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THE CONNECTION MATRIX IN MORSE-SMALE FLOWS II

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ABSTRACT. Given a connection matrix for a Morse-Smale flow on a compact manifold, if there are no periodic orbits of equal or adjacent indices related in the partial order, we show that the periodic orbits can be replaced by doubly connected rest points in such a way that the given connection matrix induces the unique connection matrix for the resulting flow. It follows that for this class of flows, all nonuniqueness in the connection matrix is a consequence of the continuation theorem for connection matrices.

1. INTRODUCTION

The purpose of this paper is to extend the results of [8] on the connection matrix for Morse-Smale flows. Recall that a Morse-Smale flow on a manifold is one where the chain recurrent set (and hence the Morse decomposition) consists of hyperbolic critical points and periodic orbits, and all stable and unstable manifolds intersect transversally. In [8] it was shown that if there are no periodic orbits, then the connection matrix is unique, but if there are periodic orbits, then the connection matrix may not be unique. However, the nonuniqueness is a consequence of the continuation theorem for the connection matrix. If we replace each periodic orbit with two doubly connected rest points in such a way that transversality is preserved and the flow is not changed except on small neighborhoods of the periodic orbits, then the (unique) connection matrix for the altered flow induces a connection matrix for the original flow. On 2-manifolds, all nonuniqueness is a consequence of this continuation property if some technical assumptions are satisfied, i.e., if Δ is a connection matrix for a given Morse-Smale flow on a 2-manifold, then it is possible to replace the periodic orbits with doubly connected rest points in such a way that the resulting flow has Δ as its unique connection matrix. In this paper we will prove a result for manifolds of dimension greater than two. We must assume there are no orbits connecting periodic orbits of adjacent indices. It is not known to what extent this hypothesis can be relaxed, but this assumption allows us to read off connection information for sets below a periodic orbit by looking at the flow on the boundary of a neighborhood of the periodic orbit (Lemma 4.1), and this lemma is not true if periodic orbits of adjacent indices are present.

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The rest of this section will be used to summarize background material. We assume the reader is familiar with Morse decompositions, the Conley index, and index filtrations. References include [1, 3, 9].

Notation. Let $\{ M_p \mid p \in (P, <) \}$ be a Morse decomposition of an isolated invariant set S. For an interval $I \subset P$, define

$$C(M_p, M_{p'}) = \{ x \mid \omega^*(x) \subset M_p, \omega(x) \subset M_{p'} \},$$

$$M(I) = \bigcup_{p \in P} M_p \cup \left(\bigcup_{p, p' \in P} C(M_p, M_{p'}) \right).$$

Then $M(I)$ is an isolated invariant set and we define

$$CH(I) = H_*(h(M(I); \mathbf{Z}_2)).$$

If (A, A^*) is an attractor-repeller pair in an isolated invariant set S, then we can find a compact triple of spaces (N_2, N_1, N_0) such that (N_2, N_0) is an index pair for S, (N_2, N_1) is an index pair for A^*, and (N_1, N_0) is an index pair for A. We consider N_2/N_1 as a pointed space with the equivalence class of N_1 as the distinguished point, and similarly for the other two pairs. Then there is a long exact sequence of pointed spaces (using $\mathbf{Z}_2$ coefficients)

$$\cdots \longrightarrow H_q(N_1/N_0) \longrightarrow H_q(N_2/N_0) \longrightarrow H_q(N_2/N_1) \xrightarrow{\partial} H_{q-1}(N_1/N_0) \longrightarrow \cdots.$$

Since this is essentially independent of the triple we write

$$\cdots \longrightarrow CH_q(A) \longrightarrow CH_q(S) \longrightarrow CH_q(A^*) \xrightarrow{\partial(A, A^*)} CH_{q-1}(A) \longrightarrow \cdots.$$

We call $\partial(A, A^*)$ the flow defined boundary map. Exactness implies that if $CH(S) = 0$, then $\partial(A, A^*)$ is an isomorphism. If $C(A^*, A) = \varnothing$, then $CH(S) \cong CH(A^*) \oplus CH(A)$ and it follows that $\partial(A, A^*) = 0$. So we have

Lemma 1.1. *If $\partial(A, A^*) \neq 0$, then $C(A^*, A) \neq \varnothing$.*

Given a Morse decomposition $\{ M_p \mid p \in P \}$, if p and p' are adjacent with $p < p'$, then there is a flow defined boundary map $\partial(p, p')$. Similarly, if (I, J) is an adjacent pair of intervals, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$, so there is an exact sequence

$$\text{(1.2.)} \qquad \cdots \longrightarrow CH_q(I) \longrightarrow CH_q(IJ) \longrightarrow CH_q(J) \xrightarrow{\partial(I, J)} CH_{q-1}(I) \longrightarrow \cdots$$

The connection matrix condenses the Morse theoretic information contained in the maps $\partial(I, J)$ into maps defined between the individual sets $\{ M_p \mid p \in P \}$. To do this, for an interval $I \subset P$, define

$$C\Delta(I) = \bigoplus_{i \in I} CH(i)$$

and let $C\Delta$ denote $C\Delta(P)$. A $\mathbf{Z}_2$-linear map $\Delta \colon C\Delta \to C\Delta$ can be thought of as a matrix

$$\left[\Delta(p, p') \colon CH(p') \to CH(p) \mid p', p \in P \right].$$

We say $\Delta = \Delta(P)$ is upper triangular if $\Delta(p, p') = 0$ for $p \not< p'$ and Δ is a boundary map if each $\Delta(p, p')$ has degree -1 and $\Delta \circ \Delta = 0$. It is not difficult to show that if Δ is an upper triangular boundary map, then so is the restriction $\Delta(I)\colon C\Delta(I) \to C\Delta(I)$. If I and J are adjacent intervals, then there is an obvious exact sequence of chain complexes

$$0 \longrightarrow C\Delta(I) \longrightarrow C\Delta(IJ) \longrightarrow C\Delta(J) \longrightarrow 0$$

which gives a long exact homology sequence

$$\cdots \longrightarrow H_q\Delta(I) \longrightarrow H_q\Delta(IJ) \longrightarrow H_q\Delta(J) \longrightarrow H_{q-1}\Delta(I) \longrightarrow \cdots. \tag{1.3}$$

