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THE DIRECT DECOMPOSITIONS OF A GROUP G WITH G/G' FINITELY GENERATED

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ABSTRACT. We consider the class $\mathscr{C}$ which consists of the groups M with M/M' finitely generated which satisfy the maximal condition on direct factors. It is well known that any $\mathscr{C}$-group has a decomposition in finite direct product of indecomposable groups, and that two such decompositions are not necessarily equivalent up to isomorphism, even for a finitely generated nilpotent group. Here, we show that any $\mathscr{C}$-group has only finitely many nonequivalent decompositions. In order to prove this result, we introduce, for $\mathscr{C}$-groups, a slightly different notion of decomposition, that we call J-decomposition; we show that this decomposition is necessarily unique. We also obtain, as consequences of the properties of J-decompositions, several generalizations of results of R. Hirshon. For instance, we have $\mathbb{Z} \times G \cong \mathbb{Z} \times H$ for any groups G, H which satisfy $M \times G \cong M \times H$ for a $\mathscr{C}$-group M.

In the present paper, the laws of the groups are written with multiplicative notation. For each group M and for each subset E of M, we denote by $\langle E \rangle$ the subgroup of M which is generated by E. For each group M and for each integer k, we write $M^k = \langle \{x^k ; x \in M\} \rangle$; we denote by $\times^k M$ the direct product of k copies of M.

We say that a group A is *cancellable* if, for any groups G, H, $A \times G \cong A \times H$ implies $G \cong H$.

For each group M and for any subgroups A, B of M, we say that B is a *supplementary* of A in M, and we write $M = A \times B$, if $M = \langle A, B \rangle$ and $A \cap B = [A, B] = 1$. For each $x \in M$, we consider the decomposition $x = \mathrm{pr}_A(x)\,\mathrm{pr}_B(x)$ with $\mathrm{pr}_A(x) \in A$ and $\mathrm{pr}_B(x) \in B$. A subgroup A of M which has a supplementary in M is a *direct factor* of M.

We say that a group M is *indecomposable* if M is nontrivial and if we have $A = 1$ or $B = 1$ for any subgroups A, B such that $M = A \times B$. We say that M is *finitely decomposable* if it is a finite direct product of indecomposable groups. Then, a *decomposition* of M is a finite sequence of indecomposable groups $A_1, \dots, A_n$ such that $M \cong A_1 \times \cdots \times A_n$. We identify two decompositions $M \cong A_1 \times \cdots \times A_m \cong B_1 \times \cdots \times B_n$ if $m = n$ and if there exists a permutation σ of $\{1, \dots, n\}$ such that $A_i \cong B_{\sigma(i)}$ for each $i \in \{1, \dots, n\}$. In particular, we say that M has a *unique decomposition* if this property is true for any two decompositions of M.

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Any group which satisfies the maximal condition on direct factors clearly has a decomposition. In this paper, we are going to prove that, if M is such a group and if M/M' is finitely generated, then M only has finitely many decompositions.

First, we recall some examples and results which illustrate how several decompositions may arise for the same group.

The infinite cyclic group J and, for each prime number p and each integer $k \geq 1$, the cyclic group $J_{p,k}$ of order p^k, are indecomposable. For each finitely generated abelian group, we have a unique decomposition $M \cong (\times^k J) \times (\times^{n(1)} J_{p(1),k(1)}) \times \cdots \times (\times^{n(r)} J_{p(r),k(r)})$ with k, r integers, $p(1), \ldots, p(r)$ distinct primes, and $k(1), \ldots, k(r)$, $n(1), \ldots, n(r) \in \mathbb{N}^*$. The classical result of R. Remak [R] states that each finite group also has a unique decomposition.

On the other hand, we have $J \times G \cong H \times K$ for the indecomposable finite-by-abelian nilpotent groups of class 2

$$G = \langle a, b, v ; a^4 = 1, b^9 = 1, [a, b] = 1, [a, v] = a^2, [b, v] = b^3 \rangle,$$

$$H = \langle c, w ; c^4 = 1, [c, w] = c^2 \rangle \quad \text{and} \quad K = \langle d, x ; d^9 = 1, [d, x] = d^3 \rangle.$$

If u is a generator of J, then we obtain an isomorphism $f: H \times K \to J \times G$ by writing $f(c) = a$, $f(w) = uv^3$, $f(d) = b$, and $f(x) = u^{-1}v^{-2}$, since we have

$$[a, uv^3] = [a, v]^3 = a^6 = a^2, \qquad [b, uv^3] = [b, v]^3 = b^9 = 1,$$

$$[a, u^{-1}v^{-2}] = [a, v]^{-2} = a^{-4} = 1, \qquad [b, u^{-1}v^{-2}] = [b, v]^{-2} = b^{-6} = b^3.$$

By [W. Theorem 2.1, p. 127], for any finitely generated finite-by-abelian groups G, H, we have $J \times G \cong J \times H$ if and only if G and H have the same finite images. According to [B1, p. 249], the two indecomposable finite-by-abelian nilpotent groups of class 2 $G = \langle c, v ; c^{25} = 1, [c, v] = c^5 \rangle$ and $H = \langle d, w ; d^{25} = 1, [d, w] = d^{10} \rangle$ satisfy these properties without being isomorphic.

Now, for each integer $c \geq 1$, let us denote by $\mathcal{N}_c$ the class of finitely generated torsion-free nilpotent group of class c. By [H3, Theorem 5, p. 158], any $\mathcal{N}_2$-group has a unique decomposition. This property is not true for $\mathcal{N}_3$-groups since, by [H3, pp. 154–155], there exist two nonisomorphic $\mathcal{N}_3$-groups G_1, G_2 such that $J \times G_1 \cong J \times G_2$. Moreover, according to [B2, p. 6], for each integer $n \geq 1$, there exist some indecomposable $\mathcal{N}_3$-groups H, H_1, H_2 such that $H_1 \times H_2 \cong (\times^n J) \times H$. In [B2], G. Baumslag also proves that, for any integers $m, n \geq 2$, there exists a $\mathcal{N}_3$-group which has a decomposition with m factors and a decomposition with n factors.

On the other hand, many surprising positive results were obtained by R. Hirshon. For instance, by [H7, Theorem 3.4, p. 361], if U is a group such that U/U' is finitely generated and $U/Z(U)$ satisfies the minimal condition on direct factors (or, equivalently, the maximal condition), then, for any groups G, H, $U \times G \cong U \times H$ implies $J \times G \cong J \times H$. Also, according to [H5, Theorem 1, p. 333], for each integer $n \geq 1$ and for any groups G, H which satisfy the maximal condition on normal subgroups, if $\times^n G$ and $\times^n H$ are isomorphic, then $J \times G$ and $J \times H$ are isomorphic.

By [O1, Theorem, p. 7], any groups G, H such that $J \times G \cong J \times H$ are elementarily equivalent. The converse is true for finitely generated finite-by-nilpotent groups, according to [O2].

