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THE REGIONALLY PROXIMAL RELATION

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ABSTRACT. Sufficient conditions for the regionally proximal relation $Q(X)$ of a minimal flow to be an equivalence relation are obtained in terms of the group $\mathscr{G}(X)$ of the flow and various groups which depend only on the acting group T.

INTRODUCTION

One of the first problems in topological dynamics was to characterize the equicontinuous structure relation $S(X)$ of a flow (X, T); i.e. to find the smallest closed equivalence relation, $S(X)$ on (X, T), such that $(X/S, T)$ is equicontinuous. A natural candidate for S is the so-called regionally proximal relation $Q(X)$ (see below for the definition). Now Q turns out to be closed, invariant, and reflexive, but not necessarily transitive. The problem was then to find conditions under which Q was an equivalence relation. Starting with Veech [V], various authors, including MacMahon [M], Ellis-Keynes [EK], came up with various sufficient conditions for Q to be an equivalence relation.

In this paper we give a sufficient condition in terms of the group of the flow (X, T), which generalizes most of the ones previously adduced (see 1.12). These groups, introduced in [E1], have begun to play a fundamental role in topological dynamics in that many dynamical properties of flows may be characterized using them. Indeed 1.10, the principal result of this paper, may be viewed as saying that $Q(X)$ is an equivalence relation if $E \subset G'\mathscr{G}(X)$ (see 1.12), a statement which refers only to the group $\mathscr{G}(X)$ of the flow (X, T) and certain groups G' and E which depend only on the acting group T.

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We begin by recalling the definition of the regionally proximal relation Q.

1.1. **Definition.** Let (X, T) be a flow. We define the *regionally proximal* relation $Q(X) \subset X \times X$ by

$$Q(X) = \{(x, y) \mid \text{there exist } x_i \to x,\ y_i \to y, \text{ and } t_i \in T \text{ such that } \lim x_i t_i = \lim y_i t_i\}.$$

It is easily checked that $Q(X)$ is symmetric, closed, and T-invariant. Moreover $Q(X)$ is trivial if and only if the flow (X, T) is equicontinuous.

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1.2. **Notation.** Let M be a fixed minimal right ideal in βT. We will write

$$J = \{v \in M \mid v^2 = v\} \quad \text{and} \quad G = Mu,$$

where $u \in J$ is fixed. Then (M, u) is a universal pointed minimal T-flow; that is, given a minimal flow (X, T) with base point $x_0 = x_0 u$, the map

$$p \to x_0 p : M \to X$$

is a homomorphism of flows. We will write Q for the regionally proximal relation $Q(M)$ and

$$Q[p] = \{q \in M \mid (p, q) \in Q\}.$$

Note that G is a group which we equip with the τ-topology. Recall that if $K \subset G$, then its closure in this topology is given by

$$\mathrm{cls}_\tau K = (K \circ u) \cap G.$$

Here

$$K \circ u = \{p \in M \mid \text{there exist } p_i \in K \text{ and } t_i \to u \text{ with } p_i t_i \to p\}.$$

A net $\{\alpha_i\} \subset G$ converges to $\alpha \in G$ in the τ-topology if and only if there exists a net $\{t_i\} \subset T$ such that $t_i \to u$ and $\alpha_i t_i \to \alpha$ in the ordinary topology on M. We denote by G' the τ-closed subgroup of G defined by

$$G' = \bigcap \{\mathrm{cls}_\tau N \mid N \text{ is a } \tau\text{-neighborhood of } u\}.$$

It follows from the definitions that $\alpha \in G'$ if and only if there exists a net in G which converges to *both* α and u in the τ-topology.

We will need the following elementary property of Q which we isolate as a lemma.

1.3. **Lemma.** *Let $p \in M$ and $(q_1, q_2) \in Q$. Then $(pq_1, pq_2) \in Q$.*

Proof. The map $q \to pq$ from $M \to M$ is a flow automorphism.

The next lemma deals with the relationship between Q and $Q(X)$ for a flow (X, T).

