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GROUPS WITH NO FREE SUBSEMIGROUPS

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ABSTRACT. We look at groups which have no (nonabelian) free subsemigroups. It is known that a finitely generated solvable group with no free subsemigroup is nilpotent-by-finite. Conversely nilpotent-by-finite groups have no free subsemigroups. Torsion-free residually finite-p groups with no free subsemigroups can have very complicated structure, but with some extra condition on the subsemigroups of such a group one obtains satisfactory results. These results are applied to ordered groups, right-ordered groups, and locally indicable groups.

1. INTRODUCTION

Let G be a group, and for any pair (a, b) of elements in G, let $S(a, b)$ denote the subsemigroup generated by a and b. We investigate properties of groups G which contain no free subsemigroup on two generators. In other words, for every pair (a, b) of elements of G, $S(a, b)$ has a relation of the form

$$a^{r_1}b^{s_1}\dots a^{r_j}b^{s_j} = b^{m_1}a^{n_1}\dots b^{m_k}a^{n_k} \tag{1}$$

where r_i, s_i, m_i, n_i are all nonnegative and r_1 and m_1 are positive integers. We shall call G a group without free subsemigroups if it has no free nonabelian subsemigroups; thus taking "free" to mean "free nonabelian." Clearly G has no free subsemigroups if and only if no two generator subgroups of G have free subsemigroup. For this reason there is no loss of generality in assuming that G is finitely generated. Our first result is the following.

Theorem 1. *Let G be a finitely generated solvable group. Then G has no free nonabelian subsemigroups if and only if it is nilpotent by finite.*

It is well known that $S(a, b)$ is not a free subsemigroup if $\langle a, b\rangle$ is a nilpotent group. In [13] Shalev showed that if $\langle a, b\rangle$ is nilpotent of class c, then it satisfies the law $u_c = v_c$ where the words $\{u_i\}$, $\{v_i\}$ on letters a, b are defined as follows: $u_0 = a$, $v_0 = b$, and for $i \geq 0$, $u_{i+1} = u_i v_i$ and $v_{i+1} = v_i u_i$. Thus if $G = \langle x, y\rangle$ is a periodic extension of a locally nilpotent group and a, b are elements in G, then $\langle a^n, b^n\rangle$ is nilpotent for some positive integer n and, hence, satisfies the law $u_c = v_c$ for some c and $\langle x, y\rangle$ does not have a free subsemigroup. The converse is not likely to be true; but with no example known to substantiate this, we leave it as an open question.

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QUESTION 1. Let $G = \langle x, y \rangle$ be a group with no free subsemigroups. Is G a periodic extension of a locally nilpotent group?

Even under additional conditions on a group G with no free subsemigroups, the structure of G can be quite complicated. Let p be a prime, F the free group of rank two, and F/R isomorphic to the Gupta-Sidki p-group constructed in [7]. Then F/R is an infinite, residually finite p-group. Thus $G = F/R'$ is a residually torsion-free solvable group. It is also a residually finite p-group. For all pairs (a, b) of elements in G, there is a relation of type (1) with $j = k = 1$. But G is not nilpotent-by-finite. In [5] Grigorchuk constructed interesting examples of finitely generated torsion-free groups of subexponential growth which are not nilpotent-by-finite. These groups, like the group $G = F/R'$ described above, are also abelian-by-periodic.

If (a, b) is a pair of elements in G satisfying a relation of type (1), then we call $j+k$ the width of the relation and the sum $r_1 + \cdots + r_j + n_1 + \cdots + n_k$ the exponent of a or $\exp(a)$ in the relation.

Theorem 2. *Suppose G is a group and there is a bound N such that for all pairs (a, b) of elements in G there is a relation of the form (1) whose width is at most N. Then G is nilpotent if it is residually torsion-free nilpotent.*

Note that the group $G = F/R'$ quoted above shows that the condition "residually torsion-free nilpotent" cannot be weakened in Theorem 2. If one looks at groups G where there is a bound N such that for all ordered pairs (a, b) of elements of G there is a relation of the form (1) where $\exp(a)$ is at most N, then one can say more about G as the next result shows.