Definition 1.4. We say the upper triangular boundary map $\Delta\colon C\Delta \to C\Delta$ is a *connection matrix* if for each interval $I \subset P$ there is a homomorphism $\Phi\colon H\Delta(I) \to CH(I)$ such that:

(1) For $p \in P$, $\Phi(p)\colon H\Delta(p) = CH(p) \to CH(p)$ is the identity.
(2) For each adjacent pair of intervals (I, J), the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_q\Delta(I) \arrow[r] \arrow[d, "\Phi(I)"] & H_q\Delta(IJ) \arrow[r] \arrow[d, "\Phi(IJ)"] & H_q\Delta(J) \arrow[r] \arrow[d, "\Phi(J)"] & H_{q-1}\Delta(I) \arrow[r] \arrow[d, "\Phi(I)"] & \cdots \\
\cdots \arrow[r] & CH_q(I) \arrow[r] & CH_q(IJ) \arrow[r] & CH_q(J) \arrow[r, "\partial(I,J)"] & CH_{q-1}(I) \arrow[r] & \cdots
\end{tikzcd}$$

where the top row is 1.3 and the bottom row is 1.2.

The existence of connection matrices was shown by Franzosa (see [4]). The first condition implies that if p and p' are adjacent, then $\Delta(p, p') = \partial(p, p')$, the flow defined boundary map. Using induction and the 5-lemma, the second condition implies that $H\Delta(I) \cong CH(I)$ for any interval I.

2. Statement of the Result

In this section we assume that the flow under consideration is a smooth (i.e., C^2) Morse-Smale flow on a smooth n-manifold. The Morse decomposition $\{M_p \mid p \in P\}$ consists of hyperbolic rest points and periodic orbits, and stable and unstable manifolds intersect transversally. If M_p is a rest point, then $h(M_p) = \Sigma^k$, the pointed k-sphere, where k is the dimension of the unstable manifold. If M_p is a periodic orbit, let k be the number of eigenvalues of the Poincaré map of modulus greater than 1. We call M_p *twisted* if the Poincaré map reverses orientation on the eigenspace corresponding to the positive eigenvalues, otherwise M_p is *untwisted.* If M_p is untwisted, then $h(M_p) = \Sigma^k \vee \Sigma^{k+1}$ (where $\vee$ means take the disjoint union and identify the distinguished point), and if M_p is twisted, then $h(M_p) = \Sigma^{k-1} \wedge \mathbf{R}P^2$ (where $X \wedge Y = (X \times Y)/(X \vee Y)$). In either case, we say the periodic orbit has index k, and its homology index is

$$CH_q(M_p) \cong \begin{cases} \mathbf{Z}_2 \text{ if } q = k,\, k+1, \\ 0 \text{ otherwise.} \end{cases}$$

(If $k = 0$, then $h(M_p)$ is actually the disjoint union of a circle and the distinguished point, but the homology is isomorphic to the homology of $\Sigma^0 \vee \Sigma^1$, so we will abuse notation.) The following result is proven in [8].

Theorem 2.1. *Suppose there are no periodic orbits. Then each nonzero map in the connection matrix is flow defined, so the connection matrix is unique.*

In the same paper we construct examples with periodic orbits where the connection matrix is not unique. We want to "replace" the periodic orbits with doubly connected rest points to obtain a unique connection matrix. The following definition makes precise the notion of "replace". Let W^u denote the unstable manifold and W^s denote the stable manifold.

Definition 2.2. Let $\mathscr{U} = \{U_1, U_2, \dots, U_m\}$ be a collection of pairwise disjoint neighborhoods of the periodic orbits $\gamma_1, \gamma_2, \dots, \gamma_m$ in the Morse-Smale flow ϕ such that $\gamma_i \subset U_i$ for $i = 1, \dots, m$ and each U_i is disjoint from every other set in the Morse decomposition. We call the flow ϕ' a $\mathscr{U}$-refinement of ϕ if

(1) ϕ' is Morse-Smale.
(2) ϕ' agrees with ϕ outside of $\bigcup_{i=1}^{k} U_i$.
(3) In each U_i, ϕ' has two rest points $\{q, p\}$ of index k and $k+1$ where $k+1$ is the dimension of the unstable manifold of γ_i. There are exactly two orbits connecting p and q and there are no other rest points or periodic orbits in U_i. Finally, $W^u(\gamma_i)$ in ϕ equals $W^u(p) \cup W^u(q)$ in ϕ' and $W^s(\gamma_i)$ in ϕ equals $W^s(p) \cup W^s(q)$ in ϕ'.

A picture of a step in a $\mathscr{U}$-refinement is given in Figure 1.

The repelling periodic orbit is replaced by a repelling fixed point, a saddle point, and two connecting orbits. The vector field generating the flow is unchanged on a neighborhood of the periodic orbit. The idea of such a refinement is due to Franks [2]. Notice that $CH(\gamma) \cong CH(M(p, q))$ where p, q are the rest points that replace γ.

Suppose $\{M_p \mid p \in P\}$ is a Morse decomposition of an isolated invariant set S in ϕ and ϕ' is a $\mathscr{U}$-refinement of ϕ. Then there is a Morse decomposition $\{M'_p \mid p \in P\}$ of S in ϕ' as follows. If M_p is a rest point, then $M'_p = M_p$ and if M_p is a periodic orbit γ, then M'_p consists of the two rest points plus the two connecting orbits which replaced γ. However, there is a finer Morse decomposition for S in ϕ', namely the rest points. Denote this Morse decomposition by $\{M''_r \mid r \in R\}$. Let Δ'' be a connection matrix for $\{M''_r \mid r \in R\}$ in the flow ϕ'. Using the isomorphism $CH(\gamma) \cong CH(M(p, q))$, Δ'' induces a map $\Delta : C\Delta(P) \to C\Delta(P)$ for the original Morse decomposition $\{M_p \mid p \in P\}$ in ϕ.