Here, we mainly consider groups M with M/M' finitely generated; we call them ABFG, which is an abbreviation for "groups whose abelian quotients are finitely generated". Taking into account the importance of the group J in the examples and results above, we introduce, for ABFG, the notions of *J-equivalence* and *J-decomposition.* We show that a ABFG has a J-decomposition if and only if it is finitely decomposable, and, also, if and only if it satisfies the maximal condition on direct factors.

The Theorem asserts the unicity of the J-decomposition for finitely decomposable ABFG. Moreover, any finitely decomposable ABFG only has finitely many decompositions in direct products of indecomposable groups, and Corollary 3 provides a bound for the number of such decompositions. Corollary 6 describes a sequence of factors which appears in each decomposition. The properties of J-decompositions also enable us to generalize several results of R. Hirshon, including the theorems mentioned above, and to simplify their proofs. These generalizations are given by Corollaries 1, 2, 4, 5. Corollary 7 provides another connexion between the J-decomposition of a finitely decomposable ABFG and its decompositions in direct products of indecomposable groups.

We say that two ABFG G, H are *J-equivalent*, and we write $G \approx_J H$, if there exist two integers $r, s \geq 0$ such that $(\times^r J) \times G \cong (\times^s J) \times H$. According to Lemma 1 below, this property is true if and only if G and H satisfy $J \times G \cong J \times H$, or $(\times^m J) \times G \cong H$ for an integer m, or $G \cong (\times^n J) \times H$ for an integer n. Any ABFG is J-equivalent to 1 if and only if it is torsion-free abelian.

Lemma 1 ([H3, p. 148]). *For any groups G, H, $J \times J \times G \cong J \times H$ implies $J \times G \cong H$.*

Proof. We show that, for each group M, for any subgroups G, H of M, and for any elements of infinite order $u, v, w \in M$, if $M = \langle u \rangle \times \langle v \rangle \times G = \langle w \rangle \times H$, then $J \times G$ and H are isomorphic. There exist some elements $u', v' \in \langle u, v \rangle$ such that $\langle u \rangle \times \langle v \rangle = \langle u' \rangle \times \langle v' \rangle$ and $w \in \langle v' \rangle \times G$. We have $\langle v' \rangle \times G = \langle w \rangle \times K$ for $K = H \cap (\langle v' \rangle \times G)$, and therefore $\langle w \rangle \times H = M = \langle u' \rangle \times \langle v' \rangle \times G = \langle u' \rangle \times \langle w \rangle \times K = \langle w \rangle \times \langle u' \rangle \times K$. But $\langle w \rangle \times H = \langle w \rangle \times \langle u' \rangle \times K$ implies $H \cong \langle u' \rangle \times K \cong \langle w \rangle \times K = \langle v' \rangle \times G$. $\square$

We say that a ABFG M is *J-indecomposable* if M is not J-equivalent to 1 and if, for any groups A, B such that $M \approx_J A \times B$, we have $A \approx_J 1$ or $B \approx_J 1$. A finitely generated abelian group is J-indecomposable if and only if it is isomorphic to $(\times^r J) \times J_{p,k}$ with $r \in \mathbb{N}$, p prime, and $k \geq 1$.

A *J-decomposition* of a ABFG M is a finite sequence of J-indecomposable ABFG $A_1, \ldots, A_n$ such that $M \approx_J A_1 \times \cdots \times A_n$. We identify two J-decompositions $M \approx_J A_1 \times \cdots \times A_m \approx_J B_1 \times \cdots \times B_n$ if $m = n$ and if there exists a permutation σ of $\{1, \ldots, n\}$ such that $A_i \approx_J B_{\sigma(i)}$ for each $i \in \{1, \ldots, n\}$. According to the proposition below, any ABFG has a J-decomposition if and only if it is finitely decomposable:

Proposition 1. *If M is a ABFG, then the following properties are equivalent:*

(1) *M satisfies the maximal condition on direct factors;*
(2) *M is finitely decomposable;*
(3) *M has a J-decomposition.*

Remark. A finitely generated group may have a strictly increasing sequence of direct factors. For instance, J. M. T. Jones proved in [J] that, for each integer

$n \geq 3$, there exists a finitely generated group A which is isomorphic to $\times^n A$ and not isomorphic to $\times^k A$ for each integer k such that $2 \leq k \leq n-1$.

The following lemma is used in the proof of Proposition 1:

Lemma 2. *For each group M and for any subgroups $A, B, C_1, \ldots, C_n$ of M such that $M = A \times B = C_1 \times \cdots \times C_n$, we have $A' = (A' \cap C_1') \times \cdots \times (A' \cap C_n')$.*

Proof. For any elements $x, y \in A$, we have $[x, y] = \prod_{1 \leq i \leq n} [\mathrm{pr}_{C_i}(x), \mathrm{pr}_{C_i}(y)]$. Moreover, for each $i \in \{1, \ldots, n\}$, we have $[\mathrm{pr}_{C_i}(x), y] \in A \cap C_i$, and therefore

$$\begin{aligned}[\mathrm{pr}_{C_i}(x), \mathrm{pr}_{C_i}(y)] &= \mathrm{pr}_{C_i}([\mathrm{pr}_{C_i}(x), y]) = \mathrm{pr}_A([\mathrm{pr}_{C_i}(x), y]) \\ &= [\mathrm{pr}_A(\mathrm{pr}_{C_i}(x)), y] \in A' \cap C_i'. \quad \square\end{aligned}$$

Proof of Proposition 1. First, we show that (2) implies (1). We suppose $M = G_1 \times \cdots \times G_r$ with $G_1, \ldots, G_r$ indecomposable. If M has a strictly increasing sequence of direct factors, then there exists a sequence of nontrivial subgroups $(H_i)_{i \geq 1}$ such that, for each integer n, $\langle H_1, \ldots, H_n \rangle = H_1 \times \cdots \times H_n$ and $\langle H_1, \ldots, H_n \rangle$ is a direct factor of M. As the groups $\times_{1 \leq i \leq n}(H_i/H_i')$ are direct factors of the finitely generated abelian group M/M', there exists an integer n such that $H_i = H_i'$ for each integer $i \geq n$.

For each integer $i \geq n$, it follows from Lemma 2 that we have

$$H_i = H_i' = (H_i' \cap G_1') \times \cdots \times (H_i' \cap G_r') \subset (H_i \cap G_1) \times \cdots \times (H_i \cap G_r),$$

and therefore $H_i = (H_i \cap G_1) \times \cdots \times (H_i \cap G_r)$. So, for each integer $i \geq n$ and each $j \in \{1, \ldots, r\}$, $H_i \cap G_j$ is a direct factor of M, and therefore a direct factor of G_j, which implies $H_i \cap G_j = 1$ or $G_j \subset H_i$ since G_j is indecomposable. Consequently, for each integer $i \geq n$, there exists an integer $j \in \{1, \ldots, r\}$ such that $G_j \subset H_i$, which implies a contradiction.