1.4. **Lemma.** *Let (X, T) be a minimal flow with base point $x_0 = x_0 u$. Let $(x, y) \in Q(X)$ with $x, y \in Xw$, where $w \in J$. Suppose that $p \in Mw$ with $x_0 p = x$. Then there exists $q \in Mw$ such that $x_0 q = y$ and $(p, q) \in Q$.*

Proof. It is well known (see [E2]) that there exist $(p_0, q_0) \in Q$ such that $x_0 p_0 = x_0 p$ and $x_0 q_0 = y$. Since $x, y \in Xw$, we may assume that $p_0, q_0 \in Mw$. (Otherwise replace p_0, q_0 by $p_0 w, q_0 w$). Now $x_0 p_0 u = x_0 p u$, so $\alpha = pu(p_0 u)^{-1} \in Mu$ satisfies

$$x_0 \alpha = x_0 \quad \text{and} \quad \alpha p_0 = \alpha p_0 u w = p u w = p.$$

It then follows from 1.3 that the pair $(\alpha p_0, \alpha q_0) = (p, \alpha q_0)$ satisfies the desired conditions.

1.5. **Definition.** We define a subset $H = H_Q$ of G by

$$H = Q[u] \cap G = \{\alpha \in G \mid (u, \alpha) \in Q\}.$$

We gather a few properties of H. Although most of them follow readily from the definitions, in the interest of completeness we include their proofs.

1.6. **Lemma.** (a) $G' \subset H$,
(b) H *is* τ*-closed,*
(c) $H = H^{-1}$,
(d) $H\beta = \beta H$ *for all* $\beta \in G$,
(e) $Hp = Q[p] \cap Gv$ *for all* $p \in Mv$.

Proof. (a) Let $\alpha \in G'$. Then there exists a net α_i in G which converges to α and to u in the τ-topology. Thus there exist nets in T

$$t_i \to u \quad \text{and} \quad s_i \to u$$

such that

$$\alpha_i t_i \to \alpha, \quad \alpha_i s_i \to u.$$

But then $\alpha s_i \to \alpha u = \alpha$; moreover

$$\alpha s_i(s_i^{-1}t_i) \to \alpha u = \alpha \quad \text{and} \quad \alpha_i s_i(s_i^{-1}t_i) \to \alpha.$$

Hence $(u, \alpha) \in Q$ and $\alpha \in H$.

(b) Let $q \in H \circ p$. Then there exist $\alpha_i \in H$ and $t_i \in T$ such that

$$t_i \to p, \quad \alpha_i t_i \to q.$$

Now $(u, \alpha_i) \in Q$, so $\lim(ut_i, \alpha_i t_i) = (p, q) \in Q$. That is, $q \in Q[p]$ and hence $H \circ p \subset Q[p]$. It follows that

$$\mathrm{cls}_\tau H = (H \circ u) \cap G \subset Q[u] \cap G = H.$$

(c) follows immediately from the fact that Q is symmetric.
(d) follows from 1.3 and the fact that Q is T-invariant.
(e) Using the proof of part (b) we have

$$Hp \subset H \circ p \subset Q[p]$$

for all $p \in M$. If $p \in Mv$, then $Hp = Hpv \subset Mv = Gv$. On the other hand if $q \in Q[p] \cap Gv$, then $(p, q) \in Q$ and hence $(pu, qu) \in Q$. It follows that

$$h = qu(pu)^{-1} \in H.$$

Thus $q = quv = hpuv = hpv = hp \in Hp$.

1.7. **Definition.** Let $p \in M$ and $v \in J$ with $pv = p$. Recall that the collection

$$\{\overline{V} \mid V \subset T \text{ with } p \in \overline{V}\}$$

forms a neighborhood base for $p \in \beta T$. We define

$$L(p) = \bigcap_{p \in \overline{V}} \mathrm{cls}_\tau \, \mathrm{int}_\tau \, \mathrm{cls}_\tau((M \cap \overline{V})v).$$

We now prove a key technical lemma regarding $L(p)$.

