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SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS

WEIYU CHEN

ABSTRACT. Let S_n be the collection of all algebraic polynomials of degree $\leq n$ with nonnegative coefficients. In this paper we discuss the extremal problem

$$\sup_{p_n(x)\in S_n} \frac{\int_a^b (p_n'(x))^2\omega(x)\,dx}{\int_a^b p_n^2(x)\omega(x)\,dx}$$

where $\omega(x)$ is a positive and integrable function. This problem is solved completely in the cases

(i) $[a, b] = [-1, 1]$, $\omega(x) = (1-x^2)^\alpha$, $\alpha > -1$;

(ii) $[a, b) = [0, \infty)$, $\omega(x) = x^\alpha e^{-x}$, $\alpha > -1$;

(iii) $(a, b) = (-\infty, \infty)$, $\omega(x) = e^{-\alpha x^2}$, $\alpha > 0$.

The second case was solved by Varma for some values of α and by Milovanović completely. We provide a new proof here in this case.

1. INTRODUCTION

In this paper we investigate the following extremal problem

$$\sup_{p_n(x)\in S_n} \frac{\int_a^b (p_n'(x))^2\omega(x)\,dx}{\int_a^b p_n^2(x)\omega(x)\,dx} \tag{1}$$

where

$$S_n = \left\{ p_n(x) : p_n(x) = \sum_{i=0}^{n} a_i x^i,\ a_i \geq 0,\ 0 \leq i \leq n \right\},$$

and $\omega(x): (a, b) \to R$ is a positive and integrable function.

In the case $[a, b) = [0, \infty)$, $\omega(x) = x^\alpha e^{-x}$, $\alpha > -1$, the extremal problem (1) was initiated and solved by Varma [10] in the cases $0 \leq \alpha \leq 1/2$ and $(\sqrt{5}-1)/2 \leq \alpha < \infty$. Later, it was solved completely by Milovanović [4] for $-1 < \alpha < \infty$.

In this note we consider the above extremal problem (1) for different weight functions on different intervals. Throughout this paper, we denote S_n the collection of all algebraic polynomials of degree $\leq n$ with nonnegative coefficients. In Section 2, we provide the complete answer to the case $[a, b] = [-1, 1]$, $\omega(x) = (1-x^2)^\alpha$, $\alpha > -1$. In the case $\alpha = 0$, this result is an analogue of a

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theorem of Lorentz [3] in the L_∞ norm. Indeed, that theorem holds for a wider class (Lorentz class) of polynomials, which was studied extensively by Scheick [7]. For some subsets of Lorentz class of polynomials, the extremal problem (1) was discussed by Milovanović and Petković [5] for the Jacobi weight.

In Section 3, we give a new proof of Milovanović's Theorem [4]. In our last section, Section 4, we consider the weight function $\omega(x) = e^{-\alpha x^2}$, $\alpha > 0$, on the interval $(-\infty, \infty)$.

The corresponding extremal problem for the unrestricted polynomials was discussed in Dörfler [1], [2], Mirsky [6] and Turán [8], which are Markov type inequalities in L_2 norm.

2. The weight $\omega(x) = (1-x^2)^\alpha$

In this section, we discuss the extremal problem in the L_2 norm under the weight function $\omega(x) = (1-x^2)^\alpha$, $\alpha > -1$, on $[-1, 1]$. For some special values of α, we obtain several corollaries corresponding to some classic weight functions. The main result in this section is the following theorem.

Theorem 2.1. *Let $p_n(x) \in S_n$, $\alpha > -1$; then*

$$\int_{-1}^{1} (p_n'(x))^2 (1-x^2)^\alpha \, dx \le \frac{2n+2\alpha+1}{2n-1} n^2 \int_{-1}^{1} p_n^2(x)(1-x^2)^\alpha \, dx \tag{2}$$

with equality when $p_n(x) = x^n$.