Now, we prove that (3) implies (1). If M has a J-decomposition, then there exists an integer r such that $(\times^r J) \times M$ is a finite direct product of J-indecomposable ABFG. The group $(\times^r J) \times M$ is also a finite direct product of indecomposable ABFG, because any J-indecomposable ABFG is a direct product of an indecomposable ABFG which is not infinite cyclic and a finite number of infinite cyclic groups. As (2) implies (1), it follows that $(\times^r J) \times M$ satisfies the maximal condition on direct factors, and the same property is true for M.

As (1) clearly implies (2), it only remains to be proved that (1) implies (3). We suppose that M has no J-decomposition. Then, there exist a sequence $(A_n)_{n \in \mathbb{N}}$ of finitely generated torsion-free abelian groups and two sequences $(G_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$ of ABFG which are not torsion-free abelian, such that $G_0 = M$ and $A_n \times G_n = H_n \times G_{n+1}$ for each $n \in \mathbb{N}$. We see by induction on n that $A_0 \times \cdots \times A_n \times M = H_0 \times \cdots \times H_n \times G_{n+1}$.

For each group K, let us consider $Q(K) = K/\langle K', Z(K) \rangle$. For each integer n, we have $Q(M) \cong Q(H_0) \times \cdots \times Q(H_n) \times Q(G_{n+1})$ since $Q(A_0 \times \cdots \times A_n) = 1$. As $Q(M)$ is finitely generated abelian, there exists an integer n such that $Q(H_i) = 1$ for each integer $i \geq n$.

For each integer $i \geq n$, we have $H_i = \langle Z(H_i), H_i' \rangle$, and therefore $H_i = \langle Z(H_i), H_i \cap M \rangle$. As $H_i/(H_i \cap M) \cong \langle H_i, M \rangle / M \subset (A_0 \times \cdots \times A_i \times M)/M$ is finitely generated and torsion-free, it follows that $H_i \cap M$ is a direct factor of

H_i. So, $H_i \cap M$ is a direct factor of $A_0 \times \cdots \times A_i \times M$, and therefore a direct factor of M.

The groups $(H_n \cap M) \times \cdots \times (H_{n+k} \cap M)$ for $k \in \mathbb{N}$ form a strictly increasing sequence of direct factors of M. $\square$

It follows from the Theorem below that any finitely decomposable ABFG has a unique J-decomposition:

Theorem. *If $G_1, \ldots, G_m$ are J-indecomposable ABFG and if $H_1, \ldots, H_n$ are ABFG such that $G_1 \times \cdots \times G_m \approx_J H_1 \times \cdots \times H_n$, then there exists a map $\sigma: \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $H_k \approx_J \times_{\sigma(i)=k} G_i$ for each $k \in \{1, \ldots, n\}$.*

The proof of the Theorem is based on three lemmas:

Lemma 3. *For any groups A, B, G, for each prime number p, and for each integer $k \geq 1$, if $J_{p,k} \times G$ is isomorphic to $A \times B$, then A or B has a direct factor which is isomorphic to $J_{p,k}$.*

Proof. It suffices to show that, for each group M, for each prime number p, for each integer $k \geq 1$, for each element $u \in M$ of order p^k, and for any subgroups A, B, G of M, if $M = \langle u \rangle \times G = A \times B$, then A or B has a cyclic direct factor of order p^k.

The elements $v = \mathrm{pr}_A(u)$, $w = \mathrm{pr}_B(u)$, and $x = \mathrm{pr}_G(\mathrm{pr}_A(u))$ belong to $Z(M)$ since u belongs to $Z(M)$. We have $u^{p^k} = v^{p^k} = w^{p^k} = x^{p^k} = 1$. There exists an integer n such that $v = u^n x$ and $w = u^{1-n} x^{-1}$. If n is not divisible by p, then we have $M = \langle v \rangle \times G$ and $A = \langle v \rangle \times (G \cap A)$. Similarly, we have $M = \langle w \rangle \times G$ and $B = \langle w \rangle \times (G \cap B)$ if $1 - n$ is not divisible by p. $\square$

Lemma 4. *The groups $J_{p,k}$ for p prime and $k \geq 1$ are cancellable.*

Proof. It suffices to show that, for each group M, for each prime number p, for each integer $k \geq 1$, for any elements $u, v \in M$ of order p^k, and for any subgroups G, H of M, if $M = \langle u \rangle \times G = \langle v \rangle \times H$, then G and H have a common supplementary in M.

Let us consider $S = \langle u, v \rangle$. We have $S = \langle u \rangle \times (S \cap G) = \langle v \rangle \times (S \cap H)$. If $S \cap G = S \cap H = 1$, then $\langle u \rangle = \langle v \rangle$ is a supplementary of G and H in M. Otherwise, there exists an integer $h \leq k$ such that $S \cap G$ and $S \cap H$ are cyclic of order p^h, we have $|S/S^p| = p^2$, and the images of $S \cap G$ and $S \cap H$ in S/S^p are cyclic of order p. Then, there exists an element $w \in S/S^p$ such that $\langle w \rangle$ is a supplementary of the images of $S \cap G$ and $S \cap H$ in S/S^p. If x is a representative of w in S, then $\langle x \rangle$ is a supplementary of $S \cap G$ and $S \cap H$ in S, and therefore a supplementary of G and H in M. $\square$

Lemma 5. *Let M be a group and let A, B, C, D be subgroups of M such that $M = A \times B = C \times D$. If A is a J-indecomposable ABFG, then A' is contained in C or in D.*

Proof. By Lemma 2, $A' \cap C'$ and $A' \cap D'$ are nontrivial if A' is not contained in C or in D. We are going to see that this property contradicts the J-indecomposability of A.

For each $x \in A$, we have $\mathrm{pr}_B(\mathrm{pr}_D(x)) = \mathrm{pr}_B(\mathrm{pr}_C(x))^{-1}$; we also have

$$\begin{aligned}\mathrm{pr}_C(\mathrm{pr}_B(\mathrm{pr}_C(x))) &= \mathrm{pr}_C(\mathrm{pr}_A(\mathrm{pr}_C(x))^{-1}\,\mathrm{pr}_C(x)) \\ &= \mathrm{pr}_C(\mathrm{pr}_A(\mathrm{pr}_C(x)))^{-1}\,\mathrm{pr}_C(x) \in \mathrm{pr}_C(A)\end{aligned}$$

and

$$\mathrm{pr}_D(\mathrm{pr}_B(\mathrm{pr}_C(x))) = \mathrm{pr}_D(\mathrm{pr}_B(\mathrm{pr}_D(x)))^{-1} \in \mathrm{pr}_D(A).$$

So, the ABFG $\mathrm{pr}_C(A)$, $\mathrm{pr}_D(A)$, and $\mathrm{pr}_B(\mathrm{pr}_C(A)) = \mathrm{pr}_B(\mathrm{pr}_D(A))$ satisfy $A \times \mathrm{pr}_B(\mathrm{pr}_C(A)) = \mathrm{pr}_C(A) \times \mathrm{pr}_D(A)$. Consequently, we can suppose that A, B, C, D are ABFG.