1.8. **Lemma.** *Let* $p \in M$. *Then*
(a) $L(p) \neq \varnothing$,
(b) $G'L(p) = L(p)$,
(c) $L(p)w \subset Q[p] \cap Gw$ *for all* $w \in J$.

Proof. (a) Let $\varnothing \neq W \subset M$ be open. We consider the left action

$$G \times M \to M$$
$$(\alpha, q) \to \alpha q.$$

A point $q \in M$ is almost periodic with respect to this action if and only if qt is almost periodic for every $t \in T$. Thus the almost periodic points for this action are dense in M, and we can find $q \in W$ which is almost periodic. Now $q = qw$ for some $w \in J$ and

$$\overline{Gq} = \overline{Gqw} = \overline{Gw}$$

is minimal with respect to the left action of G. Thus there exists a finite set $F \subset Gw$ such that

$$\overline{Gw} \subset F(W \cap \overline{Gw}).$$

But then

$$Gw \subset F(W \cap Gw) \subset F \operatorname{cls}_\tau(W \cap Gw).$$

Hence $\operatorname{int}_\tau \operatorname{cls}_\tau(W \cap Gw) \neq \varnothing$. Since the map $\beta \to \beta v : Gw \to Gv$ is a τ-homeomorphism for any idempotent $v \in J$, we have

$$\varnothing \neq (\operatorname{int}_\tau \operatorname{cls}_\tau(W \cap Gw))v = \operatorname{int}_\tau \operatorname{cls}_\tau(W \cap Gw)v \subset \operatorname{int}_\tau \operatorname{cls}_\tau Wv.$$

The desired result now follows immediately from the finite intersection property.

(b) Let $v \in J$ with $pv = p$, and let $W \subset Gv$ be a τ-open subset. We include a proof that $G' \operatorname{cls}_\tau W = \operatorname{cls}_\tau W$ although this result is found in [E2]. Let $\beta \in W$ and $\alpha \in G'$. Then there exist a net $\alpha_i \in Gv$ which converges to both αv and v in the τ-topology. Then $\alpha_i\beta$ converges to β and hence we may assume that $\alpha_i\beta \in W$ for all i. But $\alpha_i\beta$ also converges to $\alpha\beta$, so $\alpha\beta \in \operatorname{cls}_\tau W$. We have shown that

$$G'W \subset \operatorname{cls}_\tau W$$

from which it follows immediately that $G' \operatorname{cls}_\tau W = \operatorname{cls}_\tau W$. Applying this to $\operatorname{int}_\tau \operatorname{cls}_\tau((M \cap \overline{V})v)$ yields the desired result.

(c) First note that for any neighborhood N of p in M

$$\begin{aligned} L(p)w &\subset \operatorname{cls}_\tau \operatorname{int}_\tau \operatorname{cls}_\tau(Nw) \subset \operatorname{cls}_\tau(Nw) \\ &= (Nw \circ w) \cap Gw \subset ((N \circ w) \circ w) \cap Gw = (N \circ w) \cap Gw. \end{aligned}$$

Let $r \in L(p)w$, so $r \in N \circ w$ for any neighborhood in N of p. Thus there exist nets $p_i \to p$, $t_i \to w$ with $p_i t_i \to r$. On the other hand $rt_i \to rw = r$, so $(p, r) \in Q$.

1.9. **Proposition.** *Let A be a τ-closed subgroup of G such that AH is a group. Then $AHp = AHL(p)$ for all $p \in M$.*

Proof. $AHL(p)$ is the union of a collection of cosets of AH, so it suffices to show that $L(p) \subset AHp$. Let $v \in J$ with $pv = p$. Then by 1.8 and 1.6 $L(p) = L(p)v \subset Q[p] \cap Gv = Hp$.

Let (X, T) be a minimal flow with basepoint x_0 satisfying $x_0 u = x_0$. The so-called *group* of the flow

$$\mathcal{G}(X, T) = \{\alpha \in G \mid x_0\alpha = x_0\}.$$

Our main theorem gives a condition on $\mathcal{G}(X, T)$ which guarantees that $Q(X)$ is an equivalence relation.