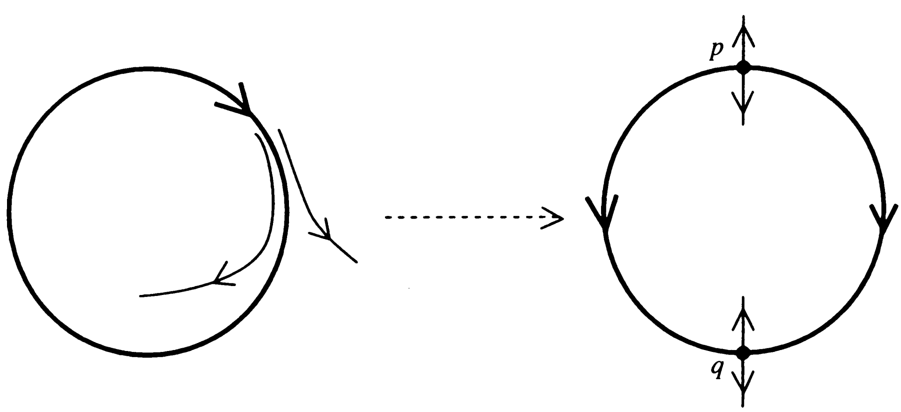

FIGURE 1

Theorem 2.3 [8]. *If the U_i in $\mathscr{U}$ are small enough, then the map Δ induced by Δ'' is a connection matrix for the Morse decomposition $\{M_p \mid p \in P\}$ in ϕ.*

The interesting question is the converse of Theorem 2.3: given a connection matrix Δ for a Morse-Smale flow and a collection $\mathscr{U}$, can one find a $\mathscr{U}$-refinement ϕ' whose (unique) connection matrix induces Δ? The answer is yes if no periodic orbits of the same or adjacent indices are comparable in the flow ordering on M_p.

Suppose γ is a periodic orbit of index one. Then the rest point q which is introduced in the $\mathscr{U}$-refinement will have two orbits in its unstable manifold. Thus we must assume the following.

(A1) If γ is a periodic orbit of index one, then there are two 1's in the column of Δ corresponding to $CH_1(\gamma)$. Similarly, if γ is a periodic orbit of index $n-2$, then there are at most two 1's in the row of Δ corresponding to $CH_{n-1}(\gamma)$.

Also, if γ is a periodic orbit of index one, and there are two 1's in the column of Δ corresponding to $CH_1(\gamma)$ we must make the following assumption.

(A2) If γ is a periodic orbit of index one, and $W^u(\gamma)$ is locally an annulus (i.e., γ is untwisted), then for any $p \in P$ with $p < \gamma$, $W^s(p)$ does not intersect both boundary circles of the annulus. Similarly, if γ is a periodic orbit of index $n-2$, and $W^s(\gamma)$ is locally an annulus, then for any $p \in P$ with $\gamma < p$, $W^u(p)$ does not intersect both boundary circles of the annulus.

These assumptions are necessary. Examples are discussed in [8].

Theorem 2.4. *Fix a collection of neighborhoods $\mathscr{U}$ of the periodic orbits of a Morse-Smale flow on a compact n-manifold. Assume* A1, A2 *and*

(A3) *If γ is a periodic orbit of index k, then there are no periodic orbits of index $k-1$, k, or $k+1$ comparable to γ in the flow defined order on $\{M_p \mid p \in P\}$.*

Then given any connection matrix Δ, there is a $\mathscr{U}$-refinement of the flow whose connection matrix induces Δ for the original flow.

It is not known to what extent assumption A3 can be relaxed. The rest of this paper is devoted to the proof of Theorem 2.4.

3. Preliminary results

In this section we collect some lemmas which will be used in our main construction.The following lemma is quite useful.

Lemma 3.1 [8]. *Suppose $\{M_p \mid p \in (P, <)\}$ is a Morse decomposition of an isolated invariant set S, $\{x_n\}$ is a sequence in S and $\omega(x_n) \subset M_p$ for each n, and $x_n \to x$. Then $\omega(x) \subset M_{p'}$ for some p' with $p \leq p'$. Similarly, if $\omega^*(x_n) \subset M_r$ for each n, then $\omega^*(x) \subset M_{r'}$ for some r' with $r' \leq r$.*

From now on we will assume that the flow is Morse-Smale on a compact manifold, and generated by a smooth vector field X.

Definition 3.2. If a periodic orbit γ of index k is untwisted, we say γ is in *standard form* if there is a tubular neighborhood V of γ with coordinates

$(\theta, x, y) \in S^1 \times \mathbf{R}^k \times \mathbf{R}^{n-k-1}$ in V such that the vector field has the form

$$X = \frac{\partial}{\partial\theta} + \sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} - \sum_{j=1}^{n-k-1} y_i \frac{\partial}{\partial y_i}.$$

If γ is twisted, then it is in standard form if it has a neighborhood whose double cover has coordinates (θ, x, y) in which the vector field has the above form. If we define the equivalence relation $\sim$ by

$$(\theta, x_1, \ldots, x_k, y_1, \ldots, y_{n-k-1}) \sim (\theta + \pi, -x_1, x_2, \ldots, x_k, -y_1, y_2, \ldots, y_{n-k-1})$$

on V, and let $\tilde{V}$ be the quotient $V/\sim$, then the vector field X induces a vector field $\tilde{X}$ on $\tilde{V}$. We say γ is in standard form if there is a neighborhood of γ diffeomorphic to $\tilde{V}$ and the diffeomorphism carries the vector field to $\tilde{X}$.

An argument of Franks [2] generalizing a result of Newhouse-Peixoto [6] shows that any Morse-Smale flow is topologically conjugate to a Morse-Smale flow whose periodic orbits are in standard form. Since topologically conjugate flows have the same connection matrices, we may assume that all of the periodic orbits are in standard form.

