(W)}{J(W)}$$

and

$$\frac{\mathscr{D}(W, V)}{J(W, V)} \times \frac{\mathscr{D}(V, W)}{J(V, W)} \longrightarrow \frac{\mathscr{D}(V)}{J(V)}$$

are surjective.

(2) $\mathscr{D}(V, W)\mathscr{D}(W, V) = \mathscr{D}(W)$ *and* $\mathscr{D}(W, V)\mathscr{D}(V, W) = \mathscr{D}(V)$.

(3) $\mathscr{E}(\mathscr{V}, \mathscr{W})\mathscr{E}(\mathscr{W}, \mathscr{V}) = \mathscr{E}(\mathscr{W})$ *and* $\mathscr{E}(\mathscr{W}, \mathscr{V})\mathscr{E}(\mathscr{V}, \mathscr{W}) = \mathscr{E}(\mathscr{V})$.

(4) $\mathscr{D}(V)$ *are* $\mathscr{D}(W)$ *are Morita equivalent.*

(5) $\mathscr{D}(V)/J(V)$ *and* $\mathscr{D}(W)/J(W)$ *are Morita equivalent.*

(1), (2), *and* (3) *are equivalent. Either implies* (4) *and* (4) *implies* (5).

Proof. The equivalence of (1), (2), and (3) stems immediately from the remarks of §3.1. The rest is routine Morita theory.

3.3. **Morita theory for simple $\mathscr{D}(V)$.**

Corollary. *Suppose V is a dense subspace of R; then the following are equivalent:*

(1) *V is primary decomposable.*

(2) *$\mathscr{D}(V)$ is Morita equivalent to $\mathscr{D}(W)$, the equivalence being induced by the progenerator $\mathscr{D}(V, W)_{\mathscr{D}(V)}$, for every primary decomposable subspace $W \subseteq R$.*

(3) *$\mathscr{D}(V)$ is Morita equivalent to $\mathscr{D}(W)$, for some primary decomposable subspace $W \subseteq R$.*

(4) *$\mathscr{D}(V)$ is a simple ring.*

(5) *$\mathscr{D}(W, V) \cdot W = V$ for every primary decomposable subspace $W \subseteq R$.*

(6) *$\mathscr{D}(W, V) \cdot W = V$ for some primary decomposable subspace $W \subseteq R$.*

Proof. Suppose V and W are primary decomposable subspaces of R. Then [CH2, Proposition 4.4] asserts that $\mathscr{D}(V)$ and $\mathscr{D}(W)$ are simple rings. Hence

$$\mathscr{D}(V, W)\mathscr{D}(W, V) = \mathscr{D}(W) \quad \text{and} \quad \mathscr{D}(W, V)\mathscr{D}(V, W) = \mathscr{D}(V),$$

since in each case the left-hand side is nonzero (for example, it contains a nonzero ideal of R). Thus $\mathscr{D}(V)$ and $\mathscr{D}(W)$ are Morita equivalent, or (1) implies (2). A priori (2) implies (3).

(3) implies that $\mathscr{D}(V)$ is Morita equivalent to a simple ring. In particular (3) implies (4).

Suppose $\mathscr{D}(V)$ is a simple ring and W is primary decomposable. We have

$$0 \neq \operatorname{Ann}_{\mathscr{D}(V)}(V/\mathscr{D}(W, V) \cdot W) \subseteq \mathscr{D}(V)$$

where the inequality occurs because the annihilator contains a nonzero ideal of R. Thus $\operatorname{Ann}_{\mathscr{D}(V)}(V/\mathscr{D}(W, V) \cdot W) = \mathscr{D}(V)$ and hence $\mathscr{D}(W, V) \cdot W = V$, proving that (4) implies (5). A priori (5) implies (6).

Finally suppose W is a primary decomposable subspace of R. Recall that $\mathscr{D}(W, V) = J(W, V) = \mathscr{D}(W, V^-)$ and hence $\mathscr{D}(W, V) * V \subseteq V^-$. Thus (6) implies (1) and the cycle is complete.

Recall (from [CH2]) that a dense subalgebra of R is primary decomposable if and only if the normalization map is injective. Thus, specialising the above result to dense subalgebras, we obtain a new proof of a theorem of [SS].