1.10. **Theorem.** *Let (X, T) be a minimal flow with group $A = \mathscr{G}(X, T)$, and assume that $H \subset AG'$. Then Q is an equivalence relation on X.*

Proof. Let $(x, y), (y, z) \in Q(X)$. By 1.6 $G' \subset H$, so the assumption that $H \subset AG'$ is equivalent to the statement $AH = AG'$. In particular AH is a group. We first show, using only the assumption that AH is a group, that $z = xhv$ for some $h \in H$ and $v \in J$. We then complete the proof (using the fact that $AH = AG'$) by showing that $(x, z) \in Q(X)$.

The flow (X, T) is minimal, so there exist $w, v \in J$ such that $xw = x$ and $zv = z$. Thus $(xw, yw) = (x, yw)$ and (yw, zw) are both in $Q(X)$. Applying 1.4 there exist $\beta_1, \beta_2, \beta_3 \in Gw$ with

$$(\beta_1, \beta_2), (\beta_2, \beta_3) \in Q \quad \text{and} \quad x_0\beta_1 = x, x_0\beta_2 = yw, x_0\beta_3 = zw.$$

We thus have

$$\beta_2\beta_1^{-1}, \beta_3\beta_2^{-1} \in Q[w] \cap Gw = Hw.$$

The fact that AH and hence AHw are groups now implies that $\beta_3\beta_1^{-1} \in AHw$ and hence $\beta_3 \in AH\beta_1 = A\beta_1 Hw$. It follows that there exists $h \in H$ with $x_0\beta_3 = x_0\beta_1 hw$. Then

$$z = zv = zwv = x_0\beta_3 v = x_0\beta_1 hwv = xhv.$$

To complete the proof we observe that

$$\begin{aligned} v = uv &\in (\beta_1^{-1}AH\beta_1)v = (\beta_1^{-1}A\beta_1)Hv \\ &= (\beta_1^{-1}A\beta_1)H(hv) = (\beta_1^{-1}A\beta_1)HL(hv) \\ &= (\beta_1^{-1}AH\beta_1)L(hv) = (\beta_1^{-1}AG'\beta_1)L(hv) \\ &= (\beta_1^{-1}A\beta_1)G'L(hv) = (\beta_1^{-1}A\beta_1)L(hv). \end{aligned}$$

Thus

$$u = vu \in (\beta_1^{-1}A\beta_1)(L(hv)u) \subset (\beta_1^{-1}A\beta_1)Q[hv],$$

and hence

$$(\beta_1^{-1}\beta\beta_1, hv) \in Q$$

for some $\beta \in A$. We then have $(\beta\beta_1, \beta_1 hv) \in Q$ by 1.3, and

$$(x, z) = (x_0\beta_1, xhv) = (x_0\beta\beta_1, x_0\beta_1 hv) \in Q(X)$$

as desired.

We shall see that the converse of 1.10 is false, but first we deduce some interesting consequences.

1.11. **Proposition.** *Let A be a τ-closed subgroup of G. Then the τ-closed subgroup generated by AH is AE where E is the group of the universal equicontinuous minimal flow.*

Proof. Let B be the τ-closed subgroup of G generated by AH. Since Q is contained in the equicontinuous structure relation on M, $H \subset E$ and hence $B \subset AE$. There exists a minimal flow (X, T) such that $\mathscr{G}(X, T) = B$ (see [E2]). Now $H \subset B = BG'$, so $Q(X)$ is an equivalence relation by 1.10. Thus $Q(X)$ is the equicontinuous structure relation on X (see [A]). In other words the maximal equicontinuous factor of X is given by

$$(*) \qquad X_{\mathrm{eq}} = X/Q(X).$$

The group $\mathscr{G}(X_{\text{eq}}, T) = BE$. On the other hand it follows from $(*)$ that

$$\begin{aligned}\mathscr{G}(X_{\text{eq}}, T) &= \{\alpha \in G \mid (x_0, x_0\alpha) \in Q(X)\}\\ &= \{\alpha \in G \mid x_0\alpha = x_0 h \text{ for some } h \in H\}\\ &= BH = B.\end{aligned}$$

Hence $BE = B$, so $AE \subset B$.