The following lemmas describe the flow defined boundary maps in some of the situations we will be interested in.

Lemma 3.3 [7]. *Suppose M_p is a rest point of index $k+1$ and M_q is a rest point of index k. If p and q are adjacent, then $\partial(q, p)$ counts the number of connecting orbits* (mod 2).

Lemma 3.4 [8]. *Suppose γ is a periodic orbit of of index k and M_q is a rest point of index k. If γ and q are adjacent, then $\partial(q, \gamma)$ counts the number of connecting orbits* (mod 2). *Similarly, if M_p is a rest point of index $k+1$ and if p and γ are adjacent, then $\partial(\gamma, p)$ counts the number of connecting orbits* (mod 2).

In the coordinates (θ, x, y) descibed in Definition 3.2, in a neighborhood V of a periodic orbit γ, the vector field has the form

$$\frac{\partial}{\partial\theta} + \sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} - \sum_{j=1}^{n-k-1} y_i \frac{\partial}{\partial y_i}.$$

Locally, $W^u(\gamma) = \{(\theta, x, 0)\}$ and $W^s(\gamma) = \{(\theta, 0, y)\}$. Following Franks [2], we can replace γ by two rest points, in the following way. Suppose first that γ is untwisted. Fix $\epsilon > 0$. Let $\rho(t)$ be a smooth function which is 0 if $t > \epsilon$ and 1 if $t < \epsilon/2$. Define a new vector field X' by

$$X' = [\rho(x^2+y^2)\sin\theta + (1-\rho(x^2+y^2))]\frac{\partial}{\partial\theta} + \sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} - \sum_{j=1}^{n-k-1} y_i \frac{\partial}{\partial y_i}. \tag{3.5}$$

Note that we have replaced the periodic orbit by 2 hyperbolic critical points, $p = (0, 0, 0)$ of index Σ^{k+1} and $q = (\pi, 0, 0)$ of index Σ^k. If the resulting flow is Morse-Smale, then this replacement is one step in a $\mathscr{U}$-refinement. If γ is twisted, then we work in the double cover and use $\sin 2\theta$ instead of $\sin\theta$ in the definition of X'. This gives us a well-defined vector field on the quotient.

Let $B^- = \{(\theta, x, 0) \mid |x| = \epsilon/2\}$. Then $B^- = S^1 \times S^{k-1} \subset W^u(\gamma)$ under the flow for X and $B^- \subset (W^u(p) \cup W^u(q))$ in the flow for X'. Note that in the flow for X', $W^u(q) \cap B^- = \{(\pi, x, 0) \mid |x| = \epsilon/2\}$. In particular, $W^u(q) \cap B^-$ represents a generator of $H_{k-1}(B^-)$. Notice also that

$$B^- = \bigcup_{r<\gamma} (W^s(r) \cap B^-) \tag{3.6}$$

and each intersection is a smooth manifold whose dimension is determined by the index of M_r. We will alter the vector field inside of B^- to arrange the appropriate connections for the rest point q.

Lemma 3.7. *Given any $(k-1)$-sphere $\mathcal{S}$ in B^- which is isotopic to $\{(\pi, x, 0) \mid |x| = \epsilon/2\}$, by adjusting the vector field in the set $\{(\theta, x, 0) \mid |x| < \epsilon/2\}$ we can arrange for $W^u(q) \cap B^- = \mathcal{S}$.*

Proof. The situation is illustrated in Figure 2.

First assume that gamma is untwisted. We will alter the vector field to

$$\alpha(\theta, x, y)\frac{\partial}{\partial \theta} + \sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} - \sum_{j=1}^{n-k-1} y_i \frac{\partial}{\partial y_i} \tag{3.8}$$

with an appropriate function α. For $\eta > 0$, define $B_\eta^- = \{(\theta, x, 0) \mid |x| = \eta\}$. For any α, the vector field has two properties that we will exploit. First, for any η, B_η^- is tranverse to the flow, and second, flowing for time t takes B_η^- diffeomorphically to $B_{\eta'}^-$ for some η' because of the form of the $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ terms in the vector field. We first define α to be

$$\alpha(\theta, x, 0) = \begin{cases} \sin\theta, & \text{if } |x| > \epsilon/3 \text{ or } |x| < \epsilon/12, \\ 0, & \text{if } x = \epsilon/4 \text{ or } x = \epsilon/6. \end{cases}$$

Now fill in $\alpha(\theta, x, 0)$ in a smooth way for $|x| \in [\epsilon/12, \epsilon/6]$ and $|x| \in [\epsilon/4, \epsilon/3]$. Let $\mathcal{S}'$ be the image of $\mathcal{S}$ in $B_{\epsilon/4}^-$ when we flow $\mathcal{S}$ backward by the flow of 3.8. Let $\mathcal{S}''$ denote $W^u(q) \cap B_{\epsilon/6}^-$. $\mathcal{S}'$ is isotopic to

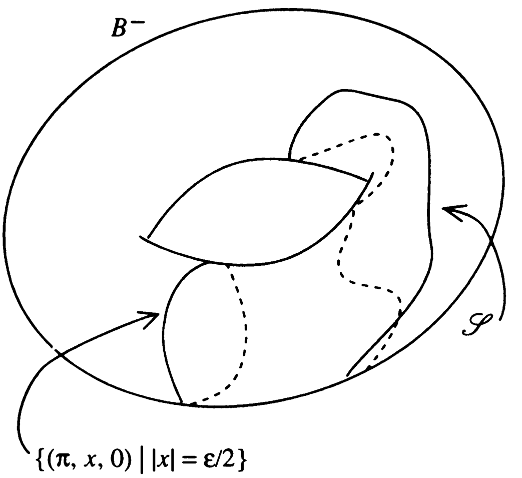

FIGURE 2

$\mathcal{S}''$ in $S^1 \times S^{k-1}$ since $\mathcal{S}$ is isotopic to $\{(\pi, x, 0) \mid |x| = \epsilon/2\}$. By the isotopy extension theorem (see, e.g., [5]), we can extend the isotopy of $\mathcal{S}'$ to $\mathcal{S}''$ to an isotopy of $S^1 \times S^{k-1}$. We use this isotopy to fill in $\alpha(\theta, x, 0)$ for $|x| \in [\epsilon/6, \epsilon/4]$ and take $\mathcal{S}''$ to $\mathcal{S}'$ under the flow. By construction, $W^u(q) \cap B^- = \mathcal{S}$.