Proof. Since $p_n(x) \in S_n$, we can write

$$p_n(x) = \sum_{i=0}^{n} a_i x^i$$

with $a_i \ge 0$, $0 \le i \le n$. Then

$$p_n'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$$

and

$$\int_{-1}^{1} p_n^2(x)(1-x^2)^\alpha dx = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \int_{-1}^{1} x^{i+j}(1-x^2)^\alpha dx,$$

$$\int_{-1}^{1} (p_n'(x))^2(1-x^2)^\alpha dx = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j ij \int_{-1}^{1} x^{i+j-2}(1-x^2)^\alpha dx.$$

Let

$$\begin{aligned} b_{ij} &= \int_{-1}^{1} x^{i+j}(1-x^2)^\alpha dx \\ &= \frac{1-(-1)^{i+j+1}}{2} B\left(\frac{i+j+1}{2}, \alpha+1\right) \end{aligned}$$

where $B(x, y)$ is the Beta function and

$$\begin{aligned} c_{ij} &= ij \int_{-1}^{1} x^{i+j-2}(1-x^2)^\alpha dx \\ &= ij \frac{1-(-1)^{i+j+1}}{2} B\left(\frac{i+j-1}{2}, \alpha+1\right) \end{aligned}$$

for $1 \le i, j \le n$, $c_{ij} = 0$ if $i = 0$ or $j = 0$. Now denote

$$B = (b_{ij})_{0 \le i, j \le n}, \quad C = (c_{ij})_{0 \le i, j \le n},$$

and

$$a = (a_0, a_1, \dots, a_n)^{\mathsf{T}};$$

then we can derive that

$$\int_{-1}^{1} p_n^2(x)(1-x^2)^\alpha dx = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j b_{ij} = a^{\mathsf{T}} B a,$$

$$\int_{-1}^{1} (p_n'(x))^2(1-x^2)^\alpha dx = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j c_{ij} = a^{\mathsf{T}} C a.$$

Now it suffices to consider the following extremal problem:

$$\sup_{a \in R_+^{n+1}} \frac{a^{\mathsf{T}} C a}{a^{\mathsf{T}} B a} \tag{3}$$

where $R_+^{n+1} = \{a : a = (a_0, a_1, \dots, a_n)^{\mathsf{T}}, \ a_i \ge 0, 0 \le i \le n\}$. Or find the least λ such that

$$\frac{a^{\mathsf{T}} C a}{a^{\mathsf{T}} B a} \le \lambda, \quad \text{for all } a \in R_+^{n+1},$$

which is

$$a^{\mathsf{T}}(\lambda B - C) a \ge 0, \quad \text{for all } a \in R_+^{n+1}. \tag{4}$$

Observe that $b_{ij} \ge 0$, $c_{ij} \ge 0$, $0 \le i, j \le n$. If we can find a smallest λ such that all the elements of $\lambda B - C$ are nonnegative, then we obtain (4) automatically. Notice also that the matrices B and C have the same structure; thus it suffices to find λ such that

$$\lambda b_{ij} - c_{ij} \ge 0, \quad \text{when } b_{ij} \ne 0,$$

i.e.,

$$\lambda \ge \frac{c_{ij}}{b_{ij}} = \frac{ij(i+j+2\alpha+1)}{i+j-1}, \qquad 1 \le i, j \le n.$$

If we consider c_{ij}/b_{ij} as a function of two continuous variables i and j, then we have

$$\frac{\partial}{\partial i}\left(\frac{ij(i+j+2\alpha+1)}{i+j-1}\right) = \frac{j[i^2 + (j-1)(2i+j+2\alpha+1)]}{(i+j-1)^2} \ge 0$$

and similarly

$$\frac{\partial}{\partial j}\left(\frac{ij(i+j+2\alpha+1)}{i+j-1}\right) = \frac{i[j^2 + (i-1)(2j+i+2\alpha+1)]}{(i+j-1)^2} \ge 0;$$

thus this is an increasing function of i and j, and we can pick up

$$\lambda = \left.\frac{ij(i+j+2\alpha+1)}{i+j-1}\right|_{i=n, j=n} = \frac{2n+2\alpha+1}{2n-1} n^2.$$

To see that λ is the best one, we can consider $p_n(x) = x^n$ or $a^{\mathsf{T}} = (0, 0, \dots, 0, 1)$. This completes the proof of the theorem. $\square$

For some special values of α, we have the following corollaries.