Theorem 3. *Suppose G is a locally indicable group and there is a bound N such that for all ordered pairs (a, b) of elements of G there is a relation of the form (1) where $\exp(a)$ is at most N. Then G is locally nilpotent-by finite.*

One place where the knowledge that G has no free subsemigroup has immediate application is when G is an orderable (O) group or a right orderable (RO) group. We refer the reader to [2] or [4] for basic results and terminology that we use. Recall that G is orderable if there exists a total order relation $\geq$ on G such that for all a, b, h, g in G, $a \geq b$ implies $hag \geq hbg$; equivalently, if there exists a normal subset P in G such that $PP = P$, $P \cup P^{-1} = G$, and $P \cap P^{-1} = \{e\}$. G is right orderable if there exists a total order relation $\geq$ on G such that for all a, b, g in G, $a \geq b$ implies $ag \geq bg$, equivalently, if there exists a subset P in G such that $PP = P$, $P \cup P^{-1} = G$, and $P \cap P^{-1} = \{e\}$. We shall show that if G is orderable and has no free subsemigroup on two generators, then all the convex subgroups are normal in G under any order on G. And if G is right orderable and has no free subsemigroup on two generators, then under any right order on G the set of convex subgroups form a series with torsion-free abelian factors; and, in particular, G is locally indicable. Our result thus extends the well-known result that nilpotent-by-finite right orderable groups are locally indicable. It also extends a recent result of Kropholler in [10] that the convex subgroups of a right-ordered supramenable group form a series with torsion-free abelian factors. This follows since supramenable groups contain no free semigroups [14, p. 189]. As corollaries of Theorems 2 and 3, we get

Theorem 4. *If G is an O-group and there is a bound N such that for all pairs (a, b) of elements in G there is a relation of the form* (1) *whose width is at most N, then G is nilpotent of class bounded by N.*

Theorem 5. *If G is an RO-group and there is a bound N such that for all ordered pairs (a, b) of elements of G there is a relation of the form* (1) *where* $\exp(a)$ *is at most N, then G is locally nilpotent-by finite.*

Recently Grigorchuk and Machi showed in [6] that the torsion-free groups of subexponential growth constructed by Grigorchuk in [5] that we referred to earlier are also right orderable. Thus a finitely generated RO-group of subexponential growth need not be nilpotent-by-finite. We do not know if a finitely generated O-group of subexponential growth must be nilpotent.

We thank Dr. Shirvani for pointing out to us that Theorem 1 was proved by Rosenblatt in [12]. His proof depends heavily on the works of Wolf [15] and Milnor [11]; our proof is short and direct. For this reason we have included the proof in this paper. There is a close similarity between our proof of Lemma 4 and that of Lemma 2 by Bass in [1].

2. Proofs

Lemma 1. *If G has no free subsemigroups, then for all a, b in G, $\langle a^{\langle b\rangle}\rangle$ is finitely generated.*

Proof. Consider the semigroup $S(b, b^a)$ generated by b and b^a. By hypothesis,

$$b^{r_1}(b^a)^{s_1}\dots b^{r_j}(b^a)^{s_j} = (b^a)^{m_1}b^{n_1}\dots(b^a)^{m_k}b^{n_k},$$

where r_i, s_i, m_i, n_i are nonnegative integers and r_1 and m_1 are positive. Hence

$$(a^{-1})^{b^{\lambda_1}}a^{b^{\lambda_2}}(a^{-1})^{b^{\lambda_3}}\dots a^{b^{\lambda_u}}b^{-\lambda_u} = (a^{-1})a^{b^{\mu_2}}(a^{-1})^{b^{\mu_3}}\dots a^{b^{\mu_{v-1}}}b^{-\mu_v}$$

where $\lambda_u < \cdots < \lambda_1 < 0$ and $\mu_v < \cdots < \mu_2 < 0$. Let $\lambda = \lambda_u$, $\mu = \mu_v$. If $\lambda \neq \mu$, then $b^{\lambda-\mu} \in \langle a^{\langle b\rangle}\rangle$, which is then finitely generated; and we are done. So assume $\lambda = \mu$. Then $a \in \langle a^{b^{-1}}, \dots, a^{b^{\lambda}}\rangle$. By replacing b with b^{-1} we similarly get $a \in \langle a^b, \dots, a^{b^{\nu}}\rangle$ for some $\nu > 0$. Thus $\langle a^{b^{\nu}}, \dots a^b, a, a^{b^{-1}}, \dots, a^{b^{\lambda}}\rangle = \langle a^{\langle b\rangle}\rangle$.