If γ is twisted, we make the above construction in a fundamental domain of the double cover and extend to the whole double cover by $\sim$ equivariance. □

Remark 3.9. Notice that α in the proof of Lemma 3.7 is only restricted in a neighborhood of $W^u(\gamma)$, and we can arrange α to be $\sin\theta$ in a neighborhood of $W^s(\gamma) = \{(\theta, 0, y)\}$. We can do a similar isotopy in the stable manifold for $|y| \in [\epsilon/12, \epsilon/3]$ without changing $W^u(q) \cap B^-$. Thus we can arrange to have $W^s(p) \cap \{(\theta, 0, y) \mid |y| = \epsilon/2\}$ to be any curve isotopic to $\{(0, 0, y) \mid |y| = \epsilon/2\}$ as well as $W^u(q) \cap B^- = \mathcal{S}$

4. The main construction

In this section we indicate how to replace a periodic orbit with two critical points to obtain the appropriate number of connecting orbits. Let ϕ be a Morse-Smale flow, let Δ be an upper triangular boundary map satisfying

(1) if p and p' are adjacent, $p < p'$, then $\Delta(p, p') = \partial(p, p')$,
(2) for any interval $I \subset P$, $H\Delta(I) \cong CH(I)$,

and assume the hypotheses of Theorem 2.4 are satisfied. Notice that Δ is like a connection matrix, but we do not require the isomorphisms $\Phi(I)$ of Definition 1.4. As noted in §1, any connection matrix satisfies these assumptions.

Let γ be a periodic orbit of index k, with coordinates and set B^- as in §3. Consider the collection of Morse sets M_r such that the entry $\Delta(r, \gamma)$ could be nonzero, i.e., $r < \gamma$ and $CH_q(r) \neq 0$ for $q = k$ or $q = k-1$. Assumption A3 implies M_r is not a periodic orbit of index k or $k-1$. If M_r is a point of index k, then r and γ are adjacent. By transversality, $W^s(r) \cap B^-$ must be a finite number of points, and Lemma 3.4 implies that $\Delta(r, \gamma)$ counts the number of connecting orbits (mod 2). If M_r is a point of index $k-1$, then $W^s(r) \cap B^-$ is a 1-manifold, i.e., a finite collection of circles and open intervals. If $W^s(r) \cap B^-$ is not a circle, then by Lemma 3.1, the endpoints of the intervals must lie on $W^s(s)$ where $r < s < \gamma$. By assumption A3 and transversality, M_s must be a rest point of index k. Thus

$$\left(\bigcup_{h(M_r)=\Sigma^k} W^s(r) \cap B^-\right) \cup \left(\bigcup_{h(M_s)=\Sigma^{k-1}} W^s(s) \cap B^-\right)$$

is a graph Γ plus some circles in B^-, where the vertices are labeled by points of index k and the edges and circles are labeled by points of index $k-1$.

Lemma 4.1. *Suppose M_r is a point of index Σ^k, M_s is a point of index Σ^{k-1}, and the set of connecting orbits $C(M_r, M_s)$ consists of j orbits. Then for each vertex in Γ labeled M_r, there are exactly j edges labeled M_s incident to it (an edge is counted twice if both ends are incident to the vertex). The matrix entry $\Delta(s, r) \equiv j \pmod 2$.*

Proof. $W^u(M_r) \cap W^s(M_s)$ consists of j components, and the intersection is transverse. Choose a small disc $D \subset B^-$ such that r is the only vertex in D

and for each s-edge e_i, $D \cap e_i$ is connected, i.e., $D \cap e_i$ is diffeomorphic to a line. For any $\epsilon > 0$, there is a T such that flowing forward for time T will map D ϵ-close (in the C^1 sense) to a disc D' in $W^u(M_r)$. For ϵ small enough, the transversality implies that the image of D intersects $W^s(M_s)$ in precisely j components, and these are the images of $D \cap e_i$ under the flow, so there are exactly j s-edges in D. Transversality and assumption A3 imply that M_r and M_s are adjacent, so Lemma 3.3 implies $\Delta(s, r) \equiv j \pmod 2$. □

Let M_s be a rest point of index $k-1$, so $W^s(s) \cap B^-$ is a 1-manifold. We will now show that one can isotop $W^u(q) \cap B^-$ (where q is the rest point of index k created when γ is replaced) so that $|C(q, s)| \equiv 1 \pmod 2$. We must then show that this procedure can be carried out simultaneously for all such s, but we first illustrate the method for a single s. Start by replacing γ by (p, q) as in 3.5. Let $\mathscr{S} = W^u(q) \cap B^-$, and make a small isotopy of $\mathscr{S}$, if necessary, to obtain transversality.

Lemma 4.2. *If M_s is a rest point or periodic orbit of index $k-1$, then $\mathscr{S}$ can be isotoped so that after the the isotopy there is transversality and $|\mathscr{S} \cap W^s(s)| \equiv \Delta(s, \gamma) \pmod 2$.*

Proof. If $k = 0$ there is nothing to prove. If $k = 1$, then the argument is similar to the one given here, but it is it is slightly different if B^- is not connected, and it is contained in [8]. So assume $k > 1$.