Theorem: Smith and Stafford. *Suppose S_1, S_2 are dense subalgebras of R such that the normalization map $\operatorname{Max} R \longrightarrow \operatorname{Max} S_1$ is injective. Then $\mathscr{D}(S_1)$ and $\mathscr{D}(S_2)$ are Morita equivalent if and only if the normalization map $\operatorname{Max} R \longrightarrow \operatorname{Max} S_2$ is injective. When this is the case $\mathscr{D}(S_1, S_2)_{\mathscr{D}(S_1)}$ is the progenerator giving rise to the Morita equivalence.*

3.4. **Morita theory for simple factors and two branch singularities.** The main idea behind the theorem of §3.2 is that one can lift Morita equivalences from the level of the finite-dimensional factors. Its corollaries may be interpreted as achieving this when the finite-dimensional factor is zero. This subsection deals with the case where the factor is simple artinian.

Theorem. *Let V and W be dense subspaces of R. Suppose that for each $m \in \max R$ either $V^+(m) = V^-(m)$ and $W^+(m) = W^-(m)$ or $\dim V/V^- = \dim V^+(m)/V^-(m)$ and $\dim W/W^- = \dim W^+(m)/W^-(m)$. Then $\mathscr{D}(V)$ and $\mathscr{D}(W)$ have simple finite-dimensional factors and are Morita equivalent via the progenerator $\mathscr{D}(V, W)_{\mathscr{D}(V)}$.*

Proof. The hypotheses ensure that

$$\begin{pmatrix} \mathscr{E}(\mathscr{V}) & \mathscr{E}(\mathscr{W}, \mathscr{V}) \\ \mathscr{E}(\mathscr{V}, \mathscr{W}) & \mathscr{E}(\mathscr{W}) \end{pmatrix} = \begin{pmatrix} \operatorname{End}(V/V^-) & \operatorname{Hom}(W/W^-, V/V^-) \\ \operatorname{Hom}(V/V^-, W/W^-) & \operatorname{End}(W/W^-) \end{pmatrix}.$$

The result is now clear by the theorem of §3.2.

Corollary. *Suppose S_1 and S_2 are subalgebras of R with integral closure R, and let $\psi_i : \operatorname{Max} R \to \operatorname{Max} S_i$ for $i = 1, 2$ be their normalisation maps. Suppose that there is a bijection $\alpha : \operatorname{Max} S_1 \to \operatorname{Max} S_2$ such that*

$$\begin{array}{ccc} \operatorname{Max} R & = & \operatorname{Max} R \\ {\scriptstyle \psi_1}\downarrow & & {\scriptstyle \psi_2}\downarrow \\ \operatorname{Max} S_1 & \xrightarrow{\alpha} & \operatorname{Max} S_2 \end{array}$$

commutes. Finally, suppose that $\mathscr{D}(S_1)/J(S_1)$ *and* $\mathscr{D}(S_2)/J(S_2)$ *are both nonzero and simple. Then* $\mathscr{D}(S_1)$ *and* $\mathscr{D}(S_2)$ *are Morita equivalent via the progenerator* $\mathscr{D}(S_1, S_2)$.

Proof. The diagonals $\mathscr{S}_1$ and $\mathscr{S}_2$ corresponding to S_1 and S_2 are each a power of $k_{\max R, T}$ for some fixed $T \subseteq \max R$. (In fact $T = \psi_i^{-1}(m_i)$, where m_i is the unique branched singular point of $\max S_i$ and with ψ_i being the appropriate normalization map.)

Corollary. (1) *Suppose* S_1 *is a subalgebra of* R *with integral closure* R *and the normalization map* $\psi_1 : \max R \longrightarrow \max S_1$ *is injective except at the points* x *and* y *of* $\max R$. *Then* $\mathscr{D}(S_1)/J(S_1) \cong \operatorname{End}(S_1/S_1^-)$.

(2) *If* S_2 *is a subring of* R *with integral closure* R *and the normalization map* $\psi_2 : \max R \longrightarrow \max S_2$ *is injective except at the two points* x *and* y *of* $\max R$, *then* $\mathscr{D}(S_1)$ *and* $\mathscr{D}(S_2)$ *are Morita equivalent via* $\mathscr{D}(S_1, S_2)$.

Proof. This is an immediate consequence of the last corollary.

Remark. The Morita class of the differential operators on an n-branch singularity is not determined by n whenever n is larger than 2. Morita classes of rings of differential operators on three branch singularities are briefly discussed in §3.7.

3.5. **Upper triangular factors.** Suppose V and W are dense subspaces and

$$\{m \in \max R : V^-(m) \neq V^+(m)\}$$
$$= \{m_1, \dots, m_n\} = \{m \in \max R : W^-(m) \neq W^+(m)\}.$$

Suppose $V \cap V^-(m_1) \subseteq \cdots \subseteq V \cap V^-(m_n)$ and $W \cap W^-(m_1) \subseteq \cdots \subseteq W \cap W^-(m_n)$ with $V \cap V^-(m_i) = V \cap V^+(m_i)$ if and only if $W \cap W^-(m_i) = W \cap W^+(m_i)$ for each i. Then $\mathscr{D}(V)$ and $\mathscr{D}(W)$ are block upper triangular and are Morita equivalent via $\mathscr{D}(V, W)$.