Corollary 2.2. *Let* $p_n(x) \in S_n$; *then*

$$\int_{-1}^{1} (p_n'(x))^2 dx \leq \frac{2n+1}{2n-1} n^2 \int_{-1}^{1} p_n^2(x) dx \tag{5}$$

with equality when $p_n(x) = x^n$.

Corollary 2.3. *Let* $p_n(x) \in S_n$; *then*

$$\int_{-1}^{1} (p_n'(x))^2 (1-x^2)^{-1/2} dx \leq \frac{2n}{2n-1} n^2 \int_{-1}^{1} p_n^2(x)(1-x^2)^{-1/2} dx \tag{6}$$

with equality when $p_n(x) = x^n$.

Corollary 2.4. *Let* $p_n(x) \in S_n$; *then*

$$\int_{-1}^{1} (p_n'(x))^2 (1-x^2)^{-1/2} dx \leq \frac{2n+2}{2n-1} n^2 \int_{-1}^{1} p_n^2(x)(1-x^2)^{-1/2} dx \tag{7}$$

with equality when $p_n(x) = x^n$.

In the case $\alpha = 1$, a similar result was proved by Varma [9] for polynomials having real roots.

3. The weight $\omega(x) = x^\alpha e^{-x}$

We give a new proof of Milovanović's Theorem [4] in this section. Indeed we use the same argument as was used in the proof of Theorem 2.1. This time, we consider the weight function $\omega(x) = x^\alpha e^{-x}$, $\alpha > -1$, on the interval $[0, \infty)$.

Theorem 3.1. *Let* $p_n(x) \in S_n$, $\alpha > -1$; *then*

$$\int_0^\infty (p_n'(x))^2 x^\alpha e^{-x} dx \leq C_n(\alpha) \int_0^\infty p_n^2(x) x^\alpha e^{-x} dx \tag{8}$$

where

$$C_n(\alpha) = \begin{cases} 1/[(2+\alpha)(1+\alpha)], & -1 < \alpha \leq \alpha_n, \\ n^2/[(2n+\alpha)(2n+\alpha-1)], & \alpha_n \leq \alpha < \infty, \end{cases}$$

and

$$\alpha_n = \tfrac{1}{2}(n+1)^{-1}[(17n^2+2n+1)^{1/2} - 3n + 1].$$

Moreover, $C_n(\alpha)$ *is the best possible constant.*

Proof. Let $p_n(x) = \sum_{i=0}^n a_i x^i$, $a_i \geq 0$, $0 \leq i \leq n$, then

$$\int_0^\infty p_n^2(x) x^\alpha e^{-x} dx = \sum_{i=0}^n \sum_{j=0}^n a_i a_j \int_0^\infty x^{i+j+\alpha} e^{-x} dx$$

$$= \sum_{i=0}^n \sum_{j=0}^n a_i a_j b_{ij} = a^\mathrm{T} B a$$

where

$$b_{ij} = \int_0^\infty x^{i+j+\alpha} e^{-x} dx = \Gamma(i+j+\alpha+1),$$

$$B = (b_{ij})_{0 \leq i, j \leq n}.$$

And similarly, we have

$$\int_0^\infty (p_n'(x))^2 x^\alpha e^{-x} dx = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j c_{ij} = a^\top C a$$

where

$$c_{ij} = \begin{cases} ij\Gamma(i+j+\alpha-1), & 1 \le i, j \le n, \\ 0, & i = 0 \text{ or } j = 0, \end{cases}$$
$$C = (c_{ij})_{0\le i, j\le n}.$$

Therefore, we need to find the least λ such that

$$\lambda b_{ij} - c_{ij} \ge 0, \quad \text{for } 1 \le i, j \le n.$$

That is, the maximum value of the function

$$f(i, j) := \frac{c_{ij}}{b_{ij}} = \frac{ij}{(i+j+\alpha)(i+j+\alpha-1)}.$$

Let $k = i + j$; then

$$\begin{aligned} f(i, j) &= \frac{ij}{(i+j+\alpha)(i+j+\alpha-1)} \\ &= \frac{i(k-i)}{(k+\alpha)(k+\alpha-1)} =: g(i, k). \end{aligned}$$