First assume that γ is untwisted. Let $i = |W^u(q) \cap B^- \cap W^s(s)|$. If we isotop $\mathscr{S} = W^u(q) \cap B^-$ across a circle or an edge in Γ, then after perturbing to achieve transversality, i will not change (mod 2). Thus, the parity of i can only change when we isotop $\mathscr{S}$ through a vertex. Also, if r is a vertex, then as we isotop $\mathscr{S}$ through r, each s edge either gains an intersection (if there was none before) or loses an intersection (if there was one before). See Figure 3.

The rays eminating from the central vertex are the edges in $W^s(s)$. Thus, as we isotop $\mathscr{S}$ through the vertex we have

$$i \text{ before isotopy } + i \text{ after isotopy } = \# \text{ of } s \text{ edges incident to } r.$$

If the number of s edges incident to r is even, then i will not change (mod 2). If the number of edges is odd, then i will change (mod 2).

It is possible to isotop $\mathscr{S}$ so that it passes through only one vertex r during the isotopy. Choose a point $x \in \mathscr{S} \setminus \Gamma$ and a smooth path τ in B^- connecting

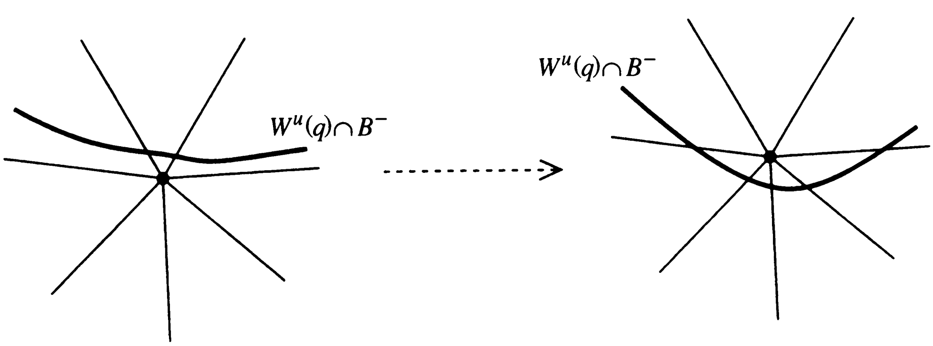

FIGURE 3

x and r which does not pass through $\mathscr{S}$ or any other vertex. Then there is a small neighborhood W of τ which does not intersect any other vertex. The isotopy fixes points in $\mathscr{S} \setminus W$. Inside W, we isotop $\mathscr{S}$ so that it stays in W and passes through the vertex. If γ is twisted, fix a fundamental domain in the double cover and choose τ to lie in this fundamental domain, then make the isotopy $\sim$ equivariant.

To finish the proof of the lemma, we must show that either there is a vertex with an odd number of s edges incident to it (to isotop through and change the parity of $|\mathscr{S} \cap W^s(s)|$ if necessary), or else $\Delta(s, \gamma) \equiv |\mathscr{S} \cap W^s(s)|$ for any isotopy class with transverse intersections. Suppose there is no vertex of odd degree. Consider the interval I generated by s and γ in P, i.e., $I = \{p \in P \mid s \le p \le \gamma\}$. Assumption A3 implies that $I = \{s, r_1, \dots, r_l, \gamma\}$ where $r_1, \dots, r_l$ are points of index Σ^k. The restriction $\Delta(I)$ of the connection matrix to I has the following form:

$$\Delta(I) = \begin{array}{c} \\ s \\ r_1 \\ \vdots \\ r_l \\ \gamma^k \\ \gamma^{k+1} \end{array} \begin{array}{c} \begin{array}{cccccc} s & r_1 & \dots & r_l & \gamma^k & \gamma^{k+1} \end{array} \\ \begin{pmatrix} & 0 & \dots & 0 & * & \\ & & & & & * \\ & & & & & \vdots \\ & & & & & * \\ & & & & & 0 \\ & & & & & \end{pmatrix} \end{array} .$$

The zeroes in the first row are there because of Lemma 4.1 and the assumption that each vertex has even degree. We have $\dim H\Delta(I) = \dim CH(I)$, and $CH(I)$ is determined by the flow. The dimension of $H\Delta_{k-1}(I)$ depends on the entry $\Delta(s, \gamma^k)$. This entry is thus determined by the flow. If we look at $I' = \{s, r_1, \dots, r_l, p, q\}$ after a replacement, we have $CH(I') \cong CH(I)$ since the filtration in the original flow gives us an index pair for $M(I')$. By Lemma 3.3 the $\Delta(s, r_j)$ entries of any I' connection matrix must be the same as those in the above connection matrix, and since $CH(I') \cong CH(I)$, in any I' connection matrix, the (s, q) entry must be the same as $\Delta(s, \gamma^k)$ regardless of how $W^u(q)$ sits in B^-. Thus for any isotopy of $\mathscr{S}$, $\Delta(s, \gamma) \equiv |\mathscr{S} \cap W^s(s)|$ (mod 2). □

Of course, an analogous statement holds for the $W^s(p)$ in the stable manifold when M_s is of index Σ^{k+2}, and the isotopy in the stable manifold is independent of the one in the unstable manifold.

If there are several sets M_s as in Lemma 4.3, then the isotoping of $\mathscr{S}$ to get the appropriate flow defined maps in the new connection matrix (i.e., the right number of connecting orbits (mod 2)) must be done in a systematic way so that the restrictions are satisfied simultaneously. Again the algebra indicates how to do this.