We leave the proof as an easy exercise.

3.6. It is possible to go yet further and model essentially all Morita equivalences of finite-dimensional algebras using differential operators.

Theorem. *Suppose* A *and* B *are finite-dimensional algebras which are Morita equivalent via* P_A *a progenerator. Then we may choose finite diagonals* $\mathscr{V}$, $\mathscr{W}$ *such that* $A \cong \operatorname{Mor}(\mathscr{V}, \mathscr{V})$ *and* $B \cong \operatorname{Mor}(\mathscr{W}, \mathscr{W})$ *and after identification* $\operatorname{Mor}(\mathscr{V}, \mathscr{V})$ *and* $\operatorname{Mor}(\mathscr{W}, \mathscr{W})$ *are Morita equivalent via* ${}_B\operatorname{Mor}(\mathscr{V}, \mathscr{W})_A \cong {}_BP_A$.

Proof. Choose $\mathscr{U}$ a diagonal such that

$$T := \begin{pmatrix} A & P^* \\ P & B \end{pmatrix} \cong \operatorname{Mor}(\mathscr{U}, \mathscr{U}).$$

If $1 = i_A + i_B$ where i_A, $i_B \in T$ are the idempotents such that $i_A T i_A = A$ and $i_B T i_B = B$, then $\mathscr{U} = \mathscr{V} \oplus \mathscr{W}$ where $\mathscr{V} = i_A \mathscr{U}$ and $\mathscr{W} = i_B \mathscr{U}$. $\mathscr{V}$ and $\mathscr{W}$ have the required properties.

For the remainder of this section suppose that $\max R$ is infinite.

Corollary. *Suppose* A_1 *and* A_2 *are finite-dimensional algebras which are Morita equivalent and* n_1, n_2 *are positive integers. Then there are algebras* S_1, S_2 *with* *integral closure* R *such that*

$$\mathscr{D}(S_i)/J(S_i) \cong \begin{pmatrix} B_i & * \\ 0 & M_{n_i}(k) \end{pmatrix}$$

where B_i *is a subalgebra of a matrix algebra isomorphic to* A_i. *Further* $\mathscr{D}(S_1)$ *is Morita equivalent to* $\mathscr{D}(S_2)$ *via* $\mathscr{D}(S_1, S_2)$.

Proof. The proof is similar to that of the corollary of §2.2.

3.7. **Morita classes of differential operators on three-branch singularities.** Suppose m_1, m_2, m_3 are distinct maximal ideals of R. In view of §§1.5, 2.1, and 3.2 we may obtain good information about the number of Morita classes of $\mathscr{D}(S)$ where S is a subalgebra of R with integral closure R such that the normalization map identifies m_1, m_2, and m_3 and is injective at all other points. If $\mathscr{V}$ and $\mathscr{W}$ have the same indecomposable summands (albeit with different multiplicities), then $\operatorname{Mor}(\mathscr{V}, \mathscr{V})$ and $\operatorname{Mor}(\mathscr{W}, \mathscr{W})$ are Morita equivalent via $\operatorname{Mor}(\mathscr{V}, \mathscr{W})$. By §1.5 there are at most 2^4 such classes since $k_{\max R, \{m_1, m_2, m_3\}}$ must appear and each of the other four indecomposables may or may not appear. Further analysis of §1.5 reveals the following result.

Theorem. (a) *There are eight Morita classes of* $\mathscr{D}(S)/J(S)$. *Their basic algebras are* $\mathscr{E}_{11111}$, $\mathscr{E}_{11110} \cong \mathscr{E}_{11101} \cong \mathscr{E}_{11011}$, $\mathscr{E}_{11100} \cong \mathscr{E}_{11010} \cong \mathscr{E}_{11001}$, $\mathscr{E}_{11000}$, $\mathscr{E}_{10111}$, $\mathscr{E}_{10110} \cong \mathscr{E}_{10101} \cong \mathscr{E}_{10011}$, $\mathscr{E}_{10100} \cong \mathscr{E}_{10010} \cong \mathscr{E}_{10001}$ *and* $\mathscr{E}_{10000}$.

(b) *There are at most sixteen Morita classes of* $\mathscr{D}(S)$.

The precise number of Morita classes of $\mathscr{D}(S)$ presumably depends on R. In view of §§1.5, 3.2 and 2.1, there are infinitely many Morita classes of rings of differential operators on curves with an n-branch singularity whenever $n \geq 4$.

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