We have $M/B' = A \times (B/B') = (C/(B' \cap C')) \times (D/(B' \cap D'))$ according to Lemma 2. So, we can assume B finitely generated abelian. We can also assume B torsion-free, because Lemma 3 and Lemma 4 allow us to cancel out the finite direct factors of B. Now, the decomposition $A \times B = C \times D$ contradicts the J-indecomposability of A since C and D are not abelian. □

Proof of the Theorem. The Theorem is clearly true for $n = 1$. We see by induction on n that is it also true for each integer $n \geq 3$ if it is true for $n = 2$.

Now, let us consider some J-indecomposable ABFG $G_1, \dots, G_m$ and two ABFG H_1, H_2 such that $G_1 \times \cdots \times G_m \approx_J H_1 \times H_2$. We may assume $G_1 \times \cdots \times G_m = H_1 \times H_2$, since it is possible to replace $G_1, \dots, G_m, H_1, H_2$ by groups which are J-equivalent to them. We can also suppose that $G_1, \dots, G_m, H_1, H_2$ have no finite abelian direct factor, since Lemma 3 and Lemma 4 allow us to reduce ourselves to this case. For each $i \in \{1, \dots, m\}$, as G_i is J-indecomposable, it follows from Lemma 5 that G_i' is contained in H_1 or H_2.

Now, let us consider $E = \{i \in \{1, \dots, m\}\,;\, G_i' \subset H_1\}$ and $F = \{1, \dots, m\} - E$. We have $H_1' = \times_{i\in E} G_i'$ and $H_2' = \times_{i\in F} G_i'$ according to Lemma 2, and therefore $H_1 \times (H_2/H_2') \cong (H_1 \times H_2)/H_2' \cong (\times_{i\in E} G_i) \times (\times_{i\in F}(G_i/G_i'))$. As H_2/H_2' and $\times_{i\in F}(G_i/G_i')$ are finitely generated abelian, it follows from Lemma 3 and Lemma 4 that H_1 and $\times_{i\in E} G_i$ are J-equivalent. Similarly, H_2 and $\times_{i\in F} G_i$ are J-equivalent. □

By [H3, Theorem 1, p. 149], if U is a group which satisfies the maximal condition for normal subgroups, then, for any groups G, H, $U \times G \cong U \times H$ implies $J \times G \cong J \times H$. In [H4, Theorem 1, p. 28], R. Hirshon generalizes this result to groups U such that U/U'' is finitely generated, $U/Z(U)$ is hopfian and $U/Z(U)$ satisfies the minimal condition on direct factors. In [H7, Theorem 3.4, p. 361], he generalizes [H4, Theorem 1, p. 28] to groups U such that U/U' is finitely generated and $U/Z(U)$ satisfies the minimal condition on direct factors (or, equivalently, the maximal condition).

If U is such a group, and if $(V_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are two sequences of direct factors of U such that $W_{n+1} = V_n \times W_n$ for each $n \in \mathbb{N}$, then $(\langle W_n, Z(U)\rangle/Z(U))_{n\in\mathbb{N}}$ is an increasing sequence of direct factors of $U/Z(U)$. So, there exists an integer k such that $\langle W_n, Z(U)\rangle = \langle W_k, Z(U)\rangle$ for each integer $n \geq k$. For each integer $n \geq k$, we have $V_n \subset Z(U)$. As U/U' is finitely generated, there exist only finitely many integers $n \geq k$ such that V_n is nontrivial. It follows that U is a finitely decomposable ABFG.

Consequently, the following result generalizes [H7, Theorem 3.4, p. 361]:

Corollary 1. *Let U be a finitely decomposable ABFG. Then, for any groups G, H, $U \times G \cong U \times H$ implies $J \times G \cong J \times H$.*

Proof. According to Lemma 1 and Lemma 4, it suffices to show that, for any groups G, H such that $U \times G \cong U \times H$, there exists a finitely generated abelian group A such that $A \times G \cong A \times H$.

There exists a finitely generated torsion-free abelian group D such that $D \times U$ is a direct product of J-indecomposable ABFG $U_1, \dots, U_k$. We see by induction on k that, if the result is true for $U_1, \dots, U_k$, then it is also true for $U_1 \times \cdots \times U_k$, and therefore true for U. So, we can suppose for the remainder of the proof that U is J-indecomposable.

Now, let us consider a group M and some subgroups G, H, V, W of M such that $M = V \times G = W \times H$ and $U \cong V \cong W$. By Lemma 5, we have $V' = W'$, or V' and W' are respectively contained in H and G. In the first case, we have $M/V' = M/W' = (V/V') \times G = (W/W') \times H$ with $V/V' \cong W/W'$ finitely generated abelian. In the second case, the groups $V/(V \cap H)$ and $W/(W \cap G)$ are finitely generated abelian. But it follows from the lemma below that $V/(V \cap H) \times W/(W \cap G) \times G$ and $V/(V \cap H) \times W/(W \cap G) \times H$ are isomorphic. □

Lemma 6 ([H1, p. 402]). *Let M be a group and let A, B, G, H be subgroups of M such that $M = A \times G = B \times H$ and $A \cong B$. Then we have $(A/S) \times (B/T) \times G \cong (A/S) \times (B/T) \times H$ for $S = A \cap H$ and $T = B \cap G$.*

Proof. We have $M/(S \times T) \cong (A/S) \times (G/T) \cong (B/T) \times (H/S)$, and therefore $A \times (A/S) \times (G/T) \cong B \times (B/T) \times (H/S)$ since A and B are isomorphic. Moreover, we have $A \times (G/T) \cong (A \times G)/T = (B \times H)/T \cong (B/T) \times H$, and therefore $A \times (A/S) \times (G/T) \cong (A/S) \times (B/T) \times H$. We also have $B \times (H/S) \cong (B \times H)/S = (A \times G)/S \cong (A/S) \times G$, and therefore $B \times (B/T) \times (H/S) \cong (A/S) \times (B/T) \times G$. It follows $(A/S) \times (B/T) \times G \cong (A/S) \times (B/T) \times H$. □

Any group which satisfies the maximal condition on normal subgroups is a finitely decomposable ABFG. So, the following result generalizes [H5, Theorem 1, p. 333]:

Corollary 2. *For each finitely decomposable ABFG G, for each group H, and for each integer $n \geq 1$, if $\times^n G$ and $\times^n H$ are isomorphic, then $J \times G$ and $J \times H$ are isomorphic.*

Proof. As G is a finitely decomposable ABFG, the same property is true for $\times^n G \cong \times^n H$, and therefore true for H, since the maximal condition on direct factors for $\times^n H$ implies the same condition for H. There exist some J-indecomposable groups $U_1, \dots, U_r, V_1, \dots, V_s$ such that $G \approx_J U_1 \times \cdots \times U_r$ and $H \approx_J V_1 \times \cdots \times V_s$. We have $\times^n G \approx_J (\times^n U_1) \times \cdots \times (\times^n U_r)$ and $\times^n H \approx_J (\times^n V_1) \times \cdots \times (\times^n V_s)$. It follows from the unicity of the J-decomposition of $\times^n G \cong \times^n H$ that $r = s$ and $|\{i \in \{1, \dots, r\}; U_i \approx_J W\}| = |\{j \in \{1, \dots, r\}; V_j \approx_J W\}|$ for each J-indecomposable group W. So, we have $G \approx_J H$.