If we consider g as a function of two continuous variables i and k, then we have

$$\frac{\partial g(i, k)}{\partial i} = \frac{k-2i}{(k+\alpha)(k+\alpha-1)}.$$

Therefore, $g(i, k)$ takes on its maximum value at $i = k/2$ if we fix k (consider it as a function of i alone). Now it suffices to consider the maximum value of the function

$$h(k) := g\left(\frac{k}{2}, k\right) = \frac{k^2}{4(k+\alpha)(k+\alpha-1)}.$$

Following the exactly same argument of Milovanović [4, p. 425], we can see that the best possible value of λ is $C_n(\alpha)$. We omit the details. This completes the proof. □

Remark. The same idea also seems to work for other L_p norms when p is an integer, but they become more and more complicated as p is bigger and bigger. We will not formulate them here. However, for the L_1 norm, the result is simple.

Theorem 3.2. *Let $p_n(x) \in S_n$, $\alpha > -1$; then*

$$\int_0^\infty p_n'(x) x^\alpha e^{-x} dx \le \lambda_n(\alpha) \int_0^\infty p_n(x) x^\alpha e^{-x} dx \tag{9}$$

where

$$\lambda_n(\alpha) = \begin{cases} 1/(1+\alpha), & -1 < \alpha \le 0, \\ n/(n+\alpha), & 0 \le \alpha < \infty. \end{cases}$$

Moreover, $\lambda_n(\alpha)$ is the best possible constant.

4. The weight $\omega(x) = e^{-\alpha x^2}$

In this section we discuss the weight function $\omega(x) = e^{-\alpha x^2}$, $\alpha > 0$, on the whole real line. The corresponding result is the following theorem.

Theorem 4.1. *Let* $p_n(x) \in S_n$, $\alpha > 0$; *then*

$$\int_{-\infty}^{\infty} (p_n'(x))^2 e^{-\alpha x^2} dx \le \frac{2\alpha}{2n-1} n^2 \int_{-\infty}^{\infty} p_n^2(x) e^{-\alpha x^2} dx \tag{10}$$

with equality when $p_n(x) = x^n$.

Proof. Let $p_n(x) = \sum_{i=0}^n a_i x^i \in S_n$; then

$$\int_{-\infty}^{\infty} p_n^2(x) e^{-\alpha x^2} dx = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j b_{ij} = a^{\mathrm{T}} B a$$

where

$$\begin{aligned} b_{ij} &= \int_{-\infty}^{\infty} x^{i+j} e^{-\alpha x^2} dx \\ &= (1-(-1)^{i+j+1})(i+j-1)!!2^{-(i+j)/2-1}\alpha^{-(i+j+1)/2}\sqrt{\pi}, \end{aligned}$$

$$B = (b_{ij})_{0\le i,j\le n},$$

and

$$\int_{-\infty}^{\infty} (p_n'(x))^2 e^{-\alpha x^2} dx = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j c_{ij} = a^{\mathrm{T}} C a$$

where

$$\begin{aligned} c_{ij} &= ij \int_{-\infty}^{\infty} x^{i+j-2} e^{-\alpha x^2} dx \\ &= (1-(-1)^{i+j+1}) ij(i+j-3)!!2^{-(i+j)/2}\alpha^{-(i+j-1)/2}\sqrt{\pi}, \end{aligned}$$

$$C = (c_{ij})_{0\le i,j\le n}.$$

For $i+j$ even, let

$$f(i, j) := \frac{c_{ij}}{b_{ij}} = 2\alpha \frac{ij}{i+j-1}, \qquad 1 \le i, j \le n;$$

then considering f as a function of two continuous variables i and j, we can obtain

$$\frac{\partial f(i,j)}{\partial i} = \frac{2\alpha j(j-1)}{(i+j-1)^2} \ge 0 \quad \text{for } 1 \le i, j \le n,$$

and

$$\frac{\partial f(i,j)}{\partial j} = \frac{2\alpha i(i-1)}{(i+j-1)^2} \ge 0, \quad \text{for } 1 \le i, j \le n.$$

Therefore, $f(i, j)$ attains its maximum value at $i = n$, $j = n$, which implies the desired result. □

Added in Proof. After this manuscript was written, the author learned that Professor A. K. Varma [11] had written a paper on the same subject. There are some overlaps between his results and our results in §§2 and 3, but we do use different methods.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1