The next result appears in [9], but we include the proof here since it is very short.

Lemma 2. *Let G be a finitely generated group. If $H \lhd G$, G/H is cyclic, and $\langle a^{\langle b\rangle}\rangle$ is finitely generated for all a, b in G, then H is finitely generated.*

Proof. For some $g \in G$ we can write $G = H\langle g\rangle$. Since G is finitely generated, there exist $h_1, \dots, h_r$ in H such that $G = \langle h_1, \dots, h_r, g\rangle$ and $H = \langle h_1, \dots, h_r\rangle^G$. For each $i = 1, \dots, r$, $\langle h_i^{\langle g\rangle}\rangle$ is finitely generated, say, $\langle h_i^{\langle g\rangle}\rangle = \langle h_{i1}, \dots, h_{id(i)}\rangle$. Then $H = \langle h_{i\ell(i)};\ 1 \le i \le r,\ 1 \le \ell(i) \le d(i)\rangle$.

Corollary 3. *Let G be a finitely generated group with no free subsemigroups. Then for every positive integer n, the nth derived subgroup $G^{(n)}$ is finitely generated. In particular if G is solvable, then it is polycyclic.*

Proof. This follows directly from Lemma 1 and Lemma 2.

Lemma 4. *Let $G = A \rtimes T$, the split extension of a finitely generated torsion-free abelian group A by infinite cyclic group $T = \langle t\rangle$. If T acts rationally irreducibly on A and G has no free subsemigroups, then G is abelian-by-finite.*

Proof. Let $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$. Then V is an irreducible $\mathbb{Q}T$-module and by Schur's Lemma, $D = \operatorname{End}_{\mathbb{Q}T} V$ is a division ring of finite dimension over $\mathbb{Q}$. Now the image of T in $\operatorname{End}_{\mathbb{Q}} V$ lies in D and generates D. Hence D is an algebraic number field. As a D-space, V is one dimensional. Let α be the image of t in D. Then we can identify V with $\mathbb{Q}(\alpha)$ under addition and the action of t on V being that of multiplication by α. If α is a root of 1, then t^n acts trivially on V and hence the subgroup $\langle A, t^n \rangle$ is abelian of finite index in G. If a is not a root of unity, then D can be embedded in $\mathbb{C}$ so that $|\alpha| < 1$ (see [8, p. 102]). By taking a power of α, if necessary, we may assume that $|\alpha| < \frac{1}{4}$.

Take any $b \neq e$ in A and consider the semigroup $S(t, t^b)$. By hypothesis there exist positive integers $p, q, r_1, \ldots, r_p, s_1, \ldots, s_{p-1}, u_2, \ldots, u_{q-1}, v_1, \ldots, v_q$ and s_p, u_q nonnegative such that

$$t^{r_1}(t^b)^{s_1} \ldots t^{r_p}(t^b)^{s_p} = (t^b)^{v_1} t^{u_2} \ldots (t^b)^{v_q} t^{u_q}.$$

Note that $\sum_{i=1}^{p} r_i + \sum_{i=1}^{p} s_i = \sum_{i=2}^{q} u_i + \sum_{i=1}^{q} v_i$ since G/A is infinite. If β corresponds to b in the isomorphism of V and $\mathbb{Q}(\alpha)$, then the above equality translates into

$$\beta \sum_{i=1}^{j} \pm\alpha^{\lambda_i} = \beta \sum_{i=1}^{k} \pm\alpha^{\mu_i}$$

where $0 < \lambda_1 < \cdots < \lambda_j$; $0 = \mu_1 < \mu_2 < \cdots < \mu_k$, and j, k are some positive integers. Since $|\alpha| < \frac{1}{4}$, $|\sum_{i=1}^{j} \pm \alpha^{\lambda_i}| \leq \sum_{i=1}^{j} (\frac{1}{4})^{\lambda_i} < \frac{1}{2}$. On the other side, $|\sum_{i=1}^{k} \pm \alpha^{\mu_i}| \geq 1 - \sum_{i=2}^{k} \frac{1}{4}^{\mu_i} > \frac{1}{2}$, giving a contradiction. Thus G is abelian-by-finite.