Lemma 4.3. *Let $s_1, \dots, s_m$ be the edges in Γ corresponding to rest points of index $k-1$. Then $\mathscr{S} = W^u(q) \cap B^-$ can be isotoped so that after the the isotopy there is transversality and for each i, $|\mathscr{S} \cap W^s(s_i)| \equiv \Delta(s, \gamma)$ (mod 2).*

Proof. Again if $k = 0$ there is nothing to prove, and if $k = 1$ the argument is given in [8]. So we assume $k > 1$ and first assume γ is untwisted. Let

$r_1, \dots, r_l$ be the rest points of index k between γ and $s_1, \dots, s_m$. We will show by induction on m that we can isotop $\mathscr{S}$ so that for each i, $|\mathscr{S} \cap W^s(s_i)| \equiv \Delta(s, \gamma) \pmod 2$. The case $m = 1$ is Lemma 4.2. So we assume that we've isotoped $\mathscr{S}$ so that the result holds for $j = 1, \dots, i-1$ and show that we can further isotop $\mathscr{S}$ so that the result holds for s_i. Let $I = \{s_1, \dots, s_i, r_1, \dots, r_l, \gamma\}$. Assumption A3 implies that I is an interval. The restriction $\Delta_k(I)$ has the form

$$\begin{array}{cc}
 & \begin{array}{ccccc} r_1 & \cdots & r_l & \quad & \gamma^k \end{array} \\
\begin{array}{c} s_1 \\ \vdots \\ \\ \vdots \\ s_{i-1} \\ s_i \end{array} &
\left(\begin{array}{cc} & \vdots \\ A & x \\ & \vdots \\ \hline \cdots \; y \; \cdots & * \end{array}\right)
\end{array}$$

Let B be the submatrix obtained by deleting the last column, i.e.,

$$B = \left(\begin{array}{c} A \\ \hline \cdots \; y \; \cdots \end{array}\right).$$

Suppose the l-vector $v \in \ker(A)$. If we isotop $\mathscr{S}$ as in Lemma 4.2 through each vertex r_t for which $v_t = 1$, then for each j, $|W^u(q) \cap B^- \cap W^s(s_j)|$ will change an even number of times since for each j, $\sum_{t=0}^{l} a_{jt} v_t = 0$ in $\mathbf{Z}_2$. We distinguish two cases.

Case 1. Rank(B) = rank(A) +1. Then there is a vector w such that $Aw = 0$ and $\sum_{t=0}^{l} y_t v_t = 1$. If we isotop $\mathscr{S}$ through the vertices r_t for which $w_t = 1$, then $|\mathscr{S} \cap W^s(s_j)|$ will be unchanged (mod 2) for $j = 1, \dots, i-1$ since $Aw = 0$, but $|\mathscr{S} \cap W^s(s_i)|$ will change by 1 (mod 2) since we isotop through an odd number of vertices with odd s_i incidence.

Case 2. Rank(B) = rank(A). We will show that the entry $\Delta(s_i, \gamma_k)$ is the same for any connection matrix. Choose the smallest collection of rows of A which sum to y, say rows $i_1, \dots, i_h$. These rows are linearly independent. Consider the interval

$$J = \{s_{i_1}, \dots, s_{i_h}, s_j, r_1, \dots, r_l, \gamma\}.$$

$\Delta_k(J)$ has the form

$$\begin{array}{c}
\begin{array}{ccccc} r_1 & \cdots & r_l & \quad & \gamma^k \end{array} \\
\left(\begin{array}{ccccc} a_{i_1 1} & \cdots & a_{i_1 l} & & x_{i_1} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i_h 1} & \cdots & a_{i_h l} & & x_{i_h} \\ y_1 & \cdots & y_l & & \Delta(s_j, \gamma) \end{array}\right)
\end{array}$$

and its rank is determined by the flow. The first h rows are linearly independent and the sum of each of the first l columns is 0, so

$$\operatorname{rank}(\Delta(J)) = \begin{cases} i_h, & \text{if } \Delta(s_j, \gamma^k) = \sum_{t=0}^{h} x_{i_t}, \\ i_{h+1}, & \text{if } \Delta(s_j, \gamma^k) \neq \sum_{t=0}^{h} x_{i_t}. \end{cases}$$

Since the rank of $\Delta(J)$ is determined by the flow, $\Delta(s_j, \gamma^k)$ must be the same for any connection matrix. After replacement by p and q, the corresponding interval

$$J' = \{s_{i_1}, \dots, s_{i_h}, s_j, r_1, \dots, r_l, q, p\}$$

has all maps counting connecting orbits since each set is a critical point. Since $H\Delta(J) \cong H\Delta(J')$, we must have $\Delta(s_j, q) = \Delta(s_j, \gamma^k)$ for any replacement, i.e., for any isotopy class of $\mathscr{S}$.

In either case we can isotop $\mathscr{S}$ to obtain the right number of connecting orbits (mod 2).

If γ is twisted, we fix a fundamental domain, do the isotopy in the fundamental domain, and make it equivariant. □

Again, there is an analogous statement for $W^s(p)$ the stable mainifold.

5. Proof of Theorem 2.4

In this section we complete the proof of the main result. Fix a collection $\mathscr{U}$ of neighborhoods of the periodic orbits, and assume A1–A3 are satisfied.

Proof of Theorem 2.4. Let Δ be an upper triangular boundary map satisfying

(1) if p and p' are adjacent, $p < p'$, then $\Delta(p, p') = \partial(p, p')$,
(2) for any interval $I \subset P$, $H\Delta(I) \cong CH(I)$.

If we replace a periodic orbit γ by two doubly connected rest points $\{p, q\}$, then Δ induces a map on the homologies of the Conley indices of the sets in the new Morse decomposition (with $CH(\gamma)$ replaced by $CH(p) \oplus CH(q)$) as noted in §2. We will show that there is a $\mathscr{U}$-refinement such that all of the maps in Δ induce flow defined boundary maps in the $\mathscr{U}$-refinement. It follows from Theorem 2.1 that Δ induces the unique connection matrix for the $\mathscr{U}$-refinement, which is the conclusion of the theorem. We proceed by induction on the number of periodic orbits.

If there is one periodic orbit, then Lemma 4.3 shows that we can make the refinement in this case. All maps involving the new critical points p and q count the number of connecting orbits (mod 2), hence they agree with the flow defined boundary maps and we are done in this case.