In the following proposition, (X, T) is a minimal flow with basepoint $x_0 = x_0u$ and $A = \mathscr{G}(X, T)$. $P(X)$ denotes the proximal relation on X, and $S(X)$ denotes the equicontinuous structure relation on X.

1.12. **Proposition.** *The following are equivalent*:
(1) AH *is a group,*
(2) $AH = AE$,
(3) $S(X) = Q(X)P(X)$,
(4) $Q(X)P(X)$ *is a closed equivalence relation.*

Proof. (1) $\Rightarrow$ (2) is an immediate consequence of Proposition 1.11, and (2) $\Rightarrow$ (1) is obvious.

(3) $\Rightarrow$ (2) It is sufficient to show that $E \subset AH$. Let $\varepsilon \in E$. Then $(x_0, x_0\varepsilon) \in S(X)$, so $(x_0, y) \in Q(X)$ and $(y, x_0\varepsilon) \in P(X)$ for some $y \in X$. Hence there exists a minimal right ideal $I \subset \beta T$ such that

$$yr = x_0\varepsilon r \quad \text{for all } r \in I.$$

Let $w \in I$ be an idempotent such that $uw = u$ and $wu = w$. (For a proof that such an idempotent exists see [E2].) Then

$$\begin{aligned}(x_0, x_0\varepsilon) &= (x_0u, x_0\varepsilon u) = (x_0uw, x_0\varepsilon uw)\\ &= (x_0w, x_0\varepsilon w) = (x_0w, yw) \in Q(X),\end{aligned}$$

and it follows that $\varepsilon \in AH$.

(2) $\Rightarrow$ (3) Clearly $Q(X)P(X) \subset S(X)$. Suppose that $(x_0, y_0) \in S(X)$ and that $v \in \beta T$ is a minimal idempotent with $y_0v = y_0$. Then

$$y_0u = x_0\varepsilon \quad \text{for some } \varepsilon \in E;$$

and since we are assuming that $AH = AE$, there exists $h \in H$ with $x_0\varepsilon = x_0h$. Now

$$(x_0, x_0h) \in Q(X) \quad \text{and} \quad (x_0h, x_0hv) \in P(X),$$

so $(x_0, y_0) = (x_0, y_0uv) = (x_0, x_0hv) \in Q(X)P(X)$. In particular note that if $y_0u = y_0$ we have shown that $(x_0, y_0) \in Q(X)$.

Now suppose that $(x, y) \in S(X)$ with $xw = x$ where $w \in \beta T$ is a minimal idempotent. Let $\alpha \in G$ such that $x\alpha = x_0$. Then as was noted above we have

$$(x_0, y\alpha) = (x\alpha, y\alpha) \in Q(X).$$

Thus

$$(x, yw) = (x_0\alpha^{-1}w, (y\alpha)\alpha^{-1}w) \in Q(X).$$

Since $(yw, y) \in P(X)$, it follows that $(x, y) \in Q(X)P(X)$.

(4) $\Rightarrow$ (3) The relation $Q(X)P(X)$ is T-invariant, contains $Q(X)$, is contained in $S(X)$, and so must be $S(X)$.

(3) $\Rightarrow$ (4) is obvious.

1.13. *Remarks.* (1) In view of 1.11, 1.10 is equivalent to: $E \subset AG'$ implies $Q(X)$ is an equivalence relation. Since this statement involves only the group A of the flow, it is natural to ask whether $Q(X)$ being an equivalence relation is a "group condition"; i.e., given that $\mathcal{G}(X, T) = \mathcal{G}(Y, T)$ and that $Q(X)$ is an equivalence relation does it follow that $Q(Y)$ is one also?

(2) In [E3] it is shown that when the almost periodic points are dense in $X \times X$, $E \subset AG'$. Thus 1.10 generalizes the result that $Q(X)$ is an equivalence relation when the almost periodic points are dense in $X \times X$. In particular, if (X, T) is point distal, then the almost periodic points are dense in $X \times X$ and it follows from 1.10 that Q is an equivalence relation for any proximal extension of (X, T).