Proof of Theorem 1. Let G be a finitely generated solvable group with no free subsemigroup. We use induction on the solvability length of G to show that G is nilpotent-by-finite. Clearly there is nothing to prove if G is abelian. Hence, using induction, we may assume that G is abelian-by-nilpotent-by-finite. Taking a subgroup of finite index in G, if necessary, we may assume that G is abelian-by-nilpotent. By Corollary 3, we know that G is polycyclic. Thus, again passing to a subgroup of finite index, if necessary, we may assume that G has a finitely generated torsion-free normal abelian subgroup A and G/A is torsion-free nilpotent. Now, there is a central series $A = A_0 \lhd A_1 \lhd \ldots \lhd A_s = G$ from A to G with infinite cyclic factors. Say $A_i = A_{i-1}\langle t_i \rangle$, $i = 1, \ldots, s$. It suffices to show that $\langle A, t_i^{n_i} \rangle$ is nilpotent for some $n_i > 0$; for then $\langle A, t_1^{n_1}, \ldots, t_s^{n_s} \rangle$ is nilpotent and of finite index in G, as is required to show.

In order to show that $\langle A, t_i^{n_i} \rangle$ is nilpotent for some $n_i > 0$, consider the series $1 = A_{i0} \lhd \ldots \lhd A_{im(i)} = A$ where A_{ij} are isolated subgroups of A, normalized by t_i, and A_{ij+1}/A_{ij} is of minimal rank. Apply Lemma 4 to $\langle A_{ij+1}, t_i \rangle / A_{ij}$ to get $[A_{ij+1}, t_i^{n_i}] \leq A_{ij}$ for some $n_i > 0$ and all $j = 0, \ldots, m(i)$. Then $\langle A, t_i^{n_i} \rangle$ is nilpotent of class at most $m(i)$. This completes the proof.

Proposition 5. *Suppose that G is a torsion-free nilpotent group. If there is a bound N such that for all pairs (a, b) of elements in G there is a relation of* *the form* (1) *whose width is at most* N, *then* G *is nilpotent whose class is bounded by a function of* N.

Proof. Let G be nilpotent of class c. Then $\gamma_{[c/2]}(G)$ is abelian, where $[c/2]$ equals $c/2$ if c is even and $c+1/2$ if c is odd. Let A denote the isolator of $\gamma_{[c/2]}(G)$. Then A is also abelian since G is torsion-free. Now, for any $a \in A$ and $g \in G$, consider $S(g, a)$ to obtain the equality

$$a^{r_1} g^{s_1} \dots a^{r_j} g^{s_j} = g^{m_1} a^{n_1} \dots g^{m_k} a^{n_k} \tag{2}$$

where $j+k \leq N$. We treat A as a $\mathbb{Z}\langle g\rangle$-module and show that $A(g-1)^N = 0$. If $g \in A$, then $A(g-1) = 0$ and we are done. So assume $g \notin A$. Now the relation (2) yields

$$a \sum_{i=1}^{j} r_i g^{\mu_i} = a \sum_{i=1}^{k} n_i g^{\lambda_i}$$

where $0 = \mu_1 < \mu_2 < \dots < \mu_j$ and $0 < \lambda_1 < \dots < \lambda_k$. Let $\sum_{i=1}^{j} r_i x^{\mu_i} - \sum_{i=1}^{k} n_i x^{\lambda_i} = q(x)$, and let $A_1 = A \otimes_{\mathbb{Z}} \mathbb{Q}$. Treat g as an operator on A_1 to get $aq(g) = 0$. Since $\langle A_1, g\rangle$ is also nilpotent of class at most c, $(g-1)^c$ annihilates a as does $q(g)$. Now if $(x-1)^e$ divides $q(x)$, then $e \leq N$ for we have $q(1) = 0$, $q'(1) = 0, \dots, q^{e-1}(1) = 0$ where $q(x) = \sum_{i=1}^{t} c_i x^{\nu_i}$ where $t \leq N$, $c_i \neq 0$. Then

$$\begin{pmatrix} 1 & \cdots & 1 \\ \nu_1 & \cdots & \nu_t \\ \nu_1^2 & \cdots & \nu_t^2 \\ & \vdots & \\ \nu_1^{e-1} & \cdots & \nu_t^{e-1} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If $e \geq N$, then the only solution to the above system is $c_i = 0$ for all $i = 1, \dots, t$ since $t \leq N$. This, in turn, would imply that $q(x) = 0$, a contradiction. Thus $a(g-1)^N = 0$ for all $a \in A$, $g \in G$. Reverting to the multiplicative notation of the group G, we have

$$[A, \underbrace{g, \dots, g}_{N}] = 1.$$

Since G is torsion-free, it follows from a result of Zelmanov (see [16, p. 166]) that A lies in $\zeta_{f(N)}(G)$, the $f(N)$th center of G, where $f(N)$ is a function of N and independent of the number of generators of G. Thus the nilpotency class of G is at most $[c/2] + f(N)$ and hence $c \leq 2f(N)$.