Now assume that for any Morse-Smale flow with $j-1$ periodic orbits, if Δ is an upper triangular boundary map satisfying properties 1 and 2, then for any $\mathscr{U}$ there is a $\mathscr{U}$-refinement such that Δ induces the unique connection matrix for the refinement. Let ϕ be a Morse-Smale flow with j periodic orbits and Δ be a connection matrix. Choose a periodic orbit γ and use the procedure of Lemma 4.3 to replace it with two points $\{p, q\}$ such that all maps $\Delta(s, q)$ and $\Delta(p, r)$ count the number of connecting orbits (mod 2) for points s of index Σ^{k-1} and r of index Σ^{k+2}. Suppose now that π is a periodic orbit of index $k-2$. If π and γ are adjacent, then the connection matrix for the interval $\{\pi, \gamma\}$ has the form

$$\begin{array}{c c} & \begin{array}{cccc} \pi^{k-2} & \pi^{k-1} & \gamma^k & \gamma^{k+1} \end{array} \\ \begin{array}{c} \pi^{k-2} \\ \pi^{k-1} \\ \gamma^k \\ \gamma^{k+1} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

where a is the only possible nonzero entry. If we do any replacement, then $\{\pi, q, p\}$ will be an interval with $CH(\{\pi, q, p\}) \cong CH(\{\pi, \gamma\})$ and the connection matrix having the same form. We must thus have the flow map $\partial(q, \pi) = a$, so Δ induces the correct map in this case. If π and γ are not adjacent, and $\Delta(\pi^{k-1}, \gamma^k) = 1$, then since $W^s(\pi) \cap B^-$ is a 2-manifold, we can isotop $\mathcal{S}$ so that it doesn't pass through a vertex and so that $\mathcal{S} \cap W^s(\pi) \neq \varnothing$. We do a similar isotopy, if necessary, for $W^s(p)$ if there are periodic orbits of index $k+2$ lying above γ in the partial order.

Now we have a flow with $j-1$ periodic orbits, and the induced map Δ satisfies $\Delta^2 = 0$. By construction, Δ is upper triangular, and Δ agrees with the flow defined boundary maps for adjacent Morse sets. Our induction will be complete if we can show $H\Delta(I) \cong CH(I)$ for any interval I in the new index set (i.e., γ replaced by p and q) with the flow defined partial order. So let I be an interval. If $\{p, q\} \subset I$ or $\{p, q\} \cap I = \varnothing$, then I is an interval in the original flow, so $H\Delta(I) \cong CH(I)$ in this case. So assume $q \in I$, $p \notin I$. $J = I \cup p$ is an interval since p and q are adjacent, and since I is an interval, p is maximal in J. It follows that

$$H\Delta_q(I) \cong H\Delta_q(J) \cong CH_q(J) \cong CH_q(I) \quad \text{for } q \neq k, k+1,$$

and $\Delta_{k+1}(J)$ has the form

$$\begin{array}{c|c} & \begin{matrix} r_1 & \cdots & r_l & p \end{matrix} \\ \begin{matrix} s_1 \\ \vdots \\ s_m \\ q \end{matrix} & \begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ * & \cdots & * & 0 \end{pmatrix} \end{array}$$

where $r_1, \ldots, r_l$ are critical points of index Σ^{k+1} and $s_1, \ldots, s_m$ are critical points of index Σ^k. Thus all of the entries in $\Delta_{k+1}(J)$ and in $\Delta_{k+1}(I)$ are flow defined maps and count the number of connections (mod 2). There are two possibilities. If the p column is in the span of the $r_1, \ldots, r_l$ columns, then

$$H\Delta_k(J) \cong H\Delta_k(I),$$
$$\dim(H\Delta_{k+1}(J)) = \dim(H\Delta_{k+1}(I)) + 1.$$

Since the maps in $H\Delta_k(J)$ are all flow defined, Theorem 2.1 implies that property 2 must hold for the homology, i.e.,

$$CH_k(J) \cong CH_k(I),$$
$$\dim(CH_k(J)) = \dim(CH_k(I)) + 1.$$

Similarly, if the p column is not in the span of the $r_1, \ldots, r_l$ columns, then

$$\dim(H\Delta_k(J)) = \dim(H\Delta_k(I)) - 1,$$
$$H\Delta_{k+1}(J) \cong H\Delta_{k+}(I),$$

and since the maps are flow defined, the same relations must hold for $CH(I)$ and $CH(J)$. Thus Δ satisfies properties 1 and 2 for the Morse decomposition with γ replaced by p and q. Now the refinement has $j-1$ periodic orbits, so by induction we can replace the remaining periodic orbits in such a way that the maps in Δ induce the flow defined maps in the final Morse decomposition, i.e., Δ induces the unique connection matrix for the refinement. $\square$

Theorem 2.4 says that for flows satisfying assumptions A1–A3, there is no unnecessary ambiguity in the set of connection matrices. Theorem 2.3 implies that every connection matrix which can be realized by a refinement must be a connection matrix for the original flow, and Theorem 2.4 implies that every connection matrix is realizable by a refinement.

References

1. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conf. Ser. in Math, no. 38, Amer. Math. Soc., Providence, R.I., 1978.
2. J. Franks, *Morse-Smale flows and homotopy theory*, Topology **18** (1979), 199–215.
3. R. Franzosa, *Index filtrations and the homology index braid for partially ordered Morse decompositions*, Trans. Amer. Math. Soc. **298** (1986), 193–213.
4. R. Franzosa, *The connection matrix theory for Morse decompositions*, Trans. Amer. Math. Soc. **311** (1989), 561–592.
5. M. Hirsch, *Differential topology*, Springer-Verlag, New York, 1976.
6. S. Newhouse and M. Peixoto, *There is a simple arc joining any two Morse-Smale flows*, Astérisque **31** (1976), 15–41.
7. C. McCord, *The connection map for attractor-repeller pairs*, Trans. Amer. Math. Soc. **308** (1988), 195–203.
8. J. Reineck, *The connection matrix in Morse-Smale flows*, Trans. Amer. Math. Soc. **322** (1990), 523–545.
9. D. Salamon, *Connected simple systems and the Conley index of isolated invariant sets*, Trans. Amer. Math. Soc. **291** (1985), 1–41.

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