Now, let us consider some finitely generated torsion-free abelian groups A, B such that $A \times G \cong B \times H$. We have $A \times (G/G') \cong B \times (H/H')$. But $\times^n G \cong \times^n H$ implies $\times^n (G/G') \cong \times^n (H/H')$ and $G/G' \cong H/H'$. So, A and B are isomorphic, and we have $J \times G \cong J \times H$ according to Lemma 1. □

For each group M and for each subgroup S of M, the *isolator* of S in M is $I_M(S) = \bigcup_{n \geq 1} \{x \in M; x^n \in S\}$. We denote it by $I(S)$ if it does not create ambiguity. The subgroup S is *isolated* in M if $I(S) = S$. We write $\Delta(M) = I(M')$.

For each ABFG M, we consider the finite abelian quotient

$$Q(M) = I(\langle Z(M), \Delta(M) \rangle)/\langle Z(M), \Delta(M) \rangle.$$

We have $Q(M \times N) = Q(M) \times Q(N)$ for any ABFG M, N.

We say that a ABFG M is *regular* if $Q(M)$ is trivial. This property is true if and only if $M = A \times N$ with A torsion-free abelian and $Z(N) \subset \Delta(N)$.

In fact, if M is regular, then there exist some elements $u_1, \dots, u_m \in Z(M)$ such that the abelian group $\langle Z(M), \Delta(M)\rangle/\Delta(M)$ is freely generated by the images of $u_1, \dots, u_m$, and some elements $v_1, \dots, v_n \in M$ such that the abelian group $M/\langle Z(M), \Delta(M)\rangle$ is freely generated by the images of $v_1, \dots, v_n$. The group $A = \langle u_1, \dots, u_m\rangle$ is torsion-free abelian. We have $M = A \times N$ and $Z(N) \subset \Delta(N)$ for $N = \langle v_1, \dots, v_n, \Delta(M)\rangle$.

We are going to prove that any finitely decomposable ABFG M only has finitely many decompositions in direct products of indecomposable groups. We shall obtain a bound for the number of decompositions of M which only depends on $Q(M)$.

In [H6], R. Hirshon shows that, for each group G with G/G' finitely generated, there exist only finitely many pairwise nonisomorphic groups H such that $J \times G \cong J \times H$. First, we give a simpler proof of Hirshon's result; this proof provides a bound, depending only on $Q(G)$, for the number of pairwise nonisomorphic groups H such that $J \times G \cong J \times H$. For each group M, we write $|\operatorname{Aut}(M)|$ for the number of automorphisms of M.

Proposition 2. *Let G be a group with $G/\Delta(G)$ finitely generated. Then, there exist at most $|\operatorname{Aut}(Q(G))|$ pairwise nonisomorphic groups H such that $J \times G \cong J \times H$.*

Proof. We consider a group M such that G has an infinite cyclic supplementary in M, and we show that there are at most $|\operatorname{Aut}(Q(G))|$ pairwise nonisomorphic subgroups of M which have infinite cyclic supplementaries.

We denote the ranks of the free abelian groups $I(\langle Z(G), \Delta(G)\rangle)/\Delta(G)$ and $G/I(\langle Z(G), \Delta(G)\rangle)$ by m and n, respectively. There exists a unique m-tuple of integers $r(1), \dots, r(m) \in \mathbb{N}^*$ such that $r(i+1)$ divides $r(i)$ for each $i \in \{1, \dots, m-1\}$ and $Q(G) \cong (J/J^{r(1)}) \times \cdots \times (J/J^{r(m)})$.

If H is a subgroup of M which has an infinite cyclic supplementary, then we have $\Delta(H) = \Delta(M)$. We also have $H/I(\langle Z(H), \Delta(H)\rangle) \cong G/I(\langle Z(G), \Delta(G)\rangle)$, $I(\langle Z(H), \Delta(H)\rangle)/\Delta(H) \cong I(\langle Z(G), \Delta(G)\rangle)/\Delta(G)$, and $Q(H) \cong Q(G)$. We choose some elements $w_{H,1}, \dots, w_{H,m}$ such that $I(\langle Z(H), \Delta(H)\rangle) = \langle w_{H,1}, \dots, w_{H,m}, \Delta(H)\rangle$ and $\langle Z(H), \Delta(H)\rangle = \langle w_{H,1}^{r(1)}, \dots, w_{H,m}^{r(m)}, \Delta(H)\rangle$. We have

$$I(\langle Z(M), \Delta(M)\rangle) = \langle w_{H,1}, \dots, w_{H,m}, Z(M), \Delta(M)\rangle$$

and

$$\langle w_{H,1}, \dots, w_{H,m}\rangle \cap \langle Z(M), \Delta(M)\rangle = \langle w_{H,1}^{r(1)}, \dots, w_{H,m}^{r(m)}\rangle.$$

Consequently, the map $w_{G,1} \to w_{H,1}, \dots, w_{G,m} \to w_{H,m}$ induces an automorphism ρ_H of $Q(M) \cong Q(G)$.

Now, we are going to see that, if H and K are subgroups of M with infinite cyclic supplementaries, and if $\rho_H = \rho_K$, then H and K are isomorphic.

For each $i \in \{1, \dots, m\}$, we have $w_{K,i}w_{H,i}^{-1} \in \langle Z(M), \Delta(M)\rangle$; we may assume for the remainder of the proof that $w_{K,i}w_{H,i}^{-1} \in Z(M)$, since it is possible to multiply $w_{K,i}$ by any element of $\Delta(M) = \Delta(K)$ without changing ρ_K. For any elements $x_1, \dots, x_n$ such that $H = \langle x_1, \dots, x_n, I(\langle Z(H), \Delta(H)\rangle)\rangle$, we have $K = \langle \mathrm{pr}_K(x_1), \dots, \mathrm{pr}_K(x_n), I(\langle Z(K), \Delta(K)\rangle)\rangle$ and $\mathrm{pr}_K(x_j)x_j^{-1} \in Z(M)$ for each $j \in \{1, \dots, n\}$. Consequently, we obtain an isomorphism $\theta\colon H \to K$ by writing $\theta(u) = u$ for each $u \in \Delta(H)$, $\theta(w_{H,i}) = w_{K,i}$ for each $i \in \{1, \dots, m\}$ and $\theta(x_j) = \mathrm{pr}_K(x_j)$ for each $j \in \{1, \dots, n\}$. □

Corollary 3. *If G is a finitely decomposable $ABFG$, then the number of decompositions of G is bounded by the product of $|\mathrm{Aut}(Q(G))|$ and the number of equivalence relations on a set with $n(G)$ elements, where $n(G)$ is the number of factors in the decomposition of $Q(G)$.*

In the proof of Corollary 3, we use the following lemma:

Lemma 7. *Let M be an $ABFG$ and let A, B, C, D be subgroups of M such that $M = A \times B = C \times D$. Let us suppose $Z(A) \subset \Delta(A)$ and D torsion-free abelian. Then A is isomorphic to a direct factor of C.*

Proof. We have $\Delta(M) = \Delta(A) \times \Delta(B) = \Delta(C)$ and $I(\langle Z(M), \Delta(M)\rangle) = \Delta(A) \times I(\langle Z(B), \Delta(B)\rangle) = I(\langle Z(C), \Delta(C)\rangle) \times D$. Consequently, $S = I(\langle Z(B), \Delta(B)\rangle) \cap I(\langle Z(C), \Delta(C)\rangle)$ is a supplementary of $\Delta(A)$ in $I(\langle Z(C), \Delta(C)\rangle)$, and we have $\Delta(B) \subset S$. Now, let us consider some elements $x_1, \dots, x_m \in A$ such that $A/\Delta(A)$ is freely generated by the images of $x_1, \dots, x_m$, and some elements $y_1, \dots, y_n \in B$ such that $B/I(\langle Z(B), \Delta(B)\rangle)$ is freely generated by the images of $y_1, \dots, y_n$. Then, we have $C = \langle \mathrm{pr}_C(x_1), \dots, \mathrm{pr}_C(x_m), \Delta(A)\rangle \times \langle \mathrm{pr}_C(y_1), \dots, \mathrm{pr}_C(y_n), S\rangle$, and $A \cong \langle \mathrm{pr}_C(x_1), \dots, \mathrm{pr}_C(x_m), \Delta(A)\rangle$. □

Proof of Corollary 3. Let us consider two integers $n \le p$ and a J-decomposition $G \approx_J G_1 \times \cdots \times G_p$ with $G_1, \dots, G_p$ indecomposable and J-indecomposable, $G_1, \dots, G_n$ not regular, and $G_{n+1}, \dots, G_p$ regular. Then we have $Q(G) \cong Q(G_1) \times \cdots \times Q(G_n)$, and therefore $n \le n(G)$.

According to the Theorem, for each decomposition $G = A_1 \times \cdots \times A_h$, there exists a map $\sigma\colon \{1, \dots, p\} \to \{1, \dots, h\}$ such that $A_i \approx_J \times_{\sigma(j)=i} G_j$ for each $i \in \{1, \dots, h\}$. For each $j \in \{n+1, \dots, p\}$, we have $\sigma^{-1}(\sigma(j)) = \{j\}$ since G_j is isomorphic to a direct factor of $A_{\sigma(j)}$ by Lemma 7. The map σ induces an equivalence relation on $\{1, \dots, p\}$ such that each element of $\{n+1, \dots, p\}$ is only equivalent to itself. The number of such relations is bounded by the number of equivalence relations on a set with $n(G)$ elements.

Now, let us consider two decompositions $G = A_1 \times \cdots \times A_h = B_1 \times \cdots \times B_k$, and let us suppose that the associated maps $\sigma\colon \{1, \dots, p\} \to \{1, \dots, h\}$ and $\tau\colon \{1, \dots, p\} \to \{1, \dots, k\}$ induce the same equivalence relation on $\{1, \dots, p\}$. Then, for each $i \in \{1, \dots, h\}$ and each $j \in \{1, \dots, k\}$ such that $\sigma^{-1}(i) = \tau^{-1}(j)$, we have $A_i \approx_J B_j$, and therefore $J \times A_i \cong J \times B_j$ since A_i and B_j are indecomposable. The relation $\sigma^{-1}(i) = \tau^{-1}(j)$ defines a bijection from $\{i \in \{1, \dots, h\}; \sigma^{-1}(i) \neq \varnothing\}$ to $\{j \in \{1, \dots, k\}; \tau^{-1}(j) \neq \varnothing\}$. Moreover, we have $A_i \cong J$ for each $i \in \{1, \dots, h\}$ such that $\sigma^{-1}(i) = \varnothing$ and $B_j \cong J$ for each $j \in \{1, \dots, k\}$ such that $\tau^{-1}(j) = \varnothing$. It follows that $|\{i \in \{1, \dots, h\}; \sigma^{-1}(i) = \varnothing\}| = |\{j \in \{1, \dots, k\}; \tau^{-1}(j) = \varnothing\}|$, since we have $G/G' \cong (A_1/A_1') \times \cdots \times (A_h/A_h') \cong (B_1/B_1') \times \cdots \times (B_k/B_k')$ and $A_i/A_i' \cong B_j/B_j'$ for each $i \in \{1, \dots, h\}$ and each $j \in \{1, \dots, k\}$ such that $\sigma^{-1}(i) = \tau^{-1}(j) \neq \varnothing$. In particular, we have $h = k$ and, by reordering $B_1, \dots, B_h$, we obtain some subgroups $C_1, \dots, C_h$ such that $J \times A_i \cong J \times C_i$ for each $i \in \{1, \dots, h\}$.

By Proposition 2, for each decomposition $G = A_1 \times \cdots \times A_h$ and for each $i \in \{1, \ldots, h\}$, there are at most $|\mathrm{Aut}(Q(A_i))|$ pairwise nonisomorphic groups B such that $J \times A_i \cong J \times B$. Consequently, the number of decompositions $G = B_1 \times \cdots \times B_h$ with $J \times A_i \cong J \times B_i$ for each $i \in \{1, \ldots, h\}$ is bounded by $|\mathrm{Aut}(Q(A_1))| \cdots |\mathrm{Aut}(Q(A_h))| \leq |\mathrm{Aut}(Q(G))|$. □

By [H3, Theorem 5, p. 158], any finitely generated torsion-free nilpotent group of class 2 has a unique decomposition. If G is such a group, then G is regular since $Z(G)$ contains $\Delta(G)$ and $G/Z(G)$ is torsion-free.

According to [H2, Theorem 1], if G is a group which satisfies the maximal condition on normal subgroups, and if the homomorphic images of G are cancellable, then G has a unique decomposition. The first condition implies that G is a finitely decomposable ABFG. The second one implies that the finitely generated torsion-free abelian group $G/\Delta(G)$ is trivial, since J is not cancellable; so, G is regular.