It is worth noting here that the method applied in the proof of Proposition 5 is general enough to be useful in other situations. We mention one such case. Suppose G is a torsion-free locally solvable group and, for some fixed positive integer N, $\langle x, y\rangle = (\langle x\rangle \langle y\rangle)^N$ for all x, y in G. Then G is nilpotent of class bounded by a function of N.

Proof of Theorem 2. Since G is residually torsion-free nilpotent, there exists a descending central series $G = G_0 > G_1 > \cdots$ where $\bigcap_{i=1}^{\infty} G_i = 1$ and G/G_i is torsion-free nilpotent for all i. By Proposition 5, there is some integer $f(N)$ such that $\gamma_{f(N)+1}(G) \subseteq G_i$ for all i and hence $\gamma_{f(N)+1}(G) = 1$.

Lemma 6. *If A is a $\mathbb{Z}$-torsion-free $\mathbb{Z}\langle g\rangle$-module, $q(x) \in \mathbb{Z}[x]$ is a polynomial of degree N, and for some $a \in A$, $a(q(g)) = a(g^n - 1) = 0$ for some $n > 0$, then there exists a positive integer $m = m(N)$, independent of n, such that $a(g^m - 1) = 0$.*

Proof. Consider $q(x)$ and $x^n - 1$ as elements of $\mathbb{Q}[x]$. Since $x^n - 1 = \prod_{d|n} \theta_d(x)$, where $\theta_d(x)$ are the dth cyclotomic polynomials of degree $\psi(d)$ which are irreducible in $\mathbb{Q}[x]$, the greatest common divisor $(q(x), x^n - 1) = \theta_{d_1}(x) \dots \theta_{d_r}(x)$ for some $d_i | n$. Now there are only finitely many integers d_i such that $\psi(d_i) \leq N$. Let $d(x)$ be the product of these $\theta_{d_i}(x)$, and let m be the least integer such that $d(x)$ divides $x^m - 1$. Then for some positive integer s, $a(s(g^m - 1)) = 0$, since A is a $\mathbb{Z}\langle g\rangle$-module rather than $\mathbb{Q}\langle g\rangle$-module. But since A is $\mathbb{Z}$-torsion-free, $s(a(g^m - 1)) = 0$ implies $a(g^m - 1) = 0$. This completes the proof.

Proposition 7. *Suppose that G is a torsion-free solvable group. If there exists a bound N such that for all ordered pairs (a, b) of elements in G there is a relation of the form (1) in which $\exp(a)$ is at most N, then the exponent of $G/\mathrm{Fitt}(G)$ and the nilpotency class of $\mathrm{Fitt}(G)$ are bounded by a function of N.*

Proof. There is no loss of generality in assuming that G is finitely generated since we shall show that $G/\mathrm{Fitt}(G)$ has bounded exponent and the nilpotency class of $\mathrm{Fitt}(G)$ is bounded. The bound on $\mathrm{Fitt}(G)$ is obtained from Proposition 5. By Theorem 1, G is nilpotent-by-finite. Let A be a normal abelian subgroup of G. Then for any $a \in A$ and $g \in G$ consider the ordered pair (g, a) to obtain the equality

$$a^{r_1} g^{s_1} \dots a^{r_j} g^{s_j} = g^{m_1} a^{n_1} \dots g^{m_k} a^{n_k} \tag{3}$$

where $\sum_{i=1}^{j} s_i + \sum_{i=1}^{k} m_i \leq N$. We consider A as $\mathbb{Z}\langle g\rangle$-module and show that $a(g^m - 1) = 0$ for some $m = m(N)$. Equation (3) in additive notation yields