(3) The proof of the crucial lemma 1.8 raises the question: Is every $p \in M$ an almost periodic point of the flow (G, M), and if not why not?

The following lemma follows from a combination of Proposition 14.14 of [E2] and Proposition 3.10 of [EGS]. In the interest of completeness we include an outline of a (different) proof.

1.14. **Lemma.** *Let D be the group of the universal distal minimal flow. Then $G'D = E$.*

Proof. First note that $G' \subset H \subset E$ and $D \subset E$ (because the universal equicontinuous flow is a factor of the universal distal flow). Thus $G'D \subset E$. Let B be any τ-closed normal subgroup of G which contains $G'D$. Then the map

$$t \to utu : T \to G/B$$

is a homomorphism. Indeed G/B is a compact topological group, so $(G/B, T)$ is an equicontinuous flow. Applying this to $B = G'D$ and $B = E$ we obtain $G/G'D \to G/E$, a homomorphism of equicontinuous flows. But G/E is the maximal equicontinuous flow, so $G'D = E$.

1.15. **Corollary.** *Let (X, T) be a minimal flow for which the proximal relation is closed. Then $Q(X)$ is an equivalence relation.*

Proof. Since the proximal relation is closed, it must be an equivalence relation (see [A]). Dividing out by this relation we see that (X, T) is a proximal extension of a distal flow. Hence $D \subset \mathcal{G}(X, T) = A$, which implies (by 1.14) that $H \subset E = DG' \subset AG'$. It then follows from 1.10 that $Q(X)$ is an equivalence relation.

We now give an example which shows that the converse of Theorem 1.10 is false.

1.16. **Example.** Let $X = S^{n-1}$ and $T = SL(n, \mathbf{R})$ with $n \geq 2$. We consider the action

$$X \times T \to X,$$
$$(x, l) \to \frac{xl}{\|xl\|}.$$

Then Q is an equivalence relation on X; in fact $(x, y) \in Q(X)$ for all $x, y \in X$. Moreover van der Waerden (see [W]) shows that the constants are the only almost periodic functions on T when the latter is provided with the discrete topology; hence $E = G$ in this case. On the other hand the identification of antipodal points gives a $\mathbf{Z}_2$ group extension:

$$X = S^{n-1} \to \mathbf{RP}^n.$$

But $(\mathbf{RP}^n, T)$ is proximal, so $\mathscr{G}(\mathbf{RP}^n, T) = G$. Therefore $G/\mathscr{G}(X, T) \cong \mathbf{Z}_2$ and $A = \mathscr{G}(X, T)$ is of index 2 in G. It follows that A is τ-clopen and $G' \subset A \neq G$. Now the fact that $Q(X) = X \times X$ implies that $AH = G$. Hence $H \not\subset A = AG'$.

References

[A] J. Auslander, *Minimal flows and their extensions*, North-Holland, Amsterdam, 1988.

[E1] R. Ellis, *Group-like extensions of minimal sets*, Trans. Amer. Math. Soc. **127** (1967), 125–135.

[E2] ——, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[E3] ——, *The Veech structure theorem*, Trans. Amer. Math. Soc. **186** (1973), 203–218.

[EGS] R. Ellis, S. Glasner, and L. Shapiro, *Proximal-isometric (PI) flows*, Adv. in Math. **17** (1975), 213–260.

[EK] R. Ellis and H. Keynes, *A characterization of the equicontinuous structure relation*, Trans. Amer. Math. Soc. **161** (1971), 171–181.

[M] D. C. McMahon, *Relativized weak disjointness and relatively invariant measures*, Trans. Amer. Math. Soc. **236** (1978), 225–237.

[V] W. A. Veech, *The equicontinuous structure relation for minimal abelian transformation groups*, Amer. J. Math. **90** (1968), 723–732.

[W] B. L. van der Waerden, *Stetigkeitssatz der halbeinfachen Lieschen Gruppen*, Math. Z. **161** (1933), 780–786.

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