Thus, [H3, Theorem 5, p. 158] and [H2, Theorem 1] are generalized by the following consequence of Corollary 3:

Corollary 4. *Any regular finitely decomposable $ABFG$ has a unique decomposition.*

In [H1], R. Hirshon proves that, if A is a group which satisfies the maximal condition on normal subgroups, and if no homomorphic image of A is isomorphic to a proper normal subgroup of itself, then A is cancellable. The first condition implies that A is a finitely decomposable ABFG. The second one implies that the finitely generated torsion-free abelian group $A/\Delta(A)$ is trivial, since J is isomorphic to J^2. So, Hirshon's result is a consequence of the corollary below:

Corollary 5. *Any finitely decomposable $ABFG$ A such that $Z(A) \subset \Delta(A)$ is cancellable. Any $ABFG$ A which is cancellable satisfies $Z(A) \subset \Delta(A)$.*

In the proof of Corollary 5, we use the two following lemmas:

Lemma 8. *Any indecomposable $ABFG$ G such that $Z(G) \subset \Delta(G)$ is J-indecomposable.*

Proof. If A is a finitely generated torsion-free abelian group, then $A \times G$ has a unique decomposition according to Corollary 4. □

Lemma 9. *Let M be a group and let A, B, C, D be subgroups of M such that $M = A \times B = C \times D$. Let us suppose that A is an $ABFG$, $A' \subset C$, and $Z(A) \subset \Delta(A)$. Then, there exists a subgroup E of M such that $M = E \times B$ and $E = \langle E \cap C, \Delta(A) \rangle$.*

Proof. Let us consider some elements $x_1, \ldots, x_n \in A$ such that $A/\Delta(A)$ is freely generated by the images of $x_1, \ldots, x_n$. For each $i \in \{1, \ldots, n\}$, let us write $x_i = y_i z_i$ with $y_i \in C$ and $z_i \in D$. Then, for each $i \in \{1, \ldots, n\}$, we have $[x_i, M] = [x_i, A] \subset A' \subset C$, and therefore $z_i \in Z(M) \subset Z(A) \times B \subset \Delta(A) \times B$. Consequently, the abelian group $M/(\Delta(A) \times B) \cong A/\Delta(A)$ is freely generated by the images of $y_1, \ldots, y_n$, and we have $M = E \times B$ for $E = \langle y_1, \ldots, y_n, \Delta(A) \rangle$. □

Proof of Corollary 5. In order to prove that any finitely decomposable ABFG A such that $Z(A) \subset \Delta(A)$ is cancellable, it suffices to show that this property is true if A is indecomposable. Then, A is J-indecomposable according to Lemma 8. We consider a group M and some subgroups C, D, G, H of M such that $M = C \times G = D \times H$ and $A \cong C \cong D$. By Lemma 5, we have $C' = D'$, or C' and D' are respectively contained in H and G.

In the first case, Lemma 9 allows us to suppose that $C = \langle C \cap D, \Delta(C) \rangle$. Then, $C/(C \cap D)$ is finite and abelian. We have

$$\begin{aligned} C \times (D/(C \cap D)) &\cong C \times (\langle C, D \rangle / C) \cong \langle C, D \rangle \\ &\cong D \times (\langle C, D \rangle / D) \cong D \times (C/(C \cap D)) \end{aligned}$$

since C and D have supplementaries in $\langle C, D \rangle$. As C and D are isomorphic and indecomposable, it follows from Lemma 3 and Lemma 4 that $C/(C \cap D)$ and $D/(C \cap D)$ are isomorphic. Moreover, we have $M/(C \cap D) \cong (C/(C \cap D)) \times G \cong (D/(C \cap D)) \times H$, and therefore $G \cong H$ according to Lemma 4.

In the second case, Lemma 9 allows us to suppose that $C = \langle C \cap H, \Delta(C) \rangle$ and $D = \langle D \cap G, \Delta(D) \rangle$. Then, $C/(C \cap H)$ and $D/(D \cap G)$ are finite abelian, and we have $(C/(C \cap H)) \times (D/(D \cap G)) \times G \cong (C/(C \cap H)) \times (D/(D \cap G)) \times H$ according to Lemma 6. It follows from Lemma 4 that G and H are isomorphic.

Now, we prove that, if an ABFG A does not satisfy $Z(A) \subset \Delta(A)$, then A is not cancellable. We consider some elements $x, x_1, \ldots, x_n \in A$, an integer $k \geq 1$, an element $y \in Z(A) - (Z(A) \cap \Delta(A))$, and an element $z \in \Delta(A)$, all of them defined in such a manner that $x^k = yz$ and the abelian group $A/\Delta(A)$ is freely generated by the images of $x, x_1, \ldots, x_n$. We also consider a prime number $p \geq 5$ which does not divide k. The groups

$$G = \langle a, b ; a^{p^2} = 1, [a, b] = a^{2p} \rangle \quad \text{and} \quad H = \langle c, d ; c^{p^2} = 1, [c, d] = c^p \rangle$$

are not isomorphic according to [B1, Lemma 1].

For any integers $\alpha, \beta \in \mathbb{Z}$, there exists a unique homomorphism $f \colon A \times G \to A \times H$ which satisfies $f(u) = u$ for each $u \in \langle x_1, \ldots, x_n, \Delta(A) \rangle$, $f(a) = c$, $f(x) = xy^\alpha d^{p\alpha}$, and $f(b) = y^\beta d^{2+p\beta}$. Moreover, f is an isomorphism if and only if f induces an isomorphism from $(A \times G)/\langle x_1, \ldots, x_n, \Delta(A), a \rangle$ to $(A \times H)/\langle x_1, \ldots, x_n, \Delta(A), c \rangle$. In $(A \times H)/\langle x_1, \ldots, x_n, \Delta(A), c \rangle$, the elements $xy^\alpha d^{p\alpha}$ and $y^\beta d^{2+p\beta}$ are respectively equivalent to $x^{1+k\alpha} d^{p\alpha}$ and $x^{k\beta} d^{2+p\beta}$. So, f is an isomorphism if the determinant $(1 + k\alpha)(2 + p\beta) - (p\alpha)(k\beta) = 2 + 2k\alpha + p\beta$ is equal to 1. But, as p and $2k$ are prime to each other, there exist some integers $\alpha, \beta \in \mathbb{Z}$ such that $2k\alpha + p\beta = 1 - 2 = -1$. □

The following result provides, for each finitely decomposable ABFG G, a sequence of indecomposable groups which appears in any decomposition of G. Moreover, in each decomposition of G, the factors H which do not belong to this sequence do not satisfy $Q(H) = 1$; consequently, the number of such factors is bounded by the number of factors in the decomposition of $Q(G)$.

Corollary 6. *Let G be a finitely decomposable ABFG. Let us consider two decompositions $G = A_1 \times \cdots \times A_m \times B_1 \times \cdots \times B_n$ and $G = C_1 \times \cdots \times C_p \times D_1 \times \cdots \times D_q$ with $Z(A_i) \subset \Delta(A_i)$ for each $i \in \{1, \ldots, m\}$, $Z(B_i) \not\subset \Delta(B_i)$ for each $i \in \{1, \ldots, n\}$, $Z(C_i) \subset \Delta(C_i)$ for each $i \in \{1, \ldots, p\}$, and $Z(D_i) \not\subset \Delta(D_i)$ for each $i \in \{1, \ldots, q\}$. Then, the following properties are true:*

(1) *We have* $m = p$ *and there exists a permutation* σ *of* $\{1, \dots, m\}$ *such that* $A_i \cong C_{\sigma(i)}$ *for each* $i \in \{1, \dots, m\}$;

(2) *The group* $B = B_1 \times \cdots \times B_n$ *is isomorphic to* $D_1 \times \cdots \times D_q$, *and there exist no nontrivial direct factors* E *of* B *such that* $Z(E) \subset \Delta(E)$.