$$a \sum_{i=1}^{j} r_i g^{\mu_i} = a \sum_{i=1}^{k} n_i g^{\lambda_i}$$

where $0 = \mu_1 < \mu_2 < \cdots < \mu_j$; $0 < \lambda_1 < \cdots < \lambda_k$, and $\mu_j + \lambda_k \leq N$. Let $q(x) = \sum_{i=1}^{j} r_i x^{\mu_i} - \sum_{i=1}^{k} n_i x^{\lambda_i}$ so that $a \cdot q(g) = 0$. Since G is nilpotent-by-finite, $a(g^n - 1)^c = 0$ for some positive integers n and c. If $c = 1$, then by Lemma 6, $a(g^m - 1) = 0$ where m depends only on N. If $c > 1$, then replace a by $a(g^n - 1)^{c-1}$ to get $a(g^n - 1)^{c-1}(g^m - 1) = 0$ and hence $a(g^m - 1)(g^n - 1)^{c-1} = 0$. Use induction on c to get $a(g^m - 1)^c = 0$.

Let $F = \mathrm{Fitt}(G)$. We show that $G^m \subseteq F$. Since G is nilpotent-by-finite by Theorem 1, F is finitely generated so that the terms of the upper central series of F are all finitely generated isolated subgroups of F. Let Z_i denote the ith center of F. Then $Z_i(g^m - 1)^{c_i} \subseteq Z_{i-1}$ where $c_i \in \mathbb{N}$. Thus F lies in some term of the upper central series of FG^m. But if $F \neq FG^m$, then let B/F be a nontrivial normal abelian subgroup of FG^m/F. Then $B \subseteq \mathrm{Fitt}(G) = F$, a contradiction. Now apply Theorem 2 to F to get a bound on the nilpotency class of F.

Proof of Theorem 3. We assume that G is finitely generated and show that it is nilpotent-by-finite. Let R be the torsion-free solvable residual of G. Then $R = \bigcap_{i=1}^{\infty} G_i$ where G/G_i is torsion-free solvable. By Proposition 7, G_iG^m/G_i is nilpotent of class n with n and m depending only on N. Thus G/R is solvable and by Corollary 3, R is finitely generated. If $R \neq 1$, then, by hypothesis, R/R' is infinite so that G/J is torsion-free solvable where J is the isolator of R' in R, and $J \neq R$, which is a contradiction.

Lemma 8. *Let $\geq$ be a (two-sided) total order on an O-group G. If G has no free subsemigroups, then the convex subgroups under $\geq$ are normal in G.*

Proof. Let C be a convex subgroup under $\geq$, and suppose that $C^g \neq C$ for some $g \in G$. We may assume that $C^g \supseteq C$ since convex subgroups are nested. Thus there exists a in C such that a^g is not in C. But then $a^{g^{n+1}}$ is not in C^{g^n} for all $n \geq 0$. But $\langle a^{\langle g \rangle} \rangle$ is finitely generated by Lemma 1. Thus $a^{g^{n+1}} \in \langle a, a^g, \dots, a^{g^n} \rangle$ for some n, and hence $a^{g^{n+1}} \in C^{g^n}$, a contradiction.

Proof of Theorem 4. Let $\geq$ be a total order on G. Let $C \rightarrowtail D$ be a jump in the set of convex subgroups of G under $\geq$; then D/C is order isomorphic to a subgroup of the additive group of reals and every g in G induces an order-preserving automorphism of D/C (see [4, p. 50]). Since the order-preserving automorphisms of an Archimedean-ordered group form a subgroup of the multiplicative group of positive reals, it follows that the centralizer $C_G(D/C)$ of D/C in G contains G' and is an isolated subgroup of G. Let J be the isolator of G' in G. Then $[D, J] \subseteq C$.

Since we shall show that G is nilpotent of class bounded by N, independent of the number of generators of G, it suffices to assume that G is finitely generated. Then by Corollary 3, J is also finitely generated. Now order J by taking the restriction of order $\geq$. Then the convex subgroups of J are $C \cap J$, where C is convex in G under $\geq$. This order on J is a G-order in that the positive cone is invariant under conjugation by elements of G. We use this order on J and extend it to an order on G by making $J \rightarrowtail G$ a convex jump. Since J is finitely generated, there is $J_1 \leq G$ such that $J_1 \rightarrowtail J$. (J_1 is simply the largest convex subgroup of G that does not contain the finite set that generates J.) Similarly J_1 is finitely generated by Corollary 3 and there is a jump $J_2 \rightarrowtail J_1$.