In the proof of Corollary 6, we use the following lemma:

Lemma 10. *Let* G *be a finitely decomposable ABFG and let* A, B, C, D *be subgroups of* G *such that* $G = A \times B = C \times D$. *Let us suppose* A *indecomposable and* $Z(A) \subset \Delta(A)$. *Then,* C *or* D *has a direct factor which is isomorphic to* A.

Proof. By Lemma 8, A is J-indecomposable. We can suppose A nonabelian according to Lemma 3. By Lemma 5, A' is contained in one of the groups C, D. Let us suppose, for instance, $A' \subset C$.

Then, we have $D' \cap A' = 1$, and therefore $D' \subset B$ according to Lemma 2. It follows $A \times B/D' \cong G/D' \cong C \times D/D'$. By cancelling out the finite direct factors of the abelian group D/D', which necessarily appear in B/D' since A is indecomposable, we obtain a relation $A \times E \cong C \times F$ with F finitely generated torsion-free abelian. So, A is isomorphic to a direct factor of C, according to Lemma 7. □

Proof of Corollary 6. We argue by induction on m. If $m = 0$, then, by Lemma 10, G has no nontrivial direct factor E such that $Z(E) \subset \Delta(E)$, and we have $p = 0$. If $m > 0$, then, according to Lemma 10, one of the groups C_i, for instance C_p, is isomorphic to A_m. By Corollary 5, it follows $A_1 \times \cdots \times A_{m-1} \times B_1 \times \cdots \times B_n \cong C_1 \times \cdots \times C_{p-1} \times D_1 \times \cdots \times D_q$, and we can apply the induction hypothesis. □

In [H8], R. Hirshon gives some conditions, for an ABFG M such that $M/Z(M)$ satisfies the minimal condition on direct factors, which imply that M has a unique decomposition. For each $s \in \mathbb{N}$, he introduces the s-terms of M, which are the direct factors B of $(\times^s J) \times M$ such that B is a direct product of an indecomposable group and a finitely generated torsion-free abelian group. He proves that there exists an integer $s(M)$, which can be expressed in terms of certain invariants of M, such that M has a unique decomposition if each $s(M)$-term of M has a unique decomposition.

This result can be generalized to finitely decomposable ABFG. If follows from the Theorem that, if M is such a group and if $M = A_1 \times \cdots \times A_m = B_1 \times \cdots \times B_n$ with $A_1, \dots, A_m$, $B_1, \dots, B_n$ indecomposable, then one of the two following properties is true:

(1) $m = n$ and there exists a permutation σ of $\{1, \dots, n\}$ such that $J \times A_i \cong J \times B_{\sigma(i)}$ for each $i \in \{1, \dots, n\}$ (some of the groups A_i, B_j may be isomorphic to J);

(2) one of the groups A_i, B_j is neither J-indecomposable nor isomorphic to J.

Consequently, M has a unique decomposition if, for each indecomposable direct factor A of M and for each $s \in \mathbb{N}$, $(\times^s J) \times A$ has a unique decomposition. It follows from Corollary 7 below applied to the indecomposable direct factors of M that this property is true for each $s \in \mathbb{N}$ if it is true when s is the number of factors in the decomposition of $Q(M)$.

For each ABFG M, we denote by $r(M)$ the rank of the free abelian group $\langle Z(M), \Delta(M)\rangle/\Delta(M)$, and $n(M)$ the number of factors in the decomposition of $Q(M)$. We have $n(M) = 0$ if and only if M is regular. For any ABFG A, B, we have $r(A \times B) = r(A) + r(B)$ and $n(A \times B) = n(A) + n(B)$.

For each ABFG M, we write $s(M) = n(M) - r(M)$ if $n(M) \geq r(M) + 1$, $s(M) = 1$ if $n(M) = r(M) \geq 1$, and $s(M) = 0$ if $n(M) \leq r(M) - 1$ or $n(M) = r(M) = 0$. In both cases, we have $s(M) \leq n(M)$.

Corollary 7. *If G is a finitely decomposable $ABFG$, then $(\times^{s(G)} J) \times G$ is a direct product of J-indecomposable $ABFG$.*

The proof of Corollary 7 rests on the following lemma:

Lemma 11. *For each $ABFG$ M, if $r(M) > n(M)$, then M has a direct factor which is isomorphic to $\times^{r(M)-n(M)} J$.*

Proof. Let us consider some elements $x_1, \dots, x_{r(M)} \in I(\langle Z(M), \Delta(M)\rangle)$ and some integers $d(1), \dots, d(r(M))$ such that $d(i-1)$ divides $d(i)$ for each $i \geq 2$, $I(\langle Z(M), \Delta(M)\rangle) = \langle x_1, \dots, x_{r(M)}, \Delta(M)\rangle$, and $\langle Z(M), \Delta(M)\rangle = \langle x_1^{d(1)}, \dots, x_{r(M)}^{d(r(M))}, \Delta(M)\rangle$. We necessarily have $d(i) = 1$ for $1 \leq i \leq r(M) - n(M)$, since there are only $n(M)$ factors in the decomposition of $Q(M)$. So, we can choose $x_1, \dots, x_{r(M)-n(M)}$ in such a way that they belong to $Z(M)$. Then, we have

$$M = \langle x_1\rangle \times \cdots \times \langle x_{r(M)-n(M)}\rangle \times \langle x_{r(M)-n(M)+1}, \dots, x_{r(M)}, y_1, \dots, y_k, \Delta(M)\rangle$$

for any elements $y_1, \dots, y_k \in M$ such that $M/I(\langle Z(M), \Delta(M)\rangle)$ is freely generated by the images of $y_1, \dots, y_k$. $\square$

Proof of Corollary 7. We consider the smallest integer h such that $M = (\times^h J) \times G$ is a direct product of J-indecomposable ABFG. There exist two integers k, m, and some ABFG $A_1, \dots, A_m$ which are both indecomposable and J-indecomposable, such that $M \cong (\times^k J) \times A_1 \times \cdots \times A_m$.

By Lemma 11, we have $r(A_i) \leq n(A_i)$ for each $i \in \{1, \dots, m\}$, and therefore $r(M) - n(M) = k + \sum_{1\leq i\leq m}(r(A_i) - n(A_i)) \leq k$. Moreover, we have $r(M) - n(M) = h + r(G) - n(G)$. It follows that $k \geq h + r(G) - n(G)$.

If $r(G) \geq n(G) + 1$ and $h \geq 1$, or if $r(G) = n(G) \geq 1$ and $h \geq 2$, then we have $k \geq 2$. If $r(G) \leq n(G) - 1$ and $h \geq n(G) - r(G) + 1$, then we have $h \geq 2$ and $k \geq 1$. In both cases, it follows from Lemma 1 that $(\times^{h-1} J) \times G \cong (\times^{k-1} J) \times A_1 \times \cdots \times A_m$, contrary to the definition of h.

If $r(G) = n(G) = 0$, then we have $Z(G) \subset \Delta(G)$, and the indecomposable direct factors of G are J-indecomposable by Lemma 8. So, G is a direct product of J-indecomposable ABFG and we have $h = 0$. $\square$

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