Continue this process, and let $K = \bigcap_{i=1}^{\infty} J_i$.

Then J/K is residually torsion-free nilpotent and hence nilpotent by Theorem 2. Thus G/K is soluble and hence nilpotent-by-finite by Theorem 1. But a nilpotent-by-finite O-group is nilpotent.

Hence G/K is nilpotent and the nilpotency class is bounded by a function of N as in Proposition 5. Thus $K = J_m$ for some m and $J_m = 1$.

Lemma 9. *Let $\geq$ be a right order on an RO-group G, and let $P = \{g \in G;\ g > e\}$. If P has no free subsemigroup, then the set of convex subgroups under $\geq$ form series from $\{e\}$ to G with torsion-free abelian factors. In particular G is locally indicable.*

Proof. Let a, b be any elements in P. We first show that $a^n b > a$ for some positive integer n. If $b > a$, then $a^n b > b > a$ for all $n \geq 0$, so assume $a > b$. By hypothesis

$$a^{r_1} b^{s_1} \dots a^{r_j} b^{s_j} = b^{m_1} a^{n_1} \dots b^{m_k} a^{n_k}$$

for some non-negative integers r_i, s_i, m_i, n_i where s_j and n_k are positive. If $a > a^n b$ for all $n \geq 0$, then $a^{r_1} b^{s_1} = (a^{r_1} b) b^{s_1 - 1} < a b^{s_1 - 1} < \cdots < ab < a$. Continue in this fashion to get $a^{r_1} b^{s_1} \dots a^{r_j} b^{s_j} < a$. On the other side $b^{m_1} a^{n_1} \dots b^{m_k} \geq e$ so that $b^{m_1} a^{n_1} \dots b^{m_k} a^{n_k} \geq a^{n_k} \geq a$, resulting in a contradiction. Now a right order $\geq$ where for each ordered pair (a, b) of elements in P there exists some $n \geq 0$ such that $a^n b > a$ is called a C-order. It was shown by Conrad in [3] that if $\geq$ is a C-order, then the set of convex subgroups of G under $\geq$ forms a system in Malcev terminology (series in P. Hall terminology) with torsion-free abelian factors. Thus every nontrivial finitely generated subgroup of G has an infinite cyclic quotient and G is locally indicable.

Proof of Theorem 5. This follows from Lemma 3 and Theorem 3.

References

1. H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc. (3) **25** (1972), 603–614.
2. R. Botto Mura and A.H. Rhemtulla, *Orderable groups*, Dekker, 1977.
3. P.F. Conrad, *Right ordered groups*, Michigan Math. J. **6** (1959), 267–275.
4. L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, 1963.
5. R. I. Grigorchuk, *On the growth degrees of p-groups and torsion-free groups*, Math. Sb. **126** (1985), 194–214; English transl. Math. USSR-Sb. **54** (1986), 347–352.
6. R. I. Grigorchuk and A. Machi, *An intermediate growth automorphism group of the real line*, preprint.
7. N. Gupta and S. Sidki, *Some infinite p-groups*, Algebra i Logika **22** (1983), 584–589.
8. E. Hecke, *Lectures on the theory of algebraic numbers*, Translated by George U. Brauer and Jay R. Goldman, Springer-Verlag, 1981.
9. Y. K. Kim and A. H. Rhemtulla, *Weak maximality condition and polycyclic groups.*
10. P. H. Kropholler, *Amenability and right orderable groups*, Bull. London Math. Soc. **25** (1993), 347–352.
11. J. Milnor, *Growth of finitely generated solvable groups*, J. Differential Geom. **2** (1968), 447–449.
12. J. M. Rosenblatt, *Invariant measures and growth conditions*, Trans. Amer. Math. Soc. **197** (1974), 33–53.
13. A. Shalev, *Combinatorial conditions in residually finite groups*, II, J. Algebra **157** (1993), 51–62.
14. S. Wagon, *The Banach-Tarski paradox*, Cambridge University Press, 1985.

15. J. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. Differential Geom. **2** (1968), 421–446.

16. E. I. Zelmanov, *On some problems of group theory and lie algebras*, Math. USSR-Sb. **66** (1990), 159